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**METASTABLE STATES,
QUASI-STATIONARY AND SOFT MEASURES,
MIXING TIME ASYMPTOTICS VIA VARIATIONAL PRINCIPLES**

ALESSANDRA BIANCHI AND ALEXANDRE GAUDILLIÈRE

ABSTRACT. We establish metastability in the sense of Lebowitz and Penrose under practical and simple hypothesis for (families of) Markov chains on finite configuration space in some asymptotic regime, including the case of configuration space size going to infinity. By comparing restricted ensemble and quasi-stationary measure, we study point-wise convergence velocity of Yaglom limits and prove asymptotic exponential exit law. We introduce soft measures as interpolation between restricted ensemble and quasi-stationary measure to prove an asymptotic exponential transition law on a generally different time scale. By using potential theoretic tools we prove a new general Poincaré inequality and give sharp estimates via two-sided variational principles on relaxation time as well as mean exit time and transition time. We also establish local thermalization on a shorter time scale and give mixing time asymptotics up to a constant factor through a two-sided variational principal. All our asymptotics are given with explicit quantitative bounds on the corrective terms.

1. METASTABILITY AFTER LEBOWITZ AND PENROSE

1.1. Phenomenology and modelization. Lebowitz and Penrose characterized *metastable thermodynamic states* by the following properties [3]:

- (a) only one thermodynamic phase is present,
- (b) a system that starts in this state is likely to take a long time to get out,
- (c) once the system has gotten out, it is unlikely to return.

We can think, for example, to freezing fog made of small droplets in which only one phase is present (liquid phase) that remains for a long time in such a state (until collision with ground or trees, forming then hard rime) and that once frozen will typically not return to liquid state before pressure or temperature have changed.

To modelize such a state they considered in [3] a deterministic dynamics with equilibrium measure μ . First, they associated with the only one phase of the metastable state a subset \mathcal{R} of the configuration space, and described this metastable state by the *restricted ensemble* $\mu_{\mathcal{R}} = \mu(\cdot|\mathcal{R})$. Second, they proved that the escape rate from \mathcal{R} of the system started in $\mu_{\mathcal{R}}$ is maximal at time $t = 0$, and that this initial escape rate is very small. Last, they used standard methods of equilibrium statistical mechanics to deal with (c). As estimate of the returning probability to the metastable state they used the fraction of member of the *equilibrium* ensemble that have configuration in \mathcal{R} and they noted ([3], Section 8):

This amounts to assuming that a system whose dynamical state has just left \mathcal{R} is no more likely to return to it than one whose dynamical state was never anywhere near \mathcal{R} . The validity of this assumption, at least in the short run, is dubious, but at least it provides us with some indication of what to expect.

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In this paper we want to give a different model for the same phenomenology that overcomes the last difficulty. We will work with stochastic processes rather than deterministic dynamics, but the Lebowitz-Penrose modelization will be our guideline. We will try to recover this phenomenology under simple and practical hypotheses only. Since the study of metastability has been considerably enriched after Lebowitz and Penrose work, we want also to incorporate in our modelization as much as possible of what was previously achieved. We will then make a brief and partial review of these achievements. On this review will depend our goals and starting ideas but not our proofs since we want to make this paper as self-contained as possible. Our model and results are presented in Section 2.

1.2. A partial review. Since Lebowitz and Penrose paper, an enormous work has been done to describe the metastability phenomenon. In particular Cassandro, Galves, Olivieri and Vares introduced the path-wise approach, which focused, in the context of stochastic processes, on time averages associated with an asymptotic exponential law [5]. This was further developed by the pioneering works of Neves and Schonmann who studied the typical paths for stochastic Ising model on a given volume in the low temperature regime ([7], [8]). This work was then extended to higher dimensions, infinite volume and fixed temperature regimes, locally conservative dynamics and probabilistic cellular automata ([12], [11], [15], [23], [29], [40]).

As developed in [24], a crucial role was played by large deviation tools inherited from Wentzell and Freidlin in their reduction procedure from continuous stochastic processes to finite configuration space Markov chains with exponentially small transition rates [6]. This is especially true for very low temperature regimes, but the same kind of reduction procedure allowed to deal in various cases with large volume rather than low temperature limits (see [14] for Curie-Weiss model under random magnetic field, see [24] for further examples).

Then, using potential theoretic rather than large deviations tools, Bovier, Eckhoff, Gaynard and Klein, developed a set of general techniques to compute sharp asymptotics of the expected value of asymptotic exponential laws associated with the metastability phenomenon, and revisited (after [4], [10]) the relation between generator spectrum and metastability ([16], [19], [22]). This allowed, for example, to go beyond logarithmic asymptotics for stochastic Ising models in the low temperature regime ([17], [25]) and to prove the first rigorous results in the fully conservative case ([35]), to deal with metastability for the random hopping time dynamics associated with the Random Energy Model ([18]), to make a detailed analysis of Sinai's random walk spectrum ([28]), to extend the study of the disordered Curie-Weiss model to the case of continuous magnetic field distribution ([30], [36]).

We then reached to an essentially complete comprehension of the metastability phenomenon in at least two classes of models: very low temperature dynamics in finite fixed volumes and large volume or continuous configuration space dynamics that can be reduced via Wentzell-Freidlin procedure to the previous case. Of course, specific and often non trivial computations have to be made for each specific model, but there exists a general approach to the problem that is developed in [24] and, as far as the potential theoretic part is concerned, [16], [19], and [22] together with [38], that bridges between potential theory and typical path description by reinforcing and generalizing the results of [9] (and it is worth to note that [38], after [33, 37], contemplates also the case of polynomially small rather than only exponentially small transition probabilities). For both classes of models, like one-dimensional metastable systems as considered in [28] or [27], a recurrence property to a very localized subset of the configuration space (single configurations identified to metastable states in the first case, small neighborhoods of the dynamics attractors in the second case) plays an important role.

Beyond these two classes of models there are many limit cases, special cases, and partial results. For example, in [18] we are far from a finite fixed volume situation but single configurations can still be identified with metastable states and have still enough mass at equilibrium for potential theoretic or renewal techniques to work. This is not the case in [35], where potential theoretic tools give only expected values of some hitting times when the system is started from some specific harmonic measures that are very different from what one would expect to be a “metastable state” (here, like in the sequel and following Lebowitz and Penrose, we mean a whole measure when referring to a metastable “state” and not to a single configuration of the configuration space). Any kind of exponential law is presently also lacking in this case. The same difficulty is faced in [30], but it is overcome in [36] by means of a specific coupling argument that gives point-wise estimates and opens the way to the exponential law. Finally, the beautiful paper by Schonmann and Shlosman [15] achieves the *tour de force* of using essentially equilibrium statistical mechanics computations to deal with the dynamical problem of metastability. In this case also the exponential law is lacking as well as sharp estimates on the transition time, even though the simple formulation of such properties is not completely obvious in this fixed temperature and vanishing magnetic field regime.

