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# Nonlinear Boundary Conditions in Kirchhoff-Love Plate Theory

Oana Iosifescu · Christian Licht · Gérard Michaille

**Abstract** Most of the derivations of the mechanical behavior of a plate as the limit behavior of a three-dimensional solid whose thickness tends to zero deal with stationary homogeneous linear boundary conditions on the lateral boundary. Here, in the framework of small strains, we rigorously determine a large class of steady-state or transient nonlinear boundary conditions which provide asymptotic kinematics of Kirchhoff-Love type.

**Keywords** Kirchhoff-Love plate theory · Nonlinear boundary conditions · Variational convergence · Trotter theory

**Mathematics Subject Classification (2000)** 74B05 · 74B20 · 74K20 · 35B40 · 74M10

## 1 Introduction

Most of the derivations of the mechanical behavior of a plate as the limit behavior of a three-dimensional solid whose thickness tends to zero deal with stationary homogeneous linear boundary conditions on the lateral boundary. Here, in the framework of small strains, we rigorously determine a large class of steady-state or transient nonlinear boundary conditions which provide asymptotic kinematics of Kirchhoff-Love type. We will first consider the equilibrium of a plate subject to body and surface forces, clamped on a part of its lateral boundary and subject to a nonlinear elastic mechanical constraint on the remaining part of

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its lateral boundary. We will deal with constraints deriving from smooth-enough surface energy densities or from confinement conditions. Next, we study the quasi-static evolution of the same plate when the previous constraints are replaced by a condition of nonlinear tangential friction with bilateral contact.

Our paper appears to be a justification, by means of rigorous asymptotic analysis (variational convergence) of unilateral phenomena in flat plate theory (see for instance [8]). Moreover, by taking into account the relative behavior of several small physical parameters, this asymptotic analysis supplies various models.

In the sequel, we will identify  $\mathbf{R}^3$  and the Euclidean physical space. For all  $\xi = (\xi_1, \xi_2, \xi_3)$  of  $\mathbf{R}^3$ ,  $\hat{\xi}$  stands for  $(\xi_1, \xi_2)$ . The Greek coordinate indices run in  $\{1, 2\}$  whereas the Latin ones run in  $\{1, 2, 3\}$ . The space of symmetric  $n \times n$  matrices is denoted by  $S^n$  and, for all  $e$  in  $S^3$ ,  $\hat{e}$  is the element of  $S^2$  such that  $\hat{e}_{\alpha\beta} = e_{\alpha\beta}$ . The Euclidean norm and the inner product in  $\mathbf{R}^n$  or  $S^n$  as well are denoted by the same symbols  $||$  and  $\cdot$  respectively;  $\text{tr}(e)$  stands for the trace of the matrix  $e$ . If  $V$  and  $W$  are two linear spaces, we denote by  $\mathcal{L}(V, W)$  the set of all linear mappings defined on  $V$  with values in  $W$ .

For any open subset  $G$  of  $\mathbf{R}^n$ ,  $H_0^1(G, \mathbf{R}^p)$  stands for the subspace of the Sobolev space  $H^1(G, \mathbf{R}^p)$  composed of the elements which vanish on  $\Gamma \subset \partial G$ . We will repeatedly use the

**Lemma 1** *Let  $G$  a domain of  $\mathbf{R}^3$  with a Lipschitz-continuous boundary  $\partial G$  and  $\Gamma$  a two-dimensional Hausdorff measurable subset of  $\partial G$ , then any bounded sequence in  $H^1(G, \mathbf{R}^3)$  contains a subsequence which converges pointwise almost everywhere in  $\Gamma$ .*

This is a straightforward consequence of the compactness of the trace mapping from  $H^1(G, \mathbf{R}^3)$  into  $L^2(\Gamma, \mathbf{R}^3)$  and the fact that any strongly convergent sequence in  $L^2(\Gamma, \mathbf{R}^3)$  contains a subsequence which pointwise converges almost everywhere in  $\Gamma$ .

If  $F$  is a subset of  $E$ , we denote the indicator function of  $F$  by  $I_F$ :

$$I_F(x) := \begin{cases} 0 & \text{if } x \in F, \\ +\infty & \text{if } x \in E \setminus F. \end{cases}$$

In the sequel,  $C$  denotes generic constants, which may vary from line to line.

## 2 The Static Case

### 2.1 Problem Setting

The reference configuration of the plate is the closure of a cylindrical domain  $\Omega^\varepsilon = \omega \times (-\varepsilon; \varepsilon)$  of  $\mathbf{R}^3$ , where  $\varepsilon$  is a small positive number and  $\omega$  a bounded open subset of  $\mathbf{R}^2$ , with a Lipschitz-continuous boundary  $\partial\omega$ . The plate is composed of a hyperelastic material with a strain energy density  $W^\varepsilon(x, e(v)(x))$ , where  $e(v)(x) := \frac{1}{2}(\nabla v(x) + (\nabla v(x))^T)$  is the linearized strain associated with the displacement field  $v$  at the point  $x$ . The plate is clamped on a part  $\Gamma_0^\varepsilon = \gamma_0 \times (-\varepsilon, \varepsilon)$  of its lateral boundary,  $\gamma_0 \subset \partial\omega$  being assumed of strictly positive length. It is also subjected to body forces of density  $f^\phi$  and surface forces of density  $g^\phi$  acting on  $\Gamma_1^\varepsilon = \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \cup \gamma_1 \times (-\varepsilon, \varepsilon)$ ,  $\Gamma_\pm^\varepsilon = \omega \times \{\pm\varepsilon\}$ ,  $\gamma_1 \subset \partial\omega$ . On the remaining part  $\Gamma_c^\varepsilon = \gamma_c \times (-\varepsilon, \varepsilon)$  of the lateral boundary  $\Gamma_{\text{lat}} = \partial\omega \times (-\varepsilon, \varepsilon)$  of  $\Omega^\varepsilon$ , the plate is subjected to an elastic mechanical constraint, whose energy density is  $h^\mu(v(x))$  or to a confinement condition.

Hence, determining the equilibrium configuration involves a problem  $\mathcal{P}^{(\varepsilon, \phi, \mu)}$  of minimization on a set of sufficiently smooth fields vanishing on  $\Gamma_0^\varepsilon$  of the total energy  $J^{(\varepsilon, \phi, \mu)}$ , which formally reads as:

$$J^{(\varepsilon, \phi, \mu)}(v) = \int_{\Omega^\varepsilon} W^\varepsilon(x, e(v)(x)) dx + \int_{\Gamma_c^\varepsilon} h^\mu(v(x)) ds - \int_{\Omega^\varepsilon} f^\phi(x) \cdot v(x) dx - \int_{\Gamma_1^\varepsilon} g^\phi(x) \cdot v(x) ds. \quad (1)$$

With standard assumptions on the data, such a problem has a solution  $u^{(\varepsilon, \phi, \mu)}$  in a suitable Sobolev space. When  $h^\mu \equiv 0$  or  $\gamma_c$  is empty we then have the starting point of a mathematical justification of the Kirchoff-Love theory of thin elastic plates by a rigorous study of the asymptotic behavior of  $u^{(\varepsilon, \phi)}$  when  $(\varepsilon, \phi)$ , *considered as parameters*, tend to a limit, say  $(0, 0)$ .

We consider three cases of mechanical constraints:

Case 1: the constraint derives from a smooth energy density  $h^\mu(\xi) = \mu h(\xi)$ ;

Case 2: the constraint is a confinement condition;

Case 3: a mixing of cases 1 and 2, which separates, as in many practical situations, the roles played by the normal and tangential components of displacement along  $\Gamma_c$ .

We will impose some slightly more general assumptions on the data  $W^\varepsilon$ ,  $f^\phi$ ,  $g^\phi$  than those used in the classical mathematical derivation of the Kirchhoff-Love theory (see [7]). To formulate these hypotheses and to carry out the following analysis it is convenient to perform a change of coordinates which corresponds to a  $1/\varepsilon$  dilation in the direction normal to the plate:

$$x = (\hat{x}, x_3) \in \overline{\Omega} := \overline{\omega \times (-1, 1)} \mapsto \Pi^\varepsilon(x) := (\hat{x}, \varepsilon x_3) \in \overline{\Omega^\varepsilon}.$$

In the sequel we drop the upper index  $\varepsilon$  for all the images by  $(\Pi^\varepsilon)^{-1}$  of the previous geometric set (e.g.,  $\Gamma_+ = (\Pi^\varepsilon)^{-1}\Gamma_+^\varepsilon$ ).

*Assumptions on  $W^\varepsilon$* : there exists  $W$  such that

1.  $W^\varepsilon(\Pi^\varepsilon x, e) = W(x, e)$ ,  $\forall x \in \Omega$ ,  $\forall e \in S^3$ ,  $\forall \varepsilon \in (0, \varepsilon_0]$ ;
2.  $W(\cdot, e)$  is measurable on  $\Omega$ ;
3.  $W(x, \cdot)$  is strictly convex and positively homogeneous of degree 2 on  $S^3$ , for almost every  $x$  in  $\Omega$ ;
4.  $\exists \alpha, \beta > 0$  such that  $\alpha|e|^2 \leq W(x, e) \leq \beta|e|^2$ ,  $\forall e \in S^3$ , a.e.  $x \in \Omega$ .

Note that these assumptions imply the existence of  $\gamma > 0$  such that

$$|W(x, e) - W(x, e')| \leq \gamma|e - e'|(|e| + |e'|), \quad \forall e, e' \in S^3, \text{ a.e. } x \text{ in } \Omega. \quad (2)$$

Actually, we note (see also [9, 11, 12]) that neither space homogeneity nor linear isotropy is necessary to justify the Kirchhoff-Love theory. This is not a gratuitous mathematical generalization: bulk energy densities satisfying the previous conditions (1–4) account for materials with different behavior in traction and compression. They can be obtained (see [2, 13, pp. 109–117]) by homogenization of micro-cracked linearly elastic media and are good representatives of the macroscopic behavior of concrete, a major component of the plates used in Civil Engineering.

Assumptions on  $f^\phi, g^\phi$ : there exists  $(f, g)$  in  $L^2((\Omega, \mathbf{R}^3)) \times L^2(\Gamma_1, \mathbf{R}^3)$  such that

$$\begin{cases} \hat{f}^\phi(\Pi^\varepsilon x) = \phi \hat{f}(x), & f_3^\phi(\Pi^\varepsilon x) = \varepsilon \phi f_3(x), & \forall x \in \Omega, \\ \hat{g}^\phi(\Pi^\varepsilon x) = \varepsilon \phi \hat{g}(x), & g_3^\phi(\Pi^\varepsilon x) = \varepsilon^2 \phi g_3(x), & \forall x \in \Gamma_1 \cap \Gamma_\pm, \\ \hat{g}^\phi(\Pi^\varepsilon x) = \phi \hat{g}(x), & g_3^\phi(\Pi^\varepsilon x) = \varepsilon \phi g_3(x), & \forall x \in \Gamma_1 \cap \Gamma_{\text{lat}}. \end{cases} \quad (3)$$

We emphasize that the magnitude of the ratio of the normal components of the forces to the tangential components, which is of the same order as the plate thickness, plays a capital role in obtaining asymptotic kinematics of Kirchhoff-Love type.

