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ON SOME LINEARIZED PROBLEMS OF SHALLOW WATER FLOWS

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Abstract. Two linearized problems of shallow water flows are solved by a technique of Semigroups of operators on Hilbert spaces.

1. INTRODUCTION

There are many papers devoted to the study of shallow water flows especially by numerical methods. In this note, we give a mathematical analysis, based on the theory of semigroups of operators on Hilbert space, of two linearized problems involving the Saint-Venant equations governing shallow water flows. The first one concerns the classical linearized Saint-Venant problem, while the second one deals with the water quality problem: the evolution of a concentration of products in a flow solution of the first problem.

2. FORMULATION OF THE LINEARIZED SAINT-VENANT PROBLEM

The linearized Saint-Venant equations which model the flow of a viscous fluid in shallow water ([1],[2]) read as follows:

$$\begin{cases} \frac{\partial \eta}{\partial t} = -h \operatorname{div} V - R\eta \\ \frac{\partial V}{\partial t} = -g\nabla\eta + \mu\Delta V - PV + f \text{ in } \Omega \times (0, T) \\ \eta(x, 0) = \eta^0(x), V(x, 0) = V^0(x), \forall x \in \Omega, \eta^0, V^0 \text{ are given in } \Omega. \end{cases} \quad (2.1)$$

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Here, Ω is a bounded domain of \mathbb{R}^2 with a Lipschitz-continuous boundary Γ , T is a positive real number, t denotes the time and η , V stand for the free surface elevation and the velocity field, respectively. The positive constants g, μ and h are the gravity acceleration, viscosity and average height of the fluid, respectively, R is a positive relaxation parameter and P is a non-negative definite matrix accounting for the Coriolis effect and numerical relaxation. Let σ denote the “stress tensor,” i.e.,

$$\sigma = -g\eta I + \mu \nabla V,$$

where I is the 2×2 identity matrix.

We assume that Γ is the union of disjoint parts Γ_i , $i = 1, \dots, 5$, n is the unit normal along Γ and outward to Ω , and consider the following boundary conditions:

$$\left\{ \begin{array}{ll} V = \bar{V}, \bar{V} \text{ is given on } \Gamma_1 \times [0, T] & (i) \\ \sigma n = \gamma, \gamma \text{ is given on } \Gamma_2 \times [0, T] & (ii) \\ V_N = \bar{V}_N, \sigma_T = \gamma_T, \bar{V}_N \text{ and } \gamma_T \text{ are given on } \Gamma_3 \times [0, T] \text{ with} & \\ V_N := V \cdot n, \sigma_T := \sigma n - \sigma_N n, \sigma_N := \sigma n \cdot n, & (iii) \\ V_T := V - V_N n = \bar{V}_T, \sigma_N = p, \bar{V}_T \text{ and } p \text{ are given on} & \\ \Gamma_4 \times [0, T] & (iv) \\ \sigma n + KV = \delta, \delta \text{ is given on } \Gamma_5 \times [0, T], & \\ K \text{ is a given } 2 \times 2 \text{ non-negative matrix.} & (v) \end{array} \right. \quad (2.2)$$

The condition on Γ_1 is an adhesion condition whereas the condition on Γ_2 expresses that the stress vector is given. On Γ_3 and Γ_4 the normal and the tangential component of the velocity and the stress vector are imposed, respectively. Finally, a friction condition is involved on Γ_5 .

3. A RESULT ON EXISTENCE AND UNIQUENESS

We will formulate this transient problem in terms of a linear evolution equation in a Hilbert space of possible states with finite energy. First we assume that:

H_1) There exists a suitable extension of \bar{V} into Ω , still denoted \bar{V} , of class $C^{1,1}([0, T]; H^1(\Omega)^2)$ such that the trace of \bar{V} on Γ_1 equals \bar{V} and the same for the normal and tangential components on Γ_3, Γ_4 respectively.