1.3. Starting ideas. In the present paper we want to elaborate some tools to describe the metastability phenomenon beyond the case of a dynamics with a recurrence property to a very localized subset of the configuration space. We will focus on exponential laws and sharp asymptotics of their expected values. We note that the exponential law itself suggests some kind of recurrence property. If it is not a recurrence property to a very localized subset, it has to be in some sense a recurrence property to a whole “spread measure”. And this measure should coincide with our metastable state. Now, following Lebowitz and Penrose, if we associate the metastable state with some subset \mathcal{R} of the configuration space \mathcal{X} , then, considering property (b), we have at least two candidates to describe our metastable state: one is the restricted ensemble $\mu_{\mathcal{R}} = \mu(\cdot|\mathcal{R})$ the other is the quasi-stationary measure

$$\mu_{\mathcal{R}}^* = \lim_{t \rightarrow +\infty} P_{\mu_{\mathcal{R}}}(X(t) \in \cdot | \tau_{\mathcal{X} \setminus \mathcal{R}} > t) \quad (1)$$

where $X(t)$ is the configuration of the system at time t and $\tau_{\mathcal{X} \setminus \mathcal{R}}$ is the exit time of \mathcal{R} (we will be more precise in the next section). The main advantage of $\mu_{\mathcal{R}}$ is that $\mu_{\mathcal{R}}$ is often an explicit measure one can compute with, while $\mu_{\mathcal{R}}^*$ is only implicitly defined. The main advantage of $\mu_{\mathcal{R}}^*$ is that the exit law of \mathcal{R} for the system started in $\mu_{\mathcal{R}}^*$ is an exponential law. Our first results will then start with a comparison between $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{R}}^*$. We will give simple and practical hypotheses to ensure that they are close in some sense, then we will be able to prove some kind of recurrence property to $\mu_{\mathcal{R}}^*$. All this will be done in the simplest possible setup: considering a Markov process on a finite configuration space in some asymptotic regime (including the possibility to send to infinity the cardinality of the configuration space).

In the present work we will essentially build on the ideas of four different papers: [3] for the formulation of the problem, [5] for the focus on exponential laws, [16] for the introduction of potential theoretic techniques in the metastability field to get sharp estimates on some mean hitting times, and Miclo’s work [34] where some concepts of local equilibrium, and “hitting times” of such equilibria, are introduced. As far as this last paper is concerned, it will only work as a source of inspiration: we will not require a full spectrum knowledge, and we will not introduce any notion of a starting point depending on local equilibrium. Finally, we note that the idea of considering quasi-stationary measures as metastable states was already contemplated in [20]. Even though some of our results echo some of [20], we were not able to make any clear comparison, essentially because of the much more analytical point of view of [20] and the many hypotheses introduced in the

results of [20]. We note that, on the one hand, [20] deals with a much more general setup than ours since the authors consider non-reversible Markov processes on a continuous configuration space, while we look at reversible Markov processes on finite configuration space. However, the reason why we assume reversibility is to be able to use potential theoretic results to get sharp estimates on mean times via variational principles, a question that is not considered in [20].

2. MODEL AND RESULTS

2.1. Quasi-stationary measure and restricted ensemble. We consider a continuous time Markov process X on a finite set \mathcal{X} with generator defined by

$$\mathcal{L}f(x) = \sum_{y \in \mathcal{X}} p(x, y)(f(y) - f(x)) \quad (2)$$

for x in \mathcal{X} and $f : \mathcal{X} \rightarrow \mathbb{R}$, and where p is such that $\sum_y p(x, y) = 1$. Since \mathcal{X} is finite, any generator can be written like in (2) up to time rescaling. We assume that X is irreducible and reversible with respect to some probability measure μ , we denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\ell^2(\mu)$, by $\|\cdot\|$ the associated 2-norm, by \mathcal{D} the Dirichlet form defined by

$$\mathcal{D}(f) = \langle f, -\mathcal{L}f \rangle = \frac{1}{2} \sum_{x, y \in \mathcal{X}} c(x, y) [f(x) - f(y)]^2 \quad (3)$$

where each conductance $c(x, y)$ is equal to

$$c(x, y) = \mu(x)p(x, y), \quad (4)$$

and by γ the spectral gap

$$\gamma = \min_{\text{Var}_\mu(f) \neq 0} \frac{\mathcal{D}(f)}{\text{Var}_\mu(f)}. \quad (5)$$

For $\mathcal{R} \subset \mathcal{X}$ we define in each $x \in \mathcal{R}$ the escape probability

$$e_{\mathcal{R}}(x) = \sum_{y \notin \mathcal{R}} p(x, y) \quad (6)$$

and we denote by $X_{\mathcal{R}}$ the *reflected process* with generator given by

$$\mathcal{L}_{\mathcal{R}}f(x) = \sum_{y \in \mathcal{R}} p_{\mathcal{R}}(x, y)(f(y) - f(x)) \quad (7)$$

for x in \mathcal{R} and $f : \mathcal{R} \rightarrow \mathbb{R}$, and where, for all x, y in \mathcal{R} ,

$$p_{\mathcal{R}}(x, y) = \begin{cases} p(x, y) & \text{if } x \neq y, \\ p(x, x) + e_{\mathcal{R}}(x) & \text{if } x = y. \end{cases} \quad (8)$$

We will only consider subsets \mathcal{R} such that both $X_{\mathcal{R}}$ and $X_{\mathcal{X} \setminus \mathcal{R}}$ are irreducible and we note that $X_{\mathcal{R}}$ inherits from X the reversibility property with respect to the restricted ensemble

$$\mu_{\mathcal{R}} = \mu(\cdot | \mathcal{R}). \quad (9)$$

We identify $\ell^2(\mu_{\mathcal{R}})$ with the subset of $\ell^2(\mu)$ of functions $f : \mathcal{X} \mapsto \mathbb{R}$ such that $f|_{\mathcal{X} \setminus \mathcal{R}} \equiv 0$ and we denote by $\langle \cdot, \cdot \rangle_{\mathcal{R}}$, $\|\cdot\|_{\mathcal{R}}$, $\mathcal{D}_{\mathcal{R}}$, $c_{\mathcal{R}}(x, y)$ and $\gamma_{\mathcal{R}}$ the associated scalar product, 2-norm, Dirichlet form, conductances for x, y in \mathcal{R} and spectral gap.

