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Abstract

An elementary situation in welding involves the perfect assembly of two adherents and a strong adhesive occupying a thin layer. The bulk energy density of the hyperelastic adherents grows superlinearly while the one of the pseudo-plastic adhesive grows linearly with a stiffness of the order of its thickness ε. We propose a simplified but accurate model by studying the asymptotic behavior, when ε goes to zero, through variational convergence methods: at the limit, the intermediate layer is replaced by a pseudo-plastic interface which allows cracks to appear.

Contents

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1 Introduction

Motivated by the mathematical modeling of a problem of welding, we revisit previous studies ([1], [8]) devoted to the asymptotic behavior of a structure made of two adherents connected by a thin and strong adhesive layer. In [8] the adherents and the adhesive were modeled as hyperelastic by bulk energy densities with the same growth exponent p laying in $(1, +\infty)$, the stiffness of the adhesive being of the order of the inverse of its thickness. Here, our first attempt to account for some fracture phenomena in soldered joint is to model the adhesive as pseudo-plastic, that is to say its behavior is described by a bulk energy density with linear growth. Hence, from the mathematical point of view, two difficulties appear: the growths of the the bulk energy in the adhesives and the adherent are different and the linear growth in the adherent will imply to work in spaces of displacement fields with free discontinuities

The paper is organized as follows. In Section 2, we describe a model problem with a simplified geometry directly connected to the study [8] where we assume that the bulk energy density of the adherents is quasiconvex and the one of the adhesive is convex. In Section 3, a variational convergence result, when the thickness of the adhesive layer goes to zero, justifies our proposal of simplified but accurate enough model. The adhesive layer is replaced by a material pseudo-plastic surface. The case when f is not quasiconvex and g is not convex is considered in Section 4. In Section 5, we use the previous results to model a more realistic situation of welding. Eventually, in the spirit of [7], [17], we consider a variational regularization of the limit functional involved by our model in Section 6.

2 Description of the model

We make no difference between $\mathbf{R}^3$ and the three dimensional euclidean physical space whose orthogonal basis is denoted by (e_1, e_2, e_3), Greek coordinate indexes will run in $\{1, 2\}$ and Latin ones in $\{1, 2, 3\}$. For all $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ of $\mathbf{R}^3$, $\hat{\zeta}$ stands for (ζ_1, ζ_2). Let S be a domain of $\mathbf{R}^2$ with a Lipschitz-continuous boundary ∂S and r a positive number. The cylindrical domain $\Omega := S \times (-r, r)$ is the reference configuration of a structure made of two adherents and an adhesive which respectively occupies $\Omega_\varepsilon^\pm := \{x \in \Omega : \pm x_3 > \varepsilon/2\}$ and $B_\varepsilon := \{x \in \Omega : |x_3| < \varepsilon/2\}$. We set $\Omega_\varepsilon := \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$. The structure is clamped on a part Γ_0 of the boundary Γ of Ω with a positive $\mathcal{H}^2$-measure and we assume that there exists $\varepsilon_0 > 0$ such that $\mathrm{dist}(\bar{\Gamma}_0, \bar{B}_{\varepsilon_0}) > 0$. The structure is subjected to body forces of density Φ and to surface forces of density φ on the complementary part Γ_φ of Γ_0. We assume that the supports of φ and Φ lay outside of $\bar{B}_{\varepsilon_0}$. Obviously, one can consider other type of boundary conditions (e.g. a combination of some components of the stress vector and of the displacement). At last, adhesive and adherents are assumed to be perfectly stuck together. In section 5 we will consider a more realistic structure (see figure 1 for the two geometrical strucures).

The stiffness of the material occupying the thin layer B_ε is assumed to be of order $1/\varepsilon$ so that the strain in B_ε is expected to be small and we will use the framework of small perturbations to model the behavior of the adhesive. To account for possible fracture phenomena inside B_ε, we consider the adhesive as pseudo-plastic. Hence the behavior of the adhesive is described by a bulk energy density like $1/\varepsilon\ g(e(u))$ where g is a convex function with linear growth of the linearized strain $e(u)$ i.e., the symmetric part $(\nabla u)_s$ of the gradient displacement ∇u. By contrast, the deformations in the adherents may be large and they are modeled as hyperelastic with a continuous quasiconvex bulk energy density f, function of the gradient displacement. More precisely, we assume that there exists $p > 1$, and two positive constants α, β such that

$$f,\ g : \mathbf{M}^{3\times 3} \to \mathbf{R};$$

$$\alpha|\xi|^p \le f(\xi) \le \beta(1 + |\xi|^p) \text{ for all } \xi \in \mathbf{M}^{3\times 3}; \tag{1}$$

$$\alpha|\xi| \le g(\xi) \le \beta(1 + |\xi|) \text{ for all } \xi \in \mathbf{M}_s^{3\times 3}. \tag{2}$$

Here and in the sequel $\mathbf{M}^{n\times n}$ and $\mathbf{M}_s^{n\times n}$ stand for the set of $n \times n$ matrices and $n \times n$ symmetric matrices with real entries respectively. It is well known that f and g satisfy the following locally Lipschitz conditions: there exists a positive constant L such that

$$|f(\xi) - f(\xi')| \le L|\xi - \xi'|\big(1 + |\xi|^{p-1} + |\xi'|^{p-1}\big) \text{ for all } \xi,\ \xi' \in \mathbf{M}^{3\times 3}; \tag{3}$$

$$|g(\xi) - g(\xi')| \le L|\xi - \xi'| \text{ for all } \xi,\ \xi' \in \mathbf{M}_s^{3\times 3}. \tag{4}$$

Thus, if $F_\varepsilon(u) := \int_{\Omega_\varepsilon} f(\nabla u)\ dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u)\ dx$ and $L(u) := \int_\Omega \Phi.u\ dx + \int_{\Gamma_\varphi} \varphi.u\ d\mathcal{H}^2$ respectively denote the total stored energy and the work of the external loading, determining the equilibrium configurations leads to the problem

$$\inf\Big\{ \int_{\Omega_\varepsilon} f(\nabla u)\ dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u))\ dx - L(u) : u \in A_\varepsilon \Big\}$$

with

$$A_\varepsilon := \big\{u \in \mathrm{LD}(\Omega, \mathbf{R}^3) : u_{\lfloor \Omega_\varepsilon} \in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbf{R}^3)\big\};$$

$$W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbf{R}^3) := \big\{u \in W^{1,p}(\Omega_\varepsilon, \mathbf{R}^3) : u = 0 \text{ on } \Gamma_0\big\};$$

$$\mathrm{LD}(\Omega, \mathbf{R}^3) := \big\{u \in L^1(\Omega, \mathbf{R}^3) : e(u) \in L^1(\Omega, \mathbf{M}_s^{3\times 3}\big\}.$$

We aim to propose a simplified but accurate model where qualitative and quantitative analysis are able to be done in an easier way than with the starting problem. For this, we will consider ε as a parameter going to zero and determine the asymptotic behavior of (approximate) solutions of the previous minimization problem by identifying the Γ-limit of F_ε extended into the fixed space $L^1(\Omega, \mathbf{R}^3)$. More precisely, we still denote by F_ε its extension outside A_ε given by:
$F_\varepsilon : L^1(\Omega, \mathbf{R}^3) \to \mathbf{R} \cup \{+\infty\}$,

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\ dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u))\ dx \text{ if } u \in A_\varepsilon \\ +\infty \text{ otherwise.} \end{cases}$$

Clearly the previous minimization problem is equivalent to the following

$$(\mathcal{P}_\varepsilon) \qquad\qquad \inf\big\{F_\varepsilon(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\big\}.$$

We will use the classical spaces

$$W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) := \big\{u \in W^{1,p}(\Omega, \mathbf{R}^3) : u = 0 \text{ on } \Gamma_0\big\};$$

$$\mathrm{BD}(\Omega, \mathbf{R}^3) := \big\{v \in L^1(\Omega, \mathbf{R}^3) : e(v) \in \mathcal{M}(\Omega, \mathbf{M}^{3\times 3}_s\big\};$$

$$\mathrm{BD}(S, \mathbf{R}^2) := \big\{v \in L^1(S, \mathbf{R}^2) : e(v) \in \mathcal{M}(S, \mathbf{M}^{2\times 2}_s\big\}$$

and the set of "horizontal rigid motions" on S, i.e.

$$\begin{aligned} \widehat{\mathcal{R}_H} \quad &:= \quad \big\{v \in \mathrm{BD}(S, \mathbf{R}^2) : e_{\alpha\beta}(v) = 0\big\} \\ &= \quad \big\{v : v(x) = (a_1 - bx_2, a_2 + bx_1),\ (a_1, a_2) \in \mathbf{R}^2,\ b \in \mathbf{R}\big\}. \end{aligned}$$

For reasons clarified in Lemma 2 below, we define the limit admissible set A_0 by

$$A_0 := \big\{u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) : \gamma_S(\hat{u}) \in \mathrm{BD}(S, \mathbf{R}^2)\big\}$$

and its subspace A^1_0 made of smooth elements

$$A^1_0 := \big\{u \in \ A_0 : \gamma_S(u) \in \mathcal{C}^1(\bar{S}, \mathbf{R}^3)\big\}.$$

For simplicity of notation γ_S will denote indifferently the trace operator from $W^{1,p}(\Omega, \mathbf{R}^i)$ into $W^{1-\frac{1}{p},p}(S, \mathbf{R}^i)$ for $i \in \{1, 2, 3\}$.

For all $\xi \in \mathbf{M}^{3\times 3}_s$, define $\hat{\xi} \in \mathbf{M}^{2\times 2}_s$ by $(\hat{\xi})_{\alpha\beta} = \xi_{\alpha\beta}$ and consider the function $g_0 : \mathbf{M}^{2\times 2}_s \to \mathbf{R}$ defined by

$$g_0(\zeta) := \min\big\{g(\xi) : \xi \in \mathbf{M}^{3\times 3}_s, \hat{\xi} = \zeta\big\}.$$

It is easily seen that g_0 is a convex function on $\mathbf{M}^{2\times 2}_s$ and it will be sometimes convenient to express g_0 as stated in the next lemma:

Lemma 1. *For all* 3×2*-matrix* ξ*,* $\inf_{\lambda\in\mathbf{R}^3} g((\xi|\lambda)_s) = g_0((\xi_{\alpha\beta})_s)$.

