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Laurentiu Hetel * **Jamal Daafouz** ** **Sophie Tarbouriech** *** **Christophe Prieur** ****

* *CNRS, LAGIS, Ecole Centrale de Lille, BP48, 59651 Villeneuve dAscq Cedex, France,* laurentiu.hetel@ec-lille.fr
** *CRAN UMR 7039 CNRS - UHP - INPL, ENSEM, 2, av. forêt de la Haye, 54516, Vandœvre-Les-Nancy, Cedex, France.* Jamal.Daafouz@ensem.inpl-nancy.fr
*** *CNRS; LAAS; 7 Avenue Colonel Roche, F-31077 Toulouse, France. Université de Toulouse; UPS; INSA; INP; ISAE; LAAS; F-31077 Toulouse, France* tarbour@laas.fr
**** *Department of Automatic Control,Gipsa-lab, 961 rue de la Houille Blanche, BP 46, 38402 Grenoble Cedex, France.* christophe.prieur@gipsa-lab.grenoble-inp.fr

Abstract: In this paper the class of linear impulsive systems is considered. These systems are those with a continuous linear dynamics for all time, except at a sequence of instants. When such a discrete time occurs, the state undergoes a jump, or more precisely follows a discrete linear dynamics. The sequence of time instants, when a discrete dynamics occurs, is nearly-periodic only, i.e. it is distant from a periodic sequence to an uncertain distance. This paper succeeds to state tractable conditions to analyze the stability, and to design reset matrices such that the hybrid system is globally asymptotically stable to the origin. The approach is based on a polytopic embedding of the uncertain dynamics. An example illustrates the main stability result.

Keywords: Impulsive systems. Reset laws. Stability Analysis. Stabilization. Uncertainty.

1. INTRODUCTION

Hybrid systems are dynamic systems subject to both continuous-time and discrete-time dynamics. The importance of hybrid systems in control systems analysis and design have been growing in the last decades, due mainly to their presence in practical systems and to overcome performance limitations of more classical controllers, i.e., regular linear or nonlinear controllers (see, e.g., Clegg [1958], Prieur et al. [2007], Zaccarian et al. [2005], Nešič et al. [2008]). A particular class of hybrid systems is represented by reset systems, see Horowitz and Rosenbaum [1975], Beker et al. [2001], Prieur et al. [2010], Barreiro and nos [2010]. An interesting sub-class of hybrid systems is the systems with finite state jump which are linear continuous-time systems for which the state undergoes finite jump discontinuities at some discrete instants of time. Such systems can be regarded as a special case of reset systems in the sense that the reset rule is done through a time condition instead of a state condition. These systems can be named impulsive dynamical linear systems Haddad et al. [2006]. See also for example Haddad et al. [2001a], Haddad et al. [2001b], Li et al. [2001] and Hespanha et al. [2008]. Lyapunov theory framework provides the main tool to test the stability of reset of impulsive systems by employing an adequate Lyapunov function (or a family of Lyapunov functions) Liberzon [2003]. The results developed in the current paper are based on the use of adequate parameter-dependent quadratic Lyapunov function (Lee and Dullerud [2006], Daafouz and Bernussou [2001]).

The paper deals with both the stability analysis and stabilization problems, for linear systems controlled by a reset compensator. The reset rule, differently from Prieur et al. [2010], is posed in function of the time. The interval between two reset instants is supposed to be uncertain. In this setup, this interval is defined as the sum of a nominal reset period and an uncertain time-varying term bounded in a given interval. Thus, in order to avoid Zenon phenomena, we assume that the nominal reset period is different from zero. The main conditions in both problems is exhibited in terms of a parametric set of linear matrix inequalities (LMI). Tractable numerical solutions are proposed for these sets of parametric LMI to be expressed as a finite number of conditions using convex embeddings.

The paper is organized as follows. Section 2 describes the class of systems under consideration and states the problems of stability analysis and reset law design. In Section 3, the main results to with both problems. Section

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4 presents two ways to solve the conditions developed in Section 3. An numerical example is detailed in Section 5 to point out the potentiality but also the difficulty of the proposed approach. Finally some concluding remarks and forthcoming issues end the paper.

Notation. For a matrix M we denote by $\|M\|$ the induced matrix norm. By $M \succ 0$ or $M \prec 0$ we mean that the symmetric matrix M is positive or negative definite respectively. We denote the transpose of a matrix M by M^T. By $\mathbf{I}_m$ we denote the $m \times m$ identity matrix. By $\mathbf{0}$ we denote the null matrix of the appropriate dimension. By $\lambda_{max}(M)$ we denote the maximum eigenvalue of a square symmetric matrix. For a given set $\mathcal{S}$, $co(\mathcal{S})$ denotes the convex hull of $\mathcal{S}$.

