# Families of quasi-rational solutions of the NLS equation as an extension of higher order Peregrine's solutions.

Pierre Gaillard

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+Pierre Gaillard, + Université de Bourgogne, Dijon, France :
e-mail: Pierre.Gaillard@u-bourgogne.fr,

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**Abstract**

We give a family of solutions of the focusing NLS equation as a product of a quotient of two wronskians of order $2N$ by an exponential. When we perform a special passage to the limit when all the periods tend to infinity, we get a family of solutions depending on $7N + 1$ parameters. We give explicit representations of solutions for $N = 1, 2, 3$. As particular cases, we recover the so called higher Peregrine's breathers labeled by the positive integer $N$.

## 1 Introduction

The nonlinear Schrödinger equation (NLS) was considered a long time by ago in the basic work by Zakharov in 1969 [13]. The first solution was discovered by Peregrine in 1983 [12].
Other families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2, 3] using Darboux transformations.
Other solutions were found for reduced self-induced transparency (SIT) integrable systems [11].

In [10], the N-phase quasi-periodic modulations of the plane waves solutions were constructed via appropriate degeneration of the finite gap periodic solutions of the NLS equation.

In this paper, we will give an extension of a representation of the solutions of the NLS equation in terms of wronskians recently founded in [9]. The solutions take the form of a quotient of two wronskians of even order $2N$ of elementary functions depending on $7N+1$ parameters.
We will call these related solutions, solutions of NLS of order N.
Then, to get quasi-rational solutions of NLS equation, we take the limit when some parameter goes to 0. We obtain here solutions depending on $5N+1$ free parameters.
As particular cases, we get for $N = 1$, the well known Peregrine's solution [12] of the focusing NLS equation. For $N = 2$, 3, we recover Akhmediev's breathers.
Here we give a new approach to get an extension of higher order Peregrine solutions different from all previous works.

# 2 Expression of solutions of NLS equation in terms of Fredholm determinant

## 2.1 Solutions of NLS equation in terms of $\theta$ functions

We use here a general formulation of the solution of the NLS equation given in ([10]), different from that used in [9]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0, \tag{1}$$

The solution is given in terms of truncated theta function by

$$v(x,t) = \frac{\theta_3(x,t)}{\theta_1(x,t)} \exp(2it - i\varphi). \tag{2}$$

The functions $\theta_r(x,t)$, $(r = 1, 3)$ are the functions defined by

$$\theta_r(x,t) = \sum_{k\in\{0;1\}^{2N}} \exp g_{r,k} \tag{3}$$

with $g_{r,k}$ given by

$$g_{r,k} = \sum_{k\in\{0;1\}^{2N}} \exp\left\{ \sum_{\mu>\nu,\, \mu,\nu=1}^{2N} \ln\left(\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu}\right)^2 k_\mu k_\nu \right. \tag{4}$$

$$+\left(\sum_{\nu=1}^{2N} i\kappa_\nu(x-x_{0\nu})-2\delta_\nu(t-t_{0\nu})+(r-1)\ln\frac{\gamma_\nu-i}{\gamma_\nu+i}+\sum_{\mu=1,\,\mu\neq\nu}^{2N}\ln\left|\frac{\gamma_\nu+\gamma_\mu}{\gamma_\nu-\gamma_\mu}\right|+\pi i\varepsilon_\nu+e_\nu\right)k_\nu\Bigg\}.$$

The solutions depend on $4N+1$ arbitrary parameters, for $N$ being a positive integer, $N\geq 1$ :
$\varphi$, $x_{0,\nu}$, $t_{0,\nu}$ for $1\leq\nu\leq 2N$;
$N$ parameters satisfying the relations $\lambda_{N+j}=-\lambda_j$ for $1\leq j\leq N$;
$2N$ parameters $e_\nu$, $1\leq\nu\leq 2N$.
The terms $\varepsilon_\nu$, $1\leq\nu\leq 2N$ are arbitrary numbers equal to 0 or 1.
The parameters $e_\nu$ are defined by the relations

$$e_\nu=ia_\nu-b_\nu,\quad 1\leq\nu\leq N,\quad e_\nu=ia_\nu+b_\nu,\quad N+1\leq\nu\leq 2N. \tag{5}$$