Proof. The conclusion is a straightforward consequence of the calculation

$$\begin{aligned} \min_{\lambda\in\mathbf{R}^3} g((\xi|\lambda)_s) \quad &= \quad \min_{\lambda\in\mathbf{R}^3} g\Big(\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+\xi_{21}}{2} & \frac{\lambda_1+\xi_{31}}{2} \\ \frac{\xi_{12}+\xi_{21}}{2} & \xi_{22} & \frac{\lambda_2+\xi_{32}}{2} \\ \frac{\lambda_1+\xi_{31}}{2} & \frac{\lambda_2+\xi_{32}}{2} & \lambda_3 \end{pmatrix}\Big) \\ &= \quad \min_{\lambda\in\mathbf{R}^3} g\Big(\begin{pmatrix} \xi_{11} & \frac{\xi_{12}+\xi_{21}}{2} & \lambda_1 \\ \frac{\xi_{12}+\xi_{21}}{2} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}\Big) \end{aligned}$$

and the definition of g_0. □

In Section 3, we establish the Γ-convergence of the sequence $(F_\varepsilon)_{\varepsilon>0}$ to the functional defined by:

$$F_0 : L^1(\Omega, \mathbf{R}^3) \to \mathbf{R} \cup \{+\infty\}$$

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S \big(g_0(e(\gamma_s(\hat{u})) \text{ if } u \in A_0 \\ +\infty \text{ otherwise} \end{cases}$$

when $L^1(\Omega, \mathbf{R}^3)$ is equipped with its strong topology. For the defintion of the scalar measure $g_0(e(v))$, $v \in \mathrm{BD}(S, \mathbf{R}^2)$, we refer the reader to [12], [17], [6]. We recall that the integral over S of the measure $g_0(e(v))$ is given by

$$\int_S g_0(e(v)) = \int_S g_0(e_a(v))\ d\hat{x} + \int_S g_0^\infty\big(\frac{e_s(v)}{|e_s(v)|}\big)|e_s(v)|$$

where $e(v) = e_a(v)\ d\hat{x} + e_s(v)$ is the Lebesgue decomposition of $e(v)$, $|e_s(v)|$ denotes the total variation of the singular measure $e_s(v)$, $\frac{e_s(v)}{|e_s(v)|}$ its Radon-Nikodym derivative, and $\xi \mapsto g_0^\infty(\xi) := \lim_{t\to+\infty} g_0(t\xi)/t$ is the recession function of g_0. In [3], Ambrosio, Coscia and Dal Maso proved that the singular measure $e_s(\hat{v})$ has the following structure: there exists a rectifiable set $S_v \subset S$ with normal ν_v and traces $v^\pm$ on both sides of S_v such that

$$e_s(v) = \frac{1}{2}\Big((v^+ - v^-) \otimes \nu_v + \nu_v \otimes (v^+ - v^-)\Big)\mathcal{H}^1 \lfloor S_v + Cv,$$

with Cv singular with respect to the Lebesgue measure and vanishing on Borel sets of σ-finite $\mathcal{H}^1$-measure. We will also denote by $[v] \otimes_s \nu_v$ the symetrical tensor product $(v^+ - v^-) \otimes \nu_v + \nu_v \otimes (v^+ - v^-)$. The set S_v will represent the macroscopic cracks whereas the support of C_v deals with the diffuse defects or fractal cracks in S towards the layer shrinks.

We start by establishing a compactness result which justifies the introduction of the limit set A_0 of admissible functions. As usual the arrows $\to$ and $\rightharpoonup$ will denote strong and weak convergences respectively.

Lemma 2 (Compactness lemma). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^1(\Omega, \mathbf{R}^3)$ such that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$. Then, there exists $u \in L^1(\Omega, \mathbf{R}^3)$ and a subsequence not relabelled such that*

i) $u_\varepsilon \rightharpoonup u$ *in* $BD(\Omega, \mathbf{R}^3)$ *and* $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$*;*

ii) $u_\varepsilon \rightharpoonup u$ *in* $W^{1,p}_{\Gamma_0}(\Omega_\eta, \mathbf{R}^3)$ *for every* $\eta > 0$*;*

iii) $\gamma_S(\hat{u}) \in BD(S, \mathbf{R}^2)$*;*

iv) $\exists r_\varepsilon \in \widehat{\mathcal{R}_H}$ *such that* $\displaystyle\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \hat{u}_\varepsilon\ dx_3 + r_\varepsilon \rightharpoonup \gamma_S(\hat{u})$ *in* $BD(S, \mathbf{R}^2)$.

Proof. From now on, we do not relabel the various subsequences obtained in the proof, and C will denote a positive constant which may vary from line to line. We divide the proof into two steps.
Step 1. We establish (i) and (ii). According to the coerciveness condition (1) and because $u_\varepsilon = 0$ on Γ_0, $(u_\varepsilon)_{\varepsilon>0}$ is clearly bounded in $\mathrm{LD}(\Omega, \mathbf{R}^3)$ so that there exist $u \in \mathrm{BD}(\Omega, \mathbf{R}^3)$ and a subsequence satisfying $u_\varepsilon \rightharpoonup u$ in $\mathrm{BD}(\Omega, \mathbf{R}^3)$ and $u_\varepsilon \to u$ in $L^1(\Omega, \mathbf{R}^3)$. The weak convergence of u_ε to u in $W^{1,p}_{\Gamma_0}(\Omega_\eta, \mathbf{R}^3)$ for every $\eta > 0$ is obvious. We are going to prove that $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$.

We extend every function $w \in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbf{R}^3)$ by reflexion on $S \times (\pm r, \pm 2r \mp \frac{\varepsilon}{2})$ so that the extended function, denoted by $\tilde{w}$, belongs to $W^{1,p}(S \times (\pm\frac{\varepsilon}{2}, \pm 2r \mp \frac{\varepsilon}{2}), \mathbf{R}^3)$ and satisfies

$$\int_{S\times(\pm\frac{\varepsilon}{2}, \pm 2r \mp \frac{\varepsilon}{2})} |\nabla \tilde{w}|^p \, dx \leq 2 \int_{\Omega_\varepsilon^\pm} |\nabla w|^p \, dx.$$

Set $\Omega^\pm := \Omega \cap [\pm x_3 > 0]$ and, for every function $w \in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbf{R}^3)$, define its ε-translate $T_\varepsilon w$ in $W^{1,p}(\Omega \setminus S, \mathbf{R}^3)$ by

$$T_\varepsilon w(\hat{x}, x_3) = \begin{cases} \tilde{w}(\hat{x}, x_3 + \varepsilon/2), & \text{if } x \in \Omega^+ \\ \tilde{w}(\hat{x}, x_3 - \varepsilon/2), & \text{if } x \in \Omega^-. \end{cases}$$

Because

$$\begin{aligned} \sup_{\varepsilon>0} \int_{\Omega\setminus S} |\nabla T_\varepsilon u_\varepsilon|^p \, dx &\leq \sup_{\varepsilon>0} \int_{S\times(\frac{\varepsilon}{2}, 2r-\frac{\varepsilon}{2})\cup S\times(-2r+\frac{\varepsilon}{2}, -\frac{\varepsilon}{2})} |\nabla \tilde{u}_\varepsilon|^p \, dx \\ &\leq 2 \sup_{\varepsilon>0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p \, dx < +\infty, \end{aligned}$$

Poincaré's inequality implies that there exist $z \in W^{1,p}(\Omega \setminus S, \mathbf{R}^3)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,p}(\Omega \setminus S, \mathbf{R}^3)$ and $u_\varepsilon \to z$ in $L^p(\Omega, \mathbf{R}^3)$. Actually $u = z$ (so that $u \in W^{1,p}(\Omega \setminus S, \mathbf{R}^3)$) since for all $\psi \in \mathcal{D}(\Omega \setminus S, \mathbf{R}^m)$,

$$\begin{aligned} \int_{\Omega\setminus S} u.\psi \, dx &= \lim_{\varepsilon\to 0} \int_{\Omega\setminus S} u_\varepsilon.\psi(\hat{x}, x_3 - \frac{\varepsilon}{2}) \, dx \\ &= \lim_{\varepsilon\to 0} \int_{\Omega\setminus S} T_\varepsilon u_\varepsilon.\psi \, dx \\ &= \int_{\Omega\setminus S} z.\psi \, dx. \end{aligned}$$

For all $w \in W^{1,p}(\Omega \setminus S, \mathbf{R}^3)$, we will denote the traces on S of w considered as a function of $W^{1,p}(\Omega^\pm, \mathbf{R}^3)$ by $w^\pm$ and its jump accross S by $[w] := w^+ - w^-$. Take $\theta \in \mathbf{C}_c^\infty(S))$, Green's formula yields

$$2 \int_{B_\varepsilon} e_{\alpha 3}(u_\varepsilon) \, \theta \, dx = \int_S \theta [T_\varepsilon u_\alpha^\varepsilon] \, d\hat{x} + I_\varepsilon \tag{5}$$

where

$$I_\varepsilon := -\int_{B_\varepsilon} \frac{\partial u_3^\varepsilon}{\partial x_\alpha}\theta\ dx.$$

The left hand side term of (5) tends to 0 by coercivity condition (2), and from

$$\begin{aligned} u_3^\varepsilon(\hat{x}, \pm|x_3|) &= u_3^\varepsilon(\hat{x}, \pm\varepsilon/2) + \int_{\pm\varepsilon/2}^{\pm|x_3|} \frac{\partial u_3^\varepsilon}{\partial x_3}(\hat{x}, t)\ dt \\ &= (T_\varepsilon u_3^\varepsilon)^\pm(\hat{x}) + \int_{\pm\varepsilon/2}^{\pm|x_3|} \frac{\partial u_3^\varepsilon}{\partial x_3}(\hat{x}, t)\ dt, \end{aligned}$$

we deduce

$$\begin{aligned} \int_{B_\varepsilon} |u_3^\varepsilon(\hat{x}, x_3)|\ dx &\leq \varepsilon \int_S \big(|(T_\varepsilon u_3^\varepsilon)^+(\hat{x})| + |(T_\varepsilon u_3^\varepsilon)^-(\hat{x})|\big)\ d\hat{x} + \varepsilon \int_{B_\varepsilon} |\frac{\partial u_3^\varepsilon}{\partial x_3}|\ dx \\ &\leq C\varepsilon(1+\varepsilon) \end{aligned} \tag{6}$$

so that

$$|I_\varepsilon| = \Big|\int_{B_\varepsilon} u_3^\varepsilon \frac{\partial \theta}{\partial x_\alpha}\ dx\Big| \leq C\varepsilon(1+\varepsilon),$$

and the right hand side term of (5) tends to $\int_S \theta[u_\alpha]\ d\hat{x}$. Going to the limit on ε in (5) yields $[u_\alpha] = 0$ a.e. on S. Similarly, by letting $\varepsilon \to 0$ in

$$\int_{B_\varepsilon} \frac{\partial u_3^\varepsilon}{\partial x_3}\theta\ dx = \int_S \theta[T_\varepsilon u_3^\varepsilon]\ d\hat{x},$$

and by using coercivity condition (2), we obtain $[u_3] = 0$ a.e. on S.