## 2.2 First Case

We first assume that  $h^\mu = \mu h$ ,  $\mu \in [0, +\infty)$  and that  $h$  is at least lower-semi-continuous on  $\mathbf{R}^3$  with values in  $[h_m, +\infty)$ ,  $h_m > -\infty$ . Proceeding to a change of unknowns (the so-called scaling operation  $S(\varepsilon, \phi)$ )

$$\widehat{S(\varepsilon, \phi)v}(x) = (1/\phi)\hat{v}(\Pi^\varepsilon x), \quad (S(\varepsilon, \phi)v)_3(x) = (\varepsilon/\phi)v_3(\Pi^\varepsilon x), \quad \forall x \in \Omega, \quad (4)$$

it is clear that considering the problem  $\mathcal{P}^{(\varepsilon, \phi, \mu)}$  is equivalent to considering the problem

$$\mathcal{P}(\varepsilon, \phi, \mu) = \text{Min}\{J(\varepsilon, \phi, \mu)(v); v \in H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)\}$$

with

$$\left\{ \begin{array}{l} J(\varepsilon, \phi, \mu) = \mathcal{W}(\varepsilon) + \mathcal{H}(\varepsilon, \phi, \mu) - L, \\ \mathcal{W}(\varepsilon)(v) = \int_{\Omega} W(x, e(\varepsilon, v)(x)) dx, \\ \hat{e}(\varepsilon, v) = \hat{e}(v), \quad e_{\alpha 3}(\varepsilon, v) = e_{3\alpha}(\varepsilon, v) = \varepsilon^{-1} e_{\alpha 3}(v), \quad e_{33}(\varepsilon, v) = \varepsilon^{-2} e_{33}(v), \\ \mathcal{H}(\varepsilon, \phi, \mu)(v) = \begin{cases} \frac{\mu}{\phi^2} \int_{\Gamma_c} h(\phi \hat{v}, \frac{\phi}{\varepsilon} v_3) ds \\ \text{if } x \mapsto h(\phi \hat{v}(x), \frac{\phi}{\varepsilon} v_3(x)) \in L^1(\Gamma_c) \\ +\infty \text{ otherwise,} \end{cases} \\ L(v) = \int_{\Omega} f \cdot v dx + \int_{\Gamma_1} g \cdot v ds. \end{array} \right. \quad (5)$$

The previous assumptions, the Korn inequality and Lemma 1 imply that  $J(\varepsilon, \phi, \mu)$  is coercive and weakly sequentially l.s.c. in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$ . Consequently,  $\mathcal{P}(\varepsilon, \phi, \mu)$  has at least one solution  $u(\varepsilon, \phi, \mu)$ . The major interest of introducing  $\Pi^\varepsilon$  and  $S(\varepsilon, \phi)$  is that, under an assumption like  $h(0) = 0$  or  $h(0)$  finite and  $\mu/\phi^2$  bounded which implies  $J(\varepsilon, \phi, \mu)(u(\varepsilon, \phi, \mu)) \leq J(\varepsilon, \phi, \mu)(0) \leq C$ ,  $u(\varepsilon, \phi, \mu)$  is bounded in the Sobolev space  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  which does not depend on  $(\varepsilon, \phi, \mu)$ . Hence the sequence  $u(\varepsilon, \phi, \mu)$  has weak- $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  cluster points  $\bar{u}$  and the aim of the next subsection is to identify the kind of variational problems  $\bar{u}$  solves. Thus the asymptotic behavior of  $u^{(\varepsilon, \phi, \mu)}$  will be obtained through the de-scaling  $S(\varepsilon, \phi)^{-1}$  of  $\bar{u}$ .

Actually, a comprehensive study of the problem, which involves a triple  $(\varepsilon, \phi, \mu)$  of parameters and the structure of  $h$ , is complex and tedious. For simplicity, we will first consider the case when  $\phi = \varepsilon$  and, in a second step, the general case when in addition  $h$  is assumed to be  $p$ -positively homogeneous ( $1 \leq p < +\infty$ ). It is worthwhile to note that this last assumption is true in many practical cases and that the surface constraint energy densities supplied by the mathematical modeling of adhesive bonding joints (see [10]) are convex and  $p$ -positively homogeneous functions.

### 2.2.1 The Case $\phi = \varepsilon$

#### (a) A convergence result

Let  $p$  be in  $[1, +\infty)$ . We assume that  $\mu/\varepsilon^2$  and  $\mu/\varepsilon^{2-p}$  have limits  $\bar{\mu}$  and  $\bar{\bar{\mu}}$  in  $[0, +\infty]$ , when  $s := (\varepsilon, \mu)$  tends to  $\bar{s} \in \{0\} \times [0, +\infty]$ . In addition to  $h$  being lower semi-continuous in  $\mathbf{R}^3$ , we make three various assumptions on  $h$  that will allow us to treat the subsequent cases, indexed by  $I \in \{1, 2, 3\}$ , in a unified way:

$$\left\{ \begin{array}{l} \mathbf{H}_1: (\bar{\mu}, \bar{\bar{\mu}}) \in [0, +\infty) \times \{0\}, \text{ (equivalent to } \mu \sim O(\varepsilon^2), \text{ if } \bar{\mu} \neq 0) \\ \quad h \in \mathbf{C}^0(\mathbf{R}; [h_m, +\infty)), \text{ where } -\infty < h_m < +\infty; \\ \mathbf{H}_2: (\bar{\mu}, \bar{\bar{\mu}}) \in \{+\infty\} \times [0, +\infty), \text{ (equivalent to } \mu \sim O(\varepsilon^{2-p}), \text{ if } \bar{\bar{\mu}} \neq 0), \\ \quad \exists \delta_1, \delta_2 > 0, \exists q \geq 1 \text{ such that } \delta_1 |a_3|^q \leq h(a) \leq \delta_2 |a|^p, \forall a \in \mathbf{R}^3, \\ \quad h(a) \geq h(\hat{a}, 0) \forall a \in \mathbf{R}^3, \\ \quad \forall \hat{a} \in \mathbf{R}^2, \forall (\hat{a}_\varepsilon)_{\varepsilon>0} \subset \mathbf{R}^2 \text{ with } \hat{a}_\varepsilon \rightarrow \hat{a}, \exists h^{0,p}(\hat{a}) \text{ such that} \\ \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-p} h(\varepsilon \hat{a}_\varepsilon, 0) = h^{0,p}(\hat{a}); \\ \mathbf{H}_3: (\bar{\mu}, \bar{\bar{\mu}}) \in \{+\infty\} \times \{+\infty\}, \text{ (equivalent to } \mu^{-1} \sim o(\varepsilon^{p-2}), \\ \quad h(0) = 0 \text{ and } \exists \delta > 0, \exists q \geq 1 \text{ such that } h(a) \geq \delta(|\hat{a}|^p + |a_3|^q), \forall a \in \mathbf{R}^3. \end{array} \right.$$

Note that  $h^{0,p}$  is necessarily  $p$ -positively homogeneous. In this section we denote  $\mathcal{P}(\varepsilon, \varepsilon, \mu)$ ,  $\mathcal{H}(\varepsilon, \varepsilon, \mu)$ ,  $J(\varepsilon, \varepsilon, \mu)$  by  $\mathcal{P}(s)$ ,  $\mathcal{H}(s)$ ,  $J(s)$  respectively. The expected asymptotic kinematics being of Kirchhoff- Love type, we introduce the space:

$$V_{KL} = \{v \in H_{\Gamma_0}^1(\Omega, \mathbf{R}^3); e_{i3}(v) = 0\} \quad (6)$$

which has an equivalent characterization (see [7]):

$$\begin{aligned} V_{KL} = \{v \in H_{\Gamma_0}^1(\Omega, \mathbf{R}^3); \exists (v^M, v^F) \in H_{\gamma_0}^1(\omega, \mathbf{R}^2) \times H_{\gamma_0}^2(\omega, \mathbf{R}) \\ \text{such that } \hat{v} = v^M - x_3 \hat{\nabla} v^F, v_3 = v^F\}. \end{aligned} \quad (7)$$

For almost every  $x$  in  $\mathbf{R}$ , the fact that  $W(x, \cdot)$  is strictly convex and coercive on  $S^3$  implies that the function

$$q \in S^2 \mapsto W_{KL}(x, q) := \text{Min}\{W(x, e); e \in S^3, \hat{e} = q\} \quad (8)$$

is a strictly convex, positively homogeneous of degree 2 and well defined function on  $S^2$ , which satisfies growth and continuity properties similar to those of  $W$ :

$$\left\{ \begin{array}{l} \alpha |q|^2 \leq W_{KL}(x, q) \leq \beta |q|^2 \quad \forall q \in S^2, \text{ a.e. } x \text{ in } \Omega \\ |W_{KL}(x, q) - W_{KL}(x, q')| \leq \gamma |q - q'| (|q| + |q'|) \quad \forall q, q' \in S^2 \text{ a.e. } x \text{ in } \Omega. \end{array} \right. \quad (9)$$

The limit total energy will involve the following functionals defined on  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  by:

$$\left\{ \begin{array}{l} \mathcal{W}_{KL}(v) = \int_{\Omega} W_{KL}(x, \hat{e}(v)(x)) dx; \\ \mathcal{H}^1(v) = \begin{cases} \bar{\mu} \int_{\Gamma_c} h(0, v_3) ds = 2\bar{\mu} \int_{\gamma_c} h(0, v^F) dl, & \text{if } v \in V_{KL}, \\ +\infty & \text{otherwise;} \end{cases} \\ \mathcal{H}^2(v) = \begin{cases} 2\bar{\bar{\mu}} \int_{\gamma_c} h^{0,p}(v^M) dl + I_{\{v^F=0 \text{ on } \gamma_c\}}, & \text{if } v \in V_{KL}, \\ +\infty & \text{otherwise;} \end{cases} \\ \mathcal{H}^3(v) = I_{\{v \in V_{KL}, v=0 \text{ on } \Gamma_c\}}; \\ J^I = \mathcal{W}_{KL} + \mathcal{H}^I - L \quad \text{with } I \in \{1, 2, 3\}. \end{array} \right. \quad (10)$$

More precisely (for notational brevity, from now on subsequences will be denoted like the sequences they are extracted from):

**Theorem 1** *The set of the solutions  $u(s)$  of  $\mathcal{P}(s)$  is sequentially weakly relatively compact in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$ . If  $\bar{u}^I$  is a cluster point, there exists a subsequence such that  $J^I(\bar{u}^I) = \lim_{s \rightarrow \bar{s}} J(s)(u(s))$ ,  $u(s)$  strongly converges in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  toward  $\bar{u}^I$  and  $\bar{u}^I$  solves the problem:*

$$\bar{\mathcal{P}}^I = \text{Min}\{J^I(v); v \in V_{KL}\}.$$

This variational convergence result is a consequence of the following three propositions:

**Proposition 1** *Any sequence in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  such that  $J(s)(u(s)) \leq C$  contains a subsequence which weakly converges in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  to some  $u$ . Moreover  $u$  belongs to  $V_{KL}$  and  $e_{i3}(u(s))$  strongly converges to 0 in  $L^2(\Omega)$ .*

*Proof* This is an obvious consequence of the Korn inequality and the fact that  $h$  is bounded from below.  $\square$

**Proposition 2** *For all  $u$  in  $V_{KL}$  and for every sequence  $u(s)$  in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  such that  $u(s)$  weakly converges to  $u$  we have:*

- (i)  $\mathcal{W}_{KL}(u) \leq \liminf_{s \rightarrow \bar{s}} \int_{\Omega} W(x, e(\varepsilon, u(s))(x)) dx$ ;
- (ii)  $\mathcal{H}^I(u) \leq \liminf_{s \rightarrow \bar{s}} \mathcal{H}(s)(u(s))$ ;
- (iii)  $J^I(u) \leq \liminf_{s \rightarrow \bar{s}} J(s)(u(s))$ .