We denote by $p_{\mathcal{R}}^*$ the sub-Markovian kernel on \mathcal{R} such that, for all x, y in \mathcal{R} ,

$$p_{\mathcal{R}}^*(x, y) = p(x, y). \quad (10)$$

We know from [2] and the Perron-Frobenius theorem that there exists $\phi_{\mathcal{R}}^* > 0$ such that $1 - \phi_{\mathcal{R}}^*$ is the spectral radius of $p_{\mathcal{R}}^*$ and that there is a unique *quasi-stationary measure* $\mu_{\mathcal{R}}^*$ such that $\mu_{\mathcal{R}}^* p_{\mathcal{R}}^* = (1 - \phi_{\mathcal{R}}^*) \mu_{\mathcal{R}}^*$. In addition we have, for all x, y in \mathcal{R} and $t \geq 0$,

$$\lim_{t \rightarrow +\infty} P_x(X(t) = y | \tau_{\mathcal{X} \setminus \mathcal{R}} > t) = \mu_{\mathcal{R}}^*(y), \quad (11)$$

$$P_{\mu_{\mathcal{R}}^*}(\tau_{\mathcal{X} \setminus \mathcal{R}} > t) = e^{-\phi_{\mathcal{R}}^* t}, \quad (12)$$

$$\mu_{\mathcal{R}}^*(e_{\mathcal{R}}) = \phi_{\mathcal{R}}^*. \quad (13)$$

In Sections 2.3 and 6 these properties will be rederived in a slightly more general context.

Our first result states that if $1/\phi_{\mathcal{R}}^*$, the mean exit time for the system started in $\mu_{\mathcal{R}}^*$, is large with respect $1/\gamma_{\mathcal{R}}$, the relaxation time of the reflected process, then the quasi-stationary measure $\mu_{\mathcal{R}}^*$ is close to the restricted ensemble $\mu_{\mathcal{R}}$. More precisely, defining, for all x in \mathcal{R} ,

$$\varepsilon_{\mathcal{R}}^* = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} \quad (14)$$

$$h_{\mathcal{R}}^*(x) = \frac{\mu_{\mathcal{R}}^*(x)}{\mu_{\mathcal{R}}(x)} \quad (15)$$

we prove the following.

Proposition 2.1. *If $\varepsilon_{\mathcal{R}}^* < 1$, then*

$$\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) = \|h_{\mathcal{R}}^* - \mathbb{1}_{\mathcal{R}}\|_{\mathcal{R}}^2 \leq \frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*} \quad (16)$$

Proof. See Section 3.1. □

Remark. When proving that $\varepsilon_{\mathcal{R}}^*$ goes to 0 in some asymptotic regime (for example when the cardinality of the configuration space goes to infinity like in [5], when some parameter of the dynamics goes to 0 like the temperature in [7] or when both happen like in [32]) one has to give upper bounds on $\phi_{\mathcal{R}}^*$ and lower bounds on $\gamma_{\mathcal{R}}$. As far as such upper bounds are concerned we will show that they can be easily obtained using test functions in some inequalities (see Theorem 2.11 below together with the following remark). In addition, since one can often easily compute with $\mu_{\mathcal{R}}$ and $e_{\mathcal{R}}$ is often explicit, one can usually estimate

$$\phi_{\mathcal{R}} = \mu_{\mathcal{R}}(e_{\mathcal{R}}) \quad (17)$$

and use the following lemma that we prove in Section 3.2.

Lemma 2.2. $\phi_{\mathcal{R}}^* \leq \phi_{\mathcal{R}}$.

Lower bounds on $\gamma_{\mathcal{R}}$ can be more difficult to obtain. However we note, first, that rough lower bounds will often be sufficient to our ends, second, that the new Poincaré inequality we will prove in this paper (Theorem 2.12) can be used to this purpose.

As a consequence of this first result we can control the convergence rate of the Yaglom limit (11). We note that, by the reversibility of X with respect to μ , $p_{\mathcal{R}}^*$ is a self-adjoint operator on $\ell^2(\mu_{\mathcal{R}})$ and has real eigenvalues. By the Perron-Frobenius theorem, this implies the existence of a spectral gap $\gamma_{\mathcal{R}}^* > 0$ equal to the difference between the first and the second largest eigenvalue of $p_{\mathcal{R}}^*$.

Proposition 2.3. *If $\varepsilon_{\mathcal{R}}^* < \frac{1}{3}$, then*

$$\frac{1}{\gamma_{\mathcal{R}}^*} \leq \frac{1}{\gamma_{\mathcal{R}}} \left\{ \frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 3\varepsilon_{\mathcal{R}}^*} \right\}. \quad (18)$$

Proof. See Section 3.3. □

Remark. Since, after the statical study made in [39], we intend to apply our results to the dynamical study of the cavity algorithm introduced in [26], for which finite effects volume are of first importance, we need to give asymptotics with quantitative control of corrective terms. This produces quite long formulas and to simplify the reading we put between graph parentheses any terms that goes to 1 in a suitable asymptotic regime.

Then we define

$$\zeta_{\mathcal{R}}^* = \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) h_{\mathcal{R}}^{*2}(x) = \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}^*(x) h_{\mathcal{R}}^*(x) \quad (19)$$

and, if $\varepsilon_{\mathcal{R}}^* < 1$, for any $\delta \in]0, 1[$,

$$T_{\delta, \mathcal{R}}^* = \frac{1}{\gamma_{\mathcal{R}}^*} \left(\ln \frac{2}{\delta(1-\delta)\zeta_{\mathcal{R}}^*} \right) \left\{ 1 + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right\} \quad (20)$$

to get point-wise mixing estimates for Yaglom limits.

Theorem 2.4 (Mixing towards quasi-stationary measure). *If $\varepsilon_{\mathcal{R}}^* < 1/3$, then for all $x, y \in \mathcal{R}$ and $\delta \in]0, 1[$,*

$$\left| \frac{\mathbb{P}_x(X(t) = y \mid \tau_{\mathcal{X} \setminus \mathcal{R}} > t)}{\mu_{\mathcal{R}}^*(y)} - 1 \right| < \delta \quad \text{as soon as} \quad t > T_{\delta, \mathcal{R}}^* \quad (21)$$

Proof. See Section 3.4. □

Remark. In words this says that either the system leaves \mathcal{R} before time $T_{\delta, \mathcal{R}}^*$ or it is described after that time by $\mu_{\mathcal{R}}^*$ in the strongest possible sense. This theorem is useful only if one can provide upper bounds on $T_{\delta, \mathcal{R}}^*$. We already estimated $\gamma_{\mathcal{R}}^*$ in Proposition 2.3 and we note that, as far as $\zeta_{\mathcal{R}}^*$ is concerned it only appears in the formula through its logarithm. Very crude estimates will then often be sufficient. One has for example the following lemma.

Lemma 2.5. *If $p(x, x) > 0$ for all $x \in \mathcal{R}$, then*

$$\zeta_{\mathcal{R}}^* \geq \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}^{*2}(x) \geq e^{-2\Delta_{\mathcal{R}} D_{\mathcal{R}}}, \quad (22)$$

where

$$\begin{aligned} \Delta_{\mathcal{R}} &= \max\{-\ln p_{\mathcal{R}}(x, y) : p_{\mathcal{R}}(x, y) > 0, \forall x, y \in \mathcal{R}\} \\ D_{\mathcal{R}} &= \min\{k \geq 0 : p_{\mathcal{R}}^k(x, y) > 0, \forall x, y \in \mathcal{R}\}. \end{aligned}$$

Proof. See Appendix A. □

We will make a special choice for the parameter δ in (20). We define

$$\delta_{\mathcal{R}}^* = \zeta_{\mathcal{R}}^* \min_{x \in \partial_- \mathcal{R}} e_{\mathcal{R}}(x) \quad (23)$$

and set

$$T_{\mathcal{R}}^* = T_{\delta_{\mathcal{R}}^*, \mathcal{R}}^*. \quad (24)$$

The next lemma is necessary for the following results to make sense.