In the preceding formula, the terms $\kappa_\nu$, $\delta_\nu$, $\gamma_\nu$ are functions of the parameters $\lambda_\nu$, $\nu=1,\ldots,2N$, and they are given by the following equations,

$$\kappa_\nu=2\sqrt{1-\lambda_\nu^2},\quad \delta_\nu=\kappa_\nu\lambda_\nu,\quad \gamma_\nu=\sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}. \tag{6}$$

We also note that

$$\kappa_{N+j}=\kappa_j,\quad \delta_{N+j}=-\delta_j,\quad \gamma_{N+j}=1/\gamma_j,\quad j=1\ldots N. \tag{7}$$

## 2.2 Relation between $\theta$ and Fredholm determinant

We know from [10] that the function $\theta_r$ defined in (3) can be written as a quotient of two different Fredholm determinants. The expression given in [10] is different from which we need in the following. We need different choices of $\varepsilon_\nu$ :

$$\begin{gathered}\varepsilon_\nu=0 \text{ if } \nu=2k,\quad \varepsilon_\nu=1 \text{ if } \nu=2k+1,\quad 1\leq\nu\leq N\\ \varepsilon_\nu=1 \text{ if } \nu=2k,\quad \varepsilon_\nu=0 \text{ if } \nu=2k+1,\quad N+1\leq\nu\leq 2N.\end{gathered} \tag{8}$$

For the next section, we change notations, replacing $\varepsilon$ by $\epsilon$ in such a way that

$$\begin{gathered}\epsilon_j=j,\quad 1\leq j\leq N\\ \epsilon_j=j+1,\quad N+1\leq j\leq 2N.\end{gathered} \tag{9}$$

It is easy to see that $(-1)^{\varepsilon_\nu} = (-1)^{\epsilon_\nu}$ for $1 \leq \nu \leq 2N$.
The function $\theta_r$ defined in (3) can be rewritten with a summation in terms of subsets of $[1, .., 2N]$

$$\theta_r(x,t) = \sum_{J\subset\{1,..,2N\}} \prod_{\nu\in J}(-1)^{\epsilon_\nu} \prod_{\nu\in J,\, \mu\notin J} \left|\frac{\gamma_\nu+\gamma_\mu}{\gamma_\nu-\gamma_\mu}\right|$$

$$\times \exp\{\sum_{\nu\in J} i\kappa_\nu(x-x_{0\nu}) - 2\delta_\nu(t-t_{0\nu}) + x_{r,\nu} + e_\nu\},$$

with

$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N, \tag{10}$$

in particular

$$x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N,$$
$$x_{r,N+j} = -(r-1)\ln\frac{\gamma_j - i}{\gamma_j + i} - (r-1)i\pi, \quad 1 \leq j \leq N. \tag{11}$$

We consider $A_r = (a_{\nu\mu})_{1\leq\nu,\mu\leq 2N}$ the matrix defined by

$$a_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu}\gamma_\nu}{\gamma_\nu+\gamma_\mu} \prod_{\eta\neq\nu}\left|\frac{\gamma_\nu+\gamma_\eta}{\gamma_\nu-\gamma_\eta}\right| \exp(i\kappa_\nu(x-x_{0\nu}) - 2\delta_\nu(t-t_{0\nu}) + x_{r,\nu} + e_\nu). \tag{12}$$

Then $\det(I + A_r)$ has the following form

$$\det(I+A_r) = \sum_{J\subset\{1,\dots,2N\}} \prod_{\nu\in J}(-1)^{\epsilon_\nu} \prod_{\nu\in J\,\mu\notin J} \left|\frac{\gamma_\nu+\gamma_\mu}{\gamma_\nu-\gamma_\mu}\right| \exp(i\kappa_\nu(x-x_{0\nu})$$
$$-2\delta_\nu(t-t_{0\nu}) + x_{r,\nu} + e_\nu). \tag{13}$$

