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► **To cite this version:**

Silvio Zilio. Dynamics of a test rigid body from a minimal action principle. *Classical and Quantum Gravity*, 2010, 27 (5), pp.55004. 10.1088/0264-9381/27/5/055004 . hal-00574016

**HAL Id: hal-00574016**

**<https://hal.science/hal-00574016>**

Submitted on 7 Mar 2011

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# Dynamics of a test rigid body from a minimal action principle

Silvio Zilio‡

Dipartimento di Fisica Università di Padova, I-35131 Padova, Italy  
I.N.F.N. Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Italy

## Abstract.

Making use of a minimal action principle, in this work we derive the dynamics of a test rigid body moving in a curved spacetime by means of a parametric invariant lagrangian formalism. In doing so we complete a line of research due to Bailey-Israel and Anandan-Dadhich-Singh. This is accomplished through the following new contributions: by fixing the lagrangian of the system, the elaboration of a complete variational procedure, the formulation of a rigidity constraint and the derivation of conserved quantities, already found, but in a very different form, in other approaches to the problem. The dynamics and the equations obtained are also generalized to all orders in the metric expansion by means of new mathematical tools. Besides, by a selection of an appropriate spatial section of the body world-tube, we obtain the simple Papapetru expression of the canonical momentum which, remaining unchanged to all orders, contributes to some reductions of complexity of the dynamics. Finally, in the quadrupolar approximation, applications of our results are presented in the form of useful observables in the context of ideal tests in general relativity

PACS numbers: 04.20.Fy

‡ zilio@pd.infn.it

## 1. Introduction

The quest for a relativistic theory of extended bodies is a long-standing problem, having important implications in general relativity and in astrophysics. In the multipole extended body formalism - originally due to Mathisson[1], developed by Papapetru[2], and Taub[3] and culminating in 1974 with the elegant Dixon theory[4] - the dynamics are obtained by integrating the conservation equation  $\nabla_\nu T^{\nu\mu} = 0$  on the body world-tube and using multipolar expansions, with various ad-hoc choices and settlements (for an historical conceptual review see Dixon[4] (1974)). In order to reduce the mathematical complexity of the Dixon theory, a very different approach to the problem was realized in 1980 by Bailey and Israel [5]. In that work the authors studied, by an 'eccentric variational' method, the equations of motion of a collection of identical point-like charged sources, which, in the case of null charge, reduced to a fluid of incoherent matter. Although the Bailey and Israel approach was more simple than the Dixon one, generalizations of the work to other more interesting non-charged matter configurations, having internal interactions, was not given by the authors and, up to now, any realistic modelization of an extended body coming from such an approach remains not carried out. This is due to the fact that the eccentric variational method of Bailey and Israel was essentially an n-body problem, a non-obvious coarse-graining procedure of average being required in order to obtain the dynamics of the collective extended body variables.

On the other hand, a general reduction of complexity of the dynamics of an extended body can be obtained by the notions of test body and rigidity. For example in the Dixon works [4] (1970a and 1979), the interdependence between the energy-momentum tensor and the gravitational self-field of a body involves many complications, so that relevant simplifications descend from the use of the test body approximation. Likewise, although no body can be actually considered as a rigid object, owing to the underlying inertial stress caused by rotation and curvature, in spite of its high abstract content, the notion of rigidity appears as another powerful concept. In the past, different constraints were imposed in order to implement the abstract idea of rigidity in general relativity (for a discussion see [4](1979) and [6]). Nevertheless, in spite of the simplifications introduced by both the notions of test body and rigidity, concrete applications of the Dixon theory have been scarcely studied (see [7] and references therein), because, notwithstanding its elegance, it appears formally intricate and, in some respects, too general.

In the restricted domain of the test body approximation, another attack to the complex dynamics of an extended body was realized by a collection of studies based on classical action principles (see Frenkel[8] and Barut[9] for the special relativistic case of spinning particles in electromagnetic fields and Kunzle[10] and Souriau[11] in Einstein-Maxwell fields). In all these works, the variational procedure was carried out without specifying the functional lagrangian form. The most important result of this approach appeared in 1975 and was again due to Bailey and Israel (BI)[12]. In that work the body was abstractly modelized as a point-like object having a spin angular momentum described by the gyration of an orthonormal tetrad  $e^a_\nu$ , defined on the particle world-line, and experiencing an arbitrary *unspecified* multipole coupling with the external gravitational field. The evolution of a spinning test body in a curved spacetime was then obtained by means of a minimal action principle applied to a parametric invariant lagrangian  $L$ , whose general functional form, also in this case, was not specified, the resulting equations of motion being, as a consequence, only

the general abstract dynamics obeyed by any model of a spinning multipole particle. No explicit model of test body derived by a minimal action principle 'a la BI' was known until the recent publication of the Anandan, Dadhich and Singh (ADS) work [13], which give some progress. In this paper appears, for the first time, the part of the lagrangian coming from the external coupling to the spacetime. However, as in the BI work, the explicit connection between the internal degrees of freedom of the lagrangian was not identified so that, using a non complete variational approach, ADS partially derived the equations of motion by a minimal action principle applied only on trajectory variation but not on the rotational quantities variation. For clarity, let us resume the main ADS formalism.

In the starting point ADS consider a general action of an extended body

$$S = \int_{\Delta} L \sqrt{-g} d^4x, \quad (1.1)$$

the integral being extended over a world tube  $\Delta$  in spacetime whose thickness  $\Lambda$  is assumed small compared to the radius of curvature  $R$ . The lagrangian  $L$  nominally encodes all the properties of a rigid body, vaguely defined as a body "...subject only to external gravitational forces and no other forces." [13]. Besides, the multipole structure of the body is expressed through the following definitions of  $2n$ -multipole moments

$$t^{\kappa_1 \dots \kappa_n \mu\nu} = \int \delta y^{\kappa_1} \dots \delta y^{\kappa_n} \sqrt{-g} T^{\mu\nu} w^\rho d\Sigma_\rho, \quad (1.2)$$

with  $\delta y^\mu = x^\mu - \gamma^\mu$ , the 4-velocity  $\dot{\gamma}^\mu = d\gamma^\mu/d\lambda = (1, 0, 0, 0)$  being the tangent vector of the reference world line  $\gamma(\lambda)$  of the body. In this expression, the integral domain is extended over an unspecified arbitrary space-like hypersurface identified by the unit normal vector field  $w^\rho$ . The ADS multipole moments are thus defined by means of normal coordinate systems related each others by linear transformations. All the final results obtained by ADS in this class of coordinate systems are then inductively extended to a general coordinate system. The progress obtained by the ADS method relies on the identification of every coupling of the gravitational field with the multipole moments (1.2), a question not answered in the BI work. In fact, the core of the ADS method is based on the expansion, around the reference world-line  $z$ , of the action (1.1) with respect to the *first order* metric variation,  $\delta g_{\mu\nu} = g_{\mu\nu, \sigma} \Big|_z \delta y^\sigma + \frac{1}{2} g_{\mu\nu, \sigma\lambda} \Big|_z \delta y^\sigma \delta y^\lambda + \dots$  and on the use of the canonical definition of the energy-momentum tensor in order to obtain, from (1.2), the multipolar expansion of the action. The final step of the ADS method consists to write down the resulting multipolar expansion of the action in a BI-like formalism

$$S = \int_{\gamma} \mathcal{L}[g, \delta g] d\lambda + O[(\delta g)^2], \quad (1.3)$$

and to extremize this action ( $\delta S = 0$ ) in order to derive the equations of motion.

In spite of this progress, the ADS strategy does not find  $\mathcal{L}[g, \delta g]$  completely, because two important quantities remain unrelated: the spin tensor and the Ricci rotation coefficients. Furthermore, ADS do not use a parametric invariant formalism, so that they define the canonical momentum up to a unknown term transversal to the four velocity, while the minimal action principle is applied only to the trajectory variation, the spin evolution being derived simply by induction from a corresponding Newtonian scheme. In addition, the following inductive procedure adopted by ADS, crucial in order to obtain a covariant generalization of the Newtonian multipole moments,

$$\begin{aligned}
& g_{00,ij} \Big|_z I^{ij} = -2R_{0i0j} \Big|_z I^{ij} \\
\rightarrow & g_{00,ij} \Big|_z t^{ij00} = -2R_{0i0j} \Big|_z I^{ij} \\
\rightarrow & g_{\alpha\beta,\mu\nu} \Big|_z t^{\mu\nu\alpha\beta} = -2R_{\alpha\mu\beta\nu} \Big|_z I^{\mu\nu} u^\alpha u^\beta,
\end{aligned} \tag{1.4}$$

inevitably does not possess an univocal character, the definition  $I^{ij} \equiv t^{ij00}$  being not used in the same way in all occurrences of  $I^{ij}$ .

The present paper is devoted to complete this BI-ADS works and to supply to various inadequacies again existing. This is useful because, in concrete applications of the test body dynamics, the BI-ADS approach is more simple and manageable than the Dixon theory. Our new contributions are obtained primarily by finding the explicit dependence of the lagrangian on all the observables and by the elaboration of a complete variational procedure applied on all the variables introduced, successively by the formulation of a mathematical well defined rigidity constraint and the derivation of conserved quantities, similar to those already found, but in a very different formalism, in other approaches to the problem[4][5]. The dynamics and the equations obtained are also generalized to all orders in the metric expansion and this is made possible thanks to a new covariant Taylor expansion of the metric field. Besides, by a space-like section of the body world-tube, selected orthogonal to the reference world-line, we obtain the simple first order Papapetru [2] expression of the canonical momentum which, remaining unchanged to all orders in our work, contributes to some reductions of complexity of the dynamics with respect to the other approaches (compare the Dixon-Ehlers-Rudolph canonical momentum [4](1979)[6] with our expression (6.5)). Finally, in the quadrupolar approximation, applications of our dynamics are presented as typical examples of quantities relevant in the context of ideal tests in general relativity.

The structure of the present work is the following. Without use of normal coordinates of ADS, in Section 2 we give a new covariant expansion of all quantities involved. This is necessary in order to adopt a clear and manageable BI parametric invariant formalism and in view of generalizations. In Section 3 we re-define the multipole moment tensors in a general coordinates system and, by a generalization of the ADS method, we formally expand the lagrangian to all orders in the metric variation. The abstract structure of the lagrangian is found in Section 4 by adding to the system, as in the BI formalism, internal degrees of freedom represented by the vectors of an orthonormal tetrad, co-rotating with the rigid body, the gyration of which simulates the angular velocity observable. In Section 5, we find univocally the explicit form of the parametric invariant lagrangian, which is formally equivalent to a multipolar particle having the Lorentz subgroup of spatial rotation  $\mathcal{L}_{rot}$  as internal symmetry. Then, by the assumption of a rigidity constraint, in Section 6 we apply a minimal action principle on a trajectory variation *and* on a tetrad variation, thus obtaining the equations of motion to all orders and without the use of the ADS Newtonian generalization (1.4). In Section 7 we derive the conserved quantities related to cyclic variables of the lagrangian. Finally, in view of applications, in Section 8 we give examples of these conserved quantities up to quadrupolar expansion, as well explicit illustrations of useful classical-like observables, such as relativistic principal axes of inertia and relativistic ellipsoid of inertia surface, both related to the relativistic analogous of the Euler equations.

Throughout the text, unless otherwise specified, use has been made of natu-

ral units:  $c = 1$ . The metric signature is +2, with determinant  $g$ . The dual of a second-rank antisymmetric tensor  $T_{\alpha\beta}$  is defined as  $*T_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}T^{\mu\nu}$ , where, in a  $\{x^\mu\}$  coordinate system, the Levi-Civita tensor is  $\eta_{\mu_1\mu_2\mu_3\mu_4} = \sqrt{-g}\delta_{\mu_1\mu_2\mu_3\mu_4}$ , with  $\delta_{\mu_1\mu_2\mu_3\mu_4} = +1$  ( $-1$ ) for an even (odd) permutation of 0123. As usual  $\epsilon_{\beta\mu\nu}$  will stand for the contraction of the Levi-Civita tensor with the velocity, i.e.  $\eta_{\beta\mu\nu\alpha}u^\alpha$ .