2. SYSTEM DESCRIPTION

Consider the following linear time-invariant system:

$$\begin{aligned} \dot{x}_p(t) &= Ax_p(t) + Bu(t),\ \forall t \in \mathbb{R}^+,\ x_p(0) = x_p^0 \in \mathbb{R}^{n_p} \\ y(t) &= Cx_p(t), \end{aligned} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ represent the system state, input and output, respectively. The matrices A, B and C are constant matrices of appropriate dimension. Associated to the system (1) we consider a reset controller:

$$\begin{aligned} \dot{\eta}(t) &= A_\eta \eta(t) + B_\eta y(t),\ \forall t \in \mathbb{R}^+ - \mathcal{T},\ \eta(0) = \eta_0 \in \mathbb{R}^{n_\eta} \\ u(t) &= C_\eta \eta(t) + D_\eta y(t), \\ \eta(t) &= R_\eta \eta(t^-) + R_y y(t^-),\ \forall t \in \mathcal{T}, \end{aligned} \tag{2}$$

where the matrices A_η, B_η, C_η, D_η, R_y, R_η are constant matrices of appropriate dimmension. $\eta \in \mathbb{R}^{n_\eta}$ is the controller state and $t^- = \lim_{\tau \to t, \tau < t} t$. The set

$$\mathcal{T} = \left\{ t_k : t_k \in \mathbb{R}^+, t_k < t_{k+1}, \forall k \in \mathbb{N}, \lim_{k \to \infty} t_k = \infty \right\} \tag{3}$$

represents the set of reset times. We denote by τ_k the interval between two reset instants,

$$\tau_k := t_{k+1} - t_k, \forall\ k \in \mathbb{N}. \tag{4}$$

We assume that the reset interval has the form

$$\tau_k = \tau_{nom} + \delta\tau_k, \tag{5}$$

where τ_{nom} represents a nominal reset period and $\delta\tau_k$ the uncertain time-varying term bounded in a given interval,

$$\delta\tau_k \in [0, \delta\tau_{max}]. \tag{6}$$

In order to avoid Zenon phenomena we assume that $\tau_{nom} \neq 0$. The problems under study are formulated as follows:

Problem 1. *Assume that the matrices A, B, C and A_η, B_η, C_η, D_η, R_η and R_y are given and constant. Provide LMI methods for checking the stability of the reset system* (1), (2) *with a reset interval* (5).

Problem 2. *Assume that the matrices A, B, C and A_η, B_η, C_η, D_η are given and constant. Design reset matrices R_η and R_y to guarantee stability of reset system* (1), (2) *with a reset interval* (5).

3. RESET STABILIZATION

We consider the following generic linear reset systems:

$$\dot{x}(t) = A_c x(t),\ \forall t \in \mathbb{R}^+ - \mathcal{T}, \tag{7}$$

$$x(t) = A_r x(t^-),\ \forall t \in \mathcal{T}. \tag{8}$$

Note that the system (1) with the control law (2) can be expressed in this form using the notation

$$x = (x_p^T \quad \eta^T)^T \in \mathbb{R}^n,\ n = n_p + n_\eta$$

$$A_c = \begin{pmatrix} A + BD_\eta C & BC_\eta \\ B_\eta C & A_\eta \end{pmatrix} \tag{9}$$

and

$$A_r = \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ R_y C & R_\eta \end{pmatrix}. \tag{10}$$

The following theorem proposes stability conditions for closed-loop system (7) with a reset law (8).

Theorem 1. Consider system (7),(8), with $\mathcal{T}$ defined in (3) such that the reset interval (4) satisfies the assumptions (5) and (6). Assume that the matrix A_r is given. If there exist symmetric positive definite matrices $P(\delta\tau)$, $\delta\tau \in [0, \delta\tau_{max}]$, and a matrix G of appropriate dimensions such that the following set of linear matrix inequalities

$$\begin{pmatrix} P(\delta\tau_a) & \left(e^{A_c(\tau_{nom}+\delta\tau_a)}\right)^T A_r^T G^T \\ GA_r e^{A_c(\tau_{nom}+\delta\tau_a)} & G + G^T - P(\delta\tau_b) \end{pmatrix} \succ 0, \tag{11}$$

is satisfied for all $\delta\tau_a, \delta\tau_b \in [0, \delta\tau_{max}]$, then there exists a reset law (8) for which the equilibrium point $x = 0$ of system (7) is asymptotically stable.