From the beginning of this section, $\tilde{\theta}$ has the same expression as in (13) so, we have clearly the equality

$$\theta_r = \det(I + A_r). \tag{14}$$

Then the solution of NLS equation takes the form

$$v(x,t) = \frac{\det(I + A_3(x,t))}{\det(I + A_1(x,t))}\exp(2it - i\varphi). \tag{15}$$

# 3 Expression of solutions of NLS equation in terms of wronkian determinant

## 3.1 Link between Fredholm determinants and wronskians

We use here the same ideas that these exposed in [9]. The proofs are the same. We don't reproduce it in this text. The reader can see the aforementioned paper.
We consider the following functions

$$\begin{array}{ll} \phi^r_\nu(y) = \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), & 1 \leq \nu \leq N, \\ \phi^r_\nu(y) = \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), & N+1 \leq \nu \leq 2N. \end{array} \tag{16}$$

For simplicity, in this section we denote them $\phi_\nu(y)$.
We use the following notations :
$\Theta_\nu = \kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1 \leq j \leq 2N.$
$W_r(y) = W(\phi_1, \dots, \phi_{2N})$ is the wronskian $W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu)_{\nu,\,\mu\in[1,\dots,2N]}]$.
We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu\in[1,\dots,2N]}$ defined by

$$\begin{array}{l} d_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu}\gamma_\nu}{\gamma_\nu+\gamma_\mu} \prod_{\eta\neq\nu} \left|\frac{\gamma_\nu+\gamma_\mu}{\gamma_\nu-\gamma_\mu}\right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu} - ie_\nu), \\ \quad 1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N, \end{array}$$

with

$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

**Theorem 3.1**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \tag{17}$$

*where*

$$k_r(y) = \frac{2^{2N}\exp(i\sum_{\nu=1}^{2N}\Theta_\nu)}{\prod_{\nu=2}^{2N}\prod_{\mu=1}^{\nu-1}(\gamma_\nu - \gamma_\mu)}.$$

**Proof : see [9]**

### 3.2 Wronskian representation of solutions of NLS equation

From the previous section, we get the following result :

**Theorem 3.2** *The function $v$ defined by*

$$v(x,t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \tag{18}$$

*is solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

**Remark 3.1** *In (18), $W_r(y) = W(\phi_1^r, \ldots, \phi_{2N}^r)$ is the wronskian $W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu^r)_{\nu,\,\mu\in[1,\ldots,2N]}]$.*
*The functions $\phi_\nu^r$ are given by (16)*

$$\phi_\nu^r(y) = \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu^r(y) = \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu), \quad N+1 \leq \nu \leq 2N.$$

*The parameters $\kappa_\nu$, $\delta_\nu$, $\gamma_\nu$ are defined by (6) for $\nu \in [1, \ldots, 2N]$*

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu\lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\}.$$

*$\lambda_\nu$ is an arbitrary parameter such that $0 < \lambda_j < 1$ and $\lambda_{N+j} = -\lambda_j$ for $j \in [1, \ldots, N]$.*
*The parameters $e_\nu$ are defined by (5)*

$$e_\nu = ia_\nu - b_\nu, \quad 1 \leq \nu \leq N, \quad e_\nu = ia_\nu + b_\nu, \quad 1 \leq \nu \leq N.$$

# 4 Construction of quasi-rational solutions of NLS equation

### 4.1 Taking the limit when the parameters $\lambda_j \to 1$ for $1 \leq j \leq N$ and $\lambda_j \to -1$ for $N+1 \leq j \leq 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a simple limiting procedure.

For simplicity, we denote $d_j$ the term $\frac{c_j}{\sqrt{2}}$.
We consider the parameter $\lambda_j$ written in the form