Lemma 2.6. *For $\varepsilon_{\mathcal{R}}^* < 1$, it holds*

$$\delta_{\mathcal{R}}^* \leq \phi_{\mathcal{R}}^* \leq \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \leq 2\varepsilon_{\mathcal{R}}^* \left(\ln \frac{\sqrt{2}}{\delta_{\mathcal{R}}^*(1-\delta_{\mathcal{R}}^*)} \right) \left\{ 1 + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right\} \quad (25)$$

Proof. See Section 4.1. □

Remark. We will often refer in the sequel to the regime $\phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \ll 1$. The lemma implies that in this regime, $\delta_{\mathcal{R}}^*$ and $\phi_{\mathcal{R}}^*$ also go to 0. In addition, it provides a sufficient and practical condition for being in such a regime: since $\min_{x \in \partial_- \mathcal{R}} e_{\mathcal{R}}(x)$ is usually an easy to estimate quantity, one has a lower bound on $\delta_{\mathcal{R}}^*$ as soon as one can estimate $\zeta_{\mathcal{R}}^*$ from below, and, since on the right hand side $\delta_{\mathcal{R}}^*$ appears only through its logarithm, rough bounds will often be enough to show that $\varepsilon_{\mathcal{R}}^*$ goes fast enough to 0.

We close this section with a first asymptotic exponential law.

Theorem 2.7 (Asymptotic exit law). *For any probability measure ν on \mathcal{R} , define $\pi_{\mathcal{R}}(\nu) = \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} < T_{\mathcal{R}}^*)$. If $\varepsilon_{\mathcal{R}}^* < 1/3$, then*

- (i)
- $$\begin{cases} \mathbb{E}_{\nu}[\tau_{\mathcal{X}\mathcal{R}}] \leq T_{\mathcal{R}}^* + \frac{1-\pi_{\mathcal{R}}(\nu)}{\phi_{\mathcal{R}}^*} \{1 + \delta_{\mathcal{R}}^*\} \leq \frac{1}{\phi_{\mathcal{R}}^*} \{1 + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* + \delta_{\mathcal{R}}^*\} \\ \mathbb{E}_{\nu}[\tau_{\mathcal{X}\mathcal{R}}] \geq \frac{1-\pi_{\mathcal{R}}(\nu)}{\phi_{\mathcal{R}}^*} \{1 + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* - \delta_{\mathcal{R}}^*\} \end{cases};$$
- (ii) for all $t \geq \phi_{\mathcal{R}}^* T_{\mathcal{R}}^*$,
- $$\begin{cases} \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*}) \leq (1 - \pi_{\mathcal{R}}(\nu))e^{-t} \{e^{\phi_{\mathcal{R}}^* T_{\mathcal{R}}^*} (1 + \delta_{\mathcal{R}}^*)\} \\ \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*}) \geq (1 - \pi_{\mathcal{R}}(\nu))e^{-t} \{e^{\phi_{\mathcal{R}}^* T_{\mathcal{R}}^*} (1 - \delta_{\mathcal{R}}^*)\} \end{cases}.$$

Proof. See Section 4.2 □

Remark. The theorem gives more than asymptotic exponential exit law. The first part shows in particular that in the regime $\phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \ll 1$, the quasi-stationary measure maximizes asymptotically the mean exit time on the set of all possible starting measures. The second part shows that, provided $\pi_{\mathcal{R}}(\nu)$ converges to some limit, the normalized mean exit time $\phi_{\mathcal{R}}^* \tau_{\mathcal{X}\mathcal{R}}$ converges in law to a convex combination between a Dirac mass in 0 and an exponential law with mean 1.

As an example of application we can consider the case of the restricted ensemble.

Lemma 2.8. *It holds*

$$\pi_{\mathcal{R}}(\mu_{\mathcal{R}}) = \mathbb{P}_{\mu_{\mathcal{R}}}(\tau_{\mathcal{X}\mathcal{R}} \leq T_{\mathcal{R}}^*) \leq \phi_{\mathcal{R}} T_{\mathcal{R}}^*. \quad (26)$$

Proof. See Section 4.3. □

Corollary 2.9. *If $\varepsilon_{\mathcal{R}}^* < 1/3$, then*

$$\frac{1}{\phi_{\mathcal{R}}^*} \{(1 - \phi_{\mathcal{R}} T_{\mathcal{R}}^*)(1 + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* - \delta_{\mathcal{R}}^*)\} \leq \mathbb{E}_{\mu_{\mathcal{R}}}[\tau_{\mathcal{X}\mathcal{R}}] \leq \frac{1}{\phi_{\mathcal{R}}^*} \{1 + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* + \delta_{\mathcal{R}}^*\} \quad (27)$$

and, for all $t > \phi_{\mathcal{R}}^* T_{\mathcal{R}}^*$,

$$e^{-t} \{(1 - \phi_{\mathcal{R}} T_{\mathcal{R}}^*)(1 - \delta_{\mathcal{R}}^*)e^{\phi_{\mathcal{R}}^* T_{\mathcal{R}}^*}\} \leq \mathbb{P}_{\mu_{\mathcal{R}}}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*}) \leq e^{-t} \{(1 + \delta_{\mathcal{R}}^*)e^{\phi_{\mathcal{R}}^* T_{\mathcal{R}}^*}\}. \quad (28)$$

This shows asymptotic exponential exit law in the regime $\phi_{\mathcal{R}} T_{\mathcal{R}}^* \ll 1$ for the system started in the restricted ensemble.