## 2. Covariant expansion of the metric

In this Section we establish in a covariant way the expansion of a parametric invariant lagrangian describing a small test body immersed in a curved space time. Preliminarily, we covariantly expand the metric tensor  $g_{\mu\nu}$  on the thin world tube  $\Delta$  of the body around a representative world line  $\gamma(\lambda)$  of the trajectory. While ADS define all multipole moments over an unspecified space-like hypersurface, we chose it orthogonally to  $\gamma(\lambda)$ . To this end we introduce some mathematical tools.

Cover  $\Delta$  by a neighborhood  $W$  of  $\gamma$  constituted by the union of normal neighborhoods of the points of  $\gamma$ [14]. Let us call  $\Sigma = \Sigma(\lambda)$  the space-like sections of  $W \cap \Delta$  spanned by all space-like geodesics  $\sigma_x(\eta)$  which stem from  $p = \sigma_x(0) = \gamma(\lambda)$  in a direction  $\mathcal{X}^\mu$  orthogonal to the vector  $\dot{\gamma}(\lambda)$ . More precisely,  $\mathcal{X}^\mu$  is the space-like tangent vector to  $\sigma_x(\eta)$  at  $p$  whose size  $\|\mathcal{X}\|$  is the measure of the proper length of  $\sigma_x$  from  $p$  to  $q = \sigma_x(\eta)$

$$\|\mathcal{X}\| = \int_0^\eta \|\dot{\sigma}_x\| d\eta, \quad (2.1)$$

Denoting by  $\mathcal{X}^\mu(q)$  the vector which satisfies (2.1) for the unique geodesic of  $\Sigma$  joining  $p$  and  $q$ , then  $\mathcal{X}^\mu(q)$  is a vector at  $p$  and a function on  $\Sigma \ni q$  which obeys the orthogonality relation

$$\mathcal{X}^\mu(q)\dot{\gamma}_\mu(p) = 0. \quad (2.2)$$

It is easy to show that

$$\mathcal{X}_{\mu_p} = -\frac{\partial}{\partial x^{\mu_p}}\Omega(p, q), \quad (2.3)$$

where  $\Omega(p, q)$  is the world-function [14] and  $x^{\mu_p} = x^\mu(\eta_p)$  are local coordinates for  $\sigma_x$ .

The first approach to a covariant Taylor expansion of the metric tensor around  $x^{\mu_p}$  seems to tentatively give it over  $\Sigma$ . This is suggested by the fact that there exists a unique tensor which reduces, at the pole  $p$  of any normal coordinate system  $\{x'^\mu\}$  centered in  $p$  (normal coordinate systems naturally induced on  $\Sigma$ ), to the  $n$ -fold repeated partial derivative of the metric tensor[15] and because the vector  $\mathcal{X}^\mu(q)$  appears to play a role similar to that of a coordinate distances  $\delta x'^\mu$  between  $p$  and  $q$ . Indicating this tensor at  $p$  as  $\mathcal{G}_{\mu_p\nu_p\kappa_{1p}\dots\kappa_{np}}$ , in a normal coordinate system we have  $\mathcal{G}_{\mu_p\nu_p\kappa'_{1p}\dots\kappa'_{np}} \equiv g_{\mu'_p\nu'_p\kappa'_{1p}\dots\kappa'_{np}}$ . In the following we will give the explicit form of this tensors up to the desired order. Thus the hypothetical Taylor expansion of the metric could be

$$g_{\mu_q\nu_q}(q) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{G}_{\mu_p\nu_p\kappa_{1p}\dots\kappa_{np}}(p) \mathcal{X}^{\kappa_{1p}}(q) \dots \mathcal{X}^{\kappa_{np}}(q). \quad (2.4)$$

On the other hand, at the l.h.s. of (2.4) we have a tensor at  $q$ , while the r.h.s. appears as a scalar at  $q$  and a tensor at  $p$ . In order to recover consistency we introduce an

operator able to 'transport' a tensorial index from a point to another. We will call it the *basis connector*.

Consider the bitensor

$$\mathcal{X}_{\mu_p \nu_q} = -\nabla_{\nu} \mathcal{X}_{\mu_p} |_{x(\eta_q)}. \quad (2.5)$$

A long calculation shows that this quantity can be expanded as follows (see [14] for details)

$$\mathcal{X}_{\mu_p \nu_q} = -g_{\mu_p \lambda_p} \Gamma_{\nu_q}^{\lambda_p} + O(R \|\mathcal{X}\| \Gamma), \quad (2.6)$$

where  $O(R \|\mathcal{X}\| \Gamma)$  is a complicated integral expression of products of the Riemann tensor, of  $\|\mathcal{X}\|$  and  $\Gamma$ , the last being the *connector*  $\Gamma_{\nu_q}^{\lambda_p}$ , the operator causing the parallel transport along the geodesic  $\sigma$  a vector from  $p$  to  $q$ , i.e.

$$\check{V}_{\mu_q} = \Gamma_{\mu_q}^{\lambda_p} V_{\lambda_p}, \quad (2.7)$$

$\check{V}_{\mu_q}$  being the parallel to  $V_{\lambda_p}$  at  $q$ . But we are interested in the change of tensors from a point to another caused uniquely by the transformation of the basis between the points, not by curvature. Then, suppose that the chart defining the coordinate system  $\{x\}$  contains a sufficiently wide region  $U$  of the body dynamics. Consider in  $U$  the flat metric  $\bar{g}_{\mu\nu}$  satisfying the property  $\bar{g}_{\mu\nu}(p) = g_{\mu\nu}(p)$  and  $\partial_{\kappa} \bar{g}_{\mu\nu}(p) = \partial_{\kappa} g_{\mu\nu}(p)$ . So we have  $\bar{\Gamma}_{\mu\nu}^{\kappa}(p) = \Gamma_{\mu\nu}^{\kappa}(p)$  but  $\bar{R}_{\mu\nu\rho\sigma}(q) = 0$ . The two metrics considered will have by construction the same normal coordinate systems centered on  $p$ . By the use of the connector operator  $\bar{\Gamma}_{\nu_q}^{\lambda_p}$  induced by  $\bar{g}_{\mu\nu}$  and using (2.6) we can define the following basis connector two point tensor

$$X_{\mu_p \nu_q} = -g_{\mu_p \lambda_p} \bar{\Gamma}_{\nu_q}^{\lambda_p} = -\bar{\nabla}_{\nu} X_{\mu_p} |_{x(\eta_q)}, \quad (2.8)$$

which is an exact operator to all order in the coordinate  $\{x\}$  ( $X$  is defined as  $\mathcal{X}$  with the use of  $\bar{g}$ ). Using (normal) cartesian coordinates  $\{x'^{\mu}\}$  centered on  $p$  we have  $\bar{\Gamma}_{\nu_q}^{\lambda_p} = \delta_{\nu_q}^{\lambda_p}$  [14]. By construction, this operator accounts only for the change of tensors caused by an arbitrary basis. This is also evident from the following interpretation of the operator action. The scalar character of  $X_{\mu_p}(x)$  at  $x = x(\eta_q)$  and definition (2.5) allow us to write  $X_{\mu_p \nu_q} = \partial X_{\mu_p} / \partial x^{\nu_q}$ . On the other hand, from the chain rule

$$\bar{\Gamma}_{\nu_q}^{\mu_p} = \bar{\Gamma}_{\nu_q}^{\lambda_{r_n}} \dots \bar{\Gamma}_{\lambda_{r_2}}^{\lambda_{r_1}} \bar{\Gamma}_{\lambda_{r_1}}^{\mu_p}, \quad (2.9)$$

with  $\eta_p < \eta_{r_1} < \eta_{r_2} \dots < \eta_n < \eta_q$  we have from (2.9), in the limit  $n \rightarrow \infty$ ,

$$X^{\mu_p \nu_q} = \prod_{i=1}^{\infty} \frac{\partial X^{\lambda_{i-1}}}{\partial x^{\lambda_i}}, \quad (2.10)$$

with  $\lambda_0 = \mu_p$  and  $\lambda_{\infty} = \nu_q$ . This relation can be regarded as an infinite product of changes of coordinate operated from the infinitesimally near points  $r_{i-1}$  to  $r_i$  of the geodesic  $\bar{\sigma}_x$  in the active interpretation point of view. In fact, the variation  $\partial X^{\lambda_{i-1}}$  of a connecting vector  $X^{\lambda_{i-1}}$  of two points  $r_{i-1}$  and  $r_i$ , on a geodesic straight line  $\bar{\sigma}_x$  ( $\bar{R}_{\mu\nu\rho\sigma} = 0$ ), due to the (partial) change of the coordinate  $x^{\lambda_{i-1}}$ , is equivalent to the effect of the change from the basis  $\{e_{\lambda_{i-1}}\}$  to the basis  $\{e_{\lambda_i}\}$ , or equal to  $\partial x'^{\lambda_i}$ , with  $\{x'\}$  coordinates of  $\{e_i^{\lambda_i}\}$  in the passive point of view. Thus in (2.10) we can use

$$\frac{\partial X^{\lambda_{i-1}}}{\partial x^{\lambda_i}} = \frac{\partial x'^{\lambda_i}}{\partial x^{\lambda_i}}, \quad (2.11)$$

from which follows the interpretation. In conclusion, if  $V^{\mu_p}$  is a tensor at  $p$ , then  $V^{\nu_q} = V^{\mu_p} X_{\mu_p}^{\nu_q}$  is the "equivalent" vector at  $q$  corrected with the entire basis history

change along the trajectory. The operator  $X_{\mu_p\nu_q}$  will be used in the the right hand side of (2.4) in order to obtain a tensor at  $q$  by saturation of both indexes at  $p$ .

Finally, taking into account (2.4), a final change is required, in order to consistently use the Taylor theorem. In fact, in its genuine formulation, all expansions and approximations depend rigorously on the variable increment  $\delta x^\mu(x) = x^\mu - x_o^\mu$  and on the coordinate position  $x$ , not on an arbitrary function of the coordinate position, like  $\mathcal{X}^\mu(x)$ . Thus, because of the tensorial character of our expansion, we must substitute  $\mathcal{X}^\mu(x)$  in (2.4) with  $X^\mu(x)$ , the only quantity which reduces, in cartesian coordinates (and without ambiguous mixing of various order of approximation in the performed metric expansions) to a coordinate increment in any point. This problematic aspect appears also in the Dixon formalism but only after the introduction of tensor-valued distributions. In fact, while we choose to Taylor expand directly on the foliation  $\bar{\Sigma}$  induced on  $\Delta$  by  $X(x)$  (note that  $X^\mu(q)\dot{\gamma}_\mu(p) = 0$ ), the Dixon approach employs as a preliminary step a Taylor-like expansion on the tangent space  $M_p$  at  $p$  using the vector  $\mathcal{X}^\mu \in M_p$  but in a formulation that requires the introduction of a tensor-valued distribution of the energy-momentum tensor  $\hat{T}^{\mu\nu}$  on  $M_p$ .

With the use of the above introduced tools the covariant expansion of the metric is thus

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{G}_{\rho_o\lambda_o\kappa_{1_o}\dots\kappa_{n_o}} X^{\rho_o}{}_{\mu} X^{\lambda_o}{}_{\nu} X^{\kappa_{1_o}} \dots X^{\kappa_{n_o}}, \quad (2.12)$$

where we have suppressed the subscript  $q$  and used the  $o$  instead of  $p$ . Note that, in a normal coordinate system  $\{x'\}$  centered on  $p$ , this expansion reduces to the ordinary Taylor expansion or

$$g_{\mu'\nu'} = \sum_{n=0}^{\infty} \frac{1}{n!} g^{\mu'_o\nu'_o\kappa'_{1_o}\dots\kappa'_{n_o}} \delta x'^{\kappa_{1_o}} \dots \delta x'^{\kappa_{n_o}}. \quad (2.13)$$

The use of the expansion (2.12) is a crucial point in order to have a clear and manageable invariant variational formulation of the multipolar test body dynamics.

### 3. Covariant expansion of the lagrangian

In this section we expand the lagrangian in the action (1.1) following the ADS approach, but in a full covariant way. This give the possibility to generalize ADS approach to all order in the metric expansion.

