HAL
open science

Improving the performance of linear systems by adding a hybrid loop

Christophe Prieur, Sophie Tarbouriech, Luca Zaccarian

To cite this version:

Christophe Prieur, Sophie Tarbouriech, Luca Zaccarian. Improving the performance of linear systems by adding a hybrid loop. IFAC WC 2011 - 18th IFAC World Congress, Aug 2011, Milan, Italy. p. 6301-6306. hal-00573991

HAL Id: hal-00573991

https://hal.science/hal-00573991v1

Submitted on 28 Mar 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Improving the performance of linear systems by adding a hybrid loop

Christophe Prieur [*,1] **Sophie Tarbouriech** [**,***,1]
Luca Zaccarian [****,2]

[*] *Department of Automatic Control, Gipsa-lab, 961 rue de la Houille Blanche, BP 46, 38402 Grenoble Cedex, France*
[**] *CNRS ; LAAS ; 7 avenue du colonel Roche, F-31077 Toulouse, France*
[***] *Université de Toulouse ; UPS, INSA, INP, ISAE ; LAAS ; F-31077 Toulouse, France*
[****] *DISP, University of Roma, Tor Vergata, Via del Politecnico 1, 00133 Roma, Italy*

Abstract:
Considering a linear system, we construct hybrid loops consisting in a (possibly non stabilizing) given linear dynamic continuous-time controller with a jump rule when some conditions of the state occur. This allows us to study the performance improvement problem. More precisely, we firstly show the interest of an hybrid loop to maximize the decay rate when the gain of the controller is limited. Secondly, we show how adding a hybrid loop can be useful to reduce the overshoot of an output. The proposed approaches are illustrated on two examples.

Keywords: hybrid systems, performance, reset controllers, detectability

1. INTRODUCTION

Even for linear control systems which follow a purely continuous dynamics, it may be useful to consider dynamic controllers having a mixed discrete/continuous dynamics. This leads to the class of hybrid control laws, and the closed-loop system turns out to be a hybrid system. Such controllers are now instrumental in many feedback control designs, for their capability to provide asymptotic stability of the closed-loop system (see e.g. Hespanha and Morse (1999); Hespanha et al. (2004)). They are also interesting for their capability to guarantee a robustness with respect to small errors in the loop, which cannot be obtained using classical (i.e. with a continuous dynamics) controllers (see e.g. Prieur (2005); Goebel and Teel (2009)).

Hybrid controllers are also instrumental to improve the performance for linear systems in presence of disturbances. See Chen et al. (2001); Beker et al. (2004); Nešić et al. (2008), where reset controllers are used to decrease the $\mathcal{L}_2$-gain between perturbations and the output. Consider also Aangenent et al. (2008) (resp. Witvoet et al. (2007)) where it is shown that reset controllers may be useful to improve the $\mathcal{L}^2$ (resp. $\mathcal{H}^2$)-stability of linear systems. See Loquen et al. (2007) for further results on reset systems with saturation in the input.

The aim of this paper is to design new hybrid strategies to improve the performance of the closed-loop system. More precisely, for linear control systems this hybrid loop may be connected to the maximization problem of the decay rate with a limited controller gain. Moreover we show that adding hybrid loops yields a solution to the reduction of the overshoot of an output, by solving a convex optimization problem. This part of our work is related to Haddad et al. (2001b) (see also Haddad et al. (2001a), and Haddad et al. (2003) for Hamiltonian systems) where Lyapunov conditions are designed for the feedback interconnections of impulsive nonlinear systems. However in our paper, an *a priori* Lyapunov-like function is used for the designs of the hybrid controllers.

These performance results are based on the derivation of a sufficient condition for the closed-loop system with the hybrid loop to be asymptotically stable. This stability analysis relies on a LaSalle invariance principle argument and on a result of Prieur et al. (2010). In this latter work, some *a priori* Lyapunov-like functions are used to design hybrid loops that stabilize the whole state or the state variable only.

The remaining part of the paper is organized as follows. The problems under consideration in this paper are introduced in Section 2. Some preliminaries are also given in this section. In Section 3, we show how we can guarantee the stability by adding a hybrid loop, this is our first main result. Section 4 contains the two other main results, namely the maximization of the decay rate with a limited jumping gain (see Section 4.1), and the reduction of the overshoot by the design of a hybrid loop (see Section 4.2). Some numerical examples illustrate the latter two results.

[1] Work supported in part by the ANR project ArHyCo, ARPEGE, contract number ANR-2008 SEGI 004 01-30011459. christophe.prieur@gipsa-lab.grenoble-inp.fr sophie.tarbouriech@laas.fr
[2] Work supported in part by ENEA-Euratom and MIUR. zack@disp.uniroma2.it

Some concluding remarks and open questions are given in Section 5. Due to space limitation, all proofs are omitted.