Step 2. We establish (iii) and (iv). Coercivity condition (2) implies

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_{B_\varepsilon} |\widehat{e(u_\varepsilon)} dx| < +\infty$$

so that there exists a subsequence of $v_\varepsilon := \dfrac{1}{\varepsilon}\displaystyle\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \hat{u}_\varepsilon\ dx_3$, $r_\varepsilon \in \widehat{\mathcal{R}_H}$ and $v \in$ $\mathrm{BD}(S, \mathbf{R}^2)$ such that $v_\varepsilon + r_\varepsilon \rightharpoonup v$ in $\mathrm{BD}(S, \mathbf{R}^2)$. It remains to prove that $v = \gamma_S(\hat{u})$. For arbitrary $\theta \in \mathcal{C}_c^\infty(S)$ and $x_3 \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, one has

$$u_\alpha^\varepsilon(\hat{x}, \pm|x_3|)\theta(\hat{x}) = (T_\varepsilon u_\alpha^\varepsilon)^\pm(\hat{x}))\theta(\hat{x}) + \int_{\pm\varepsilon/2}^{\pm|x_3|} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}(\hat{x}, t)\theta(\hat{x})\ dt$$

so that

$$\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \int_S u_\alpha^\varepsilon(\hat{x}, x_3)\theta(\hat{x})\ dx = \frac{1}{2}\int_S (T_\varepsilon u_\alpha^\varepsilon)^+(\hat{x}) + (T_\varepsilon u_\alpha^\varepsilon)^+(\hat{x}))\theta(\hat{x})\ d\hat{x} + J_\varepsilon \tag{7}$$

where

$$J_\varepsilon := \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{0} \int_S \int_{-\varepsilon/2}^{-|x_3|} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}(\hat{x}, t)\theta(\hat{x})\ dt\ dx + \frac{1}{\varepsilon}\int_0^{\frac{\varepsilon}{2}} \int_S \int_{\varepsilon/2}^{|x_3|} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}(\hat{x}, t)\theta(\hat{x})\ dt\ dx.$$

We claim that $\lim_{\varepsilon\to 0} J_\varepsilon = 0$. Indeed

$$\begin{aligned}\int_S \int_{\varepsilon/2}^{x_3} \frac{\partial u_\alpha^\varepsilon}{\partial x_3}\theta(\hat{x})\ dtd\hat{x} &= \int_S \int_{\varepsilon/2}^{x_3} \big(2e_{3\alpha}(u_\varepsilon) - \frac{\partial u_3^\varepsilon}{\partial x_\alpha}\big)\theta\ dtd\hat{x} \\ &= \int_S \int_{\varepsilon/2}^{x_3} \big(2e_{3\alpha}(u_\varepsilon)\theta + u_3^\varepsilon \frac{\partial \theta}{\partial x_\alpha}\big)\ dtd\hat{x},\end{aligned}$$

thus

$$|J_\varepsilon| \le C \int_{B_\varepsilon} \big(|e_{3\alpha}(u_\varepsilon)| + |u_3^\varepsilon)|\big)dx,$$

and the claim follows from (6) and coercivity condition (2). Letting $\varepsilon \to 0$ in (7), we obtain

$$\int_S v_\alpha \theta\ d\hat{x} = \int_S \gamma_S(u_\alpha)\theta\ d\hat{x}$$

thus $v_\alpha = \gamma_S(u_\alpha)$ a.e. in S since θ is arbitrary. □

3 The main convergence result

The main result of this section is the following theorem.

Theorem 1. *The sequence $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional F_0.*

The proof consists in establishing Proposition 1 and Proposition 2 below, corresponding to the lower bound and the upper bound in the definition of the Γ_{τ_s}-convergence.

Proposition 1 (Lower bound)**.** *For all u and all sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega, \mathbf{R}^3)$ such that $u_\varepsilon \xrightarrow{\tau_s} u$, the following inequality holds:*

$$F_0(u) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \tag{8}$$

Proof. Clearly one may assume that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$ so that from Lemma 2, u belongs to A_0, and $e\Big(\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \hat{u}_\varepsilon\ dx_3\Big) = \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \widehat{e(u_\varepsilon)}\ dx_3 \rightharpoonup e(\gamma_S(\hat{u}))$ in $\mathcal{M}(S, \mathbf{M}_s^{2\times 2})$. Thus from Jensen's inequality

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0} \Big(\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx \\ &\ge \liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx \\ &\ge \liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx \\ &\ge \int_\Omega f(\nabla u)dx + \int_S g_0(e(\gamma_S(\hat{u})).\end{aligned}$$

In the last inequality, we used the weak lower semicontinuity of the second integral functional with respect to the weak convergence of measures. For the convergence of the first integral, we proceeded as follows: take $\eta > \varepsilon$, write $\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx \geq \int_{\Omega_\eta} f(\nabla u_\varepsilon)dx$, and apply the lower semicontinuity of the integral functional $u \mapsto \int_{\Omega_\eta} f(\nabla u)dx$ for the weak convergence in $W^{1,p}(\Omega_\eta, \mathbf{R}^3)$. Then let $\eta \to 0$. $\square$

For proving the upper bound, we need to establish the following relaxation result.

Lemma 3 (Relaxation)**.** *The functional F_0 is the l.s.c. regularization for the strong topology of the functional defined on $L^1(\Omega, \mathbf{R}^3)$ by*

$$\tilde{F_0}(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S g_0(e(\gamma_S(\hat{u}))\ d\hat{x} \ \textit{if } u \in A_0^1, \\ +\infty \ \textit{otherwise.} \end{cases}$$

Proof. Step 1. By using standard lower semicontinuous arguments, it is easily seen that for every sequence $(u_n)_{n\in\mathbf{N}}$ strongly converging to u in $L^1(\Omega, \mathbf{R}^3)$, one has

$$F_0(u) \leq \liminf_{n\to+\infty} \tilde{F_0}(u_n).$$

Step 2. We assume $u \in A_0$ and we construct a sequence $(u_\delta)_{\delta>0}$ in A_0^1 such that

$$\begin{cases} u_\delta \to u \text{ strongly in } W^{1,p}(\Omega, \mathbf{R}^3); \\ \gamma_S(\hat{u}_\delta) = v_\delta \to v := \gamma_S(\hat{u}) \text{ strongly in } L^1(S, \mathbf{R}^2); \\ \displaystyle\lim_{\delta\to 0} \int_\Omega f(\nabla u_\delta)\ dx = \int_\Omega f(\nabla u)\ dx; \\ \displaystyle\lim_{\delta\to 0} \int_S g_0(e(v_\delta))\ d\hat{x} = \int_S g_0(e(v)). \end{cases}$$

Consider the open cylinder $\tilde{\Omega} := \tilde{S} \times (-r, r)$ containing Ω, where $\tilde{S}$ is an open set of $\mathbf{R}^2$ strictly containing S and extend u into a function $\tilde{u}$ in $W^{1,p}(\tilde{\Omega}, \mathbf{R}^3)$. We also extend v into a function $\tilde{v}$ in $\mathrm{BV}(\tilde{S}, \mathbf{R}^3)$. More precisely, let P_3 denote the extension operators

$$P_3 : W^{1,p}(\Omega, \mathbf{R}^3) \to W^{1,p}(\tilde{\Omega}, \mathbf{R}^3),$$

and $\gamma_{\tilde{S}}$ the trace operator associated with the Sobolev space $W^{1,p}(\tilde{\Omega}, \mathbf{R}^3)$. We define $\tilde{v}$ in $\mathrm{BV}(\tilde{S}, \mathbf{R}^3)$ by $\tilde{v} := \gamma_{\tilde{S}} \circ P_3(\hat{u})$, i.e., $\tilde{v} = \gamma_{\tilde{S}}(\hat{\tilde{u}})$.

For all $\delta > 0$, consider the open ball $B_{\eta(\delta)}(0)$ of $\mathbf{R}^2$ centered at 0 with a radius $\eta(\delta) > 0$ small enough so that $S + B_{\eta(\delta}(0) \subset \tilde{S}$. Let ϕ in $\mathcal{C}_c^\infty(\tilde{S})$ with $\phi = 1$ on a neighborhood $S + B_{\eta(\delta}(0)$ included in $\tilde{S}$ and $\rho_{\eta(\delta)}$ a standard mollifier with support $\bar{B}_{\eta(\delta)}(0)$. Set $v_\delta := \rho_{\eta(\delta)} * (\phi\tilde{v})$, $u_\delta := \rho_{\eta(\delta)} * (\phi\tilde{u})$ and choose $\eta(\delta)$ small enough so that:

$$\|v_\delta - v\|_{L^1(S,\mathbf{R}^2)} < \delta; \tag{9}$$

$$\Big| \int_{\mathbf{R}^2} g_0(\rho_{\eta(\delta)} * (\phi e(\tilde{v}))) - \int_{\mathbf{R}^2} g_0(\phi e(\tilde{v}))\Big| < \delta; \tag{10}$$

$$\|u_\delta - u\|_{W^{1,p}(\Omega,\mathbf{R}^3)} < \delta. \tag{11}$$

Estimates (9) and (11) are standard (note that the mollification of $\tilde{u}$ takes place only on the $\hat{x}$ argument). Estimate (10) is a straightforward consequence of the narrow convergence of the measure $g_0(\rho_{\eta(\delta)} * (\phi e(\tilde{v})))$ to the measure $g_0(\phi e(\tilde{v}))$ in $\mathbf{M}^+(\mathbf{R}^2)$ (see for instance Lemma 5.2 and Remark 5.1 in [17], or, if g_0 is positively homogeneous of degree 1, use Reshetnyak's continuity theorem, Theorem 2.39 in [4]).

Clearly $v_\delta \in \mathcal{C}^\infty(\bar{S}, \mathbf{R}^2)$ and, from (9), $v_\delta \to v$ in $L^1(S, \mathbf{R}^2)$. Moreover, from (10), and noticing that $\phi = 1$ on $S + B_{\eta(\delta)}(0)$,

$$\Big| \int_S g_0(e(v_\delta)) - \int_S g_0(e(v)) \Big| \leq \Big| \int_{\mathbf{R}^2} g_0(\rho_{\eta(\delta)} * (\phi e(\tilde{v})) - \int_{\mathbf{R}^2} g_0(\phi e(\tilde{v})) \Big| < \delta,$$

so that

$$\lim_{\delta \to 0} \int_S g_0(e(v_\delta)) = \int_S g_0(e(v)).$$

From (11), $u_\delta \to u$ in $W^{1,p}(\Omega, \mathbf{R}^3)$ and, from (3),

$$\lim_{\delta \to 0} \int_\Omega f(\nabla u_\delta)\ dx = \int_\Omega f(\nabla u)\ dx.$$

According to the definition of v_δ, u_δ, and from the fact that $\tilde{v} = \gamma_{\tilde{S}}(\hat{\tilde{u}})$, one has $\gamma_S(\hat{u}_\delta) = v_\delta$.