$$\lambda_j = 1 - \epsilon^2 c_j^2, \quad 1 \le j \le N. \tag{19}$$

When $\epsilon$ goes to 0, we realize limited expansions at order $p$, for $1 \le j \le N$, of the terms
$\kappa_j = 4d_j\epsilon(1-\epsilon^2 d_j^2)^{1/2}$, $\delta_j = 4d_j\epsilon(1-2\epsilon^2 d_j^2)(1-\epsilon^2 d_j^2)^{1/2}$,
$\gamma_j = d_j\epsilon(1-\epsilon^2 d_j^2)^{-1/2}$, $x_{r,j} = (r-1)\ln(1+2id_j\epsilon(1-\epsilon^2 d_j^2)^{-1/2})$,
$\kappa_{N+j} = 4d_j\epsilon(1-\epsilon^2 d_j^2)^{1/2}$, $\delta_{N+j} = -4d_j\epsilon(1-2\epsilon^2 d_j^2)(1-\epsilon^2 d_j^2)^{1/2}$,
$\gamma_{N+j} = 1/(d_j\epsilon)(1-\epsilon^2 d_j^2)^{1/2}$, $x_{r,N+j} = (r-1)\ln(1-2id_j\epsilon(1-\epsilon^2 d_j^2)^{-1/2})$,
For example, the expansions at order 1 gives :

$$\begin{array}{l}
\kappa_j = 4d_j\epsilon + O(\epsilon^2), \quad \gamma_j = d_j\epsilon + O(\epsilon^2), \quad \delta_j = 4d_j\epsilon + O(\epsilon^2),\\
x_{r,j} = (r-1)(2id_j\epsilon + O(\epsilon^2)),\\
\kappa_{N+j} = 4d_j\epsilon + O(\epsilon^2), \quad \gamma_{N+j} = 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), \quad \delta_{N+j} = -4d_j\epsilon + O(\epsilon^2),\\
x_{r,N+j} = -(r-1)(2id_j\epsilon + O(\epsilon^2)),\\
1 \le j \le N.
\end{array}$$

We choose the following notations :
$B_j = \kappa_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}) - ix_j/4 - ie_\nu$, and
$B'_j = \kappa_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}) + ix_j/4 - ie_\nu$.
Then, we realize limited expansions at order $p$ in $\epsilon$ of the functions $\phi_j^r(0)$ and $\phi_{N+j}^r(0)$, for $1 \le j \le N$ :

$$\begin{array}{l}
\phi_j^1(0) = \sin(B_j + ix_j/4) = P_j + O(\epsilon^{p+1}),\\
\phi_j^3(0) = \sin(B_j - x_j/4) = Q_j + O(\epsilon^{p+1}),\\
\phi_{N+j}^1(0) = \cos(B'_j - ix_j/4) = P'_j + O(\epsilon^{p+1}),\\
\phi_{N+j}^3(0) = \cos(B'_j + ix_j/4) = Q'_j + O(\epsilon^{p+1}).
\end{array}$$

Then we have the following statement, the same as in [**?**]. So we refer the reader to this paper.

**Theorem 4.1** *The function $v$ defined by*

$$v(x,t) = \exp(2it - i\varphi)\lim_{\epsilon\to 0}\frac{W_3(0)}{W_1(0)}. \tag{20}$$

*is a quasi-rational solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

**Remark 4.1** *In (20), $W_r(y) = W(\phi_1^r, \dots, \phi_{2N}^r)$ is the wronskian*

$$W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu^r)_{\nu,\,\mu\in[1,\dots,2N]}].$$

*$\phi_\nu^r$ are the functions defined by (16), $\kappa_\nu$, $\delta_\nu$, $\gamma_\nu$ are parameters defined by (6) and $e_\nu$ are parameters defined in (5).*

**Proof : see [9]**

## 4.2 Quasi-rational solutions of order $N$

In this section we choose $x_{0i} = 0$, $t_{0i} = 0$ for $1 \le i \le 2N$.
To get solutions of NLS equation written in the context of fiber optics (21)

$$iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0. \tag{21}$$

from these of (1), we can make the following changes of variables

$$\begin{aligned} t &\rightarrow X/2 \\ x &\rightarrow T. \end{aligned} \tag{22}$$

In the following, we give all the solutions for (1).