Notation. The Euclidian norm is denoted by $|\cdot|$ and the scalar product by $\langle\cdot,\cdot\rangle$. For positive real integers n and m, I_n (resp. $0_{n,m}$) denotes the identity matrix (resp. the null matrix) in $\mathbb{R}^{n\times n}$ (resp. in $\mathbb{R}^{n\times m}$). The subscripts may be omitted when there is no ambiguity. Moreover, for a vector x, the diagonal matrix defined by the entries of x is noted $\mathrm{diag}(x)$, and for a matrix M, $\mathrm{He}(M) = M + M'$, where M' denotes the transpose matrix of M. For any symmetric matrix, $\star$ stands for a symmetric term.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following linear plant:

$$\dot{x}_p = \bar{A}_p x_p + \bar{B}_p u, \quad y = \bar{C}_p x_p + \bar{D}_p u, \tag{1}$$

with $x_p \in \mathbb{R}^{n_p}$, in feedback interconnection with a (not necessarily stabilizing) linear dynamic controller:

$$\dot{x}_c = \bar{A}_c x_c + \bar{B}_c y, \quad u = \bar{C}_c x_c + \bar{D}_c y, \tag{2}$$

with $x_c \in \mathbb{R}^{n_c}$. Assuming that $I - \bar{D}_p\bar{D}_c$ is nonsingular, the closed-loop is well posed and described by the following linear system

$$\dot{x} = Ax := \begin{bmatrix} A_p & B_p \\ B_c & A_c \end{bmatrix} x \ , \tag{3}$$

where $x = (x_p', x_c')'$ and A_p, B_p, A_c, and B_c are matrices of appropriate dimensions uniquely defined based on the matrices in (1) and (2). In the reminder of the paper, we consider directly (3) and we do not consider anymore from which this linear system has been computed (in particular (1) and (2) will not be used anymore).

By exploiting the properties of Lyapunov functions, the aim of this paper is to construct a hybrid closed-loop system which follows the flow dynamics (3) when the state is in a set (called the flow set) and follows a suitable discrete dynamics when the state is in an other set (called the jump set). These flow and jump sets, together with the discrete dynamics, define a hybrid system and have to be designed to guarantee some performance properties of the closed-loop state x and of the plant state x_p.

More precisely, we will address the following two problems. The first problem is an optimization of the decay rate of the variable x_p by adding a hybrid loop to the system (3).

Problem 1. *Consider the closed-loop system (3). Given a limitation on the amplitude of the gain K, design a hybrid system which follows the dynamics (3) when flowing, follows the dynamics*

$$x^+ = Kx$$

when jumping, and is such that the x_p-variable have the best decay rate of the x_p-variable as possible.

The second problem is related to the reduction of the overshoot of the output y of (1) by means of a hybrid loop.

Problem 2. *Consider the closed-loop system (3). Design a hybrid system which follows the dynamics (3) when flowing and the output y of (1) converges to zero and is non-increasing along solutions.*

Both problems are solved in Theorems 2 and 3 respectively. Note that Problem 2 is solved indirectly. More precisely, to solve it, we introduce a function $V_p : \mathbb{R}^{n_p} \to \mathbb{R}_{\geq 0}$, and we design a hybrid loop such that V_p is non-increasing along the solutions of the hybrid system, and $V_p(x_p)$ is close (in a certain manner) to $|y|^2$. See Theorem 3 for more details.

Let us shortly describe the framework of the hybrid systems that is considered in this paper. For an introduction, see e.g. the recent survey Goebel et al. (2009). Such a system combines a continuous dynamics in a set F (called flow set) and discrete dynamics in a set J (called jump set), and it is formally written as

$$\begin{aligned} \dot{x} &= Ax \text{ if } x \in F \ , \\ x^+ &= Kx \text{ if } x \in J \ , \end{aligned} \tag{4}$$

where $x = (x_p', x_c')' \in \mathbb{R}^n$, $n = n_p + n_c$, A and K are two matrices of appropriate dimensions. We recall some basic ingredients on hybrid system theory, and solutions of (4). Due to mixed discrete/continuous dynamics, a solution of (4) will be defined on a mixed discrete/continuous time domain. More precisely, a set E is a *hybrid time domain* if for all $(T_0, J_0) \in E$, $E \cap ([0, T_0] \times \{0, 1, \ldots J_0\})$ is a *compact hybrid time domain*, i.e. it can be written as $\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$, for some finite sequence of times $0 = t_0 \leq t_1 \ldots \leq t_J$. A solution x to (4) consists of a hybrid time domain $\mathrm{dom}\, x$ and a function $x : \mathrm{dom}\, x \to \mathbb{R}^n$ such that $x(t, j)$ is absolutely continuous in t for a fixed j and $(t, j) \in \mathrm{dom}\, x$ satisfying

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \mathrm{dom}\, x$,

$$x(t, j) \in F, \quad \dot{x}(t, j) = Ax(t, j) \ ;$$

(S2) For all $(t, j) \in \mathrm{dom}\, x$ such that $(t, j+1) \in \mathrm{dom}\, x$,

$$x(t, j) \in J, \quad x(t, j + 1) = Kx(t, j) \ .$$

Then, the state solution x is parameterized by (t, j) where t is the ordinary time and j is an independent variable that corresponds to the number of jumps of the solution. When the state $x(t, j)$ belongs to the intersection of the flow set and of the jump set, then the solution can either flow or jump. This parameterization may be omitted when there is no ambiguity.