The sequence $(u_\delta)_{\delta > 0}$ fulfills all the conditions of the set A_0^1 except the boundary condition. By using De Giorgi's slicing method in a neighborhood of Γ_0 (see for instance Theorem 11.2.1 in [6]), one can modify u_δ into a new function $\tilde{u}_\delta \in A_0^1$ which has the same trace as its weak limit u on $\partial\Omega$, and satisfies

$$\limsup_{\delta \to 0} \int_\Omega f(\nabla \tilde{u}_\delta)\ dx \leq \int_\Omega f(\nabla u_\delta)\ dx,$$

thus, finally

$$\lim_{\delta \to 0} \int_\Omega f(\nabla \tilde{u}_\delta)\ dx = \int_\Omega f(\nabla u)\ dx.$$

Note that u_δ is not affected on a neighborhood of S by this modification because $\mathrm{dist}(\overline{\Gamma}_0, \overline{\partial B_\varepsilon \cap \Gamma}) > 0$. Thus $\tilde{u}_\delta$ that we denote now by u_δ is the expected sequence. □

Proposition 2 (Upper bound)**.** *The following inequality holds in* $L^1(\Omega, \mathbf{R}^3)$

$$(\Gamma - \limsup F_\varepsilon) \leq F_0. \tag{12}$$

Proof. Step 1. We establish $\Gamma - \limsup F_\varepsilon \leq \tilde{F}_0$.

Take $u \in A_0^1$. In what follows we set $v := \gamma_S(\hat{u})$. For every fixed ξ in $\mathcal{D}(S, \mathbf{R}^3)$, consider the function v_ε in $W^{1,p}(B, \mathbf{R}^3)$, $B = S \times (-\frac{1}{2}, \frac{1}{2})$ defined by

$$\begin{cases} v_\varepsilon^\alpha(\hat{x}, x_3) := v^\alpha(\hat{x}) + \varepsilon x_3 \xi^\alpha(\hat{x}) \\ v_\varepsilon^3(\hat{x}, x_3) := \varepsilon u^3(\hat{x}, 0) + \varepsilon^2 x_3 \xi^3(\hat{x}) \end{cases}$$

and set

$$u_\varepsilon(\hat{x}, x_3) := \begin{cases} u(\hat{x}, x_3 - \frac{\varepsilon}{2}) + \frac{\varepsilon}{2}\xi(\hat{x}) \text{ in } \Omega_\varepsilon^+ \\ v_\varepsilon(\hat{x}, \frac{x_3}{\varepsilon}) \text{ in } B_\varepsilon \\ u(\hat{x}, x_3 + \frac{\varepsilon}{2}) - \frac{\varepsilon}{2}\xi(\hat{x}) \text{ in } \Omega_\varepsilon^-. \end{cases}$$

Clearly $u_\varepsilon \in A_\varepsilon$ except the boundary condition and $u_\varepsilon \to u$ in $L^1(\Omega, \mathbf{R}^3)$. An easy calculation and the local Lipschitz conditions (3), (4), yield

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)\ dx = \int_\Omega f(\nabla u)\ dx$$

and

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{B_\varepsilon} g(e(u_\varepsilon))\ dx &= \lim_{\varepsilon \to 0} \int_B g\big((\hat{\nabla} v_\varepsilon | \frac{1}{\varepsilon}\frac{\partial v_\varepsilon}{\partial x_3})_s\big)\ dx \\ &= \int_S g\big((\hat{\nabla} v | \xi)_s\big)\ d\hat{x}, \end{aligned}$$

so that

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \int_\Omega f(\nabla u)\ dx + \int_S g\big((\hat{\nabla} v|\xi)_s\big)\ d\hat{x}.$$

By using again De Giorgi's slicing method in a neighborhood of Γ_0, one can modify u_ε into a function still denoted by u_ε, satisfying the boundary condition on Γ_0 and

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \int_\Omega f(\nabla u)\ dx + \int_S g\big((\hat{\nabla} v|\xi)_s\big)\ d\hat{x}. \tag{13}$$

According to a well known interchange result between infimum and integral (see [2]) we have

$$\int_S g_0(e(v))\ d\hat{x} = \inf_{\xi \in \mathcal{D}(S, \mathbf{R}^3)} \int_S g\big((\hat{\nabla} v|\xi)_s\big)\ d\hat{x}. \tag{14}$$

By taking the infimum over all $\xi \in \mathcal{D}(S, \mathbf{R}^3)$ in (13) and by using Lemma 1 we deduce

$$\inf \Big\{ \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \to u \text{ in } L^1(\Omega, \mathbf{R}^3) \Big\} \leq \tilde{F}_0(u),$$

i.e. $\Gamma - \limsup F_\varepsilon \leq \tilde{F}_0$.

Step 2. Taking the lower semicontinuous envelope of each two functionals for the strong topology of $L^1(\Omega, \mathbf{R}^3)$, the conclusion then follows from the lower semicontinuity of $\Gamma - \limsup F_\varepsilon$ and from Lemma 3. $\square$

According to Lemma 2 and to variational properties of the Γ-convergence, we obtain :

Corollary 1. *Let $\bar{u}_\varepsilon$ be a solution of $(\mathcal{P}_\varepsilon)$. Then there exist a subsequence of $(\bar{u}_\varepsilon)_{\varepsilon>0}$ and $\bar{u}$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ such that*

$$\begin{aligned}
&\bar{u}_\varepsilon \rightharpoonup \bar{u} \text{ in } BD(\Omega, \mathbf{R}^3);\\
&\bar{u}_\varepsilon \rightharpoonup \bar{u} \text{ in } W^{1,p}_{\Gamma_0}(\Omega_\eta, \mathbf{R}^3) \text{ for every } \eta > 0;\\
&\gamma_S(\hat{\bar{u}}) \in BD(S, \mathbf{R}^2).
\end{aligned}$$

Moreover $\bar{u}$ is solution of the minimization problem

$$(\mathcal{P}) \qquad \min\left\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\right\}$$

and

$$\min\left\{F_\varepsilon(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\right\} \to \min\left\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\right\}.$$

Thus, in this simplified case (see Section 5 for a realistic geometry), our proposal of model is given by the limit problem $(\mathcal{P})$ which describes the equilibrium of a structure made of two adherents perfectly stuck to a material surface. The reference configuration of the adherents are $\Omega^\pm := \Omega \cap [\pm x_3 > 0]$ while the one of the material surface is S. The adherents are hyperelastic with bulk energy density f and the material surface is pseudo-plastic with surface density g_0. Due to the linear growth of g_0, the displacement field solution of $(\mathcal{P})$ may present discontinuities in S which may be interpreted in terms of cracks. It is worthwhile to note that this situation with strong adhesive layer is completely different from the one considered in [15] with a soft adhesive: in the asymptotic model, the soft adhesive layer is replaced by a mechanical constraint between the adherents, whereas the strong adhesive layer is replaced by a material surface perfectly stuck to adherents. Another strategy proposed in [7] leads to a similar model.

4 The case when f is not quasiconvex and g is not convex

In this section, we drop the quasiconvex and convex assumptions on the density functions f and g respectively. This is the case when the materials undergo reversible solid/solid phase transformations, for which the density functions present a multi-well structure (for f in the large deformation setting see [10], for g in the setting of small perturbations see [13]). However we assume that f and g satisfy the locally Lipschitz conditions (3), (4) and that g is positively 1-homogeneous. In this more general situation, we would like to show that the limit energy functional is given by

$$F_0 : L^1(\Omega, \mathbf{R}^3) \to \mathbf{R} \cup \{+\infty\}$$

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega Qf(\nabla u)\, dx + \int_S SQg_0(e(\gamma_s(\hat{u})) \text{ if } u \in A_0, \\ +\infty \text{ otherwise} \end{cases}$$

where Qf is the quasiconvex envelope of f and $SQg_0 : \mathbf{M}_s^{2\times 2} \to \mathbf{R}$ is the symmetric quasiconvex envelope of g_0 defined by

$$SQg_0(\zeta) := \inf\Big\{\frac{1}{|\hat{D}|}\int_{\hat{D}} g_0(\zeta + e(\varphi))\ d\hat{x} : \varphi \in \mathcal{C}_0^\infty(\hat{D}, \mathbf{R}^2)\Big\}.$$

Let denote the operator $\zeta \mapsto \zeta_s$ from $\mathbf{M}^{2\times 2}$ into $\mathbf{M}_s^{2\times 2}$ by $\mathcal{S}_2$. Thus, for every $\zeta \in \mathbf{M}_s^{2\times 2}$, $SQg_0(\zeta) = Q(g_0 \circ \mathcal{S}_2)(\zeta)$ where $Q(g_0 \circ \mathcal{S}_2)$ is the quasiconvex envelope of $g_0 \circ \mathcal{S}_2$. Note that the right hand side term does not depend on the choice of the cube $\hat{D}$ of $\mathbf{R}^2$ and that SQg_0 is 1-homogeneous. With the notation of Section 2, the integral over S of the measure $SQg_0(e(v))$ is given by

$$\int_S SQg_0(e(v)) = \int_S SQg_0(e_a(v))\ d\hat{x} + \int_S SQg_0\big(\frac{(e_s(v)}{|e_s(v)|}\big)|e_s(v)|.$$

Like in Section 2, we prove Proposition 3 and Proposition 5 below, corresponding to the lower and the upper bound in the definition of the Γ-convergence. Unfortunately, we establish the lower bound when u belongs to the subset $\tilde{A}_0$ of A_0 defined by

$$\tilde{A}_0 := \big\{u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) : \gamma_S(\hat{u}) \in \mathrm{SBD}(S, \mathbf{R}^2)\big\}$$

where $\mathrm{SBD}(S, \mathbf{R}^2)$ denotes the set of the elements u of $\mathrm{BD}(S, \mathbf{R}^2)$ whose the Cantor part of the strain tensor $e(u)$ is zero. Then, the main result of this section is

Theorem 2. *The restriction of the Γ-limit of F_ε to the set $\tilde{A}_0$ is given by*

$$F_0(u) = \int_\Omega Qf(\nabla u)\ dx + \int_S SQg_0(e_a(\gamma_S(\hat{u})))\ d\hat{x} + \int_S SQg_0([\gamma_s(\hat{u})] \otimes_s \nu_{\gamma_S(\hat{u})}) d\mathcal{H}^1.$$

Remark 1. *If we assume that an approximate minimizer of $(\mathcal{P}_\varepsilon)$ strongly converges to some $\bar{u}$ in $L^1(\Omega, \mathbf{R}^3)$ whose distributional gradient has no Cantor part, according to the variational nature of the Γ-convergence, we deduce that $\bar{u}$ is a solution of the limit problem*

$$(\mathcal{P}) \qquad \min\big\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\big\}$$

and

$$\inf\big\{F_\varepsilon(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\big\} \to \min\big\{F_0(u) - L(u) : u \in \tilde{A}_0\big\}.$$

Therefore, under the asumption that some approximate minimizer is regular in the sense above, problem $(\mathcal{P})$ is a good model in the sense of Section 3, where the density functions are now Qf and SQg_0.