*Proof* The definition (8) of  $W_{KL}$  implies

$$\begin{aligned} \liminf_{s \rightarrow \bar{s}} \int_{\Omega} W(x, e(\varepsilon, u(s))(x)) dx &\geq \liminf_{s \rightarrow \bar{s}} \int_{\Omega} W_{KL}(x, \hat{e}(u(s))(x)) dx \\ &\geq \int_{\Omega} W_{KL}(x, \hat{e}(u)(x)) dx, \end{aligned}$$

while (ii) is a consequence of Lemma 1, Fatou's lemma and of the coercivity assumption in  $\mathbf{H}_2$  and  $\mathbf{H}_3$ . The last point is a trivial consequence of (i) and (ii).  $\square$

**Proposition 3** For all  $u$  in  $V_{KL}$ , there exists a sequence  $u(s)$  in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  which weakly converges in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  toward  $u$  and such that:

- (i)  $\limsup_{s \rightarrow \bar{s}} \int_{\Omega} W(x, e(\varepsilon, u(s))(x)) dx = \int_{\Omega} W_{KL}(x, \hat{e}(u)(x)) dx$ ;
- (ii)  $\limsup_{s \rightarrow \bar{s}} \mathcal{H}(s)(u(s)) \leq \mathcal{H}^I(u)$ ;
- (iii)  $\limsup_{s \rightarrow \bar{s}} J(s)(u(s)) \leq J^I(u)$ .

*Proof*  $W(\cdot, e)$  being measurable and  $W(x, \cdot)$  being strictly convex with quadratic growth implies that there exists  $q \in L^2(\Omega, S^3)$  such that  $\hat{q} = \hat{e}(u)$ ,  $W_{KL}(x, \hat{e}(u)(x)) = W(x, q(x))$ , a.e.  $x \in \Omega$  (cf. [6]). Consequently, there exists  $q(s)$  in  $\mathcal{D}(\Omega, S^3)$  such that

$$\int_{\Omega} |q(s) - q|^2 + |W(x, q(s)) - W(x, q)| dx \leq C\varepsilon^2. \quad (11)$$

The field  $v(s)$  defined by

$$v(s)_3(x) = \varepsilon^2 \int_0^{x_3} q(s)_{33}(\hat{x}, \tau) d\tau,$$

$$v(s)_\alpha(x) = 2\varepsilon \left[ \int_0^{x_3} \left( q(s)_{\alpha 3}(\hat{x}, t) - \frac{\varepsilon}{2} \int_0^t \partial_\alpha q(s)_{33}(\hat{x}, \tau) d\tau \right) dt \right],$$

belongs to  $H_{\Gamma_0 \cup \Gamma_c}^1(\Omega, \mathbf{R}^3)$  and  $u(s) := u + v(s)$  satisfies

$$u(s) \in H_{\Gamma_0}^1(\Omega, \mathbf{R}^3), \quad |e(\varepsilon, u(s)) - q(s)|_{L^2(\Omega, S^3)} \leq C\varepsilon.$$

Indeed from the definition of  $v(s)$ , one easily checks that  $e_{3i}(\varepsilon, u(s)) = q_{3i}(s)$  and from (11) that  $|\hat{e}(\varepsilon, u(s)) - q(s)|_{L^2(\Omega, S^3)} \leq C\varepsilon$ . Hence,  $u(s)$  strongly converges in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  toward  $u$  and

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} \int_{\Omega} W(x, e(\varepsilon, u(s))(x)) dx &= \lim_{s \rightarrow \bar{s}} \int_{\Omega} W(x, q(s)(x)) dx \\ &= \int_{\Omega} W(x, q(x)) dx = \int_{\Omega} W_{KL}(x, \hat{e}(u)(x)) dx. \end{aligned}$$

To establish (ii) and, consequently, (iii) we first note that when  $I = 3$  (ii) is true because  $\mathcal{H}(s)(s) = \frac{\mu}{\varepsilon^2} |\Gamma_c| h(x, 0) = 0$ . The Sobolev embeddings imply that for any  $u$  in  $V_{KL}$ ,  $u^F$  belongs to  $C^0(\overline{\gamma_c})$  and  $u^M$  belongs to  $L^r(\gamma_c, \mathbf{R}^2)$  for all finite  $r$ . Hence according to the Lebesgue dominated convergence theorem, (ii) is true when  $I = 2$  and when  $I = 1$  with, moreover,  $u^M$  smooth. In the last case the proof is completed by a diagonalization argument [2] taking into account the continuity of  $\mathcal{H}^I$  on  $V_{KL}$ .  $\square$

*Proof of Theorem 1* Since  $J(s)(u(s)) \leq J(s)(0) = |\Gamma_c| \frac{\mu}{\phi^2} h(0)$ , a classical combination (see [1] for a detailed proof) of the previous three propositions shows that a subsequence  $u(s)$  weakly converges in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  toward a minimizer  $\bar{u}^I$  in  $V_{KL}$  of the coercive sequential weakly lower semicontinuous functional  $J^I$  and that  $\lim_{s \rightarrow 0} J(s)(u(s)) = J^I(\bar{u}^I)$ . Furthermore, Propositions 2 and 3 imply

$$\begin{aligned} \limsup_{s \rightarrow \bar{s}} \mathcal{W}(\varepsilon)(u(s)) &\leq \lim_{s \rightarrow \bar{s}} J(s)(u(s)) + L(u) - \liminf_{s \rightarrow \bar{s}} \mathcal{H}(s)(u(s)) \\ &\leq \mathcal{W}_{KL}(\bar{u}^I) \leq \liminf_{s \rightarrow \bar{s}} \mathcal{W}(\varepsilon)(u(s)), \end{aligned}$$

$$\begin{aligned} \limsup_{s \rightarrow \bar{s}} \mathcal{H}(\varepsilon)(u(s)) &\leq \lim_{s \rightarrow \bar{s}} J(s)(u(s)) + L(u) - \liminf_{s \rightarrow \bar{s}} \mathcal{W}(\varepsilon)(u(s)) \\ &\leq \mathcal{H}^I(\bar{u}^I) \leq \liminf_{s \rightarrow \bar{s}} \mathcal{H}(s)(u(s)) \end{aligned}$$

and

$$\limsup_{s \rightarrow \bar{s}} \mathcal{W}_{KL}(u(s)) \leq \limsup_{s \rightarrow \bar{s}} \mathcal{W}(\varepsilon)(u(s)) = \mathcal{W}_{KL}(\bar{u}^I) \leq \liminf_{s \rightarrow \bar{s}} \mathcal{W}_{KL}(u(s)),$$

because of the continuity and convexity of  $\mathcal{W}_{KL}$  on  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$ . From Proposition 1 and the Korn inequality, it remains to establish that  $\hat{e}(u(s))$  strongly converges toward  $\hat{e}(\bar{u}^I)$  in  $L^2((\Omega, S^2))$ . This is a consequence of the weak convergence in  $L^2((\Omega, S^2))$  of  $\hat{e}(u(s))$  toward  $\hat{e}(\bar{u}^I)$  and the convergence:

$$\int_{\Omega} W_{KL}(x, \hat{e}(u(s))(x)) dx \rightarrow \int_{\Omega} W_{KL}(x, \hat{e}(\bar{u}^I)(x)) dx,$$

the function  $W_{KL}(x, \cdot)$  being strictly convex on  $S^2$  and positively homogeneous of degree 2 (see for instance [3] Theorem 4.9, where such an argument is used).  $\square$

### (b) An asymptotic model of behavior for a plate of thickness $2\varepsilon$

Clearly,  $\bar{u}^{I,\varepsilon} := S(\varepsilon, \varepsilon)^{-1} \bar{u}^I$  solves the problem

$$\mathcal{P}^{I,\varepsilon} = \text{Min}\{J^{I,\varepsilon}(v); v \in V_{KL}^{I,\varepsilon}\}$$

with

$$\left\{ \begin{array}{l} V_{KL}^{\varepsilon} = \{v \in H_{\Gamma_0}^1(\Omega^{\varepsilon}, \mathbf{R}^3), e_{i3}(v) = 0\} \quad \text{and} \quad V_{KL}^{1,\varepsilon} = V_{KL}^{\varepsilon}, \\ V_{KL}^{2,\varepsilon} = \{v \in V_{KL}^{\varepsilon}; v^F = 0 \text{ on } \gamma_c\}, \quad V_{KL}^{3,\varepsilon} = \{v \in V_{KL}^{\varepsilon}; v = 0 \text{ on } \Gamma_c^{\varepsilon}\}; \\ \mathcal{H}^{1,\varepsilon}(v) = 2\bar{\mu}\varepsilon^3 \int_{\gamma_c} h(0, v^F) dl, \quad \mathcal{H}^{2,\varepsilon}(v) = 2\bar{\mu}\varepsilon^{3-p} \int_{\gamma_c} h^{0,p}(v^M) dl, \\ \mathcal{H}^{3,\varepsilon}(v) = 0; \\ J^{I,\varepsilon}(v) = \int_{\Omega^{\varepsilon}} W_{KL}(x, \hat{e}(v)) dx + \mathcal{H}^{I,\varepsilon}(v) - \int_{\Omega^{\varepsilon}} f^{\varepsilon} \cdot v dx - \int_{\Gamma_1^{\varepsilon}} g^{\varepsilon} \cdot v ds \end{array} \right. \quad (12)$$

where, of course  $v^F, v^M$  are the flexural and membrane parts of  $v$  defined in a similar way to (7). Moreover  $\bar{u}^{I,\varepsilon}$  yields the asymptotic behavior of  $u^{(\varepsilon,\varepsilon,\mu)}$  because Theorem 1 implies:

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} \left( \varepsilon^{-3} \int_{\Omega^{\varepsilon}} |\hat{e}(\bar{u}^{I,\varepsilon}) - \hat{e}(u^{(\varepsilon,\varepsilon,\mu)})|^2 dx \right) &= 0, \\ \varepsilon^{-3} \int_{\Omega^{\varepsilon}} |e_{i3}(u^{(\varepsilon,\varepsilon,\mu)})|^2 dx &\text{ bounded.} \end{aligned}$$

We note that  $\varepsilon^3$  is the order of magnitude of  $J^{I,\varepsilon}(\bar{u}^{I,\varepsilon})$  if it differs from zero. Hence,  $\mathcal{P}^{I,\varepsilon}$  provides a simpler model than  $\mathcal{P}^{(\varepsilon,\varepsilon,\mu)}$ , because it involves displacement fields of Kirchhoff-Love type, but of the same accuracy.