In a general coordinate system let  $x = x(q)$  and  $x_o = x(p) = \gamma(\lambda)$ . Thus, assuming that the lagrangian  $L$  depends on the field variables and on  $g_{\mu\nu}$ , not on its derivatives, as it happens for all known reasonable matter distributions, we have

$$\begin{aligned} S &= \int_{\Delta} L_{\bar{g}} \sqrt{-\bar{g}} d^4x \\ &+ \int_{\Delta} \left( \frac{\partial L \sqrt{-g}}{\partial g_{\mu\nu}} \right)_{\bar{g}} \delta g_{\mu\nu} d^4x \\ &+ \frac{1}{2} \int_{\Delta} \left( \frac{\partial}{\partial g_{\rho\sigma}} \frac{\partial L \sqrt{-g}}{\partial g_{\mu\nu}} \right)_{\bar{g}} \delta g_{\rho\sigma} \delta g_{\mu\nu} d^4x \dots \end{aligned} \quad (3.1)$$

where  $\delta g_{\mu\nu}(x) = g_{\mu\nu}(x) - \bar{g}_{\mu\nu}(x)$  and  $L_{\bar{g}} = L|_{g(x)=\bar{g}(x)}$  and so on. In (3.1),  $L_{\bar{g}}$  cannot contains the Riemann tensor or its covariant derivative and so it encodes all



special relativistic dynamics. The perturbed lagrangian, obtained by the repeated application of the operator  $\frac{\partial}{\partial g_{\mu\nu}} \left( \cdot \right)_{\bar{g}} \delta g_{\mu\nu}$ , describes, through the covariant expansion of the metric, all general relativistic contribution to the dynamics at the order desired.

To write explicitly higher order term, we express (3.1) in a more useful form. By construction  $X^{\mu_o}{}_{\mu}$  is a bitensor in the  $\bar{g}$  metric, thus we have

$$\bar{g}_{\mu\nu} = X^{\mu_o}{}_{\mu} X^{\nu_o}{}_{\nu} = X^{\mu_o}{}_{\mu} X_{\mu_o\nu} = \bar{g}_{\mu_o\nu_o} X^{\mu_o}{}_{\mu} X^{\nu_o}{}_{\nu}. \quad (3.2)$$

In posing  $\delta g_{\mu\nu} = \delta g_{\mu_o\nu_o} X^{\mu_o}{}_{\mu} X^{\nu_o}{}_{\nu}$  with

$$\delta g_{\mu_o\nu_o} = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{G}_{\mu_o\nu_o\kappa_1_o \dots \kappa_n_o} X^{\kappa_1_o} \dots X^{\kappa_n_o}, \quad (3.3)$$

we have

$$\left( \frac{\partial L \sqrt{-g}}{\partial g_{\mu\nu}} \right)_{\bar{g}} \delta g_{\mu\nu} = \frac{\partial L_{\bar{g}} \sqrt{-\bar{g}}}{\partial \bar{g}_{\mu_o\nu_o}} \delta g_{\mu_o\nu_o} \quad (3.4)$$

and

$$\begin{aligned} & \left( \frac{\partial}{\partial g_{\rho\sigma}} \frac{\partial L \sqrt{-g}}{\partial g_{\mu\nu}} \right)_{\bar{g}} \delta g_{\rho\sigma} \delta g_{\mu\nu} = \\ & \frac{\partial}{\partial \bar{g}_{\rho_o\sigma_o}} \left( \frac{\partial L_{\bar{g}} \sqrt{-\bar{g}}}{\partial \bar{g}_{\mu_o\nu_o}} \right) \delta g_{\rho_o\sigma_o} \delta g_{\mu_o\nu_o} \end{aligned} \quad (3.5)$$

and similarly for higher order terms. We now express the measure in the integrals in a way useful to obtain a parametric invariant action. Let  $\bar{\Sigma}$  be the space-like surface section induced on  $\Delta$  by  $\bar{\sigma}_x$ . We can split the invariant proper volume as follows

$$\sqrt{-\bar{g}} d^4 x = l d\lambda d^3 \bar{\Sigma}. \quad (3.6)$$

with  $l = l_x(\lambda) = \{-\bar{g}_{\mu\nu} [x(\lambda)] \dot{x}^\mu(\lambda) \dot{x}^\nu(\lambda)\}^{1/2}$  the length factor of a congruence of time-like trajectories  $x(\lambda)$  orthogonal to  $\bar{\Sigma}(\lambda)$  at every point  $x \in \bar{\Sigma}$  ( $(\Delta, \bar{g})$  is globally flat), with  $x_o(\lambda) \equiv \gamma(\lambda)$  parametrized in such a way to have  $l_x = l_{x_o} = \{-g_{\mu\nu} [x_o(\lambda)] \dot{x}_o^\mu(\lambda) \dot{x}_o^\nu(\lambda)\}^{1/2}$  constant in  $\bar{\Sigma}$ . Thus, posing

$$\begin{aligned} & l d\lambda \mathcal{M}^{\kappa_1_o \dots \kappa_n_o}(\lambda) = \\ & 2 \int_{\bar{\Sigma}(\lambda)} L_{\bar{g}} X^{\kappa_1_o} \dots X^{\kappa_n_o} \sqrt{-\bar{g}} d^4 x \end{aligned} \quad (3.7)$$

we define all  $2n$ -multipole moments as follows

$$\begin{aligned} & \mathcal{M}^{\mu_o\nu_o\kappa_1_o \dots \kappa_n_o}(\lambda) = \\ & \frac{\partial}{\partial \bar{g}_{\mu_o\nu_o}} \mathcal{M}^{\kappa_1_o \dots \kappa_n_o}(\lambda) \end{aligned} \quad (3.8)$$

and, from them, other derived quantities, such as

$$\begin{aligned} & \mathcal{M}^{\rho_o\sigma_o\mu_o\nu_o\kappa_1_o \dots \kappa_n_o}(\lambda) = \\ & \frac{\partial}{\partial \bar{g}_{\rho_o\sigma_o}} \mathcal{M}^{\mu_o\nu_o\kappa_1_o \dots \kappa_n_o}(\lambda) \end{aligned} \quad (3.9)$$

and so on for other quantities  $\mathcal{M}^{\alpha_1_o \dots \alpha_m_o \kappa_1_o \dots \kappa_n_o}$  with  $m$  odd integer. Observe that

$$\begin{aligned} & \mathcal{M}^{\alpha_1_o \dots \alpha_m_o \kappa_1_o \dots \kappa_n_o} = \\ & \mathcal{M}^{(\alpha_1_o \alpha_2_o) \dots (\alpha_{m-1_o} \alpha_{m_o}) (\kappa_1_o \dots \kappa_n_o)} \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \mathcal{M}^{\alpha_{3_o}\alpha_{4_o}\dots\alpha_{m_o}\kappa_{1_o}\dots\kappa_{n_o}} = \\ \bar{g}_{\alpha_{1_o}\alpha_{2_o}}\mathcal{M}^{\alpha_{1_o}\alpha_{2_o}\alpha_{3_o}\alpha_{4_o}\dots\alpha_{m_o}\kappa_{1_o}\dots\kappa_{n_o}} \end{aligned} \quad (3.11)$$

and similarly for other contractions. By construction, all  $2n$ -multipole moment tensors (3.8) and all the other derived tensors, are *space-like* in the index  $\kappa_{1_o}\dots\kappa_{n_o}$  with respect to the reference world-line  $\gamma(\lambda)$ . Note that if  $L$  is linear in the metric, only the  $2n$ -multipole moment (3.8) will be employed in the expansion. Higher power dependences of  $L$  on the metric will determine the use of the other quantities introduced.

Using this apparatus, turning back to (3.1) and (3.6) we have

$$\begin{aligned} S = \int_{\gamma} \mathcal{L}(\lambda) d\lambda = \int_{\gamma} [\mathcal{L}_g(\lambda) + \mathcal{L}_{\delta_g}(\lambda) \\ + \mathcal{L}_{\delta_g^2}(\lambda) + \dots] d\lambda, \end{aligned} \quad (3.12)$$

where

$$\mathcal{L}_g = l \int_{\bar{\Sigma}(\lambda)} L_{\bar{g}} d^3\bar{\Sigma} \quad (3.13)$$

is the unperturbed lagrangian in  $\bar{\Sigma}(\lambda)$ , whose physical content must reproduce the special relativistic dynamics. Analogously,

$$\begin{aligned} \mathcal{L}_{\delta_g} = l \sum_n \frac{1}{2n!} \mathcal{M}^{\mu_o\nu_o\kappa_{1_o}\dots\kappa_{n_o}} \\ \cdot \mathcal{G}_{\mu_o\nu_o\kappa_{1_o}\dots\kappa_{n_o}}, \end{aligned} \quad (3.14)$$

where  $n$  runs over the desired  $2n$  multipolar order.  $\mathcal{L}_{\delta_g}$  is the first order multipolar expansion of the lagrangian, which encodes the influence of the gravitational field on the dynamics. Here,  $l \equiv l_{\gamma}(\lambda) = [-g_{\mu\nu}\dot{\gamma}^{\mu}(\lambda)\dot{\gamma}^{\nu}(\lambda)]^{1/2}$  is the length factor along  $\gamma$  which reduces to unity in the proper time parametrization. Using the definition of the energy-momentum tensor in (3.8) we have

$$\begin{aligned} \mathcal{M}^{\mu_o\nu_o\kappa_{1_o}\dots\kappa_{n_o}}(\lambda) = \\ \int_{\bar{\Sigma}(\lambda)} T_{\bar{g}}^{\mu\nu} X^{\mu_o}_{\mu} X^{\nu_o}_{\nu} X^{\kappa_{1_o}} \dots X^{\kappa_{n_o}} d^3\bar{\Sigma}, \end{aligned} \quad (3.15)$$

Higher order terms of the expansion are

$$\begin{aligned} \mathcal{L}_{\delta_g^m} = l \sum_N \frac{1}{\mathcal{A}_N} \mathcal{G}_{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N} \\ \cdot \mathcal{M}^{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N}, \end{aligned} \quad (3.16)$$

where  $m = M/2$  is the order of the metric expansion,  $N = n + p + \dots + q$ , while  $\mathcal{A}_N = n!p!\dots q!$ . In (3.16) we have suppressed the subscript  $_o$  and used the quantity

$$\begin{aligned} \mathcal{G}_{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N} = \mathcal{G}_{\alpha_1\alpha_2\kappa_1\dots\kappa_n} \\ \cdot \mathcal{G}_{\alpha_3\alpha_4\kappa_{n+1}\dots\kappa_{n+p}} \dots \\ \cdot \mathcal{G}_{\alpha_{M-1}\alpha_M\kappa_{N-q+1}\dots\kappa_N} \end{aligned} \quad (3.17)$$

Finally, we use the energy-momentum conservation law to obtain the symmetry properties of the multipole tensors. Consider the quantity  $\bar{\nabla}_{\mu} \left( T_{\bar{g}}^{\mu\nu} X^{\nu_o}_{\nu} \right)$ . In a particular global coordinate system in which  $X^{\nu_o}_{\mu'} = \delta^{\nu_o}_{\mu'}$  it becomes  $\bar{\nabla}_{\mu'} \left( T_{\bar{g}}^{\mu'\nu'} X^{\nu_o}_{\nu'} \right) = 0$  because of the energy-momentum conservation law. Thus in

any coordinate system we have  $\bar{\nabla}_\mu \left( T_{\bar{g}}^{\mu\nu} X^{\nu_o} \right) = 0$ . Using (2.8), this relation enables us to write

$$\begin{aligned} & \bar{\nabla}_\mu \left( T_{\bar{g}}^{\mu\nu} X^{\nu_o} X^{\mu_o} X^{\kappa_{1o}} \dots X^{\kappa_{no}} \right) \\ &= -(n+1) X^{\nu_o} X^{\mu_o} T_{\bar{g}}^{\nu(\mu} X^{\kappa_{1o}} \dots X^{\kappa_{no})}. \end{aligned} \quad (3.18)$$

Integrating this equality over  $\bar{\Sigma}(\lambda)$  with the measure  $\sqrt{-\bar{g}}d^4x$  we obtain the symmetry properties of the  $2n$ -multipole moments

$$\mathcal{M}^{\nu_o(\mu_o\kappa_{1o}\dots\kappa_{no})} = 0. \quad (3.19)$$

These symmetries were not found by ADS, while they are formally identical to that contained in the Dixon work. Because of the commutation of the operators  $\bar{\nabla}_\mu$  and  $\frac{\partial}{\partial \bar{g}_{\rho_o\sigma_o}}$  this relation is easily generalized to the other higher multipolar quantities such as (3.11)

$$\mathcal{M}^{\alpha_1 \dots (\alpha_M \kappa_1 \dots \kappa_N)} = 0. \quad (3.20)$$

Some brief comments are necessary now.