A solution x to (4) is said to be *complete* if $\mathrm{dom}\, x$ is unbounded, *Zeno* if it is complete but the projection of $\mathrm{dom}\, x$ onto $\mathbb{R}_{\geq 0}$ is bounded, and *maximal* if there does not exist an other solution x of (4) such that x is a truncation of x to some proper subset of $\mathrm{dom}\, x$. Hereafter, only maximal solutions will be considered. For more details about this hybrid systems framework, we refer the reader to Goebel et al. (2009) or Prieur et al. (2007).

Definition 1. *The hybrid system (4) is said to be*

- stable: *if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution x to (4) with $|x(0, 0)| \leq \delta$ satisfies $|x(t, j)| \leq \epsilon$ for all $(t, j) \in \mathrm{dom}\, x$;*
- attractive: *if every solution x to (4) is complete and satisfies $\lim_{t+j\to\infty} |x(t, j)| = 0$;*
- globally asymptotically stable: *if it is both stable and attractive.*

3. GUARANTEEING ASYMPTOTIC STABILITY

Before solving Problems 1 and 2, we first state a result suggesting a sufficient condition for the hybrid system (4) to be globally asymptotically stable. To do that, some

Lyapunov functions are used and since the closed-loop system (3) is linear, we restrict to the class of quadratic functions, and to sets F and J depending on the sign of a quadratic function of the state.

Moreover since the x_p-variable is the state of the plant, it is natural to impose the following structure on the gain K defining the discrete dynamics in (4): $K = \begin{bmatrix} I_p & 0 \\ K_p & 0 \end{bmatrix}$, where K_p is a given matrix in $\mathbb{R}^{n_c \times n_p}$.

More precisely given a symmetric positive definite matrix $\bar{P}_p$ in $\mathbb{R}^{n\times n}$, a matrix N in $\mathbb{R}^{n\times n}$, a matrix K_p in $\mathbb{R}^{n_c\times n_p}$, and a positive value $\bar{\alpha}$, we will consider the following structure for the hybrid system (4):

$$\begin{aligned} \dot{x} &= Ax && \text{if } x \in F \ , \\ (x_p^+, x_c^+) &= (x_p, K_p x_p) && \text{if } x \in J \ , \end{aligned} \tag{5}$$

where $F \subset \mathbb{R}^n$ and $J \subset \mathbb{R}^n$ are the closed subsets of $\mathbb{R}^n$ defined by

$$\begin{aligned} F &= \{x \in \mathbb{R}^n, \ x'Nx \le -\bar{\alpha} x_p' \bar{P}_p x_p\} \ , \\ J &= \{x \in \mathbb{R}^n, \ x'Nx \ge -\bar{\alpha} x_p' \bar{P}_p x_p\} \ , \end{aligned} \tag{6}$$

where N, $\bar{\alpha}$, $\bar{P}_p$ will be specified in the sequel.

Our first main result is based on the following proposition which gives a convergence property of the x_p-variable only:

Proposition 1. *[Prieur et al. (2010)] Consider the closed-loop system (3) and a symmetric positive definite matrix* $P = \begin{bmatrix} P_p & P_{pc} \\ \star & P_c \end{bmatrix}$ *in* $\mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$ *such that the matrix* $\bar{P}_p := P_p - P_{pc}P_c^{-1}P_{pc}'$ *satisfies*

$$\mathrm{He}(\bar{P}_p(A_p + B_pK_p)) < -\alpha \bar{P}_p \ , \tag{7}$$

for some $\alpha > 0$ *and for* $K_p := -P_c^{-1}P_{pc}'$. *Then, letting*

$$N := \mathrm{He}\left(\begin{bmatrix} \bar{P}_pA_p & \bar{P}_pB_p \\ 0 & 0 \end{bmatrix}\right) \ . \tag{8}$$

the hybrid system (5), (6) is such that the plant state x_p *converges to zero, and* $x_p \mapsto V_p(x_p) := x_p'\bar{P}_p x_p$ *is non-increasing along solutions.*

With Proposition 1, there is no guarantee that the state of the controller will converge to zero. It turns out to be sufficient that the controller dynamics is detectable from the output matrix B_p. Intuitively, this requirement corresponds to asking that any nonzero controller evolution will be detected by the plant states so that a LaSalle result can be applied to show convergence. This is the basic tool to prove the following theorem.

Theorem 1. *Consider any state stabilizing gain* K_p *such that* $A_p + B_pK_p$ *is Hurwitz and a pair* $\bar{P}_p$, α *satisfying (7). Consider now any* $\bar{\alpha} \le \alpha$ *and the flow and jump sets in (6). If the pair* (B_p, A_c) *is detectable, then the hybrid system (5) is globally exponentially stable.*

4. IMPROVING THE PERFORMANCE

In this section, we rely on the results of Theorem 1 to propose a suitable jump rule to be incorporated in a linear continuous-time control system to maximize the decay rate or to reduce the plant output overshoot (see respectively Sections 4.1 and 4.2 below). This will solve Problems 1 and 2.