### 4.2.1 Case N=1

If we consider the case $N = 1$, from (20), we realize a development at order 1 of $W_3$ and $W_1$ in $\epsilon$. The solution of NLS equation can be written as

$$v(x,t) = \frac{-16d_1^2t^2 + 16id_1^2t - 4id_1b_1 - 4d_1xa_1 - a_1^2 + 3d_1^2 + 8d_1tb_1 - b_1^2 - 4d_1^2x^2}{4d_1^2x^2 + 4d_1xa_1 + 16d_1^2t^2 - 8d_1tb_1 + a_1^2 + b_1^2 + d_1^2}\exp(2it - i\varphi).$$

Apparently, it depends on $3N + 1 = 4$ parameters.
But in fact it can be written in the form

$$v(x,t) = \frac{(4(x + \frac{a_1}{2d_1})^2 + 16(t - \frac{b_1}{4d_1})^2 - 16i(t - \frac{b_1}{4d_1}) - 3)}{(4(x + \frac{a_1}{2d_1})^2 + 16(t - \frac{b_1}{4d_1})^2 + 1))}\exp(2it - i\varphi).$$

We note that the parameter $d_1$ disappears, and the remaining parameters are only translation parameters. By denoting $X = x + \frac{a_1}{2d_1}$ and $T = t - \frac{b_1}{4d_1}$, it be rewritten as

$$v(x,t) = \frac{(4X^2 + 16T^2 - 16iT - 3)}{(4X^2 + 16T^2 + 1))} \exp(2it - i\varphi).$$

So we get the well known Peregrine's breather.
Thus, in this case $N = 1$, the parameters can be reduced to only 2 parameters of translation and $\varphi$. The changes of these parameters don't affect the aspect of the form of the representation of $|v(x,t)|$ in the $x, t$ variables.
Moreover, if we make the preceding change of variable (22), and take $a_1 = b_1 = 0$, we get exactly first order Akhmediev's solution (see [1]).
We represent here the modulus of $v$ in function of $x \in [-5; 5]$ and $t \in [-5; 5]$, for $a_1 = 1$, $b_1 = 1$ and $d_1 = 1$.

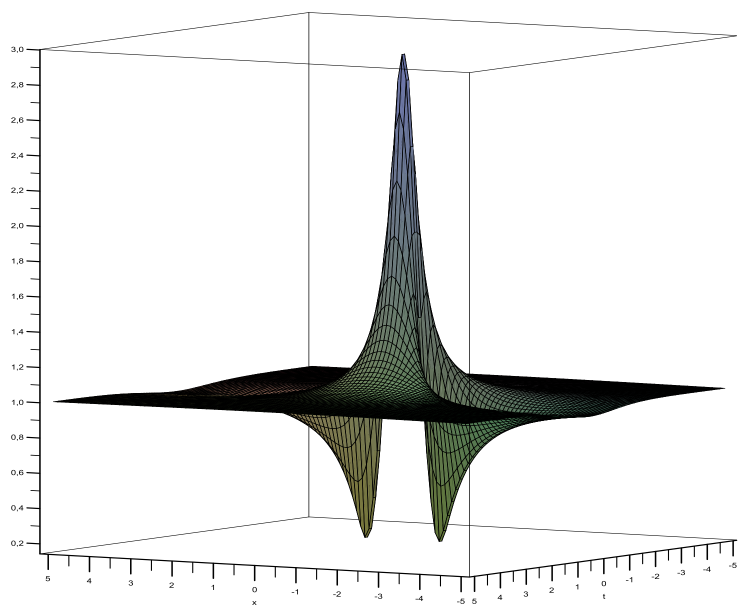

Figure 1: Solution of NLS, N=1.

### 4.2.2 Case N=2

In the case $N=2$, we realize a development at order 3 in $\epsilon$. From (20), the solution of NLS equation can be written as