Proof. The proof is based on the evaluation of the system behavior at the sampling instants t_k. The solution of the system (7) is described by the equation

$$x(t) = e^{(t-t_k)A_c} x(t_k),\ \forall t \in [t_k, t_{k+1}). \tag{12}$$

Note that since the continuous-time dynamic in between two resets instances is linear, then the evolution of the norm of the system state is upper-bounded for all $t \in [t_k, t_{k+1})$, i.e. there exist a scalar $\epsilon = \lambda_{max}\left(\frac{A_c + A_c^T}{2}\right)$ s.t. $\|x(t)\| \leq e^{\epsilon(t-t_k)}\|x(t_k)\|$ for all $t \in [t_k, t_{k+1})$. This implies that for analyzing the asymptotic stability of the system it suffice to analyze the behavior of the system for the values $t = t_k, k \in \mathbb{N}$. Note that at $t = t_{k+1}$ the system state is given by:

$$\begin{aligned} x(t_{k+1}) &= A_r e^{(t_{k+1}-t_k)A_c} x(t_k) \\ &= A_r e^{\tau_k A_c} x(t_k) \\ &= A_r e^{(\tau_{nom}+\delta\tau_k)A_c} x(t_k) \end{aligned} \tag{13}$$

The previous equation represents a discrete-time linear system with time-varying parameters $\delta\tau_k$ that appear in an exponential manner. Following generic stability results for linear system with time-varying parameters (Lee and Dullerud [2006], Daafouz and Bernussou [2001]), we consider the following class of Lyapunov functions:

$$V\left(x(t_k), \delta\tau_k\right) = x^T(t_k) P(\delta\tau_k) x(t_k). \tag{14}$$

The equilibrium point $x = 0$ is stable if the function is strictly decreasing for all $x(t_k) \neq 0$ and all variations of the parameter $\delta\tau_k$. This condition can be expressed as

$$V\left(x(t_{k+1}), \delta\tau_{k+1}\right) < V(x(t_k), \delta\tau_k) \tag{15}$$

which is the same as

$$V\left(A_r e^{(\tau_{nom}+\delta\tau_k)A_c} x(t_k), \delta\tau_{k+1}\right) < V(x(t_k), \delta\tau_k) \quad (16)$$

for all $\delta\tau_k, \delta\tau_{k+1}$ in $[0, \delta\tau_{max}]$ and for all $x(t_k) \neq 0$. Then the stability is ensured if

$$\max_{\delta\tau_b \in [0,\delta\tau_{max}]} V\left(A_r e^{(\tau_{nom}+\delta\tau_a)A_c} x, \delta\tau_b\right) < V(x, \delta\tau_a) \quad (17)$$

holds for all $\delta\tau_a \in [0, \delta\tau_{max}]$, $x \neq 0$.

Assume that there exist matrices $P(\delta\tau)$, $\delta\tau \in [0, \delta\tau_{max}]$, and a matrix G of appropriate dimensions such that the condition (11) is satisfied $\forall\ \delta\tau_a, \delta\tau_b \in [0, \delta\tau_{max}]$. Using similar arguments as in (de Oliveira et al. [1999]), one can see that multiplying the inequality (11) by

$$T := [\mathbf{I}_n \quad -(A_r e^{A_c(\tau_{nom}+\delta\tau_a)})^T]$$

on the left and by its transpose on the right, the following inequality holds true:

$$\left(e^{A_c(\tau_{nom}+\delta\tau_a}\right)^T A_r^T P(\delta\tau_b) A_r e^{A_c(\tau_{nom}+\delta\tau_a)} - P(\delta\tau_a) \prec 0, \quad (18)$$

$\forall\ \delta\tau^a, \delta\tau^b \in [0, \delta\tau_{max}]$ which guarantees that the condition (17) is satisfied. □

The following theorem proposes conditions for the design of a reset matrix A_r, as defined in (8), that stabilizes the system (7).

Theorem 2. Consider system (7) with $\mathcal{T}$ defined in (3) such that the reset interval (4) satisfies the assumptions (5) and (6). If there exist symmetric positive definite matrices $P(\delta\tau_a)$, $\delta\tau_a \in [0, \delta\tau_{max}]$, and matrices G, W of appropriate dimensions such that the following set of linear matrix inequalities

$$\begin{pmatrix} P(\delta\tau_a) & \left(e^{A_c(\tau_{nom}+\delta\tau_a}\right)^T W^T \\ W e^{A_c(\tau_{nom}+\delta\tau_a)} & G + G^T - P(\delta\tau_b) \end{pmatrix} \succ 0, \quad (19)$$

is satisfied for all $\delta\tau_a, \delta\tau_b \in [0, \delta\tau_{max}]$, then there exists a reset law (8) for which the equilibrium point $x = 0$ of system (7) is asymptotically stable. The reset law is given by (8) with $A_r = G^{-1}W$.