4.1 Maximizing the decay rate

Consider equation (7) and note that it resembles the classical state-feedback stabilization problem with guaranteed convergence rate, which is well known to be solvable via a generalized eigenvalue problem. Given a plant, a possible way to address the search for $\bar{P}_p$, K_p and α satisfying (7) is to impose a bound κ_M on the size of K_p and compute the maximum α satisfying (7) (or a conservative estimate of it) while guaranteeing $|K_p| \le \kappa_M$. This type of goal is achieved by solving the following generalized eigenvalue problem:

Find a symmetric positive matrix $\bar{Q}_p$ *in* $\mathbb{R}^{n_p\times n_p}$, *a matrix* X *in* $\mathbb{R}^{n_c\times n_p}$ *and a positive value* α *solution of*

$$\begin{aligned} \max_{\bar{Q}_p = \bar{Q}_p', \alpha, X} \quad & \alpha, \text{ s.t.} \\ \bar{Q}_p &\ge I \\ 0 &\le \begin{bmatrix} \kappa_M I & X \\ X' & \kappa_M I \end{bmatrix} \\ -\alpha \bar{Q}_p &\ge \mathrm{He}(A_p\bar{Q}_p + B_pX), \end{aligned} \tag{9}$$

where $\kappa_M > 0$ is given. The optimal solution to (9) will lead to the gain $K_p = X\bar{Q}_p^{-1}$ and to $\bar{P}_p = \bar{Q}_p^{-1}$ satisfying (7) and such that $|K_p| \le |X||Q_p^{-1}| \le \kappa_M$ because the first constraint imposes $|Q_p^{-1}| \le 1$ and the second constraint imposes $|X| \le \kappa_M$. Using this gain, we may solve Problem 1 and the following theorem holds

Theorem 2. *Consider a symmetric positive matrix* $\bar{Q}_p$ *in* $\mathbb{R}^{n_p\times n_p}$, *a matrix* X *in* $\mathbb{R}^{n_c\times n_p}$ *and a positive value* α *satisfying (9). Let* $K_p = X\bar{Q}_p^{-1}$ *and* $\bar{P}_p = \bar{Q}_p^{-1}$. *For any selection of* $0 < \bar{\alpha} \le \alpha$ *and the flow and jump sets in (6), then*

1. *for a suitable* $k > 0$, *the plant state response satisfies the following exponential bound for all* $(t, j) \in \operatorname{dom} x$:
$$|x_p(t,j)| \le k \exp\left(-\frac{\bar{\alpha}}{2}t\right)|x_p(0,0)|$$
2. *if the pair* (B_p, A_c) *is detectable, then the hybrid closed-loop is globally exponentially stable.*

The optimization problem in (9) can be used to build a curve providing the maximum decay α achievable with a certain bound κ_M on the gain K_p. Then in many practical cases, it might well be that the optimal α grows unbounded as the bound κ_M becomes arbitrarily large. This fact is illustrated in the next example study.

Example 1. Consider a one-dimensional linear plant connected in negative feedback with a one-dimensional linear controller:

$$\begin{aligned} \dot{x}_p &= a_px_p + b_px_c \\ \dot{x}_c &= a_cx_c - x_p, \end{aligned}$$

where we assume $b_p > 0$.

For this example, the solution to (9) can be computed explicitly as a function of κ_M and corresponds to $K_p = -\kappa_M$, while the optimal performance is $\alpha = -2a_p + 2b_p\kappa_M$, achieved with $\bar{Q}_p = 1$ and $X = -\kappa_M$. It is instructive to study the shape of the jump and flow sets F and J defined in (6), which are symmetric cones in the plane (x_p, x_c). Using, for example, $\bar{\alpha} = \frac{\alpha}{2} = -a_p + b_p\kappa_M$, the sets in (6) become

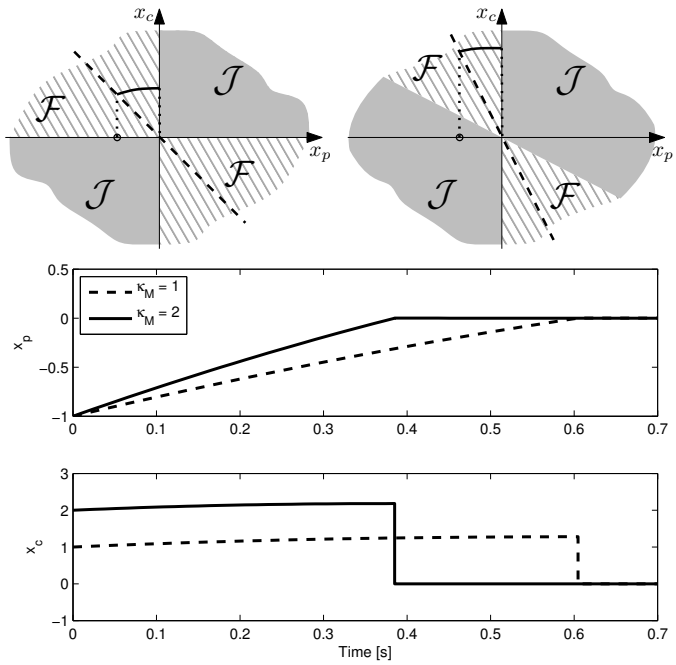

Fig. 1. Shapes of the flow and jump sets of Example 1 using $\kappa_M = 1$ (upper left) and $\kappa_M = 2$ (upper right). The lower plots show the time histories of the two trajectories shown on the state space planes.