H_2)

$$(\gamma, \gamma_T, \gamma_N, \delta) \in C^{1,1}([0, T]; L^2(\Gamma_2)^2 \times L^2(\Gamma_3)^2 \times L^2(\Gamma_4)^2) \times L^2(\Gamma_5)^2.$$

Then, the Stampacchia theorem implies that there exists a unique \bar{V}_e of class $C^{1,1}[(0, T); H^1(\Omega)^2]$ such that

$$\begin{aligned} V_e &\in \bar{V} + J, \\ &\int_{\Omega} gh(1+R)^{-1} \operatorname{div} V_e \operatorname{div} \varphi dx + \int_{\Omega} \mu \nabla V_e \cdot \nabla \varphi dx \\ &\quad + \int_{\Omega} P V_e \cdot \varphi dx + \int_{\Gamma_5} K V_e \cdot \varphi ds \\ &= \int_{\Gamma_2} \gamma \cdot \varphi ds + \int_{\Gamma_3} \gamma_T \cdot \varphi_T ds + \int_{\Gamma_4} \gamma_N \cdot \varphi_N ds + \int_{\Gamma_5} \delta \cdot \varphi ds \quad \forall \varphi \in J, \end{aligned}$$

where $J = \{V \in H^1(\Omega)^2 : V|_{\Gamma_1} = 0, V_N|_{\Gamma_3} = 0, V_T|_{\Gamma_4} = 0\}$. Let $\eta_e = -(1+R)^{-1} h \operatorname{div} V_e \in C^{1,1}([0, T]; L^2(\Omega))$ and $u_e = (\eta_e, V_e)$. Let $H := \{u = (\eta, V) \in L^2(\Omega) \times L^2(\Omega)^2\}$ be equipped with the following Hilbertian norm and inner product:

$$\begin{aligned} \|u\|_H^2 &= (u, u)_H = g \int_{\Omega} \eta^2 dx + h \int_{\Omega} |V|^2 dx, \quad \forall u = (\eta, V) \in H \\ (u, u')_H &= g \int_{\Omega} \eta \eta' dx + h \int_{\Omega} V V' dx, \quad \forall u = (\eta, V) \in H, \quad \forall u' = (\eta', V') \in H, \end{aligned}$$

and let A be the linear operator in H with domain $D(A)$, defined by

$$\begin{aligned} D(A) &= \left\{ u = (\eta, V) \in H; V \in J \text{ and } \exists! w \in L^2(\Omega) \text{ such that :} \right. \\ &\quad \left. \int_{\Omega} (g \eta \operatorname{div} \varphi - \mu \nabla V \cdot \nabla \varphi) dx = \int_{\Omega} w \cdot \varphi dx + \int_{\Gamma_5} K V \cdot \varphi ds, \forall \varphi \in J \right. \\ &\quad \left. A u = (-h \operatorname{div} V - R \eta, w - P V). \right. \end{aligned} \quad (3.1)$$

If $u_r := (\eta_r, V_r)$, $\eta_r := \eta - \eta_e$, $V_r := V - V_e$ and the following condition

$$\mathbf{H}_3 : \quad f \in C^{0,1}([0, T]; L^2(\Omega)^2),$$

holds, then it is straightforward to check that the linearized Saint-Venant problem is formally equivalent to the following Cauchy problem:

$$\frac{\partial u_r}{\partial t} = A u_r + F, \quad u_r(0) = u^0 - u_e(0) \quad (3.2)$$

with $u^0 := (\eta^0, V^0)$, $F := (\eta_e - \frac{d\eta_e}{dt}, f - \frac{dV_e}{dt})$.

From the classical results (see, e.g., [3]) of the theory of operator semi-groups we know that to prove the existence and uniqueness of a solution of (3.2), it suffices to prove that A is an m-dissipative operator.

Lemma 3.1. $(Au, u)_H \leq 0$ for all $u \in D(A)$.