The asymptotic constraint is of pure adhesion when  $I = 3$ , it is in plane elastic pull back with no flexion when  $I = 2$  and it is elastic pull back when  $I = 1$ .

## 2.2.2 The Case when $h$ is $p$ -Positively Homogeneous

Here we denote the triple  $(\varepsilon, \phi, \mu)$  of effective parameters by  $s$  and we assume that it has a limit  $\bar{s} \in \{0\} \times [0, +\infty)^2$ . In that case,  $\mathcal{H}(\varepsilon, \phi, \mu)(v)$  reads as:

$$\mathcal{H}(s)(v) = \begin{cases} (\mu/\phi^{2-p}\varepsilon^p) \int_{\Gamma_c} h(\varepsilon \hat{v}(x), v_3(x)) ds \\ \text{if } x \mapsto h(\varepsilon \hat{v}(x), v_3(x)) \in L^1(\Gamma_c), \\ +\infty \text{ otherwise,} \end{cases} \quad (13)$$

and we summarize the assumptions as follows: let  $p \in [1, +\infty)$ ; the function  $h$  is  $p$ -positively homogeneous and lower semi continuous on  $\mathbf{R}^3$  with values in  $[h_m, +\infty)$ ,  $h_m \in \mathbf{R}$ ; when  $s \rightarrow \bar{s}$ ,  $\mu/\phi^{2-p}\varepsilon^p$  and  $\mu/\phi^{2-p}$  have limits  $\bar{\mu}$ ,  $\bar{\mu}$  in  $[0, +\infty]$ , and

case  $I = 1$ :  $(\bar{\mu}, \bar{\mu}) \in [0, +\infty) \times \{0\}$ ,  $h$  is continuous on  $\mathbf{R}^3$ ;

case  $I = 2$ :  $(\bar{\mu}, \bar{\mu}) \in \{+\infty\} \times [0, +\infty)$ ,

$$\exists \delta_1, \delta_2 > 0 : \delta_1 |a_3|^p \leq h(a) \leq \delta_2 |a|^p, \quad \forall a \in \mathbf{R}^3$$

$$h(a) \geq h(\hat{a}, 0) \quad \forall a \in \mathbf{R}^3, \quad h(\cdot, 0) \text{ is continuous in } \mathbf{R}^2;$$

case  $I = 3$ :  $(\bar{\mu}, \bar{\mu}) \in \{+\infty\} \times \{+\infty\}$ ,  $\exists \delta > 0 : h(a) \geq \delta |a|^p, \quad \forall a \in \mathbf{R}^3$ .

Since the  $h^{0,p}$  function, introduced in Sect. 2.2.1, reduces to  $h(\cdot, 0)$ , it is clear that the analysis done there remains valid. Thus the statement of Theorem 1 is still true with the new notations.

As a matter of fact, the “de-scaled” field  $u^{I(\varepsilon, \phi)} := S(\varepsilon, \phi)^{-1} \bar{u}^I$  solves a problem:

$$\mathcal{P}^{I(\varepsilon, \phi)} = \text{Min}\{J^{I(\varepsilon, \phi)}(v); v \in V_{KL}^{I, \varepsilon}\}.$$

This yields a simplified but accurate model of behavior of the linearly elastic plate with thickness  $2\varepsilon$ , the effective total energy and accuracy being:

$$\left\{ \begin{array}{l} J^{I(\varepsilon, \phi)}(v) = \int_{\Omega^\varepsilon} W_{KL}(x, \hat{\varepsilon}(\varepsilon, v(x))) dx \\ \quad + \mathcal{H}^{I(\varepsilon, \phi)}(v) - \int_{\Omega^\varepsilon} f^\phi \cdot v dx - \int_{\Gamma_1^\varepsilon} g^\phi \cdot v ds; \\ \mathcal{H}^{1(\varepsilon, \phi)}(v) = 2\varepsilon \bar{\mu} \varepsilon^p \phi^{2-p} \int_{\gamma_c} h(0, v^F(\hat{x})) dl, \\ \mathcal{H}^{2(\varepsilon, \phi)}(v) = 2\varepsilon \bar{\mu} \phi^{2-p} \int_{\gamma_c} h(v^M(\hat{x}), 0) dl, \quad \mathcal{H}^{3(\varepsilon, \phi)}(v) = 0; \\ \lim_{s \rightarrow \bar{s}} \frac{1}{\varepsilon \phi^2} \left( \int_{\Omega^\varepsilon} |\hat{\varepsilon}(u^{I(\varepsilon, \phi)} - u^{(\varepsilon, \phi, \mu)})|^2 dx \right) = 0; \\ \frac{1}{\varepsilon \phi^2} \int_{\Omega^\varepsilon} |e_{i3}(u^{(\varepsilon, \phi, \mu)})|^2 dx \text{ bounded.} \end{array} \right. \quad (14)$$

We note that  $\varepsilon \phi^2$  is the order of magnitude of  $J^{I(\varepsilon, \phi)}(u^{I(\varepsilon, \phi)})$  if it does not vanish.

## 2.3 A Confinement Constraint

Let  $a, b, c$  and  $d$  be four real numbers such that  $a \leq b, 0 \leq c \leq d$ . Let  $I = [a, b]$  and  $\mathcal{K} = dK \setminus c\hat{K}$  where  $K$  is a compact subset of  $\mathbf{R}^2$  starshaped with respect to the origin of coordinates. If  $\lambda, \nu$  are two positive parameters, the confinement condition reads as:

$$v(x) \in \lambda \mathcal{K} \times \nu I \quad \text{a.e. } x \in \Gamma_c^\varepsilon. \quad (15)$$

Thus determining equilibrium configurations involves the problem:

$$\begin{aligned} \mathcal{P}^{(\varepsilon, \phi, \lambda, \nu)} = \text{Min} \left\{ \int_{\Omega^\varepsilon} W^\varepsilon(x, e(v)(x)) dx - \int_{\Omega^\varepsilon} f^\phi(x) \cdot v(x) dx \right. \\ \left. - \int_{\Gamma_1^\varepsilon} g^\phi(x) \cdot v(x) ds; v \in H_{\Gamma_0^\varepsilon}^1(\Omega^\varepsilon, \mathbf{R}^3), \right. \\ \left. v \in \lambda \mathcal{K} \times \nu I \text{ a.e. } x \text{ in } \Gamma_c^\varepsilon \right\}. \end{aligned} \quad (16)$$

Lemma 1 implies that the set  $\{v \in H_{\Gamma_0^\varepsilon}^1(\Omega^\varepsilon, \mathbf{R}^3); v \in \lambda \mathcal{K} \times \nu I \text{ a.e. } x \text{ in } \Gamma_c^\varepsilon\}$  is a sequential weakly closed subset of  $H_{\Gamma_0^\varepsilon}^1(\Omega^\varepsilon, \mathbf{R}^3)$ . Hence,  $\mathcal{P}^{(\varepsilon, \phi, \lambda, \nu)}$  has at least one solution  $u^{(\varepsilon, \phi, \lambda, \nu)}$ . To determine its behavior, when the quadruple  $s = (\varepsilon, \phi, \lambda, \nu)$  of parameters tends to some  $\bar{s} \in \{0\} \times [0, +\infty)^3$ , we assume that  $(\frac{\lambda}{\phi}, (\frac{\nu}{\phi})v)$  has a limit  $(\bar{\lambda}, \bar{\nu})$  in  $[0, +\infty)^2$ . Moreover, if  $\bar{\lambda} = 0$  or  $\bar{\nu} = 0$  we assume that the distance  $(\gamma_0, \gamma_c) > 0$  and  $\lim_{s \rightarrow \bar{s}} \frac{\nu}{\phi} = 0$  when  $\bar{\nu} = 0$ .

### 2.3.1 A Convergence Result

Using the scaling operator  $S(\varepsilon, \phi)$  defined in Sect. 2.2.2, we have that  $u(s) := S(\varepsilon, \phi) \times u^{(\varepsilon, \phi, \lambda, \nu)}$  solves:

$$\begin{aligned} \mathcal{P}(s) = \text{Min}\{J(s)(v); v \in \Lambda(s)\}, \\ \Lambda(s) := \left\{ v \in H_{\Gamma_0^\varepsilon}^1(\Omega^\varepsilon, \mathbf{R}^3), v \in \frac{\lambda}{\phi} \mathcal{K} \times \nu \left(\frac{\varepsilon}{\phi}\right) I \text{ a.e. } x \text{ in } \Gamma_c \right\}, \\ J(s)(v) = \int_{\Omega} W(x, e(\varepsilon, v)(x)) dx - L(v). \end{aligned} \quad (17)$$

The convergence result for  $u(s)$  will be a simple consequence of the following proposition:

#### Proposition 4

- (i) Any sequence  $u(s)$  in  $H_{\Gamma_0^\varepsilon}^1(\Omega, \mathbf{R}^3)$  such that  $J(s)(u(s)) \leq C$  contains a subsequence which weakly converges in  $H^1(\Omega, \mathbf{R}^3)$  toward some  $u$  and

$$u \in \bar{\Lambda}_{KL} := \{v \in V_{KL}; v(x) \in \bar{\lambda} \mathcal{K} \times \bar{\nu} I \text{ a.e. } x \text{ in } \Gamma_c\},$$

$$\bar{J}(u) := \int_{\Omega} W_{KL}(x, \hat{e}(v)(x)) dx - L(u) \leq \underline{\lim} J(s)(u(s));$$

- (ii) For all  $u$  in  $\bar{\Lambda}_{KL}$ , there exists  $(u(s))$  in  $\Lambda(s)$  which weakly converges to  $u$  in  $H^1(\Omega, \mathbf{R}^3)$  such that  $\bar{J}(u) \geq \underline{\lim} J(s)(u(s))$ .