$$\mathcal{F} = \left\{ (x_p, x_c) : \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right]' \begin{bmatrix} a_p + b_p\kappa_M & b_p \\ b_p & 0 \end{bmatrix} \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right] \leq 0 \right\}, \quad (10)$$

while $\mathcal{J}$ is the closure of the complement of $\mathcal{F}$.

According to the results in Theorem 2 the hybrid closed-loop system given by

$$\begin{cases} \dot{x}_p = a_p x_p + b_p x_c \\ \dot{x}_c = a_c x_c - x_p, \end{cases} \text{if } x \in \mathcal{F}, \qquad \begin{cases} x_p^+ = x_p \\ x_c^+ = -\kappa_M x_p, \end{cases} \text{if } x \in \mathcal{J}$$

is globally exponentially stable.

The flow set in (10) defines a symmetric cone in the (x_p, x_c) plane whose boundaries correspond to the two subspaces $\{(x_p, x_c) : x_p = 0\}$, namely the vertical axis, and $\{(x_p, x_c) : 2b_p x_c = -(a_p + b_p\kappa_M)x_p\}$, namely (by the fact that $b_p > 0$ and $\kappa_M > 0$), a line which rotates clockwise as κ_M (and, consequently, the guaranteed convergence rate $\bar{\alpha} = -a_p + b_p\kappa_M$) increases. An example corresponding to $a_p = -1$, $b_p = 1$ and $\kappa_M = 1$ is shown in the top left plane of Figure 1. In this case, the flow set (striped area) corresponds to the second and fourth quadrants and the set $x_c = K_p x_p = -\kappa_M x_p = -x_p$, where the controller state is reset, (depicted as a bold dashed line) corresponds to the bisector of the second and fourth quadrant. The top right plane in the same figure shows how these sets change when increasing the gain to $\kappa_M = 2$.

Some trajectories are also reported in Figure 1, corresponding to the case $a_c = 0$ which, in the case $\kappa_M = 1$ corresponds to a generalized version of the behavior of the so-called Clegg integrator (see Clegg (1958); Zaccarian et al. (2005)) in negative feedback interconnection with the plant $\mathcal{P}(s) = \frac{1}{s+1}$. The generalization consists in resetting the Clegg integrator state to the value of the input rather than to zero but essentially coincides with the original resetting rule where the integrator was supposed to be reset only at zero crossings of the integrator input (see Clegg (1958); Zaccarian et al. (2005) for details). Note that in the case $\kappa_M = 2$ reported to the rigth of Figure 1, the horizontal boundary of the flow set is tilted, reducing the size of the flow set, and the reset manifold $x_c = K_p x_p$ is tilted as well, to speed up the state decay rate. Nevertheless, the vertical boundary corresponding to $\{(x_p, x_c) : x_p = 0\}$, remains unchanged thus preserving the peculiarity that the response exhibits no overshoot. This property arises from the fact that the function $x_p \mapsto x_p^2$ is nonincreasing along the solutions. This fact is further illustrated by a few simulations starting from the initial conditions $(x_p(0,0), x_c(0,0)) = (-1, 0)$, reported at the bottom plot of Figure 1. In both simulations, the controller state is first reset to the reset manifold and then exhibits a flow interval until the state reaches the vertical axis, where the solution is finally reset to zero. ⋆

4.2 Overshoot reduction

Perhaps the first example study where hybrid control systems were shown to overcome intrinsic limitations of classical linear control was the one in Beker et al. (2001), were a reset controller was shown to improve upon linear control in terms of overshoot reduction. If the plant (1) is strictly proper (namely, $\bar{D}_p = 0$), a possible way to mathematically formalize the requirement of overshoot reduction in terms of the Lyapunov functions introduced here is to construct a quadratic function $x_p \mapsto V_p(x_p) := x_p' \bar{P}_p x_p$ and the stabilizing gain K_p in such a way that, to a certain extent, $V_p(x_p) \approx |y|^2$. Then achieving non-increase of V_p via the hybrid loops of this paper will induce (almost) no overshoot, namely $|y(t)|$ is (almost) non-increasing along trajectories. Let us make more explicit this informal discussion.