Proof. The definition of A implies

$$\begin{aligned}
(Au, u)_H &= g \int_{\Omega} (-h \operatorname{div} V - R\eta)\eta dx + h \int_{\Omega} (w - PV) \cdot V dx \\
&= g \int_{\Omega} (-h \operatorname{div} V - R\eta)\eta dx \\
&\quad + h \left[\int_{\Omega} (g\eta \operatorname{div} V - \mu \nabla V \cdot \nabla V - PV \cdot V) dx - \int_{\Gamma_5} KV \cdot V ds \right] \\
&= - \left[\int_{\Omega} (gR\eta\eta + h\mu \nabla V \cdot \nabla V + hPV \cdot V) dx + h \int_{\Gamma_5} KV \cdot V ds \right] \leq 0.
\end{aligned}$$

Lemma 3.2. For all $\Phi = (\Phi_1, \Phi_2) \in H$, there exists $u \in D(A)$ such that $u - Au = \Phi$.

Proof. We have $u = (\eta, V)$, where η and V satisfy

$$\eta + h \operatorname{div} V + R\eta = \Phi_1, \quad V - w + PV = \Phi_2,$$

and w satisfies (3.1). This yields

$$V \in J;$$

$$\begin{aligned}
&\int_{\Omega} (\mu \nabla V \cdot \nabla \varphi + gh(1+R)^{-1} \operatorname{div} V \operatorname{div} \varphi + (I+P)V \cdot \varphi) dx + \int_{\Gamma_5} KV \cdot \varphi ds \\
&= \int_{\Omega} (g(1+R)^{-1} \Phi_1 \operatorname{div} \varphi + \Phi_2 \cdot \varphi) dx, \quad \forall \varphi \in J.
\end{aligned}$$

The Lax-Milgram lemma implies the existence and uniqueness of V . Then, if η is the element of $L^2(\Omega)$ defined by $\eta = (1+R)^{-1}(\Phi_1 - h \operatorname{div} V)$, we can deduce that

$$\int_{\Omega} (g\eta \operatorname{div} \varphi - \mu \nabla V \cdot \nabla \varphi) dx = \int_{\Omega} ((I+P)V - \Phi_2) dx + \int_{\Gamma_5} KV \cdot \varphi ds, \quad \forall \varphi \in J,$$

which proves that $u = (\eta, V) \in D(A)$ and $u - Au = \Phi$, since $(I+P)V - \Phi_2 = w$ is associated with u in the definition of Au (see (3.1)). Therefore, we obtain the following result.

Theorem 3.3. Under the assumptions H_1 - H_3 and if $u^0 \in D(A)$, there exists a unique solution of (3.2) of class $C^1([0, T], H) \cap C^0([0, T], D(A))$.

Remark 1. When $\Gamma = \Gamma_1$ (i.e., the purely Dirichlet conditions), a solution is given in [1] by the method of linear variation evolution equations.

Remark 2. The following additional property of A could be useful to treat the nonlinear case by a fixed-point method ([4]).

Proposition 3.4. *The operator A has the following property: there exists $\theta > 0$ such that*

$$|(Au, u')_H| \leq \theta (-(Au, u)_H)^{1/2} (-(Au', u')_H)^{1/2} \quad \forall u, u' \in D(A).$$

Thus, the semigroup generated by A is analytic, so that in assumptions H_1 - H_2 , $C^{1,1}$ -regularity can be relaxed to $C^{1,\alpha}$ -regularity, with $0 < \alpha < 1$, and f can be from $C^{0,\alpha}$, which implies the existence of a classical solution of (3.2).

Proof. The computation in the proof of Lemma 3 shows that

$$\exists c_1 > 0 \text{ such that } (-Au, u)_H \geq c_1 |u|_H^2 \quad \forall u \in D(A). \quad (3.3)$$

The definition of A yields

$$\begin{aligned} & (Au, u')_H \\ &= \int_{\Omega} (-gh \operatorname{div} V \eta - gR\eta\eta' + hg\eta \operatorname{div} V' - h\mu \nabla V \cdot \nabla V' - hPV \cdot V') dx \\ & - h \int_{\Gamma_5} KV \cdot V' ds. \end{aligned}$$

The Cauchy-Schwarz inequality and the definition of $|u|_H$ imply that

$$\exists c_2 > 0 \text{ such that } |(Au, u')_H| \leq c_2 |u|_H |u'|_H. \quad (3.4)$$

Thus, the proposition is a consequence of (3.3), (3.4) and classical results on analytic semigroups.