First, note that the expansion of the lagrangian is not limited to the first order in the metric variation, which contributes to a more generality of our equations than to those of ADS. A further observation concern the expression of the multipole moments (3.15), which are only vaguely similar to those of Dixon. This is because in some case Dixon defines multipole moments containing only higher-order information, the monopole moment ( $n = 0$ ) and the dipole moment ( $n = 1$ ) being identically vanishing [4](1979). A more important discrepancy lies on the fact that the Dixon resulting expressions of the multipole moments [4](1974 and 1979) are obtained by intergrations on a space-like hypersurface orthogonal to the momentum  $P^\mu$ , while our 3-dimensional sections  $\bar{\Sigma}$  over which we define our quantities are orthogonal to the reference world line  $\gamma^\mu$ . This shows that the re-definition of the canonical momentum operated by ADS [13] (up to an unknown term transversal to the four velocity) in order to obtain formally the same dynamical equations of Dixon, is not necessary and is simply conventional, also because ADS use the Pirani  $S^{\mu\nu}u_\nu = 0$  condition[16] instead of the Tulczyjew-Dixon condition[4][17]  $S^{\mu\nu}P_\nu = 0$ . As the techniques and the definitions used are different, the present method and the Dixon theory give rise - even though formally identical - only to *similar* dynamical equations (Section 6). The present multipolar method seems to be more simple and manageable compared to that of Dixon, because of the more simple mathematical tools used, thanks of the cited orthogonality and to a simpler calculability of multipole tensors by virtue of the use of a flat metric  $\bar{g}$ . Besides, by the introduction of the sections  $\bar{\Sigma}$  we will obtain in Section 6 the simple Papapetru expression of the canonical momentum which, remaining unchanged to all orders, contributes to some reductions of complexity of the dynamics.

In the next Section we will give the explicit form of  $\mathcal{L}$  up to the desired order.

#### 4. The lagrangian structure

The purpose to find the lagrangian functional requires some care about on the order of expansions we will use in the next.

The hypothesis of a weak metric variation within the spatial body extension force us to define consistently the *spatial size*  $\Lambda$  of the body. We designate this quantity as

the maximum of  $\|X_\sigma\|$  in  $\Delta$ . Thus, for a body moving in a spacetime having radius of curvature  $R$ , we adopt the ADS assumption that the spatial size  $\Lambda$  of the body be small compared to  $R$ , or  $\Lambda/R \approx O(1)$ . The ADS method is based on a generic assumption about the weakness of the metric but, to our purposes, it is sufficient to assume a Dixon-like criterion, requiring that the *induced metric*  $h_{\mu\nu}$  in  $\bar{\Sigma}$  be weak. Thus  $\Lambda/R \approx O(1)$  means exactly that the variation of the gravitational field must be small on  $\bar{\Sigma}$  (what is, actually, the Dixon criterion[4](1979) p.190 and p.196).

The notion of spatial size  $\Lambda$  enables us to define another parameter in the expansion, which accounts for the 'surface velocity'  $v_s$  of the body, interpreted as the spatial velocity of the points of  $\partial\bar{\Sigma}$  as measured by a Fermi-Walker observer on  $\dot{\gamma}$ . We adopt the constraint that the Fermi-Walker observer will not experience a surface horizon on the region occupied by the body. Being  $v_s \approx \Lambda\Omega$  with  $\Omega$  angular velocity of the body (see the next Section), this condition is equivalent to make all expansions up to the standard  $(v_s/c)^2$ , or to consider negligible all quantities of order  $O[(\Lambda\Omega)^3]$ . Thus, our goal is to find the explicit form of the lagrangian up to this order.

Finally, because we wish to carry out exemplifications up to the quadrupole moments of the body, it is sufficient to use  $\sum_{n=1}^2$  (see [13] for details) in the multipole expansion (3.14). Thus  $m = 1$  and  $n = 1, 2$  in (3.16). In this case we will consider negligible all the quantities of order  $O[(\delta g)^2, (\Lambda\Omega)^3, (\Lambda/R)^3]$ .

The first step amounts to define in a standard way the above mentioned angular velocity  $\Omega$ . To this end we use an orthonormal tetrad field  $e^a(\lambda)$  defined on the representative world-line of the body  $\gamma(\lambda)$ :

$$\eta_{ab}e_\mu^a(\lambda)e_\nu^b(\lambda) = g_{\mu\nu}(\lambda). \quad (4.1)$$

Consider the tensor

$$\Omega_{\mu\nu} \equiv \dot{e}_{\alpha\mu}e^\alpha{}_\nu - 2e_{o[\mu}\dot{e}_{o\nu]} = \omega_{\mu\nu} - 2u_{[\mu}\dot{u}_{\nu]} \quad (4.2)$$

with  $e_a^\mu$  dual of the tetrad field and  $\omega_{\mu\nu} = \omega_{[\mu\nu]}$ , having defined the normalized four velocity as

$$u^\mu = \frac{\dot{\gamma}^\mu}{l}. \quad (4.3)$$

We have

$$\Omega^\mu{}_\nu u^\nu = 0. \quad (4.4)$$

The tensor components of  $\Omega^\mu{}_\nu$  are the Fermi rotation coefficients and represent the spatial angular velocity tensor of the body, while  $\omega_{\mu\nu}$  is the corresponding spacetime generalization. The spatial angular velocity vector is

$$\Omega^\mu \equiv {}^*\Omega^{\kappa\mu}u_\kappa. \quad (4.5)$$

Note that  $\Omega^\mu{}_\nu$  fulfills a crucial property: because of the spatial summation on the Lorentz indices

$$\Omega_{\mu\nu} = \dot{e}_{\alpha[\mu}e^\alpha{}_{\nu]} - u_{[\mu}\dot{u}_{\nu]}, \quad \alpha = 1, 2, 3 \quad (4.6)$$

then  $\Omega^\mu{}_\nu$  is invariant with respect to the Lorentz subgroup  $\mathcal{L}_{rot}$  of spatial rotation transformations  $e'^a{}_\nu = \Lambda^a{}_{rot b} e^b{}_\nu$ , provided that the Lorentz spatial rotation matrix  $\Lambda^a{}_{rot b}$  is constant along the trajectory

$$(\Lambda^a{}_{rot b})' = 0. \quad (4.7)$$

Our main purpose is to derive all the equations of motion, for the trajectory and for the spin of a test body, by means of a parametric invariant action principle  $\delta \int_{\gamma} \mathcal{L}(\lambda) d\lambda = 0$ , for a total variations of the trajectory  $\gamma^\mu(\lambda)$  and of the tetrad field  $e^a(\lambda)$  with fixed endpoints. This is just the BI formalism [12]. Thus we write the unperturbed Lagrange functional as

$$\mathcal{L}_{\bar{g}} \equiv \mathcal{L}_{\bar{g}} [l(\lambda), g(\lambda), \dot{\gamma}(\lambda), e(\lambda), \dot{e}(\lambda), \mathcal{M}(\lambda)], \quad (4.8)$$

the dot meaning the convective covariant derivative along the world-line.  $\mathcal{L}_{\bar{g}}$  must necessarily depend only on quantities defined on the reference trajectory as, e.g., on the multipole moments  $\mathcal{M}^{\mu\nu\kappa_1 \dots \kappa_n}(\lambda)$ . As we pointed out,  $\mathcal{L}_{\bar{g}}$  does not depend on derivatives of the metric tensor and must reproduce, in the non relativistic limit, the classical physics of a body in a constant gravitational field or in a free falling reference frame.

Now we assume a dependence of the lagrangian on the tetrad field  $e_a^\mu$  and  $\dot{e}_a^\mu$  is accomplished through the angular velocity tensor  $\Omega_{\mu\nu}$

$$\mathcal{L}_{\bar{g}} = \mathcal{L}_{\bar{g}} [l(\lambda), \dot{\gamma}(\lambda), \Omega(\lambda), \mathcal{M}(\lambda)]. \quad (4.9)$$

With this choice  $\mathcal{L}_{\bar{g}}$  will be invariant under the internal Lorentz subgroup of spatial rotation constant along the trajectory, the physical meaning being immediate: the spatial angular velocity tensor  $\Omega^\mu{}_\nu$  and the spatial angular velocity vector  $\Omega^\mu$  will be obviously unchanged no matter the tetrad  $e^a{}_\nu$  is linked to the body.

Another important symmetry is required: the action  $S = \int_{\gamma} \mathcal{L}(\lambda) d\lambda$  must be parametrically invariant. This is accomplished provided that  $\mathcal{L}$  is a 1st-degree functional polynomial in the  $\dot{\gamma}^\mu$ 's, or

$$\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^\mu} \dot{\gamma}^\mu. \quad (4.10)$$

Thus, the symmetries described will give the constraints required to construct the covariant model up to the order desired.

## 5. The model

First, let us search for the explicit form of the unperturbed lagrangian reproducing, in the non relativistic limit, the dynamical evolution in a constant gravitational field or in a free falling reference frame. Note that in the ADS work the part of the lagrangian corresponding to our  $\mathcal{L}_{\bar{g}}$  remains unknown because the internal degrees of freedom are uncorrelated.

We start by decomposing the unperturbed lagrangian as

$$\mathcal{L}_{\bar{g}} = \mathcal{L}_{\mathcal{M}} + \mathcal{L}_S, \quad (5.1)$$

where

$$\mathcal{L}_{\mathcal{M}} = -l\mathcal{M}, \quad (5.2)$$

is the  $O[(\Lambda\Omega)^0]$  contribution to the lagrangian,  $\mathcal{M} \equiv \mathcal{M}_\mu^\mu$  being obtained from (3.15) with  $n = 0$ . This lagrangian term manifestly satisfies (4.10).  $\mathcal{L}_{\mathcal{M}}$  is easily interpreted as the *mass lagrangian* of the body.

For the other coupling we use as a guide the Newtonian analogy. We observe that the  $O[(\Lambda\Omega)^1]$  contribution to the lagrangian obtained by means of the geometrical object  $\mathcal{M}_\mu^{\mu\kappa_1}$  is absent by dimensionality. Thus a condition on this tensor will result

as a gauge. If we assume the most simple gauge  $\mathcal{M}_\mu^{\mu\kappa_1} = 0$ , we obtain the usual condition which states that the reference world line  $\gamma$  will be the *center of mass* of the system as regards to which the subspace  $\bar{\Sigma}$  is defined and the quadrupolar tensor is referred.

The last task is to find the  $O\left[(\Lambda\Omega)^2\right]$  *spin lagrangian* contribution  $\mathcal{L}_S$ , which describes the spin of the body.

First, we introduce a natural definition of the antisymmetric spin tensor as

$$S^{\mu\nu} \equiv 2\mathcal{M}_\kappa^{[\nu\Omega^\mu]\kappa}, \quad (5.3)$$

with the spin vector being

$$S^\mu \equiv {}^*S^{\kappa\mu}u_\kappa. \quad (5.4)$$

Here,

$$\mathcal{M}^{\kappa_1\kappa_2} \equiv \mathcal{M}_\mu^{\mu\kappa_1\kappa_2} \quad (5.5)$$

and from the spatial character of the multipole moments we have the property

$$\mathcal{M}^\mu{}_\nu \dot{\gamma}^\nu = 0 \quad (5.6)$$

and consequently both the spin tensor  $S^{\mu\nu}$  and the spin vector  $S^\mu$ , like  $\Omega^{\mu\nu}$  and  $\Omega^\mu$ , will have spatial character

$$S^\mu{}_\nu u^\nu = 0. \quad (5.7)$$

Thus the spin tensor satisfies the Pirani condition [14].