Remark 2. *In the case when the deformations in the adhesive may be large, they are modeled as hyperelastic together with the deformations in the adherents.*

In this particular case, we obtain a complete description of the Γ*-limit* F_0 *in the set*

$$\tilde{A}_0' := \{u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) : \gamma_S(u) \in BV(S, \mathbf{R}^3)\}.$$

More precisely F_0 *is defined in* $\tilde{A}_0'$ *by*

$$\begin{aligned} F_0(u) = \int_\Omega Qf(\nabla u)\ dx \quad &+ \quad \int_S Qg_0(\nabla(\gamma_S(u)))\ d\hat{x} \\ &+ \quad \int_S (Qg_0)^\infty\Big(\frac{D_s\gamma_S(u)}{|D_s\gamma_S(u)|}\Big)|D_s\gamma_S(u)| d\mathcal{H}^1, \end{aligned}$$

where $Du = \nabla u\ d\hat{x} + D_s u$ *is the Lebesgue decomposition of the distributional derivative* Du*,* $g_0(\zeta) = \min\{g(\xi) : \xi \in \mathbf{M}^{3\times3}, \hat{\xi} = \zeta\}$ *for every* $\zeta \in \mathbf{R}^3$*, and* $\zeta \mapsto (Qg_0)^\infty(\zeta) := \lim_{t\to+\infty} Qg_0(t\zeta)/t$ *is the recession function of* Qg_0*. The proof uses the relaxation theorem, Theorem 11.3.1 in [6], instead of Proposition 4 below and follows point by point the claims of Propositions 3, 5 below.*

Proposition 3 (Lower bound)**.** *For all* u *in* $\tilde{A}_0$ *and all sequence* $(u_\varepsilon)_{\varepsilon>0}$ *in* $L^1(\Omega, \mathbf{R}^3)$ *such that* $u_\varepsilon \to u$*, the following inequality holds:*

$$F_0(u) \le \liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon). \tag{15}$$

Proof. One has

$$\begin{aligned} \liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to0} \Big(\int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx\Big) \\ &\ge \liminf_{\varepsilon\to0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to0} \frac{1}{\varepsilon}\int_{B_\varepsilon} g(e(u_\varepsilon))dx \\ &\ge \liminf_{\varepsilon\to0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon)dx + \liminf_{\varepsilon\to0} \frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx. \end{aligned} \tag{16}$$

For a.e. x in B, set $\hat{v}_\varepsilon(x) := \hat{v}_\varepsilon(\hat{x}, \varepsilon x_3)$ and $v^3_\varepsilon(x) = \varepsilon v^3_\varepsilon(\hat{x}, x_3)$. Set $B := S \times (-\frac{1}{2}, \frac{1}{2})$. We have

$$\liminf_{\varepsilon\to0} \frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx \ge \liminf_{\varepsilon\to0} \int_B SQg_0((\widehat{e(v_\varepsilon)})dx. \tag{17}$$

Consider the function $h : \mathbf{M}^{3\times3}_s \to \mathbf{R}$ defined for every $\xi \in \mathbf{M}^{3\times3}_s$ by $h(\xi) := SQg_0(\hat{\xi})$. It is easily seen that h is symmetric quasiconvex, i.e. satisfies the inequality:

$$h(\xi) \le \frac{1}{|D|}\int_D h(\xi + e(\varphi))\ dx.$$

for every $\varphi \in \mathcal{C}^\infty_c(D, \mathbf{R}^3)$ where D is any cube of $\mathbf{R}^3$ (see [9] for the definition). Moreover (17) can be written

$$\liminf_{\varepsilon\to0} \frac{1}{\varepsilon}\int_{B_\varepsilon} g_0(\widehat{e(u_\varepsilon)})dx \ge \liminf_{\varepsilon\to0} \int_B h(e(v_\varepsilon))dx. \tag{18}$$

Let denote by $\mathrm{BD}(B, \mathbf{R}^3)$ the space of bounded deformation on B and by $\mathcal{R}_H$ the set of rigid motions on B. According to coercivity condition (2) we have

$$\sup_{\varepsilon>0} \int_B \Big| \begin{pmatrix} e_{\alpha\beta}(\hat{v}_\varepsilon) & \frac{1}{\varepsilon} e_{\alpha 3}(v_\varepsilon) \\ \frac{1}{\varepsilon} e_{3\alpha}(v_\varepsilon) & \frac{1}{\varepsilon^2} \frac{\partial v_3^\varepsilon}{\partial v_3} \end{pmatrix} \Big| \, dx \leq \frac{1}{\alpha} \sup_{\varepsilon>0} \frac{1}{\varepsilon} \int_{B_\varepsilon} g(e(u_\varepsilon)) dx < +\infty. \quad (19)$$

Thus, by using the arguments of the proof of Lemma 2, one can easily establish the existence of $v \in \mathrm{BD}(B, \mathbf{R}^3)$ and $r_\varepsilon \in \mathcal{R}_H$ such that $v_\varepsilon + r_\varepsilon \rightharpoonup v$ in $\mathrm{BD}(B, \mathbf{R}^3)$, $\hat{v} = \gamma_S(\hat{u})$, $v_3 = 0$ and $\frac{\partial v}{\partial x_3} = 0$. Combining (16), (18), a classical results in relaxation theory (Theorem 11.2.1 in [6]) and a relaxation result in $\mathrm{BD}(S, \mathbf{R}^2)$ (see [9]), we infer

$$\begin{aligned} \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq & \int_\Omega Qf(\nabla u) \; dx + \int_S SQg_0(e_a(\gamma_S(\hat{u})) \; d\hat{x} \\ & + \int_S SQg_0([\gamma_s(\hat{u})] \otimes_s \nu_{\gamma_S(u)}(\hat{x})) d\mathcal{H}^1(\hat{x}) \end{aligned}$$

which ends the proof. $\square$

For proving the upper bound, we need to establish the following relaxation result.

Proposition 4 (Relaxation)**.** *The functional F_0 is the l.s.c. regularization for strong topology of the functional defined on $L^1(\Omega, \mathbf{R}^3)$ by*

$$\tilde{F_0}(u) := \begin{cases} \int_\Omega f(\nabla u) \; dx + \int_S g_0(e(\gamma_S(\hat{u})) \; d\hat{x} \; \textit{if } u \in A_0^1, \\ +\infty \; \textit{otherwise.} \end{cases}$$

In the proof of Proposition 4 we will use the following lemma.

Lemma 4. *Let $\eta > 0$, then for every $\zeta \in \mathbf{M}_s^{2\times 2}$,*

$$\lim_{\eta \to 0} SQ(\eta| \; . \; |^p + g_0)(\zeta) = SQg_0(\zeta).$$

Proof. From the integral representations

$$SQ(\eta| \; . \; |^p + g_0)(\zeta) = \inf_{\phi \in C_c^\infty(D, \mathbf{R}^2)} \frac{1}{|D|} \int_D (\eta| \; . \; |^p + g_0)(\zeta + e_(\phi)) d\hat{x},$$

and

$$SQg_0(\zeta) = \inf_{\phi \in C_c^\infty(D, \mathbf{R}^2)} \frac{1}{|D|} \int_D g_0(\zeta + e(\phi)) d\hat{x},$$

where D is an arbitrary cube of $\mathbf{R}^2$, we can write

$$\begin{aligned}
&\lim_{\eta\to 0}\Big(\inf_{\phi\in C_c^\infty(D,\mathbf{R}^2)}\frac{1}{|D|}\int_D(\eta|\,.\,|^p+g_0)(\zeta+e(\phi))d\hat{x}\Big)\\
=\;&\inf_{\eta>0}\Big(\inf_{\phi\in C_c^\infty(D,\mathbf{R}^2)}\frac{1}{|D|}\int_D(\eta|\,.\,|^p+g_0)(\zeta+e(\phi))d\hat{x}\Big)\\
=\;&\inf_{\phi\in C_c^\infty(D,\mathbf{R}^2)}\Big(\inf_{\eta>0}\frac{1}{|D|}\int_D(\eta|\,.\,|^p+g_0)(\zeta+e(\phi))d\hat{x}\Big)\\
=\;&\inf_{\phi\in C_c^\infty(D,\mathbf{R}^2)}\Big(\lim_{\eta\to 0}\frac{1}{|D|}\int_D(\eta|\,.\,|^p+g_0)(\zeta+e(\phi))d\hat{x}\Big).
\end{aligned}$$

We complete the proof by using the dominated convergence theorem. □

Proof of Proposition 4. Denote by $\bar{\tilde{F}}_0$ the l.s.c. regularization of $\tilde{F}_0$. By using standard l.s.c. results on integral functionals defined on Sobolev or BV-spaces, one can easily prove $F_0\le\bar{\tilde{F}}_0$ (see for instance [6], chapter 10). We only establish the converse inequality $\bar{\tilde{F}}_0\le F_0$. Its proof is not easy because of the condition $u\in A_0$ and the fact that f and g do not fulfill the same growth conditions.

Let $u\in L^1(\Omega,\mathbf{R}^3)$ such that $F_0(u)<+\infty$. Then $u\in A_0$. We have to exhibit u_n in A_0^1 strongly converging to u in $L^1(\Omega,\mathbf{R}^3)$ such that $\lim_{n\to+\infty}\tilde{F}_0(u_n)=F_0(u)$. We proceed into two steps.

Step 1. We prove the thesis when $u\in A_0^1$. For shorten notation we write $v:=\gamma_S(\hat{u})$.