Finding the above discussed pair (V_p, K_p) can be done, for example, by writing the dynamics (1) in observability canonical form, so that $x_p = (x_1', y)'$ and solving the following LMI eigenvalue problem: *Find a symmetric positive matrix $\bar{Q}_p$ in $\mathbb{R}^{n_p \times n_p}$, a matrix X in $\mathbb{R}^{n_c \times n_p}$ and positive value α, ρ_x, and ρ_y solution of*

$$\begin{aligned} \min_{\bar{Q}_p = \bar{Q}_p', \rho_x, \rho_y, X} \quad & \rho_y, \text{ s.t.} \\ \bar{Q}_p &= \begin{bmatrix} Q_1 & q_{1y} \\ q_{1y}' & q_y \end{bmatrix} > I \\ 0 &< \begin{bmatrix} \kappa_M I & X \\ X' & \kappa_M I \end{bmatrix} \\ 0 &> \mathrm{He}(A_p \bar{Q}_p + B_p X) \\ 0 &\leq \begin{bmatrix} \rho_x & 1 \\ 1 & \rho_y \end{bmatrix} \\ \rho_x &< Q_1 \\ q_y &< 1 + \rho_y, \end{aligned} \quad (11)$$

where $\kappa_M > 0$ is given. Similar to the case addressed in Section 4.1, the optimal solution to (11) will lead to the gain $K_p = X\bar{Q}_p^{-1}$ and to $\bar{P}_p = \bar{Q}_p^{-1}$ satisfying (7) with a small enough α and such that $|K_p| \leq |X||\bar{Q}_p^{-1}| \leq \kappa_M$. Moreover, the bounds given by the last three constraints can be shown to imply that smaller values of ρ_y will lead to a function $V_p(x_p) = x_p' \bar{Q}_p^{-1} x_p$ closer to $|y|^2$. This fact,

together with desirable stability properties of the hybrid closed-loop, helps us to solve Problem 2 and to establish in the following theorem.

Theorem 3. *Consider a sequence of solutions* $(\bar{Q}_p^k, \rho_y^k, \rho_x^k, X^k)_{k\in\mathbb{N}}$ *to the constraints (11) such that* $\rho_y^k \to 0$ *as* $k \to \infty$*. Then, defining* $V_p^k(x_p) = x_p' \left(\bar{Q}_p^k\right)^{-1} x_p$*, we have, for each* x_p *in* $\mathbb{R}^{n_p}$*,*

$$\lim_{k\to\infty} V_p^k(x_p) = |y|^2, \tag{12}$$

namely as ρ_y^k *approaches zero,* $V_p^k(x_p)$ *approaches* $|y|^2$*.*

Moreover, given any pair $(\bar{Q}_p, X)$ *satisfying (11), let* $K_p = X\bar{Q}_p^{-1}$ *and* $\bar{P}_p = \bar{Q}_p^{-1}$*. Then, for a small enough selection of* α*, equation (7) is satisfied. Moreover, given any selection of* $0 < \bar{\alpha} \le \alpha$*, the reset system (5) with the flow and jump sets in (6), is such that:*

1. *the plant state* x_p *converges to zero, and the function* $x_p \mapsto V_p(x_p) = x_p'\bar{P}_p x_p$ *is non-increasing along solutions;*
2. *if the pair* (B_p, A_c) *is detectable, then the hybrid closed-loop is globally exponentially stable.*

Just as in the case discussed in Section 4.1, the optimization problem in (9) can be used to build a curve providing the overshoot reduction level ρ_y^{-1} achievable with a certain bound κ_M on the gain K_p. Similar to before, it might well be that the optimal ρ_y^{-1} grows unbounded as the bound κ_M becomes arbitrarily large.

Example 2. We consider an example originally discussed in Beker et al. (2001), where a First Order Reset Element whose linear part is characterized by the transfer function $\frac{1}{s+1}$ controls via a negative unitary feedback a SISO plant whose transfer function is $\mathcal{P} = \frac{s+1}{s(s+0.2)}$. For this example, the control system involving the FORE is shown in Beker et al. (2001) to behave more desirably than the linear control system because it has only about 40% overshoot as compared to the linear closed-loop system, while retaining the rise time of the linear design (this example was also later discussed later in Nešić et al. (2005); Zaccarian et al. (to appear) where the $\mathcal{L}_2$ gain properties of the reset closed-loop were characterized). Here we show that when allowing more general resets than just the ones induced by the FORE resetting law (which essentially imposes a reset of the controller state to zero when the controller input and output have opposite signs), arbitrary small overshoot can be achieved, while retaining the same rise time. To this aim, we use the following state-space representation of the linear closed-loop:

$$\left[\begin{array}{c|c} A_p & B_p \\ \hline B_c & A_c \end{array}\right] = \left[\begin{array}{cc|c} -0.6 & 0.6 & -1 \\ -0.4 & 0.4 & 1 \\ \hline 0 & 1 & -1 \end{array}\right]. \tag{13}$$