Proof. Assume that there exist matrices $P(\delta\tau)$, $\delta\tau \in [0, \delta\tau_{max}]$, and matrices G, W of appropriate dimension such that the condition (19) is satisfied. Using the change of variables $W = GA_r$ the set of inequalities (19) leads to

$$\begin{pmatrix} P(\delta\tau_a) & \left(e^{A_c(\tau_{nom}+\delta\tau_a}\right)^T A_r^T G^T \\ G A_r e^{A_c(\tau_{nom}+\delta\tau_a)} & G + G^T - P(\delta\tau_b) \end{pmatrix} \succ 0, \quad (20)$$

$\forall\ \delta\tau_a, \delta\tau_b \in [0, \delta\tau_{max}]$. Using Theorem 1 guarantees the asymptotic stability. □

In the following corollary it is shown how the approach in Theorem 2 can be used in order to develop reset law for the particular case of system (1) with the reset law (2). Note that in this case the reset matrix A_r has a particular structure, as described in (10).

Corollary 1. Consider system (1), the reset control (2) and the closed-loop matrices (9), (10). Moreover, consider that the set $\mathcal{T}$ defined in (3) with the reset interval (4) satisfies the assumptions (5) and (6). If there exist symmetric positive definite matrices $P(\delta\tau_a)$, $\delta\tau_a \in [0, \delta\tau_{max}]$, and matrices $\bar{G}$, W_η, W_y of appropriate dimensions such that the set of linear matrix inequalities (19) is satisfied with

$$G = \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ \mathbf{0} & \bar{G} \end{pmatrix} \quad (21)$$

and

$$W = \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ W_y C & W_\eta \end{pmatrix} \quad (22)$$

for all $\delta\tau_a, \delta\tau_b \in [0, \delta\tau_{max}]$, then there exist a reset law (8) with a structured reset matrix (10) for which the equilibrium point $x = 0$ of system (1) is globally uniformly exponentially stable. The reset law is given by (8) and (10) with $R_y = \bar{G}^{-1}W_y$ and $R_\eta = \bar{G}^{-1}W_\eta$.

Proof. Assume that the exist symmetric positive definite matrices $P(\delta\tau_a)$, $\delta\tau_a \in [0, \delta\tau_{max}]$, and matrices G and W as in (21), (22) such that conditions (19) are satisfied. According to Theorem 2, there exists a stabilizing reset matrix A_r given by

$$\begin{aligned} A_r &= G^{-1}W & (23)\\ &= \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ \mathbf{0} & \bar{G}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ W_y C & W_\eta \end{pmatrix} & (24)\\ &= \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ \bar{G}^{-1}W_y C & \bar{G}^{-1}W_\eta \end{pmatrix}, & (25) \end{aligned}$$

which is of the form (10) with $R_y = \bar{G}^{-1}W_y$ and $R_\eta = \bar{G}^{-1}W_\eta$. □

The application of the previous corollary to the ideal case in which no uncertainty concerns the reset interval ($\delta\tau_{max} = 0$) is given as follows:

Corollary 2. Consider system (1), the reset control (2) and the closed-loop matrices (9), (10). Moreover, consider that the set $\mathcal{T}$ defined in (3) with a nominal reset interval τ_{nom}. If there exist a symmetric matrix P and matrices $\bar{G}$, W_η, W_y of appropriate dimension such that the set of linear matrix inequalities (19) is satisfied for $P(\delta\tau_a) = P$, $\delta\tau_a \in [0, \delta\tau_{max}]$, with G and W as defined in (21) and (22), respectively, then there exists a reset law (8) with the structured reset matrix (10) for which the equilibrium point $x = 0$ of system (1) is asymptotically stable. The reset law is given by (8) and (10) with $R_y = \bar{G}^{-1}W_y$ and $R_\eta = \bar{G}^{-1}W_\eta$.

Remark. Corollary 2 presents stabilization condition for the case of nominal reset interval. The given conditions represent a finite set of linear matrix inequalities which can be checked using classical convex optimization tools. Note that the conditions (11), (19) lead to a parametric set of linear matrix inequalities, since they depend on the different values of $\delta\tau_a, \delta\tau_b$ in the interval $[0, \delta\tau_{max}]$. Tractable numerical condition for solving this parametric set of conditions are presented in the following section.

4. NUMERICAL EVALUATION

In this section it is shown how the parametric set of LMIs presented previously, namely (11), (19), can be expressed as a finite number of conditions using convex embeddings.