4. THE INCLUSION OF A DISSIPATIVE RADIATION CONDITION ON A PART OF THE BOUNDARY.

In this section, we assume that $\Gamma = \cup_{i=1}^6 \Gamma_i$, where on Γ_i ($i = 1, \dots, 5$) we still have all the previously stated boundary conditions, and on Γ_6 we consider the following condition:

$$\frac{\partial V}{\partial t} = -M\sigma n, \quad (4.1)$$

where M is a given symmetric positive definite matrix. Such a condition may be used in numerical approximations by truncated domains when Ω is unbounded. For the sake of simplicity of exposition, we consider homogeneous boundary conditions. The trick in the formulation of the Saint-Venant problem in terms of an evolution equation consists in introducing a new variable, which is the trace on Γ_6 of the velocity. Let

$$\wp = H \times L^2(\Gamma_6), \quad U = (\eta, V, \gamma)$$

$$\begin{aligned}(U, U')_{\wp} &= g \int_{\Omega} \eta \eta' dx + h \int_{\Omega} V \cdot V' dx + \int_{\Gamma_6} M^{-1} \gamma \cdot \gamma' ds \\ (U, U)_{\wp} &= |U|_{\wp}^2\end{aligned}$$

and

$$D(\mathcal{L}) = \begin{cases} U = (\eta, V, \gamma) \in \wp; V \in J, \gamma \in V|_{\Gamma_6} \text{ and} \\ \exists (w, \theta) \in L^2(\Omega) \times L^2(\Gamma_6)^2 \text{ such that} \\ \int_{\Omega} (g\eta \operatorname{div} \varphi - \mu \nabla V \cdot \nabla \varphi) dx \\ = \int_{\Omega} w \cdot \varphi dx + \int_{\Gamma_5} KV \cdot \varphi ds - \int_{\Gamma_6} \theta \cdot \varphi ds, \forall \varphi \in J, \\ \mathcal{L}U = (-h \operatorname{div} V - R\eta, w - PV, -M\theta) \end{cases} \quad (4.2)$$

(clearly $w = \operatorname{div} \sigma$, $\theta = \sigma n|_{\Gamma_6}$). The problem can be written in the following form:

$$\frac{dU}{dt} = \mathcal{L}U, \quad U(0) = U^0 = (\eta^0, V^0, \gamma^0|_{\Gamma_6}). \quad (4.3)$$

Lemma 4.1. $(\mathcal{L}U, U)_{\wp} \leq 0 \quad \forall U \in D(\mathcal{L})$.

Proof. We have

$$\begin{aligned}(\mathcal{L}U, U)_{\wp} &= -g \int_{\Omega} (h \operatorname{div} V + R\eta) \eta dx + h \int_{\Omega} (w - PV) \cdot V dx - \int_{\Gamma_6} \theta \cdot \gamma ds \\ &= -g \int_{\Omega} (h \operatorname{div} V + R\eta) \eta dx + gh \int_{\Omega} \eta \operatorname{div} V dx \\ &\quad - h\mu \int_{\Omega} \nabla V \cdot \nabla V dx - h \int_{\Omega} PV \cdot V dx - h \int_{\Gamma_5} KV \cdot V ds - \int_{\Gamma_6} (V - \gamma) \cdot \theta ds \\ &\leq \int_{\Gamma_6} \theta (V - \gamma) ds = 0, \quad (\text{because } \gamma = V|_{\Gamma_6} \text{ if } U \in D(\mathcal{L})).\end{aligned}$$

Lemma 4.2. For all $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in \wp$, there exists $U \in D(\mathcal{L})$, such that $U - \mathcal{L}U = \Phi$.