Finally, the exact form of the rotational energy  $\mathcal{L}_S$  must reproduce the Newtonian limit. Thus we assume

$$\mathcal{L}_S = -\frac{1}{4}lS^{\mu\nu}\Omega_{\mu\nu}. \quad (5.8)$$

Note that  $\mathcal{L}_S$  can be written also as

$$\mathcal{L}_S = \frac{1}{2}l\Omega_{\sigma\kappa}\mathcal{M}^{\kappa\lambda}\Omega_\lambda{}^\sigma \quad (5.9)$$

and therefore it turns out to be the most simple scalar built out with two tensors, symmetric and antisymmetric, defined along a world line  $\gamma$ . It is easy to see that any other coupling constructed by means of the multipolar tensors and the other variables must be of higher order than  $O\left[(\Lambda\Omega)^2\right]$  or must violate the dimensionality.

Thus, collecting all the definitions, the explicit parametric invariant lagrangian satisfying the imposed symmetries is

$$\begin{aligned} \mathcal{L} = & -l\left(\mathcal{M} + \frac{1}{4}S^{\mu\nu}\Omega_{\mu\nu}\right) \\ & + l\sum_{M,N}\frac{1}{\mathcal{A}_N}\mathcal{G}_{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N}\mathcal{M}^{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N}. \end{aligned} \quad (5.10)$$

If we limit ourself to study the quadrupolar expansion, we must restrict to the  $O\left[(\delta g)^2, (\Lambda\Omega)^3, (\Lambda/R)^3\right]$  approximation. Thus, using the relations [15]

$$\begin{aligned} \mathcal{G}_{\mu\nu\rho} & \equiv 0 \\ \mathcal{G}_{\mu\nu\rho\sigma} & \equiv -\frac{2}{3}R_{\mu(\rho\sigma)\nu}, \end{aligned} \quad (5.11)$$

we get

$$\mathcal{L}_{\delta\bar{g}} = -\frac{1}{6}l\mathcal{M}^{\mu\nu\rho\sigma}R_{\mu\rho\sigma\nu}(\lambda), \quad (5.12)$$

having used the symmetry property of the quadrupolar tensor. To this order (5.10) becomes

$$\begin{aligned} \mathcal{L} = & -l\mathcal{M} - \frac{1}{4}lS^{\mu\nu}\Omega_{\mu\nu} \\ & - \frac{1}{6}l\mathcal{M}^{\mu\nu\rho\sigma}R_{\mu\rho\sigma\nu}. \end{aligned} \quad (5.13)$$

Because the tensor  $\omega^{\mu\nu} = \dot{e}_a^\mu e^{a\nu}$  depends only on the free tetrad lagrangian variables  $e_\alpha^\mu$  and  $\dot{e}_\alpha^\mu$  ( $\alpha = 1, 2, 3$ ) and on  $e_0^\mu = u^\mu$ , then from (4.3) it is a zero degree polynomial in  $\dot{\gamma}^\mu$ . From (5.3) and (4.2), also  $S^{\mu\nu}$  is of the same degree, while  $l$  is of order one. Thus, from (5.13) we easily verify that the total lagrangian  $\mathcal{L}$  is of order one and that it satisfies (4.10), as required.

In conclusion we have found the explicit form of the lagrangian of a test tiny body in a curved space time. In the next Section we will impose an analytical constraint on the structure of the multipole moments in order to give rigidity to the body. All the equations of motion will be then derived through a minimal action principle applied on the variation of the trajectory  $\gamma^\mu$  and on the variation of the rotational variable  $e_a^\mu$ .

## 6. The Minimal Action Principle

In order to apply the variational procedure to the lagrangian functional (5.10), we consider the covariant variation of the trajectory carried out by the operator

$$\delta_\gamma \equiv \delta\gamma^\kappa \nabla_\kappa. \quad (6.1)$$

Contrarily to the BI approach, we do not assume that the operator  $\delta_\gamma$  holds the tetrad fixed by parallel propagation, but that it is extended by Fermi-Walker transport in order to preserve its character. Thus  $\delta_\gamma e_a^\mu \neq 0$ . Indicating by  $\delta_e$  the operator which acts on the internal lagrangian variables  $e_a^\mu$ , the total variation will be described by

$$\delta = \delta_\gamma + \delta_e. \quad (6.2)$$

### • Variations

Applying the total variation to the lagrangian we have

$$\begin{aligned} \delta\mathcal{L} = & \frac{\partial\mathcal{L}}{\partial\dot{\gamma}^\mu}\delta\dot{\gamma}^\mu + \frac{\partial\mathcal{L}}{\partial\omega^{\mu\nu}}\delta\omega^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial\mathcal{M}}\delta\mathcal{M} \\ & + \frac{\partial\mathcal{L}}{\partial\mathcal{M}^{\mu\nu}}\delta\mathcal{M}^{\mu\nu} + \sum_{M,N} \frac{\partial\mathcal{L}}{\partial\mathcal{M}^{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N}}\delta\mathcal{M}^{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N} \\ & + \sum_{M,N} \frac{\partial\mathcal{L}}{\partial\mathcal{G}^{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N}}\delta\mathcal{G}^{\alpha_1\dots\alpha_M\kappa_1\dots\kappa_N}, \end{aligned} \quad (6.3)$$

with

$$P_\mu = \frac{\partial\mathcal{L}}{\partial\dot{\gamma}^\mu} \quad (6.4)$$

the canonical momentum of the system. In (6.3) we have used the obvious covariant variation  $\delta_\gamma g_{\mu\nu} = 0$  and assumed the condition  $\delta_e g_{\mu\nu} = \delta_e \bar{g}_{\mu\nu} = 0$  because the geometry is kept fixed (observe that  $g_{\mu\nu}(\lambda) = \bar{g}_{\mu\nu}(\lambda)$ ).

As a first step, we derive the analytical expression of  $P^\mu$ . From (5.10) we have

$$P^\mu = -\mathcal{L}u^\mu + \dot{S}^{\nu\mu}u_\nu. \quad (6.5)$$

This expression coincides with the simple Papapetru[2] but here it holds to all order. Thanks to our formalism, this Papapetru relation between the moment and the other quantities is simpler than the corresponding non-linear Dixon-Ehlers-Rudolph one (see [4](1979) and [6]).

We consider now the second term in (6.3). A direct calculation yields

$$\frac{\partial \mathcal{L}}{\partial \omega^{\mu\nu}} = -\frac{l}{2}S_{\mu\nu}. \quad (6.6)$$

Note that (6.6) would have been equivalent to a definition of the spin tensor if its explicit form (5.3) were unknown. This happened in the BI approach, where the abstract definition of the spin tensor was indeed performed without any model specification [12]. When applied to our explicit lagrangian, but with an  $(l)^{-1}$  factor in front, the BI like definition correctly gives the spin tensor,

$$\begin{aligned} S_{\mu\nu} &= (l)^{-1} \left( 2e_{a[\mu} \frac{\partial \mathcal{L}}{\partial \dot{e}_a^{\nu]}} \right) = \\ &= \frac{2}{l} e_{a[\mu} \frac{\partial \mathcal{L}}{\partial \omega^{\kappa\lambda}} \frac{\partial \omega^{\kappa\lambda}}{\partial \dot{e}_a^{\nu]}} = -\frac{2}{l} \frac{\partial \mathcal{L}}{\partial \omega^{\mu\nu}}. \end{aligned} \quad (6.7)$$

Then, with the use of the proper time parametrization  $\lambda = \tau$ , the length factor disappears and the two definitions become coincident.

The third step will amount to evaluate the dependence on the quadrupolar moments in (6.3). It is easily derived as

$$\frac{\partial \mathcal{L}}{\partial \mathcal{M}^{\mu\nu}} = \frac{1}{2} \Omega_\mu{}^\kappa \Omega_{\kappa\nu} l = \frac{1}{2} (\Omega_\mu \Omega_\nu - \Omega^2 h_{\mu\nu}) l \quad (6.8)$$

and so on for other higher order variations.

Finally, to perform the total variational procedure, we display the explicit form of the variations  $\delta \dot{x}^\mu$ ,  $\delta \omega^{\mu\nu}$ ,  $\delta \mathcal{M}^{\mu\nu}$ ,  $\delta \mathcal{M}^{\mu\nu\rho\sigma}$ , etc. Using (6.1) and (6.2), the variation of the unnormalized four-velocity yields

$$\delta \dot{\gamma}^\mu \equiv \delta_\gamma \dot{\gamma}^\mu = \delta \gamma^\kappa (\partial_\kappa \dot{\gamma}^\mu + \Gamma_{\kappa\nu}^\mu \dot{\gamma}^\nu) = (\delta \gamma^\mu)' \quad (6.9)$$

where we have made use of the first order relations

$$\delta \gamma^\kappa \partial_\kappa \dot{\gamma}^\mu = \dot{\gamma}^\mu (x') - \dot{\gamma}^\mu (x) \approx \frac{d\delta \gamma^\mu}{d\lambda} = \dot{\gamma}^\kappa \partial_\kappa \delta \gamma^\mu. \quad (6.10)$$

The variation of the spacetime angular velocity tensor gives

$$\begin{aligned} \delta \omega^{\mu\nu} &= \delta \dot{e}_a^\mu e^{a\nu} + \dot{e}_a^\mu \delta e^{a\nu} = (\delta e_a^\mu)' e^{a\nu} \\ &+ \omega^{\mu\kappa} e_{a\kappa} \delta e^{a\nu} + R^{\mu\nu}{}_{\kappa\lambda} \delta \gamma^\kappa \dot{\gamma}^\lambda, \end{aligned} \quad (6.11)$$

having employed the double derivative relation

$$\delta_\gamma \dot{e}_a^\mu = (\delta_\gamma e_a^\mu)' + R^{\mu\nu}{}_{\kappa\lambda} e_{a\nu} \delta \gamma^\kappa \dot{\gamma}^\lambda \quad (6.12)$$

and the commutation equation

$$\delta_e \dot{e}_a^\mu = (\delta_e e_a^\mu)' \quad (6.13)$$

between the covariant convective operator  $\dot{\gamma}^\mu \nabla_\mu$  and the variational derivative operator  $\delta_e$ . In fact, using  $\delta_e e_a^\mu \equiv 0$ , we have

$$\delta_e \dot{e}_a^\mu = \delta_e (e_a^\mu l) \nabla_\kappa e_a^\mu + (\delta_e e_a^\mu)' = (\delta_e e_a^\mu)' \quad (6.14)$$



The variations of the multipolar tensors are obviously at the core of the rigidity condition. Applying the total variation we have

$$\begin{aligned}\delta\mathcal{M}^{\mu\nu} &= \delta(e_a^\mu e_b^\nu \mathcal{M}^{ab}) \\ &= 2\mathcal{M}^{\kappa(\mu} \delta_{\lambda}^{\nu)} e_{a\kappa} \delta e^{a\lambda} + e_a^\mu e_b^\nu \delta\mathcal{M}^{ab}.\end{aligned}\quad (6.15)$$

For higher order term, using the symmetries properties (3.10), we have

$$\begin{aligned}\delta\mathcal{M}^{\alpha_1 \dots \alpha_M \kappa_1 \dots \kappa_N} &= \sum_{l=1}^M \delta e_{a_l}^{\alpha_l} \mathcal{M}^{\alpha_1 \dots \alpha_l \dots \alpha_M \kappa_1 \dots \kappa_N} \\ &+ N \delta e_b^{\kappa_N} \mathcal{M}^{\alpha_1 \dots \alpha_M \kappa_1 \dots \kappa_{N-1} b} \\ &+ e_{a_1}^{\alpha_1} \dots e_{a_M}^{\alpha_M} e_{b_1}^{\kappa_1} \dots e_{b_N}^{\kappa_N} \delta\mathcal{M}^{a_1 \dots a_M b_1 \dots b_N}.\end{aligned}\quad (6.16)$$

But what about  $\delta\mathcal{M}^{ab}$  or  $\delta\mathcal{M}^{a_1 \dots a_M b_1 \dots b_N}$ , the variations of the multipolar tensor in the body reference frame, induced by the variation of the trajectory?