4.1 Polytopic sets

In order to obtain a finite number of LMIs from (11) and (19), we have to deal with the *exponential uncertainty*

$e^{A_c \delta\tau_k}$ (Hetel et al. [2007]) that appears in the conditions. Note that the uncertain matrices $e^{A_c \delta\tau_k}$ are continuous with respect to $\delta\tau_k$ and that $\delta\tau_k$ belongs to a bounded set. Then, for $\delta\tau_a \in [0, \delta\tau_{max}]$, the matrix $e^{A_c \delta\tau_a}$ describes a compact subset of $\mathbb{R}^{n\times n}$:

$$\mathcal{E} = \left\{ X \in \mathbb{R}^{n\times n} : X = e^{A_c \delta\tau_a}, \delta\tau_a \in [0, \delta\tau_{max}] \right\} \quad (26)$$

The basic idea is to embed the set of matrices $\mathcal{E}$ into a polytopic set $\mathcal{Z}$, i.e. to find a set of N matrices Z_i such that

$$\mathcal{E} \subset \mathcal{Z} = co\{Z_1, Z_2, \ldots, Z_N\}. \quad (27)$$

This implies that for all $\delta\tau_a \in [0, \delta\tau_{max}]$ there exist a set of scalars $\mu_i(\delta\tau_a) \in [0,1]$, $i = 1, \ldots, N$, such that

$$e^{A_c \delta\tau_a} = \sum_{i=1}^{N} \mu_i(\delta\tau_a) Z_i, \; \sum_{i=1}^{N} \mu_i(\delta\tau_a) = 1. \quad (28)$$

In order to provide some insight about how a polytopic embedding can be approximated, we recall briefly some of the methods proposed in (Hetel et al. [2007]).

Consider an h-order Taylor series expansion of the matrix exponential $e^{A_c \delta\tau}$:

$$e^{\delta\tau_a A_c} \approx \left(\sum_{l=0}^{h} \frac{{A_c}^l}{l!} \delta\tau_a^l \right). \quad (29)$$

Then we may construct a convex polytope by considering the terms $\delta\tau^l, l = 1, 2, \ldots, h$, as independent parameters. The h-order Taylor approximation can be embedded in a matrix hypercube with 2^h vertices. However, one can exploit the relation between the different parameters to construct a convex polytope inside the hypercube. This approached is mathematically formalized in Lemma 4 (given in the Appendix). Applying Lemma 4 to the polynomial form (29) for $\delta\tau_a \in [0, \delta\tau_{max}]$, leads to a polytopic approximation (27) with $N = h + 1$ vertices given by:

$$\begin{aligned}
Z_1 &= \mathbf{I}_n, \\
Z_2 &= \delta\tau_{max} A_c + \mathbf{I}_n, \\
Z_3 &= \delta\tau_{max}^2 \frac{{A_c}^2}{2!} + \delta\tau_{max} A_c + \mathbf{I}_n, \\
&\vdots \\
Z_{h+1} &= \delta\tau_{max}^h \frac{{A_c}^h}{h!} + \delta\tau_{max}^{h-1} \frac{{A_c}^{h-1}}{(h-1)!} + \ldots \\
&\ldots + \delta\tau_{max}^2 \frac{{A_c}^2}{2!} + \delta\tau_{max} A_c + \mathbf{I}_n.
\end{aligned}$$

In the following subsection we illustrate the use of such polytopic embedding methods for reset matrices design.

4.2 Stabilization based on polytopic sets

As follows we show how the parametric set of linear matrix inequalities, such as (11), (19), can be reduced to a finite number of linear matrix inequalities. The approach is illustrated for the condition (19). The conditions proposed in Theorem 1 and Corolarry 1, can be treated in a similar manner. The reset matrix design procedure based on polytopic set (27) is formulated in the following Theorem.

Theorem 3. Consider system (7) with $\mathcal{T}$ defined in (3) such that the reset interval (4) satisfies the assumptions (5) and (6). Moreover, consider the polytopic set (27) with N vertices. If there exists symmetric positive definite matrices P_i, $i = 1, \ldots, N$, and matrices G, W of appropriate dimensions such that the following set of linear matrix inequalities

$$\begin{pmatrix} P_i & Z_i^T \left(e^{A_c \tau_{nom}}\right)^T W^T \\ W e^{A_c \tau_{nom}} Z_i & G + G^T - P_j \end{pmatrix} \succ 0, \quad (30)$$

is satisfied for all $i, j = 1, \ldots, N$, then there exists a reset law (8) for which the equilibrium point $x = 0$ of system (7) is asymptotically stable. The reset law is given by (8) with $A_r = G^{-1} W$.