Proof. We have

$$\eta + h \operatorname{div} V + R\eta = \Phi_1, \quad V - w + PV = \Phi_2, \quad \gamma + M\theta = \Phi_3.$$

Here $(w, \theta) \in L^2(\Omega) \times L^2(\Gamma_6)^2$ satisfies (4.2) and $\gamma = V|_{\Gamma_6}$. This implies that $V \in J$ and

$$\int_{\Omega} (\mu \nabla V \cdot \nabla \varphi + gh(1+R)^{-1} \operatorname{div} V \operatorname{div} \varphi + (I+P)V \cdot \varphi) dx$$

$$\begin{aligned}
& + \int_{\Gamma_5} KV \cdot \varphi ds + \int_{\Gamma_6} M^{-1}V \cdot \varphi ds \\
& = g(1 + R)^{-1} \int_{\Omega} \Phi_1 \operatorname{div} \varphi dx + \int_{\Omega} \Phi_2 \cdot \varphi dx + \int_{\Gamma_6} M^{-1}\Phi_3 \cdot \varphi ds, \quad \forall \varphi \in J.
\end{aligned}$$

It follows from the Lax-Milgram lemma that such a V exists and is unique. We note that if $\eta := (I + R)^{-1}(\Phi_1 - h \operatorname{div} V)$, then

$$\begin{aligned}
& \int_{\Omega} (g\eta \operatorname{div} \varphi - \mu \nabla V \cdot \nabla \varphi) = \int_{\Omega} ((I + P)V - \Phi_2) \cdot \varphi dx \\
& + \int_{\Gamma_5} KV \cdot \varphi ds + \int_{\Gamma_6} M^{-1}(V - \Phi_3) \cdot \varphi ds, \quad \forall \varphi \in J,
\end{aligned}$$

which proves that $U = (\eta, V, V|_{\Gamma_6}) \in D(\mathcal{L})$ and $U - \mathcal{L}U = \Phi$, because $w = (I + P)V - \Phi_2$ and $\theta = M^{-1}(\Phi_3 - V|_{\Gamma_6})$, as in the definition of $\mathcal{L}U$ by (4.2). Therefore, we have the following theorem.

Theorem 4.3. *Under the assumptions $H_1 - H_3$, if $u^0 \in D(\mathcal{L})$, then there exists a unique solution of (4.3) of class $C^1([0, T], H) \cap C^0([0, T], D(\mathcal{L}))$.*

5. A WATER QUALITY PROBLEM

Many problems dealing with water quality (salinity intrusion, transport of sediments or pollution, etc.) lead to the question of finding a field C of concentration, which satisfies

$$\begin{cases} \frac{\partial C}{\partial t} = K_c \Delta C - V \cdot \nabla C + f_c & \text{in } \Omega \times (0, T) \\ C|_{\partial\Omega} = \bar{C} \text{ and } C(x, 0) = C^0(x) & \text{in } \Omega, \end{cases} \quad (5.1)$$

where K_c is a positive diffusion coefficient and V is, as before, the velocity field solution of Saint-Venant equations in Ω . In addition, we assume, from now on, that Ω has a C^2 boundary. We are going to reformulate this transient boundary-value problem as a linear evolution equation in $L^2(\Omega)$. For this, we make the assumptions:

H_4 : The data \bar{C} is such that there exist $\sigma \in (1, 2)$ and a suitable extension into Ω of \bar{C} , still denoted by \bar{C} , of class $C^1([0, T], H^\sigma(\Omega))$ such that its trace on Γ is \bar{C} .

H_5 : $f \in C^0([0, T], L^2(\Omega)^2)$.