Figure 2 shows the overshoot reduction parameter ρ_y^{-1} as a function of the bound κ_M imposed on the gain K_p. Note that the curve has a peculiar shape for small values of κ_M and then essentially grows linearly or large values of κ_M. Table 2 reports the values of $\bar{P}_p$, K_p and ρ_y^{-1} for some selections of κ_M. Using the values in Table 2, we run several simulations of the closed-loop starting from the initial conditions $x_p(0,0) = [1,\ 1]'$, resembling the

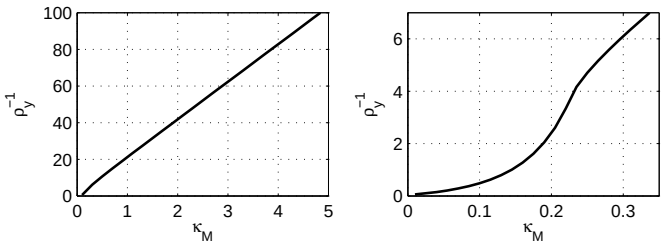

Fig. 2. Curve of the overshoot reduction parameter ρ_y^{-1} as a function of the bound κ_M imposed on the gain K_p.

κ_M	$\bar{P}_p$	K_p'	ρ_y^{-1}
0.1	$\begin{bmatrix} 0.31315 & -0.38064 \\ -0.38064 & 0.78906 \end{bmatrix}$	$\begin{bmatrix} 0.046750 \\ -0.087302 \end{bmatrix}$	0.48451
0.5	$\begin{bmatrix} 0.10170 & -0.088509 \\ -0.088509 & 0.99128 \end{bmatrix}$	$\begin{bmatrix} 0.055896 \\ -0.49023 \end{bmatrix}$	10.662
1	$\begin{bmatrix} 0.049336 & -0.045986 \\ -0.045986 & 0.99778 \end{bmatrix}$	$\begin{bmatrix} 0.057423 \\ -0.97424 \end{bmatrix}$	21.179
5	$\begin{bmatrix} 0.0097785 & -0.0096376 \\ -0.0096376 & 0.99991 \end{bmatrix}$	$\begin{bmatrix} 0.059499 \\ -4.8307 \end{bmatrix}$	103.25

Table 1. Values of the matrices $\bar{P}_p$, K_p and ρ_y^{-1} for some selections of κ_M.

step responses reported in Beker et al. (2001); Nešić et al. (2005); Zaccarian et al. (to appear). In particular, we implement the hybrid loops arising when using the values in the table within the scheme proposed in Theorem 3. Note that since in this case the pair (B_p, A_c) is observable, then according to the results of Theorem 1, the trajectories are all convergent to zero. The resulting plant input and output responses are shown in Figure 3, where we also show the linear response (thin solid) and the response obtained with the FORE used in Beker et al. (2001) (thin dashed).

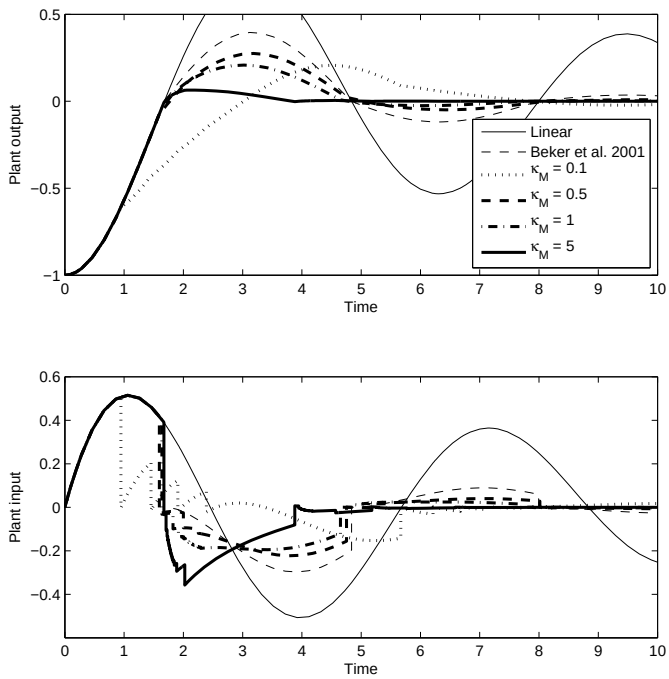

Fig. 3. Simulation results when using the linear hybrid loops discussed in Example 2. Linear response (thin solid) and response of the FORE closed-loop of Beker et al. (2001) (thin dashed) compared to the reset responses obtained with the parameters in Table 2 (bold).

The results in Figure 3 are indicative of the potential of the proposed approach with the goal of reducing overshoots in linear control systems and correspond to trajectories of the reset system (5) with the flow and jump sets in (6), with the parameters in (13) and with the gains K_p in Table 2. It should be pointed out that even though we manage to improve the overshoot reduction as compared to the FORE resetting strategy proposed in Beker et al. (2001) (see the thin dashed curve in Figure 3), this last strategy is more appealing from an implementation viewpoint because it only requires a measurement of the plant output. Conversely, the resetting strategy of the improved bold curves of the figure are obtained using a full measurement of the plant state. $\star$

5. CONCLUSIONS

The design problem of a stabilizing hybrid loop is considered in this paper. This class of system mixes discrete and continuous dynamics depending on the value of a quadratic function of the state. Adding a hybrid loop to a closed-loop linear system can been instrumental when maximizing the decay rate with an amplitude-limited controller. Finally it has been proven that adding a hybrid loop may be useful to reduce an overshoot of a given output. Some simulations highlighted the interest of the results.