Let $\eta>0$ intended to go to 0 and denote the constant involved in Korn's inequality by K:

$$\int_S|\nabla w|^p\,d\hat{x}\le K\Big(\int_S|e(w)|^p+|w|^p\Big)\,d\hat{x}$$

for all function w in $W^{1,p}(S,\mathbf{R}^2)$. From relaxation theory in Sobolev spaces, there exists a sequence of smooth functions $(u_n,v_n)_{n\in\mathbf{N}}$ in $W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3)\times W^{1,p}(S,\mathbf{R}^2)$ such that (see for instance [6], Theorem 11.2.1 and Theorem 11.4.2)

$$\begin{cases}
u_n\rightharpoonup u \text{ in } W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3),\ (|\nabla u_n|^p)_{n\in\mathbf{N}} \text{ uniformly integrable;}\\
v_n\rightharpoonup v \text{ in } W^{1,p}(S,\mathbf{R}^3);\\
\displaystyle\lim_{n\to+\infty}\int_\Omega f(\nabla u_n)dx=\int_\Omega Qf(\nabla u)dx;\\
\displaystyle\lim_{n\to+\infty}\int_S(3K\beta\eta|\,.\,|^p+g_0)(e(v_n))d\hat{x}=\int_S SQ(3K\beta\eta|\,.\,|^p+g_0)(e(v))d\hat{x}.
\end{cases}$$

The additional condition that $(|\nabla u_n|^p)_{n\in\mathbf{N}}$ may be assumed to be uniformly integrable comes from the following consideration. Consider the sequence $(\tilde{u}_n)_{n\in\mathbf{N}}$ whose gradients generate the same Young measure μ and such that $(|\nabla\tilde{u}_n|^p)_{n\in\mathbf{N}}$ is unifomly integrable (Lemma 11.4.1 in [6]). By using lower semicontinuity and continuity properties of Young measures (Proposition 4.3.3 and Theorem 4.3.3 in [6]), and standard lower semicontinuity results in Sobolev spaces, we have

$$\begin{aligned}\int_\Omega \mathcal{Q}f(\nabla u)\,dx &= \lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx\\ &\geq \int_{\Omega\times\mathbf{M}^{3\times 3}} f(\lambda)\,d\mu\\ &= \lim_{n\to+\infty}\int_\Omega f(\nabla \tilde{u}_n)\,dx\\ &\geq \int_\Omega \mathcal{Q}f(\nabla u)\,dx,\end{aligned}$$

so that

$$\lim_{n\to+\infty}\int_\Omega f(\nabla \tilde{u}_n)\,dx = \lim_{n\to+\infty}\int_\Omega f(\nabla u_n)\,dx = \int_\Omega \mathcal{Q}f(\nabla u)\,dx$$

which proves the thesis. In what follows, we still denote by $(u_n)_{n\in\mathbf{N}}$ the sequence $(\tilde{u}_n)_{n\in\mathbf{N}}$.

We start by modifying the function u_n near S so that $\gamma_S(u^n_\alpha) = v^n_\alpha$. Set $\Sigma_\eta := S\times(-\eta,\eta)$, $\Sigma_{2\eta} := S\times(-2\eta,2\eta)$, consider a cut-off function φ_η in $\mathcal{C}^1(\mathbf{R})$ satisfying

$$\varphi_\eta = 1 \text{ on } \Omega\setminus\Sigma_{2\eta},\ \varphi_\eta = 0 \text{ on } \Sigma_\eta,\ 0\leq\varphi_\eta\leq 1,\ |\frac{d\varphi_\eta}{dx_3}|\leq\frac{1}{\eta},$$

and define the function $u_{n,\eta}$ by

$$\begin{aligned}&u^{n,\eta}_\alpha := \varphi_\eta(u^n_\alpha - v^n_\alpha) + v^n_\alpha;\\ &u^{n,\eta}_3 := u^n_3.\end{aligned}$$

Clearly $u_{n,\eta}\in W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3)$ and $\gamma_S(u^{n,i}_\alpha) = v^n_\alpha$. From the growth condition satisfied by f, we have

$$\begin{aligned}\int_\Omega f(\nabla u_{n,\eta})\,dx &= \int_{\Sigma_\eta} f(\nabla u_{n,\eta})\,dx + \int_{\Sigma_{2\eta}\setminus\Sigma_\eta} f(\nabla u_{n,\eta})\,dx + \int_{\Omega\setminus\Sigma_{2\eta}} f(\nabla u_n)\,dx\\ &\leq \beta\int_{\Sigma_\eta}(1+|\nabla v_n|^p)\,dx + \beta\int_{\Sigma_{2\eta}\setminus\Sigma_\eta}(1+|\nabla v_n|^p)\,dx + C\Big[\int_{\Sigma_{2\eta}}|\nabla u_n|^p\,dx\\ &\qquad + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}}|\hat{u}_n - v_n|^p\,dx\Big] + \int_\Omega f(\nabla u_n)dx\\ &= 3\beta\eta\int_S|\nabla v_n|^p\,d\hat{x} + C\Big[\eta + \int_{\Sigma_{2\eta}}|\nabla u_n|^p\,dx + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}}|\hat{u}_n - v_n|^p\,dx\Big]\\ &\qquad + \int_\Omega f(\nabla u_n)dx.\end{aligned}$$

Thus, according to Korn's inequality (note that $\sup_{n\in\mathbf{N}}\int_S |v_n|^p \, d\hat{x} < +\infty$),

$$\begin{aligned}\int_\Omega f(\nabla u_{n,\eta}) \, dx &\leq 3\beta K\eta \int_S |(e(v_n)|^p \, d\hat{x} \\ &+ C\Big[\eta + \int_{\Sigma_{2\eta}} |\nabla u_n|^p \, dx + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u}_n - v_n|^p \, dx\Big] + \int_\Omega f(\nabla u_n)dx.\end{aligned}$$

which yields

$$\begin{aligned}\int_\Omega f(\nabla u_{n,\eta}))dx + \int_S g_0(e(v_n))d\hat{x} &\leq \int_\Omega f(\nabla u_n)dx + \int_S (3\beta K\eta| \ |^p + g_0)(e(v_n))d\hat{x} \\ &+C\Big[\eta + \int_{\Sigma_{2\eta}} |\nabla u_n|^p \, dx + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u}_n - v_n|^p \, dx\Big]\end{aligned}$$

and, from (4),

$$\begin{aligned}\limsup_{n\to+\infty} \tilde{F}_0(u_{n,\eta}) &\leq \int_\Omega Qf(\nabla u)dx + \int_S SQ(3\beta K\eta| \ |^p + g_0)(e(v_n))d\hat{x} \\ &+C\Big[\eta + \sup_{n\in\mathbf{N}}\int_{\Sigma_{2\eta}} |\nabla u_n|^p \, dx + \frac{1}{\eta^p}\int_{\Sigma_{2\eta}} |\hat{u} - v|^p \, dx\Big]\end{aligned}$$

But since $\gamma_S(\hat{u}) = v$, clearly one has

$$\int_{\Sigma_{2\eta}} |\hat{u} - v|^p dx \leq \eta^p \int_{\Sigma_{2\eta}} |\frac{\partial u}{\partial x_3}|^p dx$$

so that

$$\begin{aligned}\limsup_{n\to+\infty} \tilde{F}_0(u_{n,\eta}) &\leq \int_\Omega Qf(\nabla u)dx + \int_S SQ(3\beta K\eta| \ |^p + g_0)e(v)) \, d\hat{x} \\ &+C\Big[\eta + \sup_{n\in\mathbf{N}}\int_{\Sigma_{2\eta}} |\nabla u_n|^p \, dx + \int_{\Sigma_{2\eta}} |\frac{\partial u}{\partial x_3}|^p dx\Big]\end{aligned}$$

By letting $\eta \to 0$, from the uniform integrability of $(|\nabla u_n|^p)_{n\in\mathbf{N}}$, Lemma 4 and Lebesgue's dominated convergence theorem, we obtain

$$\limsup_{\eta\to 0}\limsup_{n\to+\infty} \tilde{F}_0(u_{n,\eta}) \leq \int_\Omega Qf(\nabla u)dx + \int_S SQg_0(e(v)) \, d\hat{x}.$$

By using a standard diagonalization argument, there exists a map $n \mapsto \eta(n)$ such that, setting $\tilde{u}_n := u_{n,\eta(n)}$,

$$\limsup_{n\to+\infty} \tilde{F}_0(\tilde{u}_n) \leq F_0(u).$$

It is easily seen that $\tilde{u}_n \to u$ in $L^1(\Omega, \mathbf{R}^3)$. Since classically $\liminf_{n\to+\infty} \tilde{F}_0(\tilde{u}_n) \geq F_0(u)$, the proof of step 1 is complete.

Step 2. We end the proof as the step 2 of the proof of Lemma 3. We only have to substitute

$$\Big| \int_{\mathbf{R}^2} SQg_0(\rho_{\eta(\delta)} * (\phi e(\tilde{v}))\ d\hat{x} - \int_{\mathbf{R}^2} SQg_0(\phi e(v)\Big| < \delta \tag{20}$$

for (10). Estimate (20) is a straightforward consequence of the weak convergence of the measure $\rho_{\eta(\delta)} * (\phi e(v)))$ to the measure $\phi e(v))$ in $\mathbf{M}(\mathbf{R}^2)$ together with $\lim_{n\to+\infty} \int_{\mathbf{R}^2} |\rho_{\eta(\delta)} * (\phi e(v)))| = \int_{\mathbf{R}^2} |\phi e(v))|$, and Reshetnyak's continuity theorem (see Theorem 2.39 in [4]). □

Then proceeding exactly like in the proof of Proposition 2 and using Lemma 4, one has

Proposition 5 (Upper bound)**.** *The following inequality holds in* $L^1(\Omega, \mathbf{R}^3)$

$$(\Gamma - \limsup F_\varepsilon) \leq F_0. \tag{21}$$

5 A modeling of a welding assembly

An elementary situation in welding can be described as follows. Let Σ^+, Σ^- and S three domains of $\mathbf{R}^2$ with Lipschitz-continuous boundaries such that $S = \Sigma^+ \cap \Sigma^-$. Let r and ε two positive numbers such that $\varepsilon << r$ and $\Omega_\varepsilon^\pm := \Sigma^\pm \times (\pm\varepsilon/2, \pm r)$, $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$, $B_\varepsilon = S \times (-\varepsilon/2, \varepsilon/2)$, and $S_\varepsilon^\pm = S^\pm \pm \varepsilon/2\ e_3$. Then $\mathcal{O}_\varepsilon := \Omega_\varepsilon \cup S_\varepsilon^+ \cup S_\varepsilon^- \cup B_\varepsilon$ is the reference configuration of a structure made of two adherents and an adhesive (the soldered joint) which respectively occupies $\Omega_\varepsilon^\pm$ and B_ε (see Figure 1). The structure is clamped on a part Γ_0 of the boundary Γ of Ω with a positive $\mathcal{H}^2$-measure and we assume that there exists $\varepsilon_0 > 0$ such that $\mathrm{dist}(\bar{\Gamma}_0, \bar{B}_{\varepsilon_0}) > 0$. The structure is subjected to body forces of density Φ and to surface forces of density φ on the complementary part Γ_φ of Γ_0. We assume that the supports of φ and Φ lay outside of $\bar{B}_{\varepsilon_0}$. Obviously one can consider other type of boundary conditions (e.g. a combination of some components of the stress vector and of the displacement). At last, adhesive and adherents are assumed to be perfectly stuck together along $S^\pm$.