*Proof* The first point is an easy consequence of the Korn inequality, the properties of  $W_{KL}$  and Lemma 1. To prove (ii), we proceed as in Proposition 3 but, in a preliminary step, we need to modify  $u$  in order to match the confinement conditions. When  $\bar{\lambda} \neq 0$  we replace  $\hat{u}$  by  $(\lambda/\phi)\hat{u}/\bar{\lambda}$  and  $u_3$  by  $\nu(\varepsilon/\phi)u_3/\bar{\nu}$  if  $\bar{\nu} \neq 0$ . When  $\bar{\lambda} = 0$  we add  $((\lambda/\phi)w, 0)$  to  $u$  where  $w$  is any extension to  $H_{\Gamma_0^\varepsilon}^1(\Omega, \mathbf{R}^3)$  of the constant field equal on  $\Gamma_c$  to some element of  $\mathcal{K}$ ; if  $\bar{\nu} = 0$  we add  $(0, \nu(\varepsilon/\phi)z)$  to  $u$ ,  $z$  being any extension to  $H_{\Gamma_0^\varepsilon}^1(\Omega, \mathbf{R}^3)$  of the constant function equal on  $\Gamma_c$  to some element of  $I$ .  $\square$

Since, there exists a constant  $C$  such that

$$J(s)(u(s)) \leq J(s)((\lambda/\phi)w, v(\varepsilon/\phi)z) \leq C,$$

similarly to Theorem 1 we can prove:

**Theorem 2** *The set of solutions  $u(s)$  of  $\mathcal{P}(s)$  is weakly relatively compact in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$ . If  $\bar{u}$  is a cluster point, there exists a subsequence such that  $\bar{J}(\bar{u}) = \lim_{s \rightarrow \bar{s}} J(s)(u(s))$  and  $u(s)$  strongly converges toward  $\bar{u}$  in  $H^1(\Omega, \mathbf{R}^3)$  which solves:*

$$\bar{\mathcal{P}} = \text{Min}\{\bar{J}(v); v \in \bar{\Lambda}_{KL}\}.$$

### 2.3.2 An Asymptotic Model of Behavior for the Confined Plate of Thickness $2\varepsilon$

Clearly,  $\bar{u}^s := S(\varepsilon, \phi)\bar{u}$  solves the problem:

$$\bar{\mathcal{P}}^s = \text{Min}\{\bar{J}^s(v); v \in \Lambda_{KL}^s\},$$

with

$$\begin{aligned} \Lambda_{KL}^s &:= \{v \in V_{KL}^s; v(x) \in \phi\lambda\mathcal{K} \times (\phi/\varepsilon)\bar{v}I \text{ a.e. } x \in \Gamma_c^s\}, \\ \bar{J}^s(v) &:= \int_{\Omega^\varepsilon} W_{KL}(x, \hat{e}(\varepsilon, v(x)))dx - \int_{\Omega^\varepsilon} f^\phi \cdot v dx - \int_{\Gamma_i^s} g^\phi \cdot v ds. \end{aligned} \quad (18)$$

For the reasons explained in Sects. 2.2.1(b) and 2.2.2, this yields a simpler but accurate model.

### 2.4 A Mixed Case (Bilateral Contact with Tangential Pull-back)

Here we assume that the surface energy density  $h^\mu$  is a function of the sole tangential component  $v_T$  of the displacement  $v$  along  $\Gamma_c$ , while the normal component  $v_N$  is assumed to be zero (a so-called condition of bilateral contact). We recall that if  $n$  denotes the outward unit normal then  $v_N = v \cdot n$  and  $v_T = v - v_N n$  and note that  $n = (v, 0)$  on  $\Gamma_c$ . We make the same assumption on  $h^\mu$  as in Sect. 2.2.2 and the problem of equilibrium reduces to:

$$\begin{aligned} \mathcal{P}^{(\varepsilon, \phi, \mu)} = \text{Min} \left\{ \int_{\Omega^\varepsilon} W^\varepsilon(x, e(v)(x))dx + \mu \int_{\Gamma_c^\varepsilon} h(v_T(x))ds \right. \\ \left. - \int_{\Omega^\varepsilon} f^\phi(x) \cdot v(x)dx - \int_{\Gamma_i^s} g^\phi(x) \cdot v(x)ds; v \in H_{\Gamma_c^s N}^1(\Omega^\varepsilon, \mathbf{R}^3) \right\}, \end{aligned}$$

which clearly has a solution,  $H_{\Gamma_c^s N}^1(\Omega^\varepsilon, \mathbf{R}^3) = \{v \in H_{\Gamma_0}^1(\Omega^\varepsilon, \mathbf{R}^3); v_N(x) = 0 \text{ a.e. } x \in \Gamma_c^\varepsilon\}$  being a closed subspace of  $H_{\Gamma_0}^1(\Omega^\varepsilon, \mathbf{R}^3)$ . Determining the asymptotic behavior of  $u^{(\varepsilon, \phi, \mu)}$  is a variant of the previous analysis and we will only state the result which will be used in Sect. 3.

**Theorem 3** *Let  $s = (\varepsilon, \phi, \mu)$ ,*

$$\mathcal{P}(s) = \text{Min}\{J(s)(v); v \in H_{\Gamma_c^s N}^1(\Omega, \mathbf{R}^3)\}$$

where  $H_{\Gamma_{cN}}^1(\Omega, \mathbf{R}^3) = \{v \in H_{\Gamma_0}^1(\Omega, \mathbf{R}^3); v_N(x) = 0 \text{ a.e. } x \text{ in } \Gamma_c\}$  and

$$J(s)(v) = \int_{\Omega} W(x, e(\varepsilon, v)(x)) dx + \frac{\mu}{\phi^{2-p} \varepsilon^p} \int_{\Gamma_c} h(\varepsilon \hat{v}_T(x), v_{T3}(x)) ds - L(v).$$

Then,  $u(s) = S(\varepsilon, \phi)u^{(\varepsilon, \phi, \mu)}$  solves  $\mathcal{P}(s)$ . The set  $u(s)$  is sequentially weakly relatively compact in  $H_{\Gamma_{cN}}^1(\Omega, \mathbf{R}^3)$  and for every cluster point  $\bar{u}^l$  there exists a subsequence  $u(s)$  which strongly converges in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  to  $\bar{u}^l$  with  $J^l(\bar{u}^l) = \lim_{s \rightarrow \bar{s}} J(s)(u(s))$ ,

$$\begin{aligned} J^l(v) &= \int_{\Omega} W_{KL}(x, \hat{e}(v)(x)) dx + \mathcal{H}^l(v) - L(v); \\ \mathcal{H}^l(v) &= \begin{cases} 2\bar{\mu} \int_{\gamma_c} h(0, v^F(\hat{x})) dl, & \text{if } v \in V_{KL}, \\ +\infty, & \text{otherwise}; \end{cases} \\ \mathcal{H}^2(v) &= \begin{cases} 2\bar{\mu} \int_{\gamma_c} h(v_T^M(\hat{x})) dl & \text{if } v \in V_{KL}, \\ +\infty & \text{otherwise,} \end{cases} \quad \mathcal{H}^3(v) = 0. \end{aligned}$$

Moreover  $\bar{u}^l$  solves:

$$\bar{p}^l = \text{Min}\{J^l(v); v \in V_{KL N}^l\},$$

with

$$\begin{aligned} V_{KL N}^1 &= V_{KL N} = \{v \in V_{KL}; v_N = 0 \text{ a.e. } x \text{ in } \gamma_c\} \\ &= \{v \in V_{KL}; v^M \cdot v = \partial_v v^F = 0 \text{ a.e. } x \text{ in } \gamma_c\}, \\ V_{KL N}^2 &= \{v \in V_{KL}; v_N = 0, v^F = 0 \text{ a.e. } x \text{ in } \gamma_c\} \\ &= \{v \in V_{KL}; v^M \cdot v = v^F = \partial_v v^F = 0 \text{ a.e. } x \text{ in } \gamma_c\}, \\ V_{KL N}^3 &= \{v \in V_{KL N}; v = 0 \text{ a.e. } x \text{ in } \gamma_c\} \\ &= \{v \in V_{KL}; v^M = v^F = \partial_v v^F = 0 \text{ a.e. } x \text{ in } \gamma_c\}. \end{aligned}$$

Of course, a de-scaling of  $\bar{u}^l$  yields a suitable simplified and accurate model of behavior of the real plate.

## 2.5 Concluding Remarks

First, as previously announced we have established that neither space homogeneity, nor linear isotropy is necessary to justify the Kirchhoff-Love theory. When  $W(x, e)$  reduces to  $W_{\lambda, \mu}(e) = \frac{\lambda}{2}(\text{tr } e)^2 + \mu|e|^2$  this corresponds to homogeneous isotropic linearized elasticity (the so-called Hooke's law). In this case it has been shown (see [7]) that the asymptotic behavior of the plate involves a reduced bulk energy acting on Kirchhoff-Love fields which reads as:

$$\hat{e} \in S^2 \mapsto \widehat{W}_{\lambda, \mu}(\hat{e}) = \frac{2\lambda\mu}{\lambda + 2\mu}(\text{tr } \hat{e})^2 + 2\mu|\hat{e}|^2.$$

A straightforward calculation gives that  $(W_{\lambda, \mu})_{KL} = \widehat{W}_{\lambda, \mu}$ .

More generally, if  $W(x, e) = \frac{1}{2}A(x)e \cdot e$  with  $A(x) \in \mathcal{L}(S^3, S^3)$ , the decomposition of  $S^3$  into  $S^2$  and its orthogonal complement  $S^\perp$  induces a decomposition of  $A$  into  $A^{22} \in \mathcal{L}(S^2, S^2)$ ,  $A^{\perp 2} \in \mathcal{L}(S^\perp, S^2)$ ,  $A^{2\perp} \in \mathcal{L}(S^2, S^\perp)$ ,  $A^{\perp\perp} \in \mathcal{L}(S^\perp, S^\perp)$ .

Classically we know that  $W_{KL}(x, \hat{e}) = \frac{1}{2}A_{KL}(x)\hat{e} \cdot \hat{e}$ , with  $A_{KL} = A^{22} - A^{2\perp}(A^{\perp\perp})^{-1}A^{\perp 2}$  being the Schur complement of  $A^{22}$  in the decomposition  $(S^2, S^\perp)$  of  $S^3$ . Note that a derivation by  $\Gamma$ -convergence of the Kirchhoff-Love plate theory when  $\Gamma_c = \emptyset$  and  $W(x, e) = W_{\lambda, \mu}(e)$  has been done in [4].

Next, we have stated sufficient conditions on the mechanical constraint in order for the asymptotic kinematics to be of a Kirchhoff-Love type, with an asymptotic bulk energy of ‘‘Kirchhoff-Love type’’ (say given by  $W_{KL}$  as previously defined by (8)) and an asymptotic constraint obtained as the limit of the three-dimensional one. Due to the Korn inequality and Lemma 1, no convexity conditions for the constraint have to be included in the assumptions. But if the mechanical constraint is convex, then, by the strict convexity of the genuine bulk energy density  $W(x, \cdot)$ , the problem  $\mathcal{P}(s)$  has a unique solution  $u(s)$  and the whole sequence  $u(s)$  strongly converges in  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  to the unique solution of the limit problem. We can easily consider the case of nonhomogeneous constraints by assuming that  $h^\mu$  is such that  $h^\mu = \mu h$  with  $h$  defined on  $\Gamma_c \times \mathbf{R}^3$ ,  $h(\cdot, e)$  measurable and  $h(x, \cdot)$  satisfying assumptions **H**<sub>1</sub>. Considering nonhomogeneous confinement conditions is more involved and tedious but feasible.

It is worthwhile to note that the whole previous analysis still works if  $W(x, \cdot)$  is assumed to be strictly convex and  $q$ -positively homogeneous,  $q \in (1, \infty)$ , the basic space  $H_{\Gamma_0}^1(\Omega, \mathbf{R}^3)$  being replaced by  $W_{\Gamma_0}^{1,q}(\Omega, \mathbf{R}^3)$ . Moreover, as in practice  $\varepsilon, \phi, \mu, \lambda, \nu$  are not parameters at all, but the corresponding values for thickness, strength, stiffness of the pull-back and size of the confinement, we believe that a reasonable proposal for a simpler and efficient model of the constrained plate is to replace the limits  $\bar{\mu}, \overline{\bar{\mu}}, \bar{\lambda}, \bar{\nu}$  by their actual values  $\nu/\phi^{2-p}\varepsilon^p, \nu/\phi^{2-p}, \lambda/\phi, (\varepsilon/\phi)\nu$  in the expressions of the total energy in our asymptotic models.