2.2. (κ, λ) -capacities, mean exit times and a new Poincaré inequality. In this section we introduce a new object which extends the notion of capacity between sets. For any $\kappa, \lambda > 0$ and $A, B \subset \mathcal{X}$, we first extend the electrical network (\mathcal{X}, c) , with $c(x, y) = \mu(x)p(x, y) = \mu(y)p(y, x)$ for all distinct $x, y \in \mathcal{X}$, into a larger electrical network $(\tilde{\mathcal{X}}, \tilde{c})$ by attaching a dangling edge (a, \bar{a}) with conductance $\kappa\mu(a)$ to each $a \in A$ and a dangling edge (b, \bar{b}) with conductance $\lambda\mu(b)$ to each $b \in B$. More precisely, we add $|A| + |B|$ nodes and edges to the network by setting

$$\tilde{\mathcal{X}} = \mathcal{X} \cup \{\bar{a} : a \in A\} \cup \{\bar{b} : b \in B\}$$

and, for all distinct $\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}$ we define

$$\tilde{c}(\tilde{x}, \tilde{y}) = \begin{cases} c(x, y) & \text{if } (\tilde{x}, \tilde{y}) = (x, y) \in \mathcal{X} \times \mathcal{X} \\ \kappa\mu(a) & \text{if } (\tilde{x}, \tilde{y}) = (a, \bar{a}) \text{ for some } a \in A \\ \lambda\mu(b) & \text{if } (\tilde{x}, \tilde{y}) = (b, \bar{b}) \text{ for some } b \in B \\ 0 & \text{otherwise} \end{cases}. \quad (29)$$

Definition 2.10. The (κ, λ) -capacity, $C_\kappa^\lambda(A, B)$, is defined as the capacity between the sets \bar{A} and \bar{B} in the electrical network $(\tilde{\mathcal{X}}, \tilde{c})$, and then is given by

$$\begin{aligned} C_\kappa^\lambda(A, B) &= \min_{\tilde{f}: \tilde{\mathcal{X}} \rightarrow \mathbb{R}} \left\{ \frac{1}{2} \sum_{\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}} \tilde{c}(\tilde{x}, \tilde{y}) [\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{y})]^2; \tilde{f}|_{\bar{A}} = 1, \tilde{f}|_{\bar{B}} = 0 \right\} \\ &= \min_{f: \mathcal{X} \rightarrow \mathbb{R}} \left\{ \mathcal{D}(f) + \kappa \sum_{a \in A} \mu(a) [f(a) - 1]^2 + \lambda \sum_{b \in B} \mu(b) [f(b) - 0]^2 \right\} \\ &= \min_{f: \mathcal{X} \rightarrow \mathbb{R}} \left\{ \mathcal{D}(f) + \kappa \mu(A) \mathbb{E}_{\mu_A} [(f|_A - 1)^2] + \lambda \mu(B) \mathbb{E}_{\mu_B} [(f|_B - 0)^2] \right\}. \end{aligned} \quad (30)$$

Remarks.

- i) Since all the points of \bar{A} and \bar{B} are at potential 1 and 0 respectively in formula (30), they are electrically equivalent and we could have defined the (κ, λ) -capacity between A and B by adding just two nodes to the electrical network (\mathcal{X}, c) . However, our definition with dangling edges will be more useful in the sequel.
- ii) A (κ, λ) -capacity is in some sense easy to estimate since it satisfies a two-sided variational principle. On one hand, by definition, it is the infimum of some functional, and any test function will provide an upper bound. On the other hand it is the supremum of another functional on flows from \bar{A} to \bar{B} , which are anti-symmetric function of oriented edges with null divergence in \mathcal{X} , i.e., on functions $\tilde{\psi}: \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ such that for all $x \in \mathcal{X} \setminus (\bar{A} \cup \bar{B})$, $\text{div}_x \tilde{\psi} = \sum_{\tilde{x} \in \tilde{\mathcal{X}}} \tilde{\psi}(x, \tilde{x}) = 0$. This is Thomson principle that goes back to [1], Chapter 1, Appendix A (see also lecture notes [31] for a more modern presentation or textbook [13] for the proof of an almost equivalent result). Letting

$$\tilde{\mathcal{D}}(\tilde{\psi}) = \frac{1}{2} \sum_{\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}} \frac{\tilde{\psi}(\tilde{x}, \tilde{y})}{c(\tilde{x}, \tilde{y})},$$

be the energy dissipated by the flow $\tilde{\psi}$ in the network $(\tilde{\mathcal{X}}, \tilde{c})$, and $\tilde{\phi}_1(\bar{A}, \bar{B})$ the set of unitary flows from \bar{A} to \bar{B} , that is, the set of flows $\tilde{\psi}$ from \bar{A} to \bar{B} such that

$$\sum_{\bar{a} \in \bar{A}} \text{div}_{\bar{a}} \tilde{\psi} = \sum_{\bar{a} \in \bar{A}} \sum_{\tilde{x} \in \tilde{\mathcal{X}}} \tilde{\psi}(\bar{a}, \tilde{x}) = 1 = - \sum_{\bar{b} \in \bar{B}} \text{div}_{\bar{b}} \tilde{\psi} = - \sum_{\bar{b} \in \bar{B}} \sum_{\tilde{x} \in \tilde{\mathcal{X}}} \tilde{\psi}(\bar{b}, \tilde{x}), \quad (31)$$

or, if $\tilde{\mathcal{D}}(\tilde{\psi}) < +\infty$, such that

$$- \sum_{\bar{a} \in \bar{A}} \tilde{\psi}(a, \bar{a}) = 1 = \sum_{\bar{b} \in \bar{B}} \tilde{\psi}(b, \bar{b}), \quad (32)$$

we have

$$\begin{aligned} C_\kappa^\lambda(A, B) &= \max_{\tilde{\psi} \in \tilde{\phi}_1(\bar{A}, \bar{B})} \tilde{\mathcal{D}}(\tilde{\psi}) \\ &= \max_{\psi \in \phi_1(A, B)} \left\{ \mathcal{D}(\psi) + \sum_{a \in A} \frac{(\text{div}_a \psi)^2}{\kappa \mu(a)} + \sum_{b \in B} \frac{(\text{div}_b \psi)^2}{\lambda \mu(b)} \right\} \\ &= \max_{\psi \in \phi_1(A, B)} \left\{ \mathcal{D}(\psi) + \frac{\mu(A)}{\kappa} \mathbb{E}_{\mu_A} \left[\left(\frac{\text{div} \psi}{\mu} \right)_{|A}^2 \right] + \frac{\mu(B)}{\lambda} \mathbb{E}_{\mu_B} \left[\left(\frac{\text{div} \psi}{\mu} \right)_{|B}^2 \right] \right\}, \end{aligned} \quad (33)$$

where $\phi_1(A, B)$ is the set of unitary flows ψ from A to B and

$$\mathcal{D}(\psi) = \frac{1}{2} \sum_{x, y \in \mathcal{X}} \frac{\psi(x, y)}{c(x, y)}.$$

Then, any test flow provides a lower bound on $C_\kappa^\lambda(A, B)$.

- iii) We know ([13], [31]) that the infimum and supremum in (30) and (33), are realized, respectively, by the equilibrium potential $V_\kappa^\lambda = \mathbb{P}_{(\cdot)}(G_A(\sigma_\kappa) < G_B(\sigma_\lambda))$, where G_A and G_B are the right continuous inverses of the local times in A and B , while σ_κ and σ_λ are independent exponential times with rates κ and λ , and by its associated normalized current

$$-\frac{c\nabla V_\kappa^\lambda}{C_\kappa^\lambda(A,B)} : (x, y) \in \mathcal{X} \times \mathcal{X} \mapsto \frac{c(x,y)}{C_\kappa^\lambda(A,B)}(V_\kappa^\lambda(x) - V_\kappa^\lambda(y)). \quad (34)$$

We will say more on such quantities in the next section.