The adherents and the adhesive are modeled as in section 2 so that determining the equilibrium configuration leads to the problem

$$(\mathcal{P}_\varepsilon) \qquad\qquad \inf \left\{F_\varepsilon(u) - L(u) : u \in A_\varepsilon\right\}$$

where F_ε and L have the same expression as in Section 2 but with the new definitions of Ω_ε and B_ε, whereas A_ε now reads as:

$$A_\varepsilon := \left\{u \in \mathrm{LD}(\mathcal{O}_\varepsilon, \mathbf{R}^3) : u_{\lfloor\Omega_\varepsilon} \in W^{1,p}_{\Gamma_0}(\Omega_\varepsilon, \mathbf{R}^3)\right\}.$$

Again, to propose a simplified but accurate model we consider ε as a parameter and study the asymptotic behavior, when ε goes to zero, of (approximate) solution of $(\mathcal{P}_\varepsilon)$. The essential difference from the model problem of Section 2 is that here the structure occupies a domain $\mathcal{O}_\varepsilon$ which *varies with* ε, which from

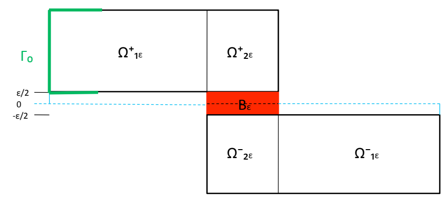

Figure 1: the reference configuration $\mathcal{O}_\varepsilon$. The reference configuration involved in Section 2 is $\Omega^+_{2\varepsilon} \cup \Omega^-_{2\varepsilon} \cup B_\varepsilon \cup S^+_\varepsilon \cup S^-_\varepsilon$.

the mathematical point of view is only of technical nature by simply modifying the kinds of convergences. That is why we have preferred to consider the model problem in whole details and to confine to state the sole results about this realistic problem of welding.

Let $\Omega := \Sigma^+ \times (0, r) \cup S \cup \Sigma^- \times (-r, 0)$ the set to which $\mathcal{O}_\varepsilon$ "converges". Let $I_\varepsilon := F_\varepsilon - L$ and $I_0 := F_0 - L$ with $F_0 : L^1(\Omega, \mathbf{R}^3) \to \mathbf{R} \cup \{+\infty\}$,

$$F_0(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + \int_S \big(g_0(e(\gamma_s(\hat{u}))\ \text{if}\ u \in A_0 \\ +\infty\ \text{otherwise,} \end{cases}$$

where we keep the same definition for γ_S and A_0 but with the new definition of Ω. Doing the same for the definition of the operator T_ε, our asymptotic model is supplied by:

Theorem 3. *Let $\bar{u}_\varepsilon$ be a solution of $(\mathcal{P}_\varepsilon)$. Then there exist a subsequence of $(\bar{u}_\varepsilon)_{\varepsilon>0}$ and $\bar{u}$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ such that*

$$\begin{aligned} &T_\varepsilon \bar{u}_\varepsilon \to \bar{u}\ \textit{weakly in}\ W^{1,p}(\Omega \setminus S, \mathbf{R}^3); \\ &\bar{u}_{\varepsilon \lfloor S\times(-r,r)} \to \bar{u}\ \textit{weakly in}\ BD(S \times (-r, r), \mathbf{R}^3); \\ &\gamma_S(\hat{\bar{u}}) \in BD(S, \mathbf{R}^2). \end{aligned}$$

Moreover $\bar{u}$ is solution of the minimization problem

$$(\mathcal{P}) \qquad \min\big\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\big\}$$

and

$$\min\big\{F_\varepsilon(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\big\} \to \min\big\{F_0(u) - L(u) : u \in L^1(\Omega, \mathbf{R}^3)\big\}.$$

Sketch of the proof. The proof follows the line of the proof of Section 2 by considering simultaneously $T_\varepsilon \bar{u}_\varepsilon$ and its restriction to the fixed domain $S \times (-r, r)$. $\square$

For the mechanical interpretation see the end of Section 3.

6 A variational regularization of the limit functional

For the numerical solving of the optimization problem

$$\inf \left\{ F_0(u) - L(u) \right\}$$

obtained in Section 3, we approximate, in a variational way, the functional of measure $u \mapsto \int_S g_0(e(\gamma_S(\hat{u}))$ by a suitable functional defined in the Sobolev space $W^{1,q}(S, \mathbf{R}^2)$, where q is close to 1 in the spirit of Norton-Hoff regularisation (cf [17]). The mathematical technics used here is an adaptation from that of [7]. In order to simplify the proofs, we assume that S is a finite union of cubes in $\mathbf{R}^2$.

We denote the limit density g_0 by h that we assume to be positively 1-homogeneous and fulfilling the growth conditions $\alpha|\xi| \leq h(\xi) \leq \beta|\xi|$. We consider a sequence $(h_q)_{q \in (1,p)}$ satisfying the following three conditions:

$i_q)$ $h_q : \mathbf{M}_s^{2\times 2} \to \mathbf{R}^+$ is convex and positively homogeneous of degree q;

$ii_q)$ $h_q \to h$ pointwise in $\mathbf{M}_s^{2\times 2}$;

$iii_q)$ there exists $a > 0$ such that for all $q > 1$ close enough to 1,

$$h_q(\xi) \geq h(\xi) \text{ for all } \xi \in \mathbf{M}_s^{2\times 2},\ |\xi| \geq a. \tag{22}$$

For instance, when $h = |.|$, $h_q = |.|^q$ satisfies these conditions with $a = 1$. Another natural example consists in choosing $h_q := h^q$. Condition $iii_q)$ is then satisfyed by taking $a = \frac{1}{\alpha}$. In these two examples, h_q satisfies uniform growth conditions with respect to q (for the second example, $\frac{\alpha}{2}|\xi|^q \leq h_q(\xi) \leq 2\beta|\xi|^q$ for q closed to 1). Note that, according to $iii)_q$, h_q fulfills the equi-coerciveness condition:

$$h_q(\xi) \geq \alpha|\xi|\ \forall \xi,\ |\xi| \geq a. \tag{23}$$

In what follows, the function h_q is not assumed to satisfy a uniform upper growth condition.

We consider the functional $\mathcal{F}_q : W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) \to \mathbf{R}^+ \cup \{+\infty\}$ defined by:

$$\mathcal{F}_q(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S h_q(e(\gamma_S(\hat{u})))d\hat{x} \text{ if } u \in \mathcal{B}_q, \\ +\infty \text{ otherwise} \end{cases}$$

where
$$\mathcal{B}_q := \big\{u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) : h_q \circ (e(\gamma_S(\hat{u}))) \in L^1(\Omega)\big\}.$$
We are going to establish the Γ-convergence of $\mathcal{F}_q$ when $q \to 1$, when the space $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ is equipped with its weak topology. The expected limit is the functional $\mathcal{F}_0 : W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) \to \mathbf{R}^+ \cup \{+\infty\}$ defined in the previous section, more precisely its restriction to $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ defined by

$$\mathcal{F}_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\ dx + \int_S h(e(\gamma_S(\hat{u}))) \text{ if } u \in \mathcal{B} \\ +\infty \text{ otherwise,} \end{cases}$$

where
$$\mathcal{B} := \big\{u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) : \gamma_S(\hat{u}) \in \mathrm{BD}(S, \mathbf{R}^2)\big\}.$$

Lemma 5 (Compactness lemma)**.** *Consider a sequence $(u_q)_{q\in]1,p]}$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ such that $\sup_{q\in(1,p)} \mathcal{F}_q(u_q) < +\infty$. Then there exist a subsequence of $(u_q)_{q\in]1,p]}$ and $u \in \mathcal{B}$ such that $(u_q, \gamma_S(\hat{u}_q)) \rightharpoonup (u, \gamma_S(\hat{u}))$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) \times BD(S, \mathbf{R}^2)$ when q goes to 1.*

Proof. Since $\sup_{q\in(1,p)} \mathcal{F}_q(u_q) < +\infty$, one has $u_q \in \mathcal{B}_q$ and

$$\mathcal{F}_q(u_q) = \int_\Omega f(\nabla u_q))\ dx + \int_S h_q(e(v_q) d\hat{x},\ v_q := \gamma_S(\hat{u}_q).$$

Thus, from (23)

$$\begin{aligned} \sup_{q\in(1,p)} \int_S |e(v_q)|\ d\hat{x} &\leq \sup_{q\in(1,p)} \Big(a|S| + \int_{[|e(v_q)|\geq a]} |e(v_q)|\ d\hat{x}\Big) \\ &\leq a|S| + \frac{1}{\alpha} \sup_{q\in(1,p)} \int_S h_q(e(v_q))\ d\hat{x} < +\infty. \end{aligned} \tag{24}$$

On the other hand from the coercivity condition fulfilled by f, there exists a subsequence and $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ such that $u_q \rightharpoonup u$ in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$. According to the continuity of the trace operator γ_S, we deduce that $v_q \to \gamma_S(\hat{u})$ in $L^p(S, \mathbf{R}^3)$, thus strongly in $L^1(S, \mathbf{R}^2)$ which, combined with (24), yields $v_q \rightharpoonup v = \gamma_S(\hat{u})$ in $\mathrm{BD}(S, \mathbf{R}^2)$ and $u \in \mathcal{B}$. $\square$

Proposition 6 (lower bound)**.** *For every sequence $((u_q)_{q\in]1,p]}$ converging to u in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$, we have*

$$\mathcal{F}_0(u) \leq \liminf_{q\to 1} \mathcal{F}_q(u_q).$$

Lemma 6. *Let $V_q \in L^q(S, \mathbf{M}^{2\times 2}_s)$ and assume that the measure $\mu_q = V_q\ d\hat{x}$ weakly converges to μ in $\mathcal{M}(S, \mathbf{M}^{2\times 2}_s)$ and that $|\mu_q| = |V_q|\ d\hat{x}$ weakly converges to ν in $\mathcal{M}^+(S)$. Then for all open subset ω of S such that $\nu(\partial\omega) = 0$, one has*

$$\liminf_{q\to 1} \int_\omega h_q(V_q)\ d\hat{x} \geq h\Big(\int_\omega d\mu\Big).$$

Proof. From Jensen's inequality and since h_q is positively q-homogeneous,

$$\int_\omega h_q(V_q)\ d\hat{x} \geq |\omega| h_q\Big(\frac{1}{|\omega|}\int_\omega V_q\ d\hat{x}\Big) = |\omega|^{1-q} h_q\Big(\int_\omega V_q\ d\hat{x}\Big).$$

It remains to establish

$$\liminf_{q\to 1} h_q\Big(\int_\omega V_q\ d\hat{x}\Big) \geq h\Big(\int_\omega d\mu\Big). \tag{25}$$