### 3 The Quasi-Static Case

#### 3.1 Problem Setting

Now, we assume that the plate is made of a linearly elastic material and is in bilateral contact with friction along  $\Gamma_c^\varepsilon$  with a rigid body; moreover the forces are time-dependent. We are going to formulate the determination of the quasi-static evolution of the plate from the initial state  $u_0^s$  in terms of a nonlinear evolution equation in a Hilbert space of possible states with finite energy. The study of the asymptotic behavior, when the thickness tends to zero, is easily carried out by simply using the results obtained in the static case, Sect. 2.4. This occurs through a suitable nonlinear extension of the Trotter theory of convergence of semi-groups of operators, acting on variable spaces (see Appendix). Hence we will make use of the notations of Sect. 2.4 and of some classical ones of Convex Analysis:  $J^*$  and  $\partial J(v)$  respectively stands for the Legendre-Fenchel conjugate of  $J$  and the sub-differential of the convex function  $J$  at  $v$ , with domain  $D(J)$ .

The strain energy density is such that:

$$\begin{aligned} W(x, e) &= \frac{1}{2}A(x)e \cdot e \quad \text{a.e. } x \in \Omega, \quad \forall e \in S^3 \quad \text{with} \\ A &\in L^\infty(\Omega, \mathcal{L}(S^3, S^3)), \quad A(x)e \cdot e \geq \alpha|e|^2, \quad \text{a.e. } x \in \Omega, \quad \forall e \in S^3. \end{aligned} \quad (19)$$

The constraint along  $\Gamma_c$  reads as:

$$-\sigma_T^s \in \partial h^\mu(\dot{u}_T^s), \quad u_N^s = 0, \quad (20)$$

where  $\sigma_T^s := \sigma^s n - (\sigma^s n \cdot n)n$  denotes the tangential component of the stress vector  $\sigma^s n$ , the convex function  $h^\mu$  satisfies the same assumptions as in Sect. 2.2.2, the overdot “ $\dot{\cdot}$ ” denotes the time derivative. Note that  $h^\mu = \mu h$ ,  $\mu \in [0, +\infty)$ , with  $h = |\cdot|^p$  corresponds to a Norton ( $p > 1$ ) or to a Tresca ( $p = 1$ ) friction.

The principle of virtual power leads us to seek  $u^s$  smooth enough such that:

$$\begin{aligned} & \int_{\Omega^\varepsilon} A((\Pi^\varepsilon)^{-1}x)e(u^s)(x, t) \cdot (e(v)(x) - e(\dot{u}^s)(x, t))dx \\ & + \int_{\Gamma_c^s} h^\mu(v_T(x))ds - \int_{\Gamma_c^s} h^\mu(\dot{u}_T^s(x, t))ds \\ & \geq \int_{\Omega^\varepsilon} f^\phi(x, t) \cdot (v(x) - \dot{u}^s(x, t))dx + \int_{\Gamma_1^\varepsilon} g^\phi(x, t) \cdot (v(x) - \dot{u}^s(x, t))ds \\ & \forall v \in \{v \in H_{\Gamma_0^\varepsilon}^1(\Omega^\varepsilon, \mathbf{R}^3); v_N = 0 \text{ a.e. in } \Gamma_c^\varepsilon\}. \end{aligned} \quad (21)$$

Keeping the same assumptions on the magnitude of the forces  $f^\phi$ ,  $g^\phi$  as in the static case, the scaled displacement  $u(s) = S(\varepsilon, \phi)u^s$  has to satisfy:

$$\begin{aligned} & \int_{\Omega} A(x)e(\varepsilon, u(s)) \cdot e(\varepsilon, v - \dot{u}(s))dx \\ & + \frac{\mu}{\phi^{2-p}\varepsilon^p} \left( \int_{\Gamma_c} h(\varepsilon \hat{v}_T, v_{3T})ds - \int_{\Gamma_c} h(\varepsilon \hat{u}_T^s(s), \dot{u}_{3T}^s(s))ds \right) \\ & \geq L(t)(v - \dot{u}(s)), \quad \forall v \in H_{\Gamma_{cN}}^1(\Omega, \mathbf{R}^3), \end{aligned} \quad (22)$$

where

$$L(t)(v) := \int_{\Omega} f(x, t) \cdot v(x)dx + \int_{\Gamma_1} g(x, t) \cdot v(x)ds$$

and  $H_{\Gamma_{cN}}^1(\Omega, \mathbf{R}^3)$  was defined in Sect. 2.4. We denote by  $V(s)$  this space equipped with the scalar product

$$(u, v)_s := \int_{\Omega} A(x)e(\varepsilon, u) \cdot e(\varepsilon, v)dx, \quad (23)$$

which, associated with the scaled strain energy, depends on  $s$ . Due to the Korn inequality, it is equivalent to the usual one, for all strictly positive fixed value of  $\varepsilon$ .

We make the additional assumption:

$$(f, g) \in H^1(0, T; L^2(\Omega, \mathbf{R}^3) \times L^2(\Gamma_1, \mathbf{R}^3)), \quad (24)$$

so that there exists a unique solution  $u^e(s)$  in  $H^1(0, T; V(s))$  of the equilibrium problem:

$$(u^e(s), v)_s = L(t)(v), \quad \forall v \in V(s), \quad \forall t \in [0, T] \quad (25)$$

and we define on  $V(s)$  the lower semi-continuous and convex function

$$\mathcal{H}(s)(v) = \begin{cases} (\mu/\phi^{2-p}\varepsilon^p) \int_{\Gamma_c} h(\varepsilon \hat{v}_T(x), v_{T3}(x)) ds \\ \text{if } x \mapsto h(\varepsilon \hat{v}_T(x), v_{T3}(x)) \in L^1(\Gamma_c) \\ +\infty \quad \text{otherwise.} \end{cases} \quad (26)$$

Thus, (22) yields

$$-u(s) + u^e(s) \in \partial \mathcal{H}(s)(\dot{u}(s)), \quad (27)$$

or

$$\dot{u}(s) \in \partial(\mathcal{H}(s))^*(u^e(s) - u(s)). \quad (28)$$

Hence, if

$$\begin{aligned} \bar{u}(s) &:= u(s) - u^e(s); \\ \check{\mathcal{H}}(s)(v) &:= \mathcal{H}(s)(-v), \quad \forall v \in V(s); \\ \bar{u}(s)_0 &:= u(s)_0 - u^e(s)(0), \end{aligned} \quad (29)$$

where  $u(s)_0 := S(\varepsilon, \phi)u_0^s$  is the scaled initial displacement of the plate, the problem can be written in terms of an evolution equation in  $V(s)$ :

$$\begin{cases} \dot{\bar{u}}(s) + \partial(\check{\mathcal{H}}(s))^*(\bar{u}(s)) \ni \dot{u}^e(s), \\ \bar{u}(s)(0) = \bar{u}(s)_0. \end{cases} \quad (30)$$

From a classical result [5], we deduce the

**Proposition 5** *For all  $\bar{u}(s)_0 \in D((\check{\mathcal{H}}(s))^*)$ , (30) has a unique solution in  $H^1(0, T; V(s))$ .*

Note that the condition  $\bar{u}(s)_0 \in D((\check{\mathcal{H}}(s))^*)$ , equivalent to:

$$\exists M > 0; \quad (u(s)_0, \varphi)_s - L(0)(\varphi) - \check{\mathcal{H}}(s)(\varphi) \leq M \quad \forall \varphi \in V(s), \quad (31)$$

is a compatibility assumption between the initial state and the loading at time  $t = 0$ :

$$\begin{aligned} \operatorname{div} \sigma_0^s + f^\phi(0) &= 0, & \sigma_0^s &= Ae(u_0^s) \quad \text{in } \Omega^\varepsilon, \\ \sigma_0^s n &= g(0) \quad \text{on } \Gamma_1^\varepsilon, & \sigma_0^s T &\in L^p(\Gamma_c^\varepsilon, \mathbf{R}^3), & u_{0N}^s &= 0 \quad \text{on } \Gamma_c^\varepsilon. \end{aligned}$$

As in the static case, we will derive an asymptotic model for the behavior of the given plate of thickness  $2\varepsilon$  by studying the asymptotic behavior of  $\bar{u}(s)$  when  $s$  goes to  $\bar{s} \in \{0\} \times [0, +\infty]^2$ .

### 3.2 A Convergence Result

We will show that  $\bar{u}(s)$  converges toward  $\bar{u}^I$  which solves a nonlinear evolution equation in the Hilbert space  $\bar{V}^I$  equal to  $V_{KL N}^I$ ,  $I \in \{1, 2, 3\}$ , equipped with the scalar product:

$$(u, v)_{\bar{V}^I} = \int_{\Omega} A_{KL}(x) \hat{e}(u) \cdot \hat{e}(v) dx, \quad (32)$$

where  $V_{KL}^I$  and  $A_{KL}$  have been introduced in Sects. 2.4 and 2.5, respectively. The previous bilinear form can be extended into  $V(s)$  and the definition (8) of  $W_{KL}$  implies:

$$|v|_{\bar{s}}^2 := (v, v)_{\bar{s}} \leq |v|_s^2 := (v, v)_s \quad \forall v \in V(s). \quad (33)$$

The properties of  $A$  make it possible, by the Lax-Milgram lemma, to define  $P(s) \in \mathcal{L}(\bar{V}^I, V(s))$  by

$$v \in \bar{V}^I \mapsto P(s)v \in V(s), \quad (P(s)v, w)_s = (v, w)_{\bar{s}}, \quad \forall w \in V(s), \quad (34)$$

which is useful to “compare” elements of  $V(s)$  and  $\bar{V}^I$ .

**Proposition 6** *For all  $v \in \bar{V}^I$  we have:*

- (i)  $|P(s)v|_s \leq |v|_{\bar{s}}$ ;
- (ii)  $\lim_{s \rightarrow \bar{s}} |P(s)v|_s = |v|_{\bar{s}}$ .

*Proof* First, (i) stems from (33) and (34) with  $w = P(s)v$ . Next, because  $P(s)v$  minimizes the functional  $\frac{1}{2} | \cdot |_s^2 - (v, \cdot)_{\bar{s}}$  on  $V(s)$ , (ii) is a mere consequence of Theorem 3 by choosing  $\mu \equiv 0$  and replacing the linear form  $L$  by  $(v, \cdot)_{\bar{s}}$ .  $\square$

We will say that  $v(s) \in V(s)$  converges toward  $v \in \bar{V}^I$  in the sense of Trotter if  $\lim_{s \rightarrow \bar{s}} |P(s)v - v(s)|_s = 0$ . It is useful to relate this convergence to some classical ones.

**Proposition 7** *For all  $v$  in  $\bar{V}^I$  and every sequence  $v(s)$  in  $V(s)$  the two properties are equivalent:*

- (i)  $\lim_{s \rightarrow \bar{s}} |P(s)v - v(s)|_s = 0$ ;
- (ii)  $v(s)$  strongly converges in  $H^1(\Omega, \mathbf{R}^3)$  toward  $v$  and  $\lim_{s \rightarrow \bar{s}} |v(s)|_s = |v|_{\bar{s}}$ .