### • Relativistic Rigidity Condition

Let  $\gamma^\mu(\lambda, \epsilon)$  be a 1-parameter family of time-like curves with both definitions, of the orthonormal tetrad field  $e^{a\nu}(\lambda, \epsilon)$  and of the multipolar tensor field  $\mathcal{M}^{\alpha_1 \dots \alpha_M \kappa_1 \dots \kappa_N}(\lambda, \epsilon)$ , extended on them,  $\epsilon$  being an infinitesimal parameter. Then  $e^{a\nu}(\lambda, \epsilon)$  is the Fermi-Walker transported of  $e^{a\nu}(\lambda, 0)$  along  $\gamma^\mu(\lambda, \epsilon)$ . A slightly varied curve  $\gamma'^\mu(\lambda)$  with respect to the reference trajectory  $\gamma^\mu(\lambda) \equiv \gamma^\mu(\lambda, \epsilon = 0)$  with end-fixed points is given by the smooth arbitrary function of re-parametrization  $\epsilon(\lambda)$ , such that  $\gamma'^\mu(\lambda) \equiv \gamma^\mu[\lambda, \epsilon(\lambda)]$ . Consider an arbitrary variation of the reference curve  $\gamma^\mu(\lambda)$

$$\delta\gamma^\mu \equiv \gamma'^\mu(\lambda) - \gamma^\mu(\lambda) = \frac{\partial\gamma'^\mu}{\partial\epsilon} \delta\epsilon.\quad (6.17)$$

Then

$$\begin{aligned}\delta_\gamma \mathcal{M}^{a_1 \dots a_M b_1 \dots b_N} &= \frac{\partial\gamma'^\mu}{\partial\epsilon} \nabla_\mu \mathcal{M}^{a_1 \dots b_N} \delta\epsilon = \\ &\left[ \left( \frac{\partial\gamma'^\mu}{\partial\lambda} + \frac{\partial\gamma'^\mu}{\partial\epsilon} \frac{d\epsilon}{d\lambda} \right) - \frac{d\gamma^\mu}{d\lambda} \right] \nabla_\mu \mathcal{M}^{a_1 \dots a_M b_1 \dots b_N} \delta\lambda \\ &+ [(\mathcal{M}^{a_1 \dots a_M b_1 \dots b_N})_{\cdot\gamma'^\mu} - (\mathcal{M}^{a_1 \dots a_M b_1 \dots b_N})_{\cdot\gamma^\mu}] \delta\lambda,\end{aligned}\quad (6.18)$$

where we have used  $\partial\gamma'^\mu/\partial\lambda \equiv d\gamma'^\mu/d\lambda$  and the subscript in the dot derivative denotes along which curve the covariant convective derivative is computed.

The last relation suggests to assume

$$(\mathcal{M}^{a_1 \dots a_M b_1 \dots b_N})_{\cdot\gamma'^\mu(\lambda)} = 0, \quad \forall\gamma'^\mu(\lambda),\quad (6.19)$$

or

$$\delta_\gamma \mathcal{M}^{a_1 \dots a_M b_1 \dots b_N} = 0.\quad (6.20)$$

which is equivalent to the total variational condition

$$\delta\mathcal{M}^{a_1 \dots a_M b_1 \dots b_N} = 0\quad (6.21)$$

in (6.16), because of  $e_a^\mu \delta e^{a\nu} = 0$ . All this relations can be obtained from the following *relativistic rigidity condition* on the  $2n$ -multipole moments

$$\delta\mathcal{M}^{a_1 a_2 b_1 \dots b_N} = 0.\quad (6.22)$$

or

$$(\mathcal{M}^{a_1 a_2 b_1 \dots b_N})^\cdot = 0. \quad (6.23)$$

In fact (6.21) can be easily obtained from (6.23) using (3.11) and the relations  $\delta_e \bar{g}_{a_i a_j} = \delta_\gamma \bar{g}_{a_i a_j} = 0$ . Formally our relativistic rigidity conditions on the multipole moments coincide with that of Dixon [4]. Contracting (6.21) we have also  $\delta \mathcal{M}^{ab} = 0$  and the condition of the constance of the 'rest mass'  $\delta \mathcal{M} = 0$ . The interpretation is straightforward: because of rigidity, in the body reference frame every non inertial stress caused by rotation, or curvature, leaves unchanged all tensors  $\mathcal{M}^{a_1 \dots b_N}$ . This considerably simplifies the variational procedure. On the contrary, in the case of a non rigid body, it will be necessary to specify an equation of state describing the internal structure of the body and the evolution of the energy momentum tensor along the trajectory.

### • Equations of motion

Applying the action principle to the lagrangian, and using the equations obtained in the previous subsection, after a tedious calculation we have

$$\delta \mathcal{L} = X_\mu \delta \gamma^\mu + E_{\mu a} \delta e^{a\mu} + S \quad (6.24)$$

with

$$\begin{aligned} X_\mu = & - \left( \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^\mu} \right)^\cdot + \frac{\partial \mathcal{L}}{\partial \omega^{\kappa\nu}} R^{\kappa\nu}{}_{\mu\lambda} \dot{\gamma}^\lambda \\ & + l \sum_{M,N} \frac{1}{\mathcal{A}_N} \mathcal{M}^{\alpha_1 \dots \kappa_N} \nabla_\mu \mathcal{G}_{\alpha_1 \dots \kappa_N} \end{aligned} \quad (6.25)$$

and

$$\begin{aligned} E_{\mu\nu} = & - \left( \frac{\partial \mathcal{L}}{\partial \omega^{\mu\nu}} \right)^\cdot + 2 \frac{\partial \mathcal{L}}{\partial \omega^{\kappa[\mu} \omega^{\kappa}{}_{\nu]}} + \\ & 2 \frac{\partial \mathcal{L}}{\partial \mathcal{M}^{\kappa[\mu} \mathcal{M}^{\kappa}{}_{\nu]}} + l \sum_{M,N} \frac{1}{\mathcal{A}_N} \sum_{cp} \left( \mathcal{G}_{[\mu|\alpha_2 \dots \kappa_N | \mathcal{M}^{\cdot}{}_{\nu]}^{\alpha_2 \dots \kappa_N}} \right) \\ & + l \sum_{M,N} \frac{N}{\mathcal{A}_N} \mathcal{G}_{\alpha_1 \dots \kappa_{N-1} [\mu} \mathcal{M}^{\alpha_1 \dots \kappa_{N-1} \cdot}{}_{\nu]} \end{aligned} \quad (6.26)$$

S being a surface term. In (6.26)  $\sum_{cp}$  means the sum over the cyclic permutation of the position of the indexes of the tensors included in the sum (for example  $\sum_{cp} A^\mu{}_{\nu\sigma} B^{\rho\kappa\lambda} = A^\mu{}_{\nu\sigma} B^{\rho\kappa\lambda} + A_{\nu\sigma}{}^\mu B^{\kappa\lambda\rho} + A_{\sigma\mu}{}^\nu B^{\lambda\rho\kappa}$ ).

Because of the independence of the variations  $\delta \gamma^\mu$  and  $\delta e^{a\mu}$  the minimal action principle gives

$$X_\mu = 0, \quad E_{\mu\nu} = 0. \quad (6.27)$$

On the other hand, using in what follows an affine parametrization, from (6.6), (6.8), (4.4) and (6.5) we have

$$\begin{aligned} & 2 \frac{\partial \mathcal{L}}{\partial \omega^{\kappa[\mu} \omega^{\kappa}{}_{\nu]}} + 2 \frac{\partial \mathcal{L}}{\partial \mathcal{M}^{\kappa[\mu} \mathcal{M}^{\kappa}{}_{\nu]}} \\ & = \dot{S}_{\kappa[\nu} u_{\mu]} u^\kappa = P_{[\nu} u_{\mu]}. \end{aligned} \quad (6.28)$$

Thus, with the aid of (6.4) and (6.6) and the previous equations, in an affine parametrization the constraints (6.27) become the equations of motion to all order

$$\dot{P}_\mu = \frac{1}{2} R_{\kappa\nu\sigma\mu} S^{\kappa\nu} u^\sigma + \sum_{M,N} \frac{1}{\mathcal{A}_N} \mathcal{M}^{\alpha_1 \dots \kappa_N} \nabla_\mu \mathcal{G}_{\alpha_1 \dots \kappa_N} \quad (6.29)$$

and

$$\begin{aligned} \dot{S}^{\mu\nu} &= 2P^{[\mu} u^{\nu]} + \\ &2 \sum_{M,N} \frac{1}{\mathcal{A}_N} \sum_{cp} \left( \mathcal{G}^{[\nu}{}_{\alpha_2 \dots \kappa_N} \mathcal{M}^{\mu] \alpha_2 \dots \kappa_N} \right) \\ &+ 2 \sum_{M,N} \frac{N}{\mathcal{A}_N} \mathcal{G}_{\alpha_1 \dots \kappa_{N-1}}^{[\nu} \mathcal{M}^{|\alpha_1 \dots \kappa_{N-1}| \mu]}. \end{aligned} \quad (6.30)$$

By defining, similarly to Dixon [4], the following tensor

$$\mathcal{J}_\kappa{}^{\rho\nu}{}_\sigma = \mathcal{M}_{[\kappa}^{[\nu\rho]}{}_{\sigma]} \quad (6.31)$$

and using  $M = 1$  and  $N = 2$  we have

$$\begin{aligned} &2 \frac{\partial \mathcal{L}}{\partial \mathcal{M}^{[\mu|\kappa\rho\sigma]}} \mathcal{M}^{\nu] \kappa \rho \sigma} + 2 \frac{\partial \mathcal{L}}{\partial \mathcal{M}^{\kappa\rho\sigma[\mu}} \mathcal{M}^{|\kappa\rho\sigma|\nu]} = \\ &-\frac{2}{3} R_{[\mu|\rho\sigma\kappa]} \mathcal{J}^{|\kappa\rho|\nu]\sigma}, \end{aligned} \quad (6.32)$$

so, in the quadrupolar approximation, the general equations (6.29) and (6.30) become

$$\dot{P}_\mu = \frac{1}{2} R^{\kappa\nu\sigma}{}_\mu S_{\kappa\nu} u^\sigma + \frac{1}{6} \mathcal{J}^{\kappa\rho\sigma\nu} \nabla_\mu R_{\nu\kappa\rho\sigma} \quad (6.33)$$

and

$$\dot{S}^{\mu\nu} = 2P^{[\mu} u^{\nu]} + \frac{4}{3} R^{\mu}{}_{\rho\sigma\kappa} \mathcal{J}^{\nu]\sigma\kappa\rho}. \quad (6.34)$$

As previously stated, equations (6.33) and (6.34) formally coincide with those of Dixon (apart from a sign convention and the metric signature). The present formalism is suitable in searching for interesting implications when we have to deal with cyclic variables.

## 7. Conserved quantities

In this Section we derive the expressions of the conserved quantities coming from the existence to cyclic variables in the lagrangian. We obtain this result to all order in the expansion of the lagrangian.