$$v(x,t)=\frac{N(x,t)}{D(x,t)}\exp(2it-i\varphi),$$

with

$$\begin{aligned}
N(x,t) = {}& -3744d_1^4d_2^4t^2-(720i)d_1^6td_2^2-9a_2^2d_1^2-9a_1^2d_2^2-45d_1^6d_2^2-45d_2^6d_1^2+90d_1^4d_2^4-9b_2^2d_1^2+(6144i)d_1^2x^2d_2^6t^3\\
&-9b_1^2d_2^2+1872d_1^6t^2d_2^2-64d_1^6x^6d_2^2+128d_1^4x^6d_2^4+72d_1^3d_2^2tb_1-72d_2d_1^4tb_2-72d_1d_2^4tb_1+384d_1^4t^3d_2b_2\\
&+1536d_1^4x^4d_2^4t^2-768d_1^6x^4t^2d_2^2-3072d_1^6x^2t^4d_2^2-384d_1^3t^3d_2^2b_1-384b_2d_1^2d_2^3t^3+18d_1b_2d_2b_1-11520d_1^4x^2d_2^4t^2\\
&-108d_1^2xd_2^3a_2+72d_2^3td_1^2b_2-48d_1^2x^3d_2^3a_2+5760d_1^2x^2d_2^6t^2+6144d_1^4x^2t^4d_2^4+108a_1d_1d_2^4x+108d_1^4xd_2a_2\\
&+5760d_1^6x^2t^2d_2^2-108d_1^3xd_2^2a_1+384d_1d_2^4t^3b_1+48d_1^4x^3d_2a_2-3072d_1^2x^2d_2^6t^4-768d_1^2x^4d_2^6t^2+48a_1d_1d_2^4x^3\\
&+18a_1d_2d_1a_2-48a_1d_1^3x^3d_2^2-576d_1^4xt^2d_2a_2-4096d_2^6t^6d_1^2+8448d_2^6t^4d_1^2-4096d_1^6t^6d_2^2+8192d_1^4t^6d_2^4+8448d_1^6t^4d_2^2\\
&-16896d_1^4t^4d_2^4+180d_1^2x^2d_2^6-288b_1d_2^4x^2d_1t+576a_1d_1^3t^2d_2^2x+288d_1^3x^2d_2^2tb_1-576a_1d_1d_2^4xt^2+576d_1^2xd_2^3t^2a_2\\
&-288d_1^4x^2d_2tb_2+288d_1^2x^2d_2^3tb_2+144d_1^2x^4d_2^6+180d_1^6x^2d_2^2+144d_1^6x^4d_2^2-360d_1^4x^2d_2^4-288d_1^4x^4d_2^4+1872d_2^6d_1^2t^2\\
&-64d_1^2x^6d_2^6+(576i)d_1^2t^2d_2^3b_2+(2304i)d_1^4x^2d_2^4t-(720i)d_1^2td_2^6+(6144i)d_1^6x^2t^3d_2^2-(1152i)d_1^6x^2td_2^2+(576i)b_1d_1^3t^2d_2^2\\
&+(144i)d_1^4x^2d_2b_2-(1536i)d_1^4x^4d_2^4t+(144i)b_1d_1d_2^4x^2-(576i)d_1^4t^2d_2b_2-(1152i)d_1^2x^2d_2^6t-(144i)b_1d_1^3x^2d_2^2\\
&+(36i)d_1^4d_2b_2-(576i)b_1d_1d_2^4t^2-(576i)d_1^2xd_2^3ta_2+(1536i)d_2^6d_1^2t^3+(1440i)d_1^4td_2^4+(1536i)d_1^6t^3d_2^2+(576i)a_1d_1d_2^4xt\\
&-(36i)d_1^3d_2^2b_1+(12288i)d_1^2t^5d_2^6+(36i)b_1d_1d_2^4+(12288i)d_1^6t^5d_2^2-(36i)d_2^3d_1^2b_2+(768i)d_1^2x^4d_2^6t-(24576i)d_1^4t^5d_2^4\\
&-(3072i)d_1^4t^3d_2^4+(768i)d_1^6x^4td_2^2+(576i)d_1^4xtd_2a_2-(12288i)d_1^4x^2d_2^4t^3-(576i)a_1d_1^3xtd_2^2-(144i)d_1^2x^2d_2^3b_2
\end{aligned}$$