*Proof* (ii)  $\Rightarrow$  (i) By definition of  $P(s)$  we have:

$$\begin{aligned} |P(s)v - v(s)|_s^2 &= |P(s)v|_s^2 - 2(P(s)v, v(s))_s + |v(s)|_s^2 \\ &= |P(s)v|_s^2 - 2(v, v(s))_{\bar{s}} + |v(s)|_s^2, \end{aligned}$$

which, according to Proposition 6, tends to zero.

(i)  $\Rightarrow$  (ii) Using the remark made in the second part of the proof of Proposition 6, we have that  $P(s)v$  converges strongly in  $H^1(\Omega, \mathbf{R}^3)$  toward  $v$ . Moreover

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} |e(P(s)v - v(s))|_{L^2(\Omega, S^3)} &\leq \lim_{s \rightarrow \bar{s}} |e(\varepsilon, P(s)v - v(s))|_{L^2(\Omega, S^3)} \\ &\leq \lim_{s \rightarrow \bar{s}} (\alpha^{-1/2} |P(s)v - v(s)|_s) = 0, \end{aligned}$$

and so Korn’s inequality implies that  $v(s)$  strongly converges toward  $v$  in  $H^1(\Omega, \mathbf{R}^3)$ . Finally from Proposition 6(i) we have:

$$\lim_{s \rightarrow \bar{s}} |v(s)|_s = \lim_{s \rightarrow \bar{s}} |P(s)v|_s = |v|_{\bar{s}}. \quad \square$$

Now we define a priori a limit evolution equation in  $\bar{V}^I$  which will describe the asymptotic behavior of  $\bar{u}(s)$ :

$$\begin{cases} \dot{\bar{u}}^I + \partial(\check{\mathcal{H}}^I)^*(\bar{u}^I) \ni \dot{u}^{Ie}, \\ \bar{u}^I 0 = \bar{u}_0^I, \end{cases} \quad (35)$$

with  $\check{\mathcal{H}}^I := \mathcal{H}^I(-v)$ ,  $\mathcal{H}^I$  being defined in Sect. 2.4 and  $u^{Ie}$  the unique element of  $H^1(0, T; \bar{V}^I)$  such that:

$$(u^{Ie}, w)_{\bar{V}} = L(t)(w), \quad \forall w \in \bar{V}^I, \quad \forall t \in [0, T]. \quad (36)$$

The initial condition  $\bar{u}_0^I$  will be defined later (see (37), (38)). As for (30), a classical result of [5] gives the:

**Proposition 8** *For all  $\bar{u}_0^I \in D((\check{\mathcal{H}}^I)^*)$ , (36) has a unique solution in  $H^1(0, T; \bar{V}^I)$ .*

According to our nonlinear version of the Trotter theory of approximation of semi-groups acting on variable spaces (see Appendix), to prove the convergence, in the sense of Trotter, of  $\bar{u}(s)$  towards  $\bar{u}^I$  it suffices to establish the following last two propositions:

**Proposition 9**

$$\forall f \in \bar{V}^I: \quad \lim_{s \rightarrow \bar{s}} |P(s)(I + \partial(\check{\mathcal{H}}^I)^*)^{-1}(f) - (I + \partial(\check{\mathcal{H}}(s))^*)^{-1}(P(s)f)|_s = 0.$$

**Proposition 10**

$$\lim_{s \rightarrow \bar{s}} \int_0^T |P(s)\dot{u}^{Ie}(t) - (\dot{u}^e(s))(t)|_s dt = 0,$$

$$\lim_{s \rightarrow \bar{s}} |P(s)u^{Ie}(t) - (u^e(s))(t)|_s = 0 \quad \text{uniformly on } [0, T].$$

*Proof of Proposition 9* Classically,  $(I + \partial(\check{\mathcal{H}}^I)^*)^{-1}(f) = \xi^I + f$ ,  $(I + \partial(\check{\mathcal{H}}(s))^*)^{-1} \times (P(s)f) = \xi(s) + P(s)f$  where  $\xi^I$  and  $\xi(s)$  satisfy

$$\xi^I + f + \partial\check{\mathcal{H}}^I(\xi^I) \ni 0, \quad \xi(s) + P(s)f + \partial\check{\mathcal{H}}(s)(\xi(s)) \ni 0.$$

Hence the definition of  $P(s)$  implies that  $\xi^I$  and  $\xi(s)$  are the unique minimizers of the convex lower semi-continuous functions

$$\mathcal{J}^I(\cdot) = \frac{1}{2} |\cdot|_{\bar{V}}^2 + \mathcal{H}^I(\cdot) + (f, \cdot)_{\bar{V}} \quad \text{and} \quad \mathcal{J}(s)(\cdot) = \frac{1}{2} |\cdot|_s^2 + \mathcal{H}(s)(\cdot) + (f, \cdot)_{\bar{V}}$$

and it suffices to establish  $\lim_{s \rightarrow \bar{s}} |P(s)\xi^I - \xi(s)|_s = 0$ . Obviously, the assertions of Theorem 3 are still true when the linear form  $L$  is replaced by  $(f, \cdot)_{\bar{V}}$ . Thus  $\xi(s)$  converges strongly in  $H^1(\Omega, \mathbf{R}^3)$  toward  $\xi^I$  and  $\lim_{s \rightarrow \bar{s}} \mathcal{J}(s)(\xi(s)) = \mathcal{J}^I(\xi^I)$ . Arguing as in the proof of Theorem 1, we deduce  $\lim_{s \rightarrow \bar{s}} |\xi(s)|_s^2 = |\xi^I|_s^2$  which, according to Proposition 7, ends the proof.  $\square$

*Proof of Proposition 10* First, the definition of  $u^e$  and Proposition 6(i) imply:

$$\begin{aligned} |P(s)\dot{u}^{le}(t) - (\dot{u}^e(s))(t)|_s &\leq |\dot{u}^{le}(t)|_{\bar{s}} + |(\dot{u}^e(s))(t)|_s \\ &\leq C(|\dot{f}(t)|_{L^2(\Omega, \mathbf{R}^3)} + |\dot{g}(t)|_{L^2(\Gamma_1, \mathbf{R}^3)}). \end{aligned}$$

As previously observed, since  $u^{le}$  and  $u^e(s)$  are minimizers of  $\frac{1}{2}|\cdot|_{\bar{s}}^2 - L(\cdot)$  and  $\frac{1}{2}|\cdot|_s^2 - L(\cdot)$ , we have

$$\begin{aligned} \lim_{s \rightarrow \bar{s}} |P(s)\dot{u}^{le}(t) - (\dot{u}^e(s))(t)|_s &= \lim_{s \rightarrow \bar{s}} |P(s)u^{le}(t) - (u^e(s))(t)|_s = 0 \\ &\text{a.e. on } [0, T]. \end{aligned}$$

Thus the first part of the proposition is a consequence of the Lebesgue dominated convergence theorem while the second is due to the uniform equi-continuity of  $t \in [0, T] \mapsto |P(s)u^{le}(t) - (u^e(s))(t)|_s$  which stems from (24).  $\square$

We make an additional assumption on the initial states:

$$\begin{aligned} \exists u_0^l &\in u^{le}(0) + D((\check{\mathcal{H}}^l)^*); & u(s)_0 &\in u_s^e(0) + D((\check{\mathcal{H}}(s))^*), \\ \lim_{s \rightarrow \bar{s}} |P(s)u_0^l - u(s)_0|_s &= 0. \end{aligned} \tag{37}$$

The first condition is a compatibility condition between the initial limit state and the initial loading; the second is a convergence condition which, due to Proposition 9, is satisfied by:

$$u(s)_0 = u_s^e(0) + (I + \partial(\check{\mathcal{H}}(s))^*)^{-1}(P(s)(I + \partial(\check{\mathcal{H}}^l)^*)(u_0^l - u^{le}(0))).$$

Hence, from the nonlinear Trotter theorem (see Appendix) we deduce the uniform convergence on  $[0, T]$ , in the sense of Trotter, of the solution of (30) toward the solution of (34) with  $\bar{u}_0^l = u_0^l - u^{le}(0)$ . Taking into account the previous propositions, we may rephrase this convergence result in the more explicit form:

**Theorem 4** *The solution  $u(s)$  of (27) with an initial data  $u(s)_0$  converges in  $C^0([0, T]; H^1(\Omega, \mathbf{R}^3))$  toward the solution of*

$$\begin{cases} -u^l + u^{le} \in \partial\mathcal{H}^l(\dot{u}^l), \\ u^l(0) = u_0^l, \end{cases} \tag{38}$$

with  $\lim_{s \rightarrow \bar{s}} \int_{\Omega} W(e(\varepsilon, u(s))) dx = \int_{\Omega} W_{KL}(\hat{e}(u^l)) dx$  uniformly on  $[0, T]$ .

Note that a more explicit writing of (38)<sub>1</sub> is:

$$\begin{aligned} &\int_{\Omega} A_{KL} \hat{e}(u^l(t)) \cdot \hat{e}(v - \dot{u}^l(t)) dx + \mathcal{H}^l(v) - \mathcal{H}^l(\dot{u}^l) \\ &\geq \int_{\Omega} f \cdot (v - \dot{u}^l(t)) dx + \int_{\Gamma_1} g \cdot (v - \dot{u}^l(t)) ds, \\ &\forall v \in V_{KL N}^l \text{ a.e. } t \in [0, T]. \end{aligned}$$

### 3.3 An Asymptotic Model for the Behavior of the Plate

As in the static case, a de-scaling of  $u^I$  yields a suitable simplified and accurate model for the behavior of the real plate:

$$u^{I(\varepsilon, \phi)} := S(\varepsilon, \phi)^{-1} u^I \in H^1(0, T; V_{KL N}^{I, \varepsilon}) \quad \text{satisfies a.e. } t \text{ in } [0, T],$$

$$\int_{\Omega^\varepsilon} A_{KL} \hat{e}(u^{I(\varepsilon, \phi)}(t)) \cdot \hat{e}(v - \dot{u}^{I(\varepsilon, \phi)}(t)) dx + \mathcal{H}^{I(\varepsilon, \phi)}(v_T) - \mathcal{H}^{I(\varepsilon, \phi)}(\dot{u}_T^{I(\varepsilon, \phi)})$$

$$\geq \int_{\Omega^\varepsilon} f \cdot (v - \dot{u}^{I(\varepsilon, \phi)}(t)) dx + \int_{\Gamma_1^i} g \cdot (v - \dot{u}^{I(\varepsilon, \phi)}(t)) ds, \quad \forall v \in V_{KL N}^{I, \varepsilon},$$

where  $\mathcal{H}^{I(\varepsilon, \phi)}$  are defined in (14) and  $V_{KL N}^{I, \varepsilon}$  are the analogues of  $V_{KL N}^I$ , defined in Sect. 2.4, with  $\Omega^\varepsilon$  in place of  $\Omega$ . The accuracy of the model is due to the fact that  $\lim_{s \rightarrow \bar{s}} \frac{1}{\varepsilon \phi^2} \int_{\Omega^\varepsilon} |\hat{e}(u^{I(\varepsilon, \phi)} - u^{(\varepsilon, \phi, \mu)})|^2 dx = 0$ , uniformly on  $[0, T]$ .

Eventually, note that our model corresponds to a condition of bilateral contact on  $\Gamma_c^\varepsilon$  associated with flexural friction when  $I = 1$ , tangential membrane friction with no flexion when  $I = 2$  and pure adhesion when  $I = 3$ .