- iv) The previous definitions and observations extend to the case when κ and λ are equal to $+\infty$. In that case we identify \bar{A} to A in the extended network if $\kappa = +\infty$, or \bar{B} to B if $\lambda = +\infty$, and we drop the infinite upper or lower index in the notation, so that, for example, $C_\kappa(A, B) = C_\kappa^\infty(A, B)$. However, when κ and λ are both equal to $+\infty$, to avoid any ambiguity we need to require that $A \cap B = \emptyset$. In that case the notation becomes $C(A, B) = C_\infty^\infty(A, B)$ and we recover indeed the usual notion of capacity.

We then get sharp asymptotics on mean exit times for the system started in the quasi-stationary measure.

Theorem 2.11 (Mean exit time estimates). *If $\varepsilon_{\mathcal{R}}^* < 1/3$, then, for all $\kappa > 0$,*

$$\frac{C_\kappa(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})} \left\{ 1 - \varepsilon_{\mathcal{R}}^* - \frac{\kappa}{\gamma_{\mathcal{R}}} \right\} \leq \phi_{\mathcal{R}}^* \leq \frac{C_\kappa(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})} \left\{ 1 + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* + \delta_{\mathcal{R}}^* + \frac{\phi_{\mathcal{R}}^*}{\kappa} \right\} \quad (35)$$

Proof. See Section 5. □

Remarks.

- i) In the regime $\varepsilon_{\mathcal{R}}^* + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \ll 1$ one can choose κ such that $\phi_{\mathcal{R}}^* \ll \kappa \ll \gamma_{\mathcal{R}}$ and get matching bounds.
- ii) Both bounds are in some sense easy to estimate since capacities satisfies a two-sided variational principal. Moreover, compared with the formula for mean exit time provided by potential theoretic techniques (see, e.g., [16]), the above inequalities require no residual mean potential estimates. (Such estimates, as well as some harmonic measures will only play a role in the *proof* of the theorem.)
- iii) Resolving the upper bound in $\phi_{\mathcal{R}}^*$ ($\phi_{\mathcal{R}}^*$ appears also in the right hand side of the inequality) one can get good bounds on $\phi_{\mathcal{R}}^*$ and prove that it decreases fast enough to guarantee $\phi_{\mathcal{R}}^* T_{\mathcal{R}}^* \ll 1$.

Our (κ, λ) -capacities provide also spectral gap estimates and a new general Poincaré inequality. For $\kappa, \lambda > 0$ and $A, B \subset \mathcal{X}$ we set

$$\phi_\kappa^\lambda(A, B) = \frac{C_\kappa^\lambda(A, B)}{\mu(A)\mu(B)} = \phi_\lambda^\kappa(B, A). \quad (36)$$

Theorem 2.12 (Relaxation time estimates). *For all $\kappa, \lambda > 0$ and any $\mathcal{R} \subset \mathcal{X}$ such that $X_{\mathcal{R}}$ and $X_{\mathcal{X} \setminus \mathcal{R}}$ are both irreducible Markov processes,*

$$\left\{ \begin{array}{l} \frac{1}{\gamma} \geq \frac{1}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} \left\{ 1 - \frac{C_\kappa(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\kappa\mu(\mathcal{R})} - \frac{C^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\lambda\mu(\mathcal{X} \setminus \mathcal{R})} \right\}^2 \\ \frac{1}{\gamma} \leq \frac{1}{\phi_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} \left\{ 1 + \max \left(\frac{\kappa + \phi_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\gamma_{\mathcal{R}}}, \frac{\lambda + \phi_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\gamma_{\mathcal{X} \setminus \mathcal{R}}} \right) \right\} \end{array} \right\}. \quad (37)$$

Proof. See Section 5. □

Remarks.

- i) Without loss of generality, we can assume $\mu(\mathcal{R}) \leq \mu(\mathcal{X} \setminus \mathcal{R})$ so that, by (36), $\phi_\kappa^\lambda \leq 2C_\kappa^\lambda(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})/\mu(\mathcal{R})$. Then, as a consequence of the previous theorem and of the monotonicity in κ and λ of (κ, λ) -capacities, we get matching bounds over $1/\gamma$ in the regime $\varepsilon_{\mathcal{R}}^* + \varepsilon_{\mathcal{X} \setminus \mathcal{R}}^* + \phi_{\mathcal{R}}^*/\gamma_{\mathcal{X} \setminus \mathcal{R}} \ll 1$. One can indeed choose κ such that

- $\phi_{\mathcal{R}}^* \ll \kappa \ll \gamma_{\mathcal{R}}$, as for Theorem 2.11 (Remark i)), and λ such that $\phi_{\mathcal{R}}^*, \phi_{\mathcal{X} \setminus \mathcal{R}}^* \ll \lambda \ll \gamma_{\mathcal{X} \setminus \mathcal{R}}$. In addition and like previously, all the relevant quantities can be estimated by two-sided variational principles.
- ii) The lower bound is a generalization of the classical isoperimetrical estimate that is recovered for $\kappa = \lambda = +\infty$.
 - iii) The upper bound is a new Poincaré inequality. This inequality, or an easy to derive version when one divides the configuration space into more than two subsets, echoes Poincaré inequalities given in [21]. We are not able to compare in full generality our result with that of [21] but we note that because of the presence of some global parameter called γ in [21] one gets generally in our metastable situation an extra factor $1/\min(\gamma_{\mathcal{R}}, \gamma_{\mathcal{X} \setminus \mathcal{R}})$ by applying the results of [21].

2.3. Soft measures, local thermalization, transition and mixing times. We address now the difficulty raised by Lebowitz and Penrose. Whatever the measure we choose to describe our metastable state, restricted ensemble or quasi-stationary measure, it is associated with some subset \mathcal{R} of the configuration space. Then there is an ambiguity when one looks at property (b): what is “getting out” of the metastable state? One is tempted to say that it corresponds in our model to the exit of \mathcal{R} . But doing so we are very unlikely to modelize in any satisfactory way property (c): we can expect that “on the edge”, when the system just exited \mathcal{R} , it has probabilities of the same order to “proceed forward” and thermalize in $\mathcal{X} \setminus \mathcal{R}$ and to go “backward” and thermalize in \mathcal{R} . Then we would like to define what would be a “true escape” from \mathcal{R} . Theorem 2.4 suggests an answer in the regime $\phi_{\mathcal{X} \setminus \mathcal{R}}^* T_{\mathcal{X} \setminus \mathcal{R}}^* \ll 1$. We could define the true escape has the first excursion of length $T_{\mathcal{X} \setminus \mathcal{R}}^*$ inside $\mathcal{X} \setminus \mathcal{R}$. Since time randomization is almost always a good idea, we are led to the following definitions.