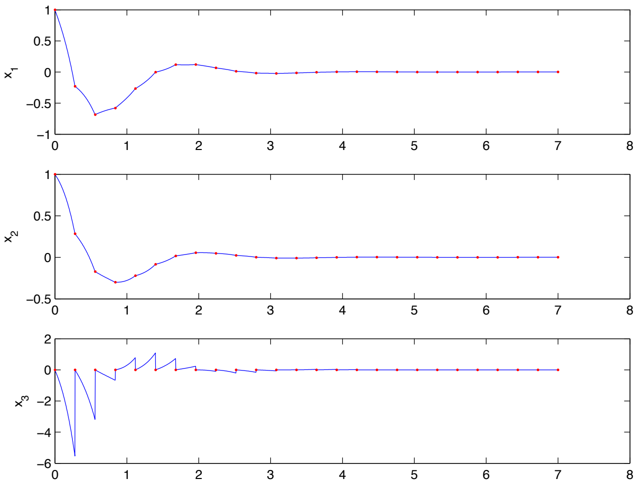

Fig. 1. Illustration of system evolution with a periodic reset interval $\tau_{nom}^1 = 0.28$.

Proof. Assume that there exist a set of matrices P_i, $i = 1, \ldots, N$ and matrices G and R such that the set of inequalities (30) hold true. Then the condition

$$\begin{pmatrix} \sum_{i=1}^N \mu_i P_i & \sum_{i=1}^N \mu_i Z_i^T \left(e^{A_c \tau_{nom}}\right) W^T \\ \sum_{i=1}^N \mu_i W e^{A_c \tau_{nom}} Z_i & G + G^T - \sum_{i=1}^N \mu_j P_j \end{pmatrix} \succ 0, \quad (31)$$

is satisfied for any set of scalars $\mu_i, \mu_j \in [0,1]$, $i, j = 1, \ldots, N$, such that $\sum_{i=1}^N \mu_i = \sum_{i=1}^N \mu_j = 1$. Note that $e^{A_c \delta\tau_a} \in \mathcal{Z}$ for all $\delta\tau_a \in [0, \delta\tau_{max}]$. Then the previous condition implies the existences of matrices $P(\delta\tau_a)$ such that the condition (19) holds with

$$P(\delta\tau_a) = \sum_{i=1}^{N} \mu_i(\delta\tau_a) P_i, \quad (32)$$

$$P(\delta\tau_b) = \sum_{j=1}^{N} \mu_j(\delta\tau_b) P_j, \quad (33)$$

where $\mu_i(\delta\tau_a), \mu_j(\delta\tau_b) \in [0,1]$, $i, j = 1, \ldots, N$, represent the barycentric coordinates of $e^{A_c \delta\tau_a}$ and $e^{A_c \delta\tau_b}$ in the polytope $\mathcal{Z}$. $\square$

5. NUMERICAL EXAMPLES

Consider a reset system (7) with

$$A_c = \begin{pmatrix} 0 & -3 & 1 \\ 1.4 & -2.6 & 0.6 \\ 8.4 & -18.6 & 4.6 \end{pmatrix}, \; A_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (34)$$

The matrix A_c has the eigenvalues

$$\lambda_1 = -1.44, \lambda_2 = 3.44, \lambda_3 = 0. \quad (35)$$

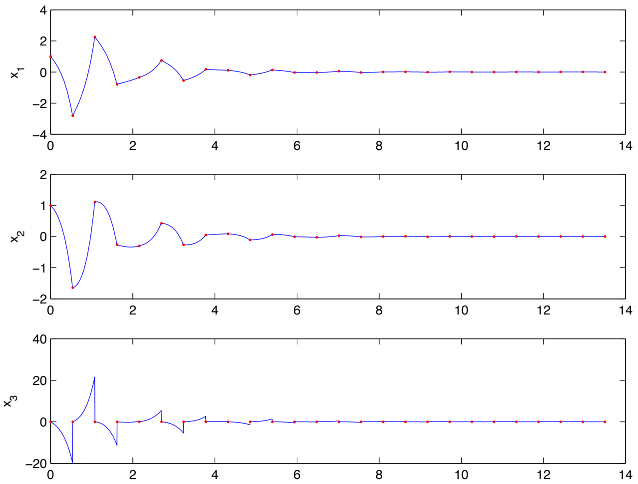

Fig. 2. Illustration of system evolution with a periodic reset interval $\tau_{nom}^2 = 0.54$.