and

$$\begin{aligned}
D(x,t) = {}& -3168d_1^4d_2^4t^2+9a_2^2d_1^2+9a_1^2d_2^2+9d_1^6d_2^2+9d_2^6d_1^2-18d_1^4d_2^4+9b_2^2d_1^2+9b_1^2d_2^2+1584d_1^6t^2d_2^2+216d_1^3d_2^2tb_1\\
&-216d_2d_1^4tb_2-216d_1d_2^4tb_1-384d_1^4t^3d_2b_2+384d_1^3t^3d_2^2b_1+384b_2d_1^2d_2^3t^3-18d_1b_2d_2b_1+216d_2^3td_1^2b_2\\
&-384d_1d_2^4t^3b_1-18a_1d_2d_1a_2+4096d_2^6t^6d_1^2+6912d_2^6t^4d_1^2+4096d_1^6t^6d_2^2-8192d_1^4t^6d_2^4+6912d_1^6t^4d_2^2-13824d_1^4t^4d_2^4\\
&+1584d_2^6d_1^2t^2+(576a_1d_1d_2^4t^2-576a_1d_1^3t^2d_2^2+576d_1^4t^2d_2a_2-36d_1^2d_2^3a_2-36d_1^3d_2^2a_1+36d_1^4d_2a_2-576d_1^2d_2^3t^2a_2\\
&+36a_1d_1d_2^4)x+(64d_1^6d_2^2-128d_1^4d_2^4+64d_2^6d_1^2)x^6+(48d_2^6d_1^2+768d_2^6d_1^2t^2-96d_1^4d_2^4-1536d_1^4d_2^4t^2+48d_1^6d_2^2\\
&+768d_1^6t^2d_2^2)x^4+(48d_1^3d_2^2a_1+48d_1^2d_2^3a_2-48d_1^4d_2a_2-48a_1d_1d_2^4)x^3+(-216d_1^4d_2^4+3072d_2^6t^4d_1^2+2304d_1^4d_2^4t^2\\
&-288d_1^3d_2^2tb_1+108d_1^6d_2^2-1152d_2^6d_1^2t^2-6144d_1^4t^4d_2^4+108d_2^6d_1^2-1152d_1^6t^2d_2^2+3072d_1^6t^4d_2^2-288d_2^3td_1^2b_2\\
&+288d_1d_2^4tb_1+288d_2d_1^4tb_2)x^2.
\end{aligned}$$

This solution depends on $3N+1=7$ parameters. If we make the preceding changes of variables defined by (22), and take $a_1=a_2=b_1=b_2=0$ it can be reduced exactly at the second order Akhmediev's solution (see [1]).

Conversely to the case $N = 1$, in this case it remains two parameters interesting on the 7 which are different from parameters of translation. It is the crucial point. With these parameters the shape of the curve of $|v|$ change radically as we prove it in the following. We recover as well Peregrine's breather as well the three sisters.
We represent the modulus of $v$ in function of $x \in [-5; 5]$ and $t \in [-5; 5]$.
If we take $d_1 = 1$, $d_2 = 2$, $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 1$,, we get the well known Peregrine's breather of order 2 :

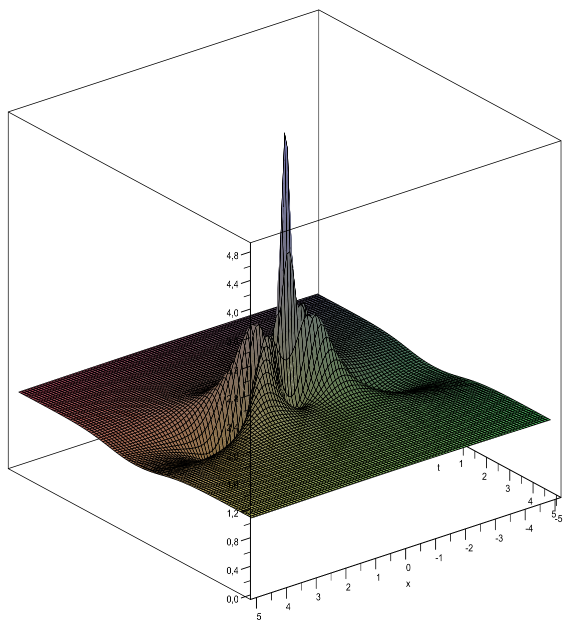


Figure 2: Solution of NLS, N=2.