For any $A \subset \mathcal{X}$ we call

$$L_A(t) = \int_0^t \mathbb{1}_A(X(s)) ds \quad (38)$$

the local time associated with A up to time t and we denote by G_A the right-continuous inverse of L_A :

$$G_A(t) = \inf\{s \geq 0 : L_A(s) > t\}. \quad (39)$$

For σ_λ an exponential time with mean $1/\lambda$ that is independent from X we define for all x and y in \mathcal{R}

$$p_{\mathcal{R},\lambda}^*(x, y) = \mathbb{P}_x(X(\tau_{\mathcal{R}}^+) = y, L_{\mathcal{X} \setminus \mathcal{R}}(\tau_{\mathcal{R}}^+) \leq \sigma_\lambda) \quad (40)$$

with $\tau_{\mathcal{R}}^+$ the return time in \mathcal{R} after the first clock ring τ , i.e., $\tau_{\mathcal{R}}^+ = \tau + \tau_{\mathcal{R}} \circ \theta_\tau$ with θ the usual shift operator. (One can see the process X as a process updated according to its discrete version with transition probability matrix p at each ring of a Poissonian clock with intensity 1.) We also define, for all x in \mathcal{R} ,

$$e_{\mathcal{R},\lambda}(x) = \mathbb{P}_x(L_{\mathcal{X} \setminus \mathcal{R}}(\tau_{\mathcal{R}}^+) > \sigma_\lambda) = 1 - \sum_{y \in \mathcal{R}} p_{\mathcal{R},\lambda}^*(x, y) \quad (41)$$

and for all x and y in \mathcal{R}

$$p_{\mathcal{R},\lambda}(x, y) = \begin{cases} p_{\mathcal{R},\lambda}^*(x, y) & \text{if } x \neq y, \\ p_{\mathcal{R},\lambda}^*(x, x) + e_{\mathcal{R},\lambda}(x) & \text{if } x = y. \end{cases} \quad (42)$$

The Markov process $X_{\mathcal{R},\lambda}$ on \mathcal{R} with generator defined by

$$\mathcal{L}_{\mathcal{R},\lambda} f(x) = \sum_{y \in \mathcal{R}} p_{\mathcal{R},\lambda}(f(y) - f(x)) \quad (43)$$

is reversible with respect to $\mu_{\mathcal{R}}$ and has spectral gap

$$\gamma_{\mathcal{R},\lambda} = \min_{\text{Var}_{\mu_{\mathcal{R}}}(f) \neq 0} \frac{\mathcal{D}_{\mathcal{R},\lambda}(f)}{\text{Var}_{\mu_{\mathcal{R}}}(f)} \quad (44)$$

where

$$\mathcal{D}_{\mathcal{R},\lambda}(f) = \frac{1}{2} \sum_{x,y} c_{\mathcal{R},\lambda}(x,y) (f(x) - f(y))^2 \quad (45)$$

with

$$c_{\mathcal{R},\lambda}(x,y) = \mu_{\mathcal{R}}(x) p_{\mathcal{R},\lambda}(x,y) = p_{\mathcal{R},\lambda}(y,x) \mu_{\mathcal{R}}(y). \quad (46)$$

In addition we define

$$\tau_{\mathcal{X} \setminus \mathcal{R},\lambda} = L_{\mathcal{R}}(G_{\mathcal{X} \setminus \mathcal{R}}(\sigma_{\lambda})). \quad (47)$$

We know by the Perron-Frobenius theorem that the spectral radius of $p_{\mathcal{R},\lambda}^*$ is a simple positive eigenvalue that is smaller than or equal to 1 and has left and right eigenvectors with positive coordinates. We call it $1 - \phi_{\mathcal{R},\lambda}^*$ and denote by $\mu_{\mathcal{R},\lambda}^*$ the unique associated left eigenvector that is also a probability measure on \mathcal{R} .

Lemma 2.13. *It holds*

- i) $\phi_{\mathcal{R},\lambda}^* = \mu_{\mathcal{R},\lambda}^*(e_{\mathcal{R},\lambda})$;
- ii) $\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_{\mathcal{X} \setminus \mathcal{R},\lambda} > t) = e^{-t\phi_{\mathcal{R},\lambda}^*}, \quad \forall t \geq 0$;
- iii) $\lim_{t \rightarrow \infty} \mathbb{P}_x(X \circ G_{\mathcal{R}}(t) = y \mid \tau_{\mathcal{X} \setminus \mathcal{R},\lambda} > t) = \mu_{\mathcal{R},\lambda}^*, \quad \forall x, y \in \mathcal{R}$.

Proof. See Section 6.1. □

We say that $\mu_{\mathcal{R},\lambda}^*$ is a quasi-stationary measure associated with a soft barrier, or a soft quasi-stationary measure, or, more simply, a *soft measure*. Indeed, $\mu_{\mathcal{R},\lambda}^*$ is the limiting distribution of the process conditioned to survival when it is killed at rate λ outside \mathcal{R} . So, the hardest quasi-stationary measure associated with \mathcal{R} , corresponding to $\lambda = +\infty$, is the quasi-stationary measure $\mu_{\mathcal{R}}^*$, while the softest measure, corresponding to $\lambda = 0$, is the restricted ensemble $\mu_{\mathcal{R}}$ ($\phi_{\mathcal{R},0}^* = 0$ and $\mu_{\mathcal{R},0}^*$ is the equilibrium measure associated with $p_{\mathcal{R},0}^* = p_{\mathcal{R},0}$, which is reversible with respect to $\mu_{\mathcal{R}}$). More precisely we have the following.

Lemma 2.14. *The function $\lambda \in [0, +\infty] \mapsto \mu_{\mathcal{R},\lambda}^* \in \ell^2(\mu_{\mathcal{R}})^*$ is a continuous interpolation between the restricted ensemble $\mu_{\mathcal{R}}$ and the quasi-stationary distribution $\mu_{\mathcal{R}}^*$. In particular, for any $\lambda_0 \in [0, +\infty]$ and $y \in \mathcal{R}$, we have*

$$\lim_{\lambda \rightarrow \lambda_0} \mu_{\mathcal{R},\lambda}(y) = \mu_{\mathcal{R},\lambda_0}(y) \quad (48)$$

and for all $x \in \mathcal{R}$ it holds the commutative limite property

$$\lim_{\lambda \rightarrow \lambda_0} \lim_{t \rightarrow \infty} \mathbb{P}_x(X \circ G_{\mathcal{R}}(t) = y \mid \tau_{\mathcal{X} \setminus \mathcal{R},\lambda} > t) = \lim_{t \rightarrow \infty} \lim_{\lambda \rightarrow \lambda_0} \mathbb{P}_x(X \circ G_{\mathcal{R}}(t) = y \mid \tau_{\mathcal{X} \setminus \mathcal{R},\lambda} > t). \quad (49)$$

Proof. See Section 6.2. □

Analogously to what was done in the case $\lambda = +\infty$ we set $\varepsilon_{\mathcal{R},\lambda}^* = \phi_{\mathcal{R},\lambda}^*/\gamma_{\mathcal{R},\lambda}$, $h_{\mathcal{R},\lambda}^* = \mu_{\mathcal{R},\lambda}^*/\mu_{\mathcal{R}}$ and we call $\gamma_{\mathcal{R},\lambda}^*$ the gap between the largest and the second eigenvalue of $p_{\mathcal{R},\lambda}^*$ (since $p_{\mathcal{R},\lambda}^*$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ it has only real eigenvalues). We also define $\phi_{\mathcal{R},\lambda} = \mu_{\mathcal{R}}(e_{\mathcal{R},\lambda})$.