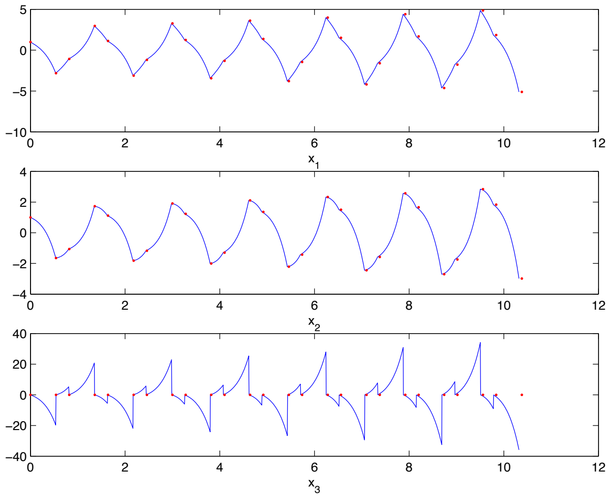

Fig. 3. Illustration of unstable system behavior when the reset interval switches among two values $\tau_{nom} \in \{0.28, 0.54\}$.

For this system the matrix $A_r e^{\tau_{nom}}$ is Schur for $\tau_{nom} \in [0, 0.58]$. This implies that the system is stable if the reset occurs periodically, with constant reset interval τ_{nom} in $[0, 0.58]$ and $\delta\tau_k = 0$. However, variation of the reset interval may induce instability. While the system is stable for $\tau_{nom}^1 = 0.28$ and $\tau_{nom}^2 = 0.54$ (see Fig. 1 and Fig. 2, respectively), the matrix $A_r e^{\tau_{nom}^1} A_r e^{\tau_{nom}^2}$ has eigenvalues outside the unit circle. Therefore, resetting the system with the pattern $\tau_{nom}^1 \to \tau_{nom}^2 \to \tau_{nom}^1 \to \tau_{nom}^2 \dots$, leads to an unstable behavior. This phenomena is illustrated in Fig. 3.

Consider now that $\tau_{nom} = 0.1$ and that $\delta\tau_{max} = 0.2$. In order to illustrate graphically the construction of the polytopic embedding $\mathcal{Z}$ in (27), consider the Jordan normal form of A_c:

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3. \end{pmatrix} \quad (36)$$

with $\lambda_i = 1, \dots, 3$, given in (35). For this particular case, the exponential uncertainty $e^{\delta\tau_k J}$ has the form

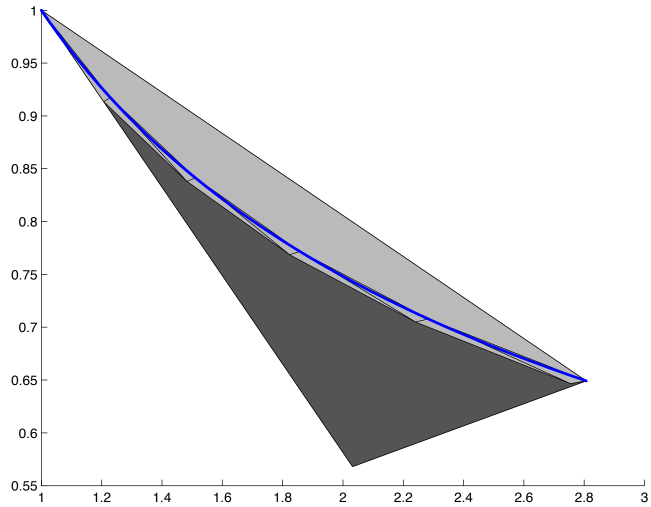

Fig. 4. Representation of exponential uncertainty and of construction of convex polytopes in the $\phi_1(\delta\tau_a) - \phi_2(\delta\tau_b)$ plane, for $\tau_{nom} = 0.1$, $\delta\tau_a \in [0, 0.2]$. The polytopic embedding in dark gray is obtained based on a $8th$ order Taylor approximation. In light gray we represent a polytopic embedding obtained based on a $8th$ order Taylor approximation applied on 5 subintervals.

$$e^{\delta\tau_k J} = T \begin{pmatrix} \phi_1(\delta\tau_k) & 0 & 0 \\ 0 & \phi_2(\delta\tau_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} T^{-1} \quad (37)$$

with two uncertain scalar parameters $\phi_i(\delta\tau_k) = e^{\delta\tau_k \lambda_i}$, $i = 1, 2$, and $\delta\tau_k \in [0, 0.2]$. In Fig. 4, we illustrate the curve described by the exponential uncertainty in the $\phi_1(\delta\tau_k) - \phi_2(\delta\tau_k)$ plane as well as the obtained polytopic embedding based on a 8th order Taylor series expansion (in dark gray). Note that the polytopic embedding procedure can be refined by dividing the interval $[0, \delta\tau_{max}]$ in several subintervals, and applying the Taylor method locally. Then, the extreme point of the global polytope can be determined among the vertex of each local embedding using classical convex hull algorithms. An illustration based on 5 subinterval is given in Fig. 4 (light gray). Using such an embedding and adapting Theorem 1 similarly to Theorem 3, we can show that the system is robustly stable for $\tau_{nom} = 0.1$ with variations of the reset interval of a maximum amplitude $\delta\tau_{max} = 0.2$. An example of system evolution according to a random variation of the reset interval is given in Fig. 5.