Then, there exists a unique C_e in $C^1([0, T], H^\sigma(\Omega))$, such that

$$C_e \in \bar{C} + H_0^1(\Omega); \quad \int_{\Omega} K_c \nabla C_e \cdot \nabla \varphi dx = 0, \quad \forall \varphi \in H_0^1(\Omega).$$

Thus, $C_r := C - C_e$ satisfies the following evolution equation in $L^2(\Omega)$:

$$\frac{dC_r}{dt} = -BC_r + \Phi(t)C_r + F_c(t), \quad C_r(0) = C^0 - C_e(0), \quad (5.2)$$

where the operator B on $L^2(\Omega)$ is defined by $D(B) = \{\varphi \in H_0^1(\Omega) : \exists w \in L^2(\Omega) \text{ such that } \int_{\Omega} \nabla \varphi \cdot \nabla \Psi dx = - \int_{\Omega} \frac{w}{K_c} \Psi dx, \forall \Psi \in H_0^1(\Omega)\}$, $B\varphi = w$. It is a classical result that, since B is self-adjoint and positive, B^s is well defined for all $s \geq 0$ and $-B$ generates an analytic semigroup of strongly continuous operators $S_{-B}(t)$ on $L^2(\Omega)$ ($t \geq 0$).

Note that the first two terms of

$$F_c := -\frac{dC_e}{dt} + f_c + V \cdot \nabla C_e$$

belong to $C^0([0, T]; L^2(\Omega))$. By the Sobolev embedding theorems we have $\nabla C_e \in C^1([0, T]; L^{q(\sigma)}(\Omega))$, with $q(\sigma) = \frac{2}{2-\sigma}$, and since

$$V \in C^0([0, T]; H^1(\Omega)^2),$$

we have $V \in C^0([0, T]; L^{p(\sigma)}(\Omega))$ with $\frac{1}{p(\sigma)} = \frac{1}{2} - \frac{1}{q(\sigma)}$, so that

$$F_c \in C^0([0, T]; L^2(\Omega)).$$

Finally, the operator $\Phi(t)$ is defined in $D(B^{s/2})$, for every $s \in (1, 2)$, by

$$\varphi \in D(B^{s/2}) \mapsto \Phi(t)\varphi := -V \cdot \nabla \varphi \in L^2(\Omega).$$

In fact, from the classical result $D(B^{s/2}) \subset H^s(\Omega)$ and the Sobolev embedding we have

$$\begin{aligned} |V \cdot \nabla \varphi|_{L^2(\Omega)} &\leq |V|_{L^{p(s)}} | \nabla \varphi |_{L^{q(s)}} \leq |V|_{H^1(\Omega)} | \varphi |_{H^s(\Omega)} \\ &\leq |V|_{H^1(\Omega)} | B^{s/2} \varphi |_{L^2(\Omega)}, \end{aligned}$$

with $\frac{1}{q(s)} = 1 - \frac{s}{2}$, $\frac{1}{p(s)} = \frac{s-1}{2}$, so that $\Phi(t) \in C^0([0, T], Lin(D(B^{s/2}), L^2(\Omega)))$, with $Lin(D(B^{s/2}), L^2(\Omega))$ being the space of bounded linear operators from $D(B^{s/2})$ to $L^2(\Omega)$. Finally, $S_{-B}(t)$ being a strongly continuous analytic semigroup on $L^2(\Omega)$ generated by the self adjoint negative operator $-B$, $S_{-B}(t)$ is a strongly continuous semigroup on $D(B^{s/2})$ for $t \geq 0$ and

$$|S_{-B}(t)|_{Lin(L^2(\Omega), D(B^{s/2}))} \leq ct^{-s/2} \text{ for } t \in (0, 1].$$

Hence, a classical fixed-point method ([4]) and the Gronwall lemma imply the following.

Theorem 5.1. *Under the assumptions H_1-H_5 and if $C^0 - C_e(0) \in D(B^{s/2})$, the equation*

$$C_r(t) = S_B(t)(C^0 - C_e(0)) + \int_0^t S_B(t - \tau)(F_c(\tau) + \Phi(\tau)C_r(\tau))d\tau$$

has a unique solution in $C^0([0, T]; H^s(\Omega))$, which is a mild solution of (5.2).

Remark 3. The regularity assumption on Γ may be dropped and various other mixed boundary conditions may be considered in the situation where the induced operator B satisfies $D(B^{s/2}) \subset H^s(\Omega)$.

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