Let  $x^c$  be a cyclic variable of the metric. Thus

$$\begin{aligned} \partial_c g_{\mu\nu} &= 0 \\ \Gamma_{c\sigma}^\lambda A^\sigma{}_\lambda &= \Gamma_{c\sigma}^\lambda A^{[\sigma}{}_\lambda]. \end{aligned} \quad (7.1)$$

Consider the first term in the right hand side of the equation of motion (6.29) having posed  $\mu = c$ . We have

$$\begin{aligned} &\frac{1}{2} R^{\kappa}{}_{\nu\sigma c} S_\kappa{}^\nu u^\sigma = \frac{1}{2} (\partial_\sigma \Gamma_{c\nu}^\kappa + \Gamma_{\rho\sigma}^\kappa \Gamma_{c\nu}^\rho - \Gamma_{\rho c}^\kappa \Gamma_{\nu\sigma}^\rho) S_\kappa{}^\nu u^\sigma \\ &= \frac{1}{2} \frac{d}{d\lambda} \Gamma_{c\nu}^\kappa S_\kappa{}^\nu + \frac{1}{2} \Gamma_{c\nu}^\rho (\Gamma_{\rho\sigma}^\kappa S_\kappa{}^\nu u^\sigma - \Gamma_{\kappa\sigma}^\nu S_\rho{}^\kappa u^\sigma) \\ &= \frac{1}{2} \frac{d}{d\lambda} (\Gamma_{c\nu}^\kappa S_\kappa{}^\nu) - \frac{1}{2} \Gamma_{c\nu}^\kappa \frac{D}{D\lambda} S_\kappa{}^\nu. \end{aligned} \quad (7.2)$$

Using the last term and the equation of motions (6.30) and the last of (7.2) we obtain

$$\begin{aligned}
 & \frac{1}{2} \Gamma_{c\nu}^{\kappa} \frac{D}{D\lambda} S_{\kappa}^{\nu} = \Gamma_{c\nu}^{\kappa} P_{\kappa} u^{\nu} \\
 & - \sum_{M,N} \frac{1}{\mathcal{A}_N} \Gamma_{c\nu}^{\kappa} \Sigma_{l=1}^M \left( \delta_{\kappa}^{\alpha_l} \mathcal{G}_{\alpha_1 \dots \alpha_l \dots \kappa_N} \right) \left( \delta_{\alpha_l}^{\nu} \mathcal{M}^{\alpha_1 \dots \alpha_l \dots \kappa_N} \right) \\
 & - \sum_{M,N} \frac{N}{\mathcal{A}_N} \Gamma_{c\nu}^{\kappa} \mathcal{G}_{\alpha_1 \dots \kappa_{N-1} \kappa} \mathcal{M}^{\alpha_1 \dots \kappa_{N-1} \nu}, \tag{7.3}
 \end{aligned}$$

$\alpha_l$  being summed within the same round brackets. On the other hand the last term in (6.29), in the case of the cyclic component, becomes

$$\begin{aligned}
 & \sum_{M,N} \frac{1}{\mathcal{A}_N} \mathcal{M}^{\alpha_1 \dots \kappa_N} \nabla_c \mathcal{G}_{\alpha_1 \dots \kappa_N} = \\
 & - \sum_{M,N} \frac{1}{\mathcal{A}_N} \Sigma_{l=1}^M \left( \mathcal{M}^{\alpha_1 \dots \alpha_l \dots \kappa_N} \delta_{\alpha_l}^{\nu} \right) \left( \Gamma_{c\nu}^{\alpha_l} \mathcal{G}_{\alpha_1 \dots \alpha_l \dots \kappa_N} \right) \\
 & - \sum_{M,N} \frac{N}{\mathcal{A}_N} \mathcal{M}^{\alpha_1 \dots \kappa_N} \Gamma_{c\kappa_1}^{\kappa} \mathcal{G}_{\alpha_1 \dots \kappa_{N-2} \kappa_1 \kappa_N}. \tag{7.4}
 \end{aligned}$$

Collecting (7.2) and (7.3), the equations of motion (6.29) give

$$\frac{D}{D\lambda} P_c = \frac{1}{2} \frac{d}{d\lambda} \left( \Gamma_{c\nu}^{\kappa} S_{\kappa}^{\nu} \right) - \Gamma_{c\nu}^{\kappa} P_{\kappa} u^{\nu} \tag{7.5}$$

or

$$\frac{d}{d\lambda} \left( P_c + \frac{1}{2} \Gamma_{c\nu}^{\kappa} S_{\kappa}^{\nu} \right) = 0 \tag{7.6}$$

by which we obtain the conserved quantities along the trajectory of the rigid test body corresponding to the cyclic coordinate  $x^c$

$$Q_c = P_c + \frac{1}{2} \Gamma_{c\nu}^{\kappa} S_{\kappa}^{\nu} \tag{7.7}$$

that, using (6.5), explicitly are

$$Q_c = -\mathcal{L}u_c + S_c^{\kappa} \dot{u}_{\kappa} + \frac{1}{2} \Gamma_{c\nu}^{\kappa} S_{\kappa}^{\nu}. \tag{7.8}$$

Note that these conserved quantities, not found by BI-ADS, hold to all order in the multipolar expansion.

In order to give concrete examples of our results useful in applications, in the next Section we will investigate some conserved quantities and some properties of the relativistic moment of inertia tensor by the introduction of the principal axes of inertia and of the axial moment of inertia of a rigid test body.

## 8. Relativistic dynamics of a spinning rigid test body

In what follows we give some examples of the dynamical evolution of a test rigid body in the quadrupolar approximation.

- **Conserved quantities**

As an example of (7.8) we consider the conserved quantity  $E = -\frac{1}{2}cQ_t$  associated to the cyclic temporal coordinate  $x^t$  in the Schwarzschild metric in the polar coordinates. In standard units, using  $c$  as the speed of light, we have

$$E = -\frac{1}{2}cQ_t = E_M + E_T + E_{SO}, \quad (8.1)$$

with

$$\begin{aligned} E_M &= c\mathcal{L}u_t \\ E_T &= -cS_t^\kappa \dot{u}_\kappa \\ E_{SO} &= -\frac{1}{2}cS^{\kappa\lambda} \partial_\kappa g_{t\lambda}. \end{aligned} \quad (8.2)$$

Consider the field  $k = (-g_{tt})^{-\frac{1}{2}} \partial_t$  of static observers in the space time outside the horizon. It will use the tetrad

$$\begin{aligned} \partial_{\hat{t}} &= (-g_{tt})^{-\frac{1}{2}} \partial_t \\ \partial_{\hat{r}} &= (g_{rr})^{-\frac{1}{2}} \partial_r \\ \partial_{\hat{\theta}} &= (g_{\theta\theta})^{-\frac{1}{2}} \partial_\theta \\ \partial_{\hat{\phi}} &= (g_{\phi\phi})^{-\frac{1}{2}} \partial_\phi. \end{aligned} \quad (8.3)$$

Thus we have

$$u_t = -\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} c \quad (8.4)$$

where  $v$  is interpreted as the local velocity of the body as measured by  $k$ . Using the quadrupolar expansion (5.13) of the lagrangian, in the weak gravitational region of space-time where  $r \gg 2m = 2\frac{GM_\odot}{c^2}$ , neglecting the term of  $O(v^3/c^3)$ , the first term in (8.2) becomes

$$\begin{aligned} E_M &= \mathcal{M}c^2 + \frac{1}{2}\mathcal{M}v^2 - G\frac{M_\odot\mathcal{M}}{r} \\ &+ \frac{1}{4}S^{\mu\nu}\Omega_{\mu\nu} + \frac{1}{6}\mathcal{M}^{\mu\nu\rho\sigma}R_{\mu\rho\sigma\nu}c^2 \end{aligned} \quad (8.5)$$

having used the standard units redefinitions  $S^{\mu\nu} \rightarrow cS^{\mu\nu}$  and  $\Omega^{\mu\nu} \rightarrow c\Omega^{\mu\nu}$ . In the r.h.s. of (8.5), the first three terms are the barycentric energy contribution of the rigid body. The remaining terms encode effects coming from the internal structure of the body: the spin rotational energy and the gravitational energy due to quadrupolar distribution of the "internal masses" within the small space-like region  $\bar{\Sigma}(\tau)$  occupied by the body and comoving with it. Thus  $E_M$  is the mechanical energy.

The origin of  $E_T$  in (8.2) is equally well justified: it is the Thomas precession energy contribution of the spin caused by the non inertial motion of the body.

The last term seems more obscure but it can be rapidly calculated. Using the Schwarzschild metric, equation (5.4) and the tetrad (8.3), we have

$$\begin{aligned} E_{SO} &= -\frac{1}{2c}\sqrt{-g}\delta_{tr\rho\sigma}u^\rho S^\sigma \partial_r g_{tt} \\ &= c\sin\theta (u^\theta S^\phi - u^\phi S^\theta) \\ &= -\frac{m}{r} (\omega^{\hat{\theta}} S^{\hat{\theta}} + \omega^{\hat{\phi}} S^{\hat{\phi}}) \end{aligned} \quad (8.6)$$

where we have defined the local angular orbital velocity  $\omega^{\hat{\theta}} = -c\dot{\theta}$  and  $\omega^{\hat{\phi}} = c\dot{\phi}\sin\theta$ . By defining the spatial angular momentum vector measured by the observer  $K$  at

infinity as  $\vec{l} = \mathcal{M}r^2\vec{\omega}$  and the zero order geodesic potential  $V = -\frac{GM_{\odot}\mathcal{M}}{r}$  we finally have

$$E_{so} = \frac{1}{\mathcal{M}^2c^2} \frac{1}{r} \frac{\partial V}{\partial r} \vec{S} \cdot \vec{l}. \quad (8.7)$$

It is the spin-orbital coupling of gravitational origin: the right hand of equation (8.7) is formally double of the analog quantity in the special relativistic case of an atom immersed in a magnetic field, the factor 2 being of general relativistic origin. Summing all the terms we have a complete description of the conserved total energy.

Analogous examples could be derivable from the angular conserved quantity coming from the cyclic variable  $x^\phi$  in the Schwarzschild metric.

### • Spin dynamics

Now we briefly study the intrinsic rotational dynamics of the body. In a proper time parametrization, (4.5) and (5.3)-(5.6) give

$$S^\mu = (\mathcal{M}^\kappa{}_\nu \delta^\mu{}_\nu - \mathcal{M}^\mu{}_\nu) \Omega^\nu. \quad (8.8)$$

The quantity in round brackets is formally identical to the Newtonian expression of the moment of inertia tensor in cartesian coordinates[18]. This suggests the following definition of the *relativistic moment of inertia tensor*  $\mathcal{I}^{\mu\nu}$

$$\mathcal{I}^{\mu\nu} \equiv (\mathcal{M}^\kappa{}_\nu g^{\mu\nu} - \mathcal{M}^{\mu\nu}). \quad (8.9)$$

By the use of this quantity, the spin vector has the form

$$S^\mu = \mathcal{I}^\mu{}_\nu \Omega^\nu. \quad (8.10)$$

In a general coordinate system and in completely relativistic domain, the rigidity condition (6.19) holds as

$$\dot{\mathcal{I}}^{\mu\nu} = 2\mathcal{I}^\kappa{}_\nu (\omega^\nu{}^\kappa), \quad (8.11)$$

which is formally similar to the classical one [18], apart from the use of a covariant convective derivative and the occurrence of the spacetime angular velocity tensor  $\omega^{\nu\kappa}$  instead of the space angular velocity tensor  $\Omega^{\nu\kappa}$ . The simple use of the fluid  $T^{\mu\nu} = \rho u^\mu u^\nu$  in the definition of multipolar tensors would tell us that (8.9) is indeed a good definition.

In order to go on and to introduce the principal axis and axial moments of inertia of the body, we specialize our quantities in a tetrad.  $\mathcal{M}^{\mu\nu}$  it is a real symmetric tensor, then it is an orthogonal matrix and consequently diagonalizable. It has 10 independent components or degrees of freedom. Consider the tetrad  $t_a^\mu$  and the decomposition  $\mathcal{M}^{\mu\nu} = \mathcal{M}^{ab} t_a^\mu t_b^\nu$ .  $\mathcal{M}^{ab}$  is diagonalizable too. Using the  $\mathcal{L}_{rot}$  symmetry of the model, we operate the ordinary diagonalization procedure with a change of the tetrads  $t_a^\mu \rightarrow e_a^\mu$ , where the use of the symbol  $e_a^\mu$  is in order to assign a special character to this tetrad:  $e_a^\mu$  is the tetrad which diagonalizes the Lorentz mass tensor  $\mathcal{M}^{ab}$  and therefore the Lorentz moment of inertia tensor  $\mathcal{I}^{ab}$ . Thus we have

$$\mathcal{M}^{\mu\nu} = \sum_{a=0}^3 \mathcal{M}^a e_a^\mu e_a^\nu, \quad (8.12)$$

with  $\mathcal{M}^a$  a Lorentz vector, with 4 degrees of freedom. Because the four bivectors  $e_a^\mu e_a^\nu$  are constructed with the use of a tetrad  $e_a^\nu$ , they possess 6 degree of freedom: 16 degrees belonging from the vectors components of the tetrad, reduced to 6 by the

imposition of the 10 orthonormal conditions (4.1). Thus the decomposition (8.12) contains the 10 exact degrees of freedom of a symmetric tensor.