6. CONCLUSION

Focusing on linear impulsive systems, we stated tractable conditions to analyze the stability. Also some sufficient conditions are derived to compute reset matrices such that the hybrid system is global asymptotically stable to the origin. It is assumed that the jump instants are not periodic but only nearly-periodic, i.e. with an uncertain distance to a periodic sequence.

This paper lets many questions open. The first one may be the generalization of this work to nonlinear systems. Lyapunov techniques may be applied in this context (see e.g. Goebel et al. [2009]). Also the performance issues may be considered either for linear impulsive systems or for nonlinear ones. One criterion to be optimized may be the rejection of perturbations and the gain perturba-

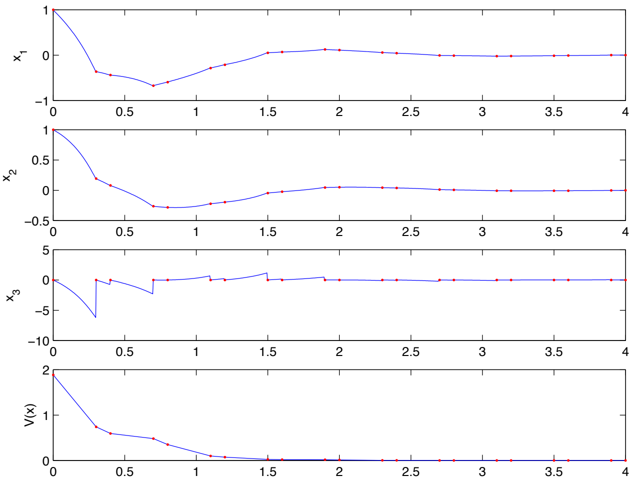

Fig. 5. Example of system evolution for $\tau_{nom} = 0.1$, $\delta\tau \in [0, 0.2]$, based on an arbitrary sequence of reset intervals.

tions/output to be made as low as possible. The connection with the Input-to-State Stability (ISS) of impulsive systems as in Hespanha et al. [2008] may be fruitful.

APPENDIX

Lemma 4. (Hetel et al. [2007]) Consider a polynomial matrix

$$L(\rho) = L_0 + \rho L_1 + \rho^2 L_2 + \ldots + \rho^{\,p} L_p, \qquad (38)$$

where the parameter $\rho \in \mathbb{R}$ and $L_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, p$. For each upper bound $\bar{\rho} \geq 0$ on ρ there exist matrices $U_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, p+1$, s.t. the following property holds:
For all $\rho \in [0, \bar{\rho}]$ there exist parameters $\mu_i(\rho)$, $i = 1, \ldots, p+1$, with

$$\sum_{i=1}^{p+1} \mu_i(\rho) = 1, \text{ and } \mu_i(\rho) \geq 0, \quad i = 1, \ldots, p+1,$$

s.t.

$$L(\rho) = \sum_{i=1}^{p+1} \mu_i(\rho) U_i. \qquad (39)$$

In particular, U_i, $i = 1, \ldots, p$, can be chosen as

$$\begin{aligned} U_1 &= L_0, \\ U_2 &= \overline{\rho} L_1 + L_0, \\ U_3 &= \overline{\rho}^{\,2} L_2 + \overline{\rho} L_1 + L_0, \\ &\vdots \\ U_{p+1} &= \overline{\rho}^{\,p} L_p + \overline{\rho}^{\,p-1} L_{p-1} + \ldots + \overline{\rho}^{\,2} L_2 + \overline{\rho} L_1 + L_0. \end{aligned}$$

The relations between the parameters ρ and μ are given by:

$$\begin{aligned} \mu_1(\rho) &= 1 - \frac{\rho}{\overline{\rho}}, \\ \mu_i(\rho) &= \frac{\rho^{\,i-1}}{\overline{\rho}^{\,i-1}} - \frac{\rho^{\,i}}{\overline{\rho}^{\,i}}, \; i = 2 \ldots p, \qquad (40) \\ \mu_{p+1}(\rho) &= 1 - \frac{\rho^{\,p}}{\overline{\rho}^{\,p}}. \end{aligned}$$

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