## Appendix

Most of the boundary value problems obtained by mathematical modeling in Continuum Mechanics and Physics are parametrized, the parameters being the coefficients of the partial differential equations, the domain, the boundary, etc. . . Thus the Trotter theory (see [14]) of approximation of semi-groups of operators acting on *variable* Hilbert spaces is useful and powerful for questions of convergence in transient boundary value problems. Here we propose a nonlinear extension of the theory of Trotter.

**Theorem 5** *Let  $H_n, H$  be Hilbert spaces and let  $A^n : H_n \rightarrow 2^{H_n}$ ,  $A : H \rightarrow 2^H$  be maximal monotone multivoque operators. Define  $P_n : (H, |\cdot|) \rightarrow (H_n, |\cdot|_n)$  such that  $P_n \in \mathcal{L}(H, H_n)$  and*

1.  $|P_n(x)|_n \leq C|x|$ ,  $\forall x \in H$ , with  $C$  constant, independent of  $n$ ;
2.  $|P_n(x)|_n \rightarrow |x|$ ,  $\forall x \in H$ .

*Let  $f_n \in L^1(0, T; H_n)$  and  $f \in L^1(0, T; H)$ ,  $u_n^0 \in \overline{D(A^n)}$  and  $u^0 \in \overline{D(A)}$ . Let  $u_n$  and  $u$  be the weak solutions of the equations:*

$$\begin{cases} \frac{du_n}{dt} + A^n u_n \ni f_n, \\ u_n(0) = u_n^0, \end{cases} \quad \begin{cases} \frac{du}{dt} + Au \ni f, \\ u(0) = u^0. \end{cases}$$

*If  $|P_n u^0 - u_n^0|_n \rightarrow 0$ ,  $\int_0^T |P_n f(t) - f_n(t)|_n dt \rightarrow 0$ ,*

$$|(I + \lambda A^n)^{-1} P_n z - P_n (I + \lambda A)^{-1} z|_n \rightarrow 0 \quad \text{when } n \rightarrow \infty, \forall \lambda \geq 0, \forall z \in H,$$

*then*

$$|P_n u(t) - u_n(t)|_n \rightarrow 0 \quad \text{when } n \rightarrow \infty, \text{ uniformly on } [0, T].$$

*Proof* In ([5] Theorem 3.16) an analogous result is given for the case  $H_n = H$  and consequently  $P_n$  the identity on  $H$ . Our proof is a slight variation of the previous one in this more general case, by simply proceeding to a suitable insertion of the linear operators  $P_n$  which compare elements of the variable space  $H_n$  and the limit space  $H$ .

(i) *First step:*  $f_n = f = 0$ .

Let  $\lambda > 0$  be fixed,  $y = (I + \sqrt{\lambda}A)^{-1}u^0$  and  $y_n = (I + \sqrt{\lambda}A^n)^{-1}P_nu^0$ . Let  $v$ , respectively,  $v^\lambda$ ,  $v_n$ ,  $v_n^\lambda$  be the solution of the following problems:

$$\frac{dv}{dt} + Av \ni 0; \quad v(0) = y,$$

$$\frac{dv^\lambda}{dt} + A_\lambda v^\lambda \ni 0; \quad v^\lambda(0) = y,$$

$$\frac{dv_n}{dt} + A^n v_n \ni 0; \quad v_n(0) = y_n,$$

$$\frac{dv_n^\lambda}{dt} + A_\lambda^n v_n^\lambda \ni 0; \quad v_n^\lambda(0) = y_n,$$

where

$$A_\lambda := \frac{I - (I + \lambda A)^{-1}}{\lambda}, \quad A_\lambda^n := \frac{I - (I + \lambda A^n)^{-1}}{\lambda}.$$

Then

$$\begin{aligned} |u_n(t) - P_nu(t)|_n &\leq |u_n(t) - v_n(t)|_n + |v_n(t) - v_n^\lambda(t)|_n + |v_n^\lambda(t) - P_nv^\lambda(t)|_n \\ &\quad + |P_nv^\lambda(t) - P_nv(t)|_n + |P_nv(t) - P_nu(t)|_n. \end{aligned}$$

Since  $A$  and  $A_n$  generate semigroups of contractions we have:

$$\begin{aligned} (1) \quad |u_n(t) - v_n(t)|_n &\leq |u_n^0 - y_n|_n = |u_n^0 - (I + \sqrt{\lambda}A^n)^{-1}P_nu^0|_n \\ &\leq |u_n^0 - P_n(I + \sqrt{\lambda}A)^{-1}u^0|_n \\ &\quad + |(I + \sqrt{\lambda}A^n)^{-1}P_nu^0 - P_n(I + \sqrt{\lambda}A)^{-1}u^0|_n \\ &\leq |u_n^0 - P_nu^0|_n + C |u^0 - (I + \sqrt{\lambda}A_n)^{-1}u^0|_n \\ &\quad + |(I + \sqrt{\lambda}A^n)^{-1}P_nu^0 - P_n(I + \sqrt{\lambda}A)^{-1}u^0|_n, \end{aligned}$$

where the first and third term of the last inequality go to 0 when  $n \rightarrow \infty$ ,

$$(2) \quad |P_nv(t) - P_nu(t)|_n \leq C|v(t) - u(t)| \leq C|u^0 - (I + \sqrt{\lambda}A)^{-1}u^0|.$$

It is proven in [5] that:

$$|v^\lambda(t) - v(t)| \leq \sqrt{\lambda T}|A^0y|,$$

where  $A^0y := \text{Argmin}\{|\xi|; \xi \in Ay\}$ ,

$$|v^\lambda(t) - v(t)| \leq \sqrt{T}|u^0 - (I + \sqrt{\lambda}A)^{-1}u^0|.$$

Hence we have:

$$(3) \quad |P_n v^\lambda(t) - P_n v(t)|_n \leq C\sqrt{T}|u^0 - (I + \sqrt{\lambda}A)^{-1}u^0|.$$

By the same argument we obtain:

$$(4) \quad |v_n(t) - v_n^\lambda(t)|_n \leq \sqrt{T}|P_n u^0 - (I + \sqrt{\lambda}A^n)^{-1}P_n u^0|_n \\ \leq \sqrt{T}[C|u^0 - (I + \sqrt{\lambda}A)^{-1}u^0| \\ + |(I + \sqrt{\lambda}A^n)^{-1}P_n u^0 - P_n(I + \sqrt{\lambda}A)^{-1}u^0|_n],$$

where the second term of the last inequality goes to zero when  $n \rightarrow \infty$ . Since

$$v^\lambda(t) = v^\lambda(0) + \int_0^t A_\lambda v^\lambda(\tau) d\tau, \quad v_n^\lambda(t) = v_n^\lambda(0) + \int_0^t A_\lambda^n v_n^\lambda(\tau) d\tau,$$

a trivial computation gives:

$$(5) \quad |v_n^\lambda(t) - P_n v^\lambda(t)|_n \leq |P_n y - y_n|_n + \int_0^t |A_\lambda^n v_n^\lambda(\tau) - P_n A_\lambda v^\lambda(\tau)|_n d\tau \\ \leq |P_n y - y_n|_n + \int_0^t |A_\lambda^n v_n^\lambda(\tau) - A_\lambda^n P_n v^\lambda(\tau)|_n d\tau \\ + \int_0^t |A_\lambda^n P_n v^\lambda(\tau) - P_n A_\lambda v^\lambda(\tau)|_n d\tau \\ \leq |P_n y - y_n|_n + \frac{1}{\lambda} \int_0^t |v_n^\lambda(\tau) - P_n v^\lambda(\tau)|_n d\tau \\ + \int_0^t |A_\lambda^n P_n v^\lambda(\tau) - P_n A_\lambda v^\lambda(\tau)|_n d\tau,$$

because  $A_\lambda^n$  is a Lipschitz operator of constant  $\frac{1}{\lambda}$ . Then, the Gronwall inequality implies:

$$|v_n^\lambda(t) - P_n v^\lambda(t)|_n \leq \left[ |P_n y - y_n|_n + \int_0^t |A_\lambda^n P_n v^\lambda(\tau) - P_n A_\lambda v^\lambda(\tau)|_n d\tau \right] e^{\frac{t}{\lambda}}$$

and finally

$$\int_0^t |A_\lambda^n P_n v^\lambda(\tau) - P_n A_\lambda v^\lambda(\tau)|_n d\tau \\ \leq \frac{1}{\lambda} \int_0^t |(I + \sqrt{\lambda}A^n)^{-1}P_n v^\lambda(\tau) - P_n(I + \sqrt{\lambda}A)^{-1}v^\lambda(\tau)|_n d\tau.$$

By Lebesgue's dominated convergence theorem the last term goes to 0 when  $n \rightarrow \infty$  and

$$|P_n y - y_n|_n = |P_n(I + \sqrt{\lambda}A)^{-1}u_0 - (I + \sqrt{\lambda}A^n)^{-1}P_n u_0|_n \rightarrow 0 \\ \text{when } n \rightarrow \infty.$$

Thus, for a fixed  $\lambda$  we have:

$$\lim_{n \rightarrow \infty} \|u_n(\cdot) - P_n u(\cdot)\|_{L^\infty(0,T)} \leq (C + C\sqrt{T})|u^0 - (I + \sqrt{\lambda}A)^{-1}u^0|.$$

The conclusion follows since the right-hand side of the last inequality is arbitrarily small when  $\lambda \rightarrow 0$ .

(ii) *Second step: general case.*

Let  $g$  be a step function on  $[0, T]$ . Let  $w_n, w$  be the solutions of:

$$\begin{cases} \frac{dw_n}{dt} + A^n w_n \ni P_n g, & \frac{dw}{dt} + Au \ni g, \\ w_n(0) = u_n^0, & w(0) = u^0. \end{cases}$$

The previous result, applied for each interval of  $[0, T]$ , where  $g$  is constant and for the operators  $\mathcal{L}^n u_n := A^n u_n - P_n g$  and  $\mathcal{L} u := Au - g$ , which are obviously maximal monotone, gives:

$$|P_n w(t) - w_n(t)|_n \rightarrow 0 \quad \text{uniformly on } [0, T].$$

Then we have

$$\begin{aligned} |P_n u - u_n|_{L^\infty(O, T, H_n)} &\leq |P_n u - P_n w|_{L^\infty(O, T, H_n)} + |P_n w - w_n|_{L^\infty(O, T, H_n)} \\ &\quad + |w_n - u_n|_{L^\infty(O, T, H_n)} \\ &\leq C|u - w|_{L^\infty(O, T, H)} + |P_n g - f_n|_{L^1(O, T, H_n)} \\ &\quad + |P_n w - w_n|_{L^\infty(O, T, H_n)} \\ &\leq C|f - g|_{L^1(O, T, H)} + C|f - g|_{L^1(O, T, H)} \\ &\quad + |P_n f - f_n|_{L^1(O, T, H_n)} + |P_n w - w_n|_{L^\infty(O, T, H_n)}. \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} |P_n u - u_n|_{L^\infty(O, T, H_n)} \leq 2C|f - g|_{L^1(O, T, H)}.$$

The conclusion follows since the last term is arbitrarily small.  $\square$

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