One may assume $\Big|\int_\omega d\mu\Big| > 0$, otherwise $\int_\omega d\mu = 0$ and (25) is trivially satisfied. Since $\nu(\partial\omega) = 0$, one has $\lim_{q\to 1}\int_\omega V_q\ d\hat{x} = \int_\omega d\mu$ (cf Corollary 4.2.1 in [6]). Then for $0 < \sigma < \Big|\int_\omega d\mu\Big|$, there exists q_0, $1 < q_0 \leq p$ such that, for all q, $1 < q \leq q_0$, $|\int_\omega V_q\ d\hat{x}| \geq \sigma$. One has

$$h_q\Big(\int_\omega V_q\ d\hat{x}\Big) = (\frac{\sigma}{a})^q h_q(\Big(\frac{a}{\sigma}\int_\omega V_q\ d\hat{x}\Big)$$

with $\Big|\frac{a}{\sigma}\int_\omega V_q\ d\hat{x}\Big| \geq a$, so that from assumption (22),

$$\begin{aligned} h_q\Big(\int_\omega V_q\ d\hat{x}\Big) &\geq (\frac{\sigma}{a})^q h(\Big(\frac{a}{\sigma}\int_\omega V_q\ d\hat{x}\Big) \\ &= (\frac{\sigma}{a})^{q-1} h(\Big(\int_\omega V_q\ d\hat{x}\Big). \end{aligned}$$

Letting $q \to 1$, (25) follows from the lower semicontinuity of h and the fact that $\lim_{q\to 1}\int_\omega V_q\ d\hat{x} = \int_\omega d\mu$. $\square$

Proof of Proposition 6. One may assume $\mathcal{F}_q(u_q) < +\infty$ so that, from Lemma 5, one has $u \in \mathcal{B}$. We write v_q for $\gamma_S(\hat{u}_q)$ and v for $\gamma_S(\hat{u})$. According to a standard lower semicontinuity result in Sobolev spaces we have

$$\liminf_{q\to 1}\mathcal{F}_q(u_q) \geq \int_\Omega f(\nabla u)dx + \liminf_{q\to 1}\int_S h_q(e(v_q))\ d\hat{x}$$

and it remains to establish

$$\liminf_{q\to 1}\int_S h_q(e(v_q))\ d\hat{x} \geq \int_S h(e(v)). \tag{26}$$

For $\delta > 0$ intended to go to 0, consider a standard mollifier ρ_δ and θ_δ in $\mathcal{C}_c^\infty(S)$ satisfying $0 \leq \theta_\delta \leq 1$, $\theta_\delta \to 1$ a.e. in S. For a subsequence not relabelled on q, clearly $\rho_\delta * \theta_\delta |e(v_q)|\ d\hat{x}$ weakly converges to some measure ν_δ in $\mathcal{M}^+(S)$. Moreover $\rho_\delta * \theta_\delta e(v_q)\ d\hat{x}$ weakly converges to the measure $\rho_\delta * \theta_\delta e(v)$ in $\mathcal{M}(S, \mathbf{M}_s^{2\times 2})$. For $\eta > 0$, consider a finite family $(\omega_i)_{i\in I_\eta}$ of pairwise disjoint open subsets of S, $|\omega_i| < \eta$, such that $|S \setminus \cup_{i\in I_\eta}\omega_i| = 0$ and a family $(\tilde{\omega}_\eta)_{i\in I_\eta}$ satisfying

$$\begin{aligned} &|S \setminus \bigcup_{i\in I_\eta}\tilde{\omega}_i| \leq \eta; \\ &\tilde{\omega}_i \subset \omega_i; \\ &\nu_\delta(\partial\tilde{\omega}_i) = 0. \end{aligned}$$

Such a family exists (use Lemma 4.2.1 of [6]). Note that this family depends on δ. By using the q-homogeneity of h_q, Jensen's inequality and standard convex duality principle, we have

$$\begin{aligned}
\int_S h_q(e(v_q))\ d\hat{x} &\geq \int_S \theta_\delta^q h_q(e(v_q))\ d\hat{x} \\
&= \int_S h_q(\theta_\delta\ e(v_q))\ d\hat{x} \\
&\geq \int_S h_q(\rho_\delta * (\theta_\delta e(v_q))\ d\hat{x} \\
&\geq \sum_{i\in I_\eta} \int_{\tilde{\omega}_i} h_q(\rho_\delta * (\theta_\delta e(v_q)))\ d\hat{x}.
\end{aligned}$$

Thus, from Lemma 6, 1-homogeneity and Lipschitz property of h,

$$\begin{aligned}
\liminf_{q\to 1} \int_S h_q(e(v_q))\ d\hat{x} &\geq \liminf_{q\to 1} \sum_{i\in I_\eta} \int_{\tilde{\omega}_i} h_q(\rho_\delta * (\theta_\delta e(v_q)))\ d\hat{x} \\
&\geq \sum_{i\in I_\eta} h\Big(\int_{\tilde{\omega}_i} \rho_\delta * (\theta_\delta e(v))\Big) \\
&\geq \sum_{i\in I_\eta} |\omega_i| h\Big(\frac{1}{|\omega_i|}\int_{\omega_i} \rho_\delta * (\theta_\delta e(v))\Big) \\
&\qquad -L|\rho_\delta * (\theta_\delta e(v))|(S\setminus \bigcup_{i\in I_\eta} \tilde{\omega}_i).
\end{aligned}$$

The first term of the second member is a Riemann sum. Since moreover $\rho_\delta * \theta_\delta e(v)$ is a smooth function, by letting $\eta\to 0$, we obtain

$$\liminf_{q\to 1} \int_S h_q(e(v_q))\ d\hat{x} \geq \int_S h(\rho_\delta * \theta_\delta e(v)).$$

Noticing that $\rho_\delta * \theta_\delta e(v) \rightharpoonup e(v)$ in $\mathcal{M}(S, \mathbf{M}_s^{2\times 2})$, estimate (26) is obtained by letting $\delta\to 0$. $\square$

Proposition 7 (Upper bound)**.** *For every $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ there exists u_q weakly converging to u in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ such that*

$$\limsup_{q\to 1} \mathcal{F}_q(u_q) \leq \mathcal{F}_0(u),$$

or, equivalently, $\Gamma - \limsup \mathcal{F}_q \leq \mathcal{F}_0$.

Proof. Consider the following subset $\mathcal{B}^1$ of $\mathcal{B}$:

$$\mathcal{B}^1 := \big\{u\in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3) : \gamma_S(u) \in \mathcal{C}^1(\bar{S}, \mathbf{R}^3)\big\}$$

and the functional $\tilde{\mathcal{F}}_0 : W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3) \to \mathbf{R}^+\{+\infty\}$ defined by

$$\tilde{\mathcal{F}}_0(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + \int_S h(e(\gamma_S(u)))\, d\hat{x} \text{ if } u \in \mathcal{B}^1 \\ +\infty \text{ otherwise.} \end{cases}$$

Take $u \in \mathcal{B}^1$ and set $v := \gamma_s(u)$. Let $(v_n)_{n\in\mathbf{N}^*}$ be a sequence of continuous piecewise affine functions satisfying $\|v_n - v\|_{W^{1,1}(S,\mathbf{R}^2)} \le 1/n$, and consider a sequence $(u_n)_{n\in\mathbf{N}}$ weakly converging to u in $W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3)$ satisfying

$$\begin{array}{l} \displaystyle\lim_{n\to+\infty} \int_\Omega f(\nabla u_n)\, dx = \int_\Omega f(\nabla u)\, dx; \\ \gamma_S(\hat{u}_n) = v_n. \end{array}$$

Such a sequence exists from step 1 of the proof of Proposition 4 in previous section and u_n belongs to $\mathcal{B}_q$. Writting $e(v_n) = \sum_{i\in I_n} a_{i,n} 1_{S_{i,n}}$ where $(S_{i,n})_{i\in I_n}$ is a finite partition of S, and $a_{i,n} \in \mathbf{M}^{2\times 2}_s$, the following estimate holds:

$$\begin{aligned} \lim_{q\to 1} \Big(\int_\Omega f(\nabla u_n)\, dx + \int_S h_q(e(v_n))\, d\hat{x} \Big) &= \int_\Omega f(\nabla u_n)\, dx + \lim_{q\to 1} \sum_{i\in I_n} h_q(a_{i,n})|S_{i,n}| \\ &= \int_\Omega f(\nabla u_n)\, dx + \sum_{i\in I_n} h(a_{i,n})|S_{i,n}| \\ &= \int_\Omega f(\nabla u_n)\, dx + \int_S h(e(v_n))\, d\hat{x}. \quad (27) \end{aligned}$$

Letting $n \to +\infty$, (27) yields

$$\lim_{n\to+\infty} \lim_{q\to 1} \Big(\int_\Omega f(\nabla u_n)\, dx + \int_S h_q(e(v_n))\, d\hat{x} \Big) = \int_\Omega f(\nabla u)\, dx + \int_S h(e(v))\, d\hat{x}.$$

Then, by using a standard diagonalization argument, there exists a map $q \mapsto n(q)$ such that

$$\begin{array}{l} \lim_{q\to 1} \mathcal{F}_q(u_{n(q)}) = \tilde{\mathcal{F}}_0(u); \\ u_{n(q)} \rightharpoonup u, \end{array}$$

which implies

$$\inf \big\{ \limsup_{p\to 1} \mathcal{F}_q(u_q) : u_q \rightharpoonup u \text{ in } W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3) \big\} \le \tilde{\mathcal{F}}_0(u)$$

for all $u \in W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3)$. Thus $\Gamma - \limsup \mathcal{F}_q \le \tilde{\mathcal{F}}_0$. The conclusion of Proposition 7 follows by taking the lower semi-continuous envelop of each two functionals for the weak topology of $W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3)$ and by using Lemma 3. $\square$

Corollary 2. *Assume that h_q satisfies the additional coerciveness condition: there exists $\alpha_q > 0$ such that $\alpha_q|\xi|^q \le h_q(\xi)$ for all $\xi \in \mathbf{M}^{2\times 2}_s$. Then*

i) The problem $\min\big\{\mathcal{F}_q(u) - L(u)\big\}$ *possesses at least a solution $\bar{u}_q$;*

ii) There exists a subsequence of $(\bar{u}_q)_{q\in(1,p)}$ *and* $\bar{u}$ *solution of the problem* $\min\left\{\mathcal{F}_0(u) - \langle L, u\rangle\right\}$ *such that* $\bar{u}_q \rightharpoonup \bar{u}$ *in* $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$.

Proof. The first assertion is obtained by using the direct method in the calculus of variations. The second assertion is a straightforward consequence of variational properties of the Γ-convergence. □

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