But what are the principal axes? It is easy to show that, being identically  $\mathcal{M}^0 = 0$ , the transformation

$$\begin{aligned} \mathcal{M}^a &\equiv \frac{1}{2} (\mathcal{I}^{a+1} + \mathcal{I}^{a+2} - \mathcal{I}^a) \\ a + i &= 1, 2, 3 \text{ mod } (3) \end{aligned} \quad (8.13)$$

realizes the diagonalization of the moment of inertia tensor

$$\mathcal{I}^{\mu\nu} \equiv \sum_{a=1}^3 \mathcal{I}^a e_a^\mu e_a^\nu - \frac{1}{2} \sum_{a=1}^3 \mathcal{I}^a u^\mu u^\nu. \quad (8.14)$$

Notice the separation between the spatial and the temporal components. The form of (8.14) suggests that the triad  $e^{a\mu}$  ( $a = 1, 2, 3$ ), is the triad of the *principal axis of inertia*. The diagonal form of (8.14) shows also that (8.13) implicitly defines the *axial moments of inertia*  $\mathcal{I}^a$ , which behave as a Lorentz spatial vector in the tetrad utilized. Using (1.2) we can invert (8.14) obtaining

$$\begin{aligned} \mathcal{I}^a &= \int_{\Sigma(\lambda)} T_{\bar{g}^\kappa}^\kappa \left[ (X^{a+1})^2 + (X^{a+2})^2 \right] d^3\Sigma \\ a + i &= 1, 2, 3 \text{ mod } (3), \end{aligned} \quad (8.15)$$

from which we easily recover the well known Newtonian expression in the non relativistic limit. Besides, the use of the rigidity condition (8.11) gives, after a short manipulation, the rigidity property of the axial moment of inertia vector

$$(\mathcal{I}^a)^\cdot = 0. \quad (8.16)$$

With the aid of (8.14) the spatial spin vector, (8.10) takes the form

$$S^\mu = \sum_{a=1}^3 \mathcal{I}^a e_a^\mu e_a^\nu \Omega_\nu. \quad (8.17)$$

In the case of the most simple model of a rigid body, the *spherical rigid body*, it is obvious to define it by imposing

$$\mathcal{I}^1 = \mathcal{I}^2 = \mathcal{I}^3 \equiv \mathcal{I} \quad (8.18)$$

In this case, being  $\sum_{a=1}^3 e_a^\mu e_a^\nu = h^{\mu\nu}$ , (8.17) becomes

$$S^\mu = \mathcal{I} \Omega^\mu. \quad (8.19)$$

showing that *in the case of a spherical body, the spatial spin vector  $S^\mu$  and the spatial angular velocity vector  $\Omega^\mu$  are aligned*, as expected.

We now focus our attention to the spin vector evolution. Using (5.4) and (6.34) it is easy to prove that

$$\dot{S}^\mu = S^\kappa \dot{u}_\kappa u^\mu + \frac{2}{3} R^\lambda{}_{\rho\sigma\kappa} \mathcal{J}^{\kappa\rho\nu\sigma} \epsilon^\mu{}_{\lambda\nu} \quad (8.20)$$

or that the usual special relativistic Fermi-Walker transport of the spin is broken by the curvature term. From this evolution equation, using the spin expression (8.10) and the rigidity condition (8.11) after a short calculation we have also

$$\begin{aligned} \mathcal{I}^{\mu\nu} \left( \dot{\Omega}_\nu - \Omega^\kappa \dot{u}_\kappa u_\nu \right) &= \\ \epsilon^{\kappa\mu\sigma} \left( \mathcal{I}_\sigma{}^\nu \Omega_\nu \Omega_\kappa + \frac{2}{3} R^\rho{}_{\kappa\lambda\nu} \mathcal{J}^{\nu\rho\sigma\lambda} \right). \end{aligned} \quad (8.21)$$

These are the equations of spin evolution in the language of the moment of inertia tensor. We now translate this equation in a form containing the axial moment of inertia vector. Employing (8.14), (6.31) and saturating the previous equation with the principal axis of inertia  $e_{a\mu}$  we have (Latin repeated index are not summed)

$$\begin{aligned} \mathcal{I}_a (\Omega_a) \cdot - \Omega_b \Omega_c (\mathcal{I}_b - \mathcal{I}_c) - \frac{2}{3} R_b^{\alpha\beta\gamma} \mathcal{M}_{\gamma[c\alpha]\beta} &= 0, \\ c = b + 1 \text{ mod } (3) = a + 2 \text{ mod } (3), a = 1, 2, 3. \end{aligned} \quad (8.22)$$

We call (8.22) the *Relativistic Euler Equations* because they are formally identical to the classical Newtonian ones in the case of flat spacetime[18].

Finally, consider the rotational energy  $-\mathcal{L}_s$

$$-\mathcal{L}_s = -\frac{1}{2} (\Omega_\mu \Omega_\nu - \Omega^2 h_{\mu\nu}) \mathcal{M}^{\mu\nu}. \quad (8.23)$$

Solving (8.9) with respect to the mass tensor,

$$\mathcal{M}^{\mu\nu} = \left[ \frac{1}{2} (h_{\kappa\lambda} \mathcal{I}^{\kappa\lambda}) g^{\mu\nu} - \mathcal{I}^{\mu\nu} \right] \quad (8.24)$$

and using (8.14), after a straightforward calculation (8.23) takes the form

$$-\mathcal{L}_s = \frac{1}{2} \sum_{a=1}^3 \mathcal{I}_a \Omega_a^2. \quad (8.25)$$

Projecting the spin vector (8.17) along the principal axes,

$$S_a = \mathcal{I}_a \Omega_a, \quad (8.26)$$

(repeated index are not summed) and collecting (8.25) and (8.26), we obtain

$$\sum_{a=1}^3 \frac{S_a^2}{2\mathcal{I}_a (-\mathcal{L}_s)} = 1. \quad (8.27)$$

On the other hand, for the spin vector in the tetrad components we can write

$$\sum_{a=1}^3 \frac{S_a^2}{S^2} = 1. \quad (8.28)$$

Let us now consider the constraint (4.10) of parametric invariance, or  $\mathcal{L} = P_\mu u^\mu$ . Using (6.5) and (5.7) and the equation of motion (6.34) we have

$$\dot{\mathcal{L}} = \frac{1}{6} \mathcal{J}^{\kappa\rho\sigma\nu} \dot{R}_{\nu\kappa\rho\sigma}. \quad (8.29)$$

From (5.13) and the rigidity condition (6.23), equation (8.29) gives

$$\dot{\mathcal{L}}_s = \frac{1}{6} \dot{\mathcal{J}}^{\kappa\rho\sigma\nu} R_{\nu\kappa\rho\sigma}. \quad (8.30)$$

On the other hand, from (8.20) we have

$$(S^2) \cdot = \frac{4}{3} R^\lambda_{\rho\sigma\kappa} \mathcal{J}^{\kappa\rho\nu\sigma} \epsilon^\mu_{\lambda\nu} S_\mu \quad (8.31)$$

Thus, in the trivial case of flat spacetime, we have  $\dot{S} = \dot{\mathcal{L}}_s = 0$  and using the rigidity condition  $\dot{\mathcal{I}}_a = 0$ , equation (8.27) describes the *Relativistic Binet ellipsoid* in the variables  $S_a$ . Intersecting this ellipsoid with the spherical surface of the constant modulus of the spin, equation (8.28), we obtain the complete *trajectory of the spin*



*spatial vector* in the principal axis system of reference, provided we have an initial condition to assign.

The outcome of the naive case of zero curvature tensor everywhere can be extended if the spacetime is maximally symmetric, i.e., in the constant scalar curvature case, namely when

$$R_{\mu\nu\rho\sigma} = \frac{R}{12} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \quad (8.32)$$

Employing (8.30) we have in fact

$$\dot{\mathcal{L}}_S = \frac{R}{72} \left( \dot{\mathcal{J}}^{\kappa\lambda}{}_{\kappa\lambda} - \dot{\mathcal{J}}^{\kappa}{}_{\kappa}{}^{\lambda}{}_{\lambda} \right) = \frac{R}{36} \dot{\mathcal{J}}^{ab}{}_{ab} = 0. \quad (8.33)$$

having used the symmetry properties in (6.31) and the rigidity condition. Similarly, from (8.31) we have

$$\begin{aligned} (S^2) \cdot &= \frac{R}{9} \left( \dot{\mathcal{J}}^{\kappa}{}_{\kappa}{}^{\nu\lambda} - \dot{\mathcal{J}}^{\lambda}{}_{\lambda}{}^{\nu\kappa} \right) \epsilon^{\mu}{}_{\lambda\nu} S_{\mu} \\ &= \frac{2}{9} R \dot{\mathcal{J}}^{\kappa}{}_{\kappa}{}^{\nu\lambda} \epsilon^{\mu}{}_{\lambda\nu} S_{\mu} \end{aligned} \quad (8.34)$$

Using (6.31) we finally have

$$(S^2) \cdot = \frac{4}{9} R \dot{\mathcal{M}}^{(\nu}{}^{\lambda)\kappa} \epsilon^{\mu}{}_{\nu\lambda} S_{\mu} = 0 \quad (8.35)$$

thus obtaining the same conclusion on the trajectory of the spin previously deduced in the flat manifold.

Finally, we can ask whether the most general case in which we can infer the same result is that of a rigid body with high degree of symmetry or of a body for which

$$\mathcal{M}_{[\kappa}{}^{[\nu\rho]}{}_{\sigma]} = \mathcal{J}_{\kappa}{}^{\rho\nu}{}_{\sigma} = 0. \quad (8.36)$$

But we have also  $\mathcal{M}_{[\kappa}{}^{(\nu\rho)}{}_{\sigma]} = 0$  because of (3.19) and therefore  $\mathcal{M}_{[\kappa}{}^{\nu\rho}{}_{\sigma]} = 0$ . Saturating the first two indexes we have  $\mathcal{M}_{\kappa}{}^{\kappa\rho\sigma} = \mathcal{M}_{\kappa}{}^{(\rho|\kappa|\sigma)}$ . On the other hand, saturating the first two indexes in (3.19) we have  $\mathcal{M}_{\kappa}{}^{\kappa\rho\sigma} = -2\mathcal{M}_{\kappa}{}^{(\rho|\kappa|\sigma)}$  or  $\mathcal{M}_{\kappa}{}^{\kappa\rho\sigma} = 0$ . Then (5.5) and (8.9)-(8.10) give  $\mathcal{I}^{\mu\nu} = 0$  and  $S^{\mu} = 0$  and therefore a trivial case. Inversely, the weaker condition  $\mathcal{M}_{\kappa}{}^{\kappa\rho\sigma} = 0$  or  $\mathcal{I}^{\mu\nu} = 0$  and  $S^{\mu} = 0$  cannot give (8.36); however, for ordinary matter, such as  $T^{\mu\nu} = \rho \dot{x}^{\mu} \dot{x}^{\nu}$ ,  $\dot{x}^{\mu}$  a congruence on  $\Delta$ , it is easy to see that (8.36) will be also true.

## 9. Conclusions

Applying a minimal action principle, in this work we have completed a line of research due to Bailey-Israel and Anandan-Dadhich-Singh concerning the variational formulation of the dynamics of a test rigid body immersed in a curved spacetime. In particular we have found the dependence of the lagrangian from the internal degrees of freedom and, employing a rigidity constraint, we have derived, by a complete variational procedure, the equations of motion to all orders as well as conserved quantities coming from cyclic variables. The equations obtained give rise to some simplifications on the dynamics of an extended rigid test body in general relativity with respect to the Dixon theory. Finally, in view of applications, we have found explicit expressions of these conserved quantities, of relativistic Euler equations and of some other classical-like observables implied in the dynamics up to quadrupolar order.

## 10. Acknowledges

I am indebted to Francesco Sorge for helpful discussions.

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