If we take $d_1 = 1$, $d_2 = 2$, $a_1 = 0$, $a_2 = 0$, $b_1 = 1000$, $b_2 = 1000$,, we get the case of the three sisters :

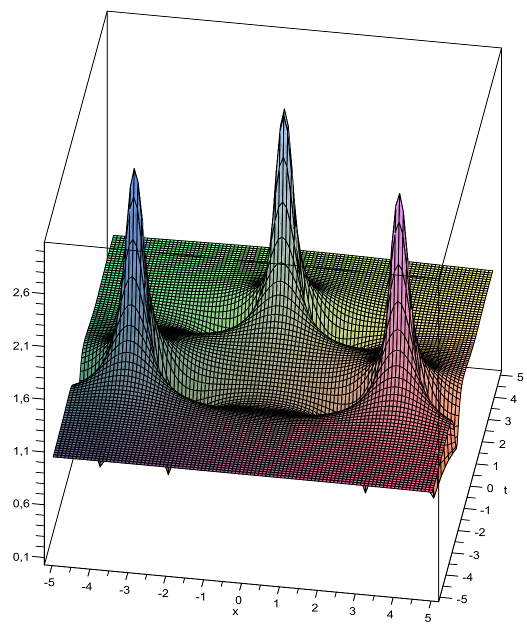


Figure 3: Solution of NLS, N=2.

#### 4.2.3 Case N=3

In the case $N = 3$, we make a development at order 5 in $\epsilon$ and get from (20), the solution of NLS equation (21) in the form

$$v(x,t) = \frac{N(x,t)}{D(x,t)} \exp(2it - i\varphi).$$

In this case, the analytical expression takes about 36 pages of usual format. We can't reproduce it in this text. We obtain a solution depending on $3N + 1 = 10$ parameters.
We only give the following graphic for the modulus of $v$ in function of $x \in [-5; 5]$ and $t \in [-5; 5]$.
If we take the following parameters : $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, we get the Peregrine's breather of order 3 :

If we take other parameters, the shape of the modulus of $v$ in the $x, t$

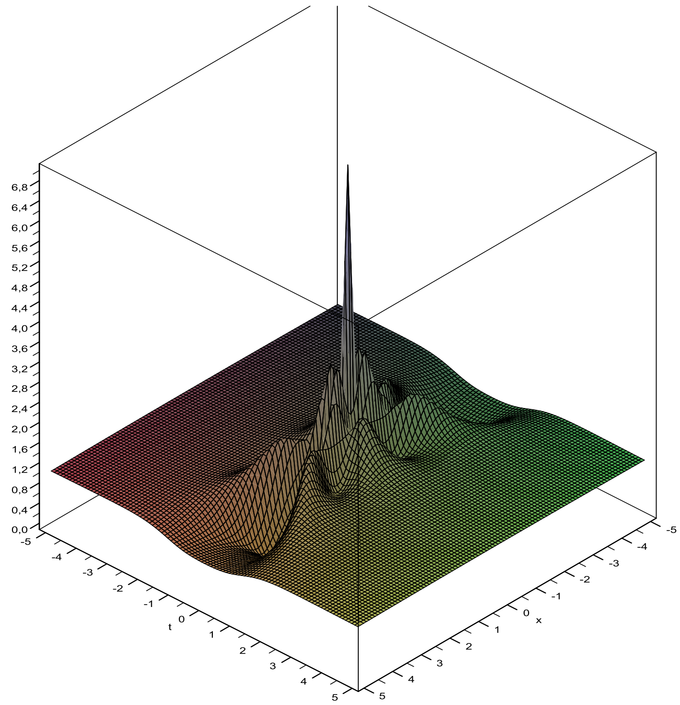


Figure 4: Solution of NLS, N=3.

coordinates change to get for example 6 described like in the approach of Krichever (see [6, 7]).

If we make the preceding changes of variables defined by (22), and take $d_1 = d_2 = d_3 = 0$, we still recover the solution given recently by Akhmediev [1].

This method can be extended to get an infinite family of quasi-rational solutions of NLS equation at any order.

# 5 Conclusion

We have given here an extension of a previous result exposed in [9]. These solutions also are written as a quotient of wronskians. An other approach

has been given in [6].
This method described in the present paper provides a powerful tool to get explicitly solutionsof the NLS equation. This new formulation gives an infinite set of non singular solution of NLS equation.

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