REFERENCES

Aangenent, W., Witvoet, G., Heemels, W., van de Molengraft, M., and Steinbuch, M. (2008). An LMI-based L2 gain performance analysis for reset control systems. In *Proc. of American Control Conference (ACC)*, 2248–2253. Seattle, WA.

Beker, O., Hollot, C., and Chait, Y. (2001). Plant with integrator: an example of reset control overcoming limitations of linear feedback. *IEEE Transactions on Automatic Control*, 46(11), 1797–1799.

Beker, O., Hollot, C., and Chait, Y. (2004). Fundamental properties of reset control systems. *Automatica*, 40(6), 905–915.

Chen, Q., Chait, Y., and Hollot, C. (2001). Analysis of reset control systems consisting of a FORE and second order loop. *J. Dynamic Systems, Measurement and Control*, 123, 279–283.

Clegg, J. (1958). A nonlinear integrator for servomechanisms. *Trans. A. I. E. E.*, 77 (Part II), 41–42.

Goebel, R., Sanfelice, R., and Teel, A.R. (2009). Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2), 28–93.

Goebel, R. and Teel, A.R. (2009). Direct design of robustly asymptotically stabilizing hybrid feedback. *ESAIM: Control Optim. Calc. Var.*, 15(1), 205–213.

Goebel, R. and Teel, A. (2010). Preasymptotic stability and homogeneous approximations of hybrid dynamical systems. *SIAM Review*, 52(1), 87–109.

Haddad, W.M., Chellaboina, V., and Kablar, N.A. (2001a). Non-linear impulsive dynamical systems. Part I: Stability and dissipativity. *International Journal of Control*, 74(17), 1659–1677.

Haddad, W.M., Chellaboina, V., and Kablar, N.A. (2001b). Non-linear impulsive dynamical systems. Part II: Stability of feedback interconnections and optimality. *International Journal of Control*, 74(17), 1631–1658.

Haddad, W.M., Nersesov, S.G., and Chellaboina, V. (2003). Energy-based control for hybrid port-controlled Hamiltonian systems. *Automatica*, 39(8), 1425–1435.

Hespanha, J., Liberzon, D., and Morse, A. (2004). Hysteresis-based switching algorithms for supervisory. *Automatica*, 39(2), 263–272.

Hespanha, J. and Morse, A. (1999). Stabilization of nonholonomic integrators via logic-based switching. *Automatica*, 35(3), 385–393.

Loquen, T., Tarbouriech, S., and Prieur, C. (2007). Stability analysis for reset systems with input saturation. In *46th IEEE Conference on Decision and Control (CDC)*, 3272–3277. New Orleans, USA.

Nešić, D., Zaccarian, L., and Teel, A. (2005). Stability properties of reset systems. In *IFAC World Congress*. Prague, Czech Republic.

Nešić, D., Zaccarian, L., and Teel, A. (2008). Stability properties of reset systems. *Automatica*, 44(8), 2019–2026.

Nešić, D., Teel, A., and Zaccarian, L. (2010, submitted). Stability and performance of SISO control systems with First Order Reset Elements. *IEEE Transactions on Automatic Control*.

Nešić, D., Teel, A., and Zaccarian, L. (February 2010). Stability and performance of SISO control systems with First Order Reset Elements. Technical report, RR-10.82, Dipart. di Inf., Sist. e Prod., University of Rome, Tor Vergata.

Prieur, C. (2005). Asymptotic controllability and robust asymptotic stabilizability. *SIAM J. Control Opt.*, 43(5), 1888–1912.

Prieur, C., Goebel, R., and Teel, A. (2007). Hybrid feedback control and robust stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 52(11), 2103–2117.

Prieur, C., Tarbouriech, S., and Zaccarian, L. (2010). Guaranteed stability for nonlinear systems by means of a hybrid loop. In *IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, 72–77. Bologna, Italy.

Sanfelice, R., Goebel, R., and Teel, A. (2007). Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12), 2282–2297.

Witvoet, G., Aangenent, W., Heemels, W., van de Molengraft, M., and Steinbuch, M. (2007). H2 performance analysis of reset control systems. In *46th IEEE Conference on Decision and Control (CDC)*, 3278–3284. New Orleans, LA.

Zaccarian, L., Nešić, D., and Teel, A. (to appear). Analytical and numerical Lyapunov functions for SISO linear control systems with First Order Reset Elements. *Int. J. Nonlin. Rob. Contr.*

Zaccarian, L., Nešić, D., and Teel, A. (2005). First order reset elements and the Clegg integrator revisited. In *American Control Conference*, 563–568. Portland (OR), USA.