Proposition 2.15. *The parameters $\gamma_{\mathcal{R},\lambda}$, $\phi_{\mathcal{R},\lambda}^*$, $\varepsilon_{\mathcal{R},\lambda}^*$ and $\phi_{\mathcal{R},\lambda}$ depend continuously on λ . In addition, when λ decreases to 0, so do $\phi_{\mathcal{R},\lambda}^*$, $\varepsilon_{\mathcal{R},\lambda}^*$ and $\phi_{\mathcal{R},\lambda}$, while $\gamma_{\mathcal{R},\lambda}$ increases.*

Proof. See Section 6.3. □

The proofs of Sections 3 carry over this more general setup, and we get the following theorem.

Theorem 2.16 (Mixing towards soft measures). *For all $\lambda \geq 0$, $\phi_{\mathcal{R},\lambda}^* \leq \phi_{\mathcal{R}}^*$, $\gamma_{\mathcal{R},\lambda} \geq \gamma_{\mathcal{R}}$ and $\varepsilon_{\mathcal{R},\lambda}^* \leq \varepsilon_{\mathcal{R}}^*$, Proposition 2.1, Proposition 2.3, Theorem 2.4 and Lemma 2.5 hold with an extra index λ and writing $X \circ G_{\mathcal{R}}$ instead of X .*

Remark. By continuity and monotonicity, the hypothesis $\varepsilon_{\mathcal{R},\lambda}^* < 1$ and $\varepsilon_{\mathcal{R},\lambda}^* < 1/3$ are always satisfied for λ small enough.

We are now ready to deal with local thermalization. We define inductively, for $\kappa, \lambda \geq 0$, the stopping times τ_i for $i \geq 0$:

$$\tau_0 = 0, \quad (50)$$

$$\tau_1 = G_{\mathcal{R}}(\sigma_{\kappa}) \wedge G_{\mathcal{X}\setminus\mathcal{R}}(\sigma_{\lambda}), \quad (51)$$

$$\tau_{i+1} = \tau_i + \tau_1 \circ \theta_{\tau_i} \quad (52)$$

Then for $\delta \in (0, 1)$ we call i_0 the smallest $i \geq 1$ such that one of the two following conditions holds,

$$i) \quad X(\tau_i) \in \mathcal{R} \text{ and } L_{\mathcal{R}}(\tau_i) - L_{\mathcal{R}}(\tau_{i-1}) > T_{\delta, \mathcal{R}, \lambda}^*, \quad (53)$$

$$ii) \quad X(\tau_i) \notin \mathcal{R} \text{ and } L_{\mathcal{X}\setminus\mathcal{R}}(\tau_i) - L_{\mathcal{X}\setminus\mathcal{R}}(\tau_{i-1}) > T_{\delta, \mathcal{X}\setminus\mathcal{R}, \lambda}^*, \quad (54)$$

and we set $\tau_{\delta} = \tau_{i_0}$.

Theorem 2.17 (Local thermalization). *For any $\delta \in]0, 1[$ and any probability measure ν on \mathcal{X} , if $\varepsilon_{\mathcal{R},\lambda}^* < 1/3$ and $\varepsilon_{\mathcal{X}\setminus\mathcal{R},\kappa}^* < 1/3$, then $\forall x \in \mathcal{X}$,*

$$\max \left(\left| \frac{\mathbb{P}_{\nu}(X(\tau_i) = x \mid X(\tau_{\delta}) \in \mathcal{R})}{\mu_{\mathcal{R},\lambda}^*(x)} - 1 \right|, \left| \frac{\mathbb{P}_{\nu}(X(\tau_i) = x \mid X(\tau_{\delta}) \notin \mathcal{R})}{\mu_{\mathcal{X}\setminus\mathcal{R},\kappa}^*(x)} - 1 \right| \right) < \delta. \quad (55)$$

Moreover if $\xi = \max \left(e^{\kappa T_{\delta, \mathcal{R}, \lambda}^*} - 1, e^{\lambda T_{\delta, \mathcal{X}\setminus\mathcal{R}, \lambda}^*} - 1 \right) < 1$, it holds

$$\mathbb{P}_{\nu} \left(T - \delta > t \left(\frac{1}{\kappa} + \frac{1}{\lambda} \right) \right) \leq e^{-t} \left\{ \frac{1}{1 - \xi} \right\}. \quad (56)$$

Proof. See Section 6.4. □

Remark. For κ and λ small enough, we have $\varepsilon_{\mathcal{R},\lambda}^* < 1/3$ and $\varepsilon_{\mathcal{X}\setminus\mathcal{R},\kappa}^* < 1/3$. Then, when κ and λ decrease to 0, we have non increasing upper bounds on $T_{\delta, \mathcal{R}, \lambda}^*$ and $T_{\delta, \mathcal{X}\setminus\mathcal{R}, \lambda}^*$. As a consequence, the condition $\xi < 1$ will always be satisfied for κ and λ small enough and the theorem says that starting from any configuration the system is close to a random mixture of two states ($\mu_{\mathcal{R},\lambda}^*$ and $\mu_{\mathcal{X}\setminus\mathcal{R},\kappa}^*$, close to $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{X}\setminus\mathcal{R}}$ respectively) after a time of order $T_{\delta, \mathcal{R}, \lambda}^* + T_{\delta, \mathcal{X}\setminus\mathcal{R}, \lambda}^*$.

As previously we make a special choice for the parameter δ in previous theorems. We set

$$\delta_{\mathcal{R},\lambda}^* = \zeta_{\mathcal{R},\lambda}^* \times \min_{x \in \partial_{-}\mathcal{R}} e_{\mathcal{R}}(x), \quad T_{\mathcal{R},\lambda}^* = T_{\delta_{\mathcal{R},\lambda}^*, \mathcal{R}, \lambda}^*. \quad (57)$$

Note that here we use $\min_{x \in \partial_{-}\mathcal{R}} e_{\mathcal{R}}(x)$ and not $\min_{x \in \partial_{-}\mathcal{R}} e_{\mathcal{R},\lambda}(x)$ for the sake of concreteness and practical estimates : there is no simple way to give lower bounds on $e_{\mathcal{R},\lambda}(x)$ while $\min_{x \in \partial_{-}\mathcal{R}} e_{\mathcal{R}}(x)$ is usually an easy to control quantity. In doing so we have no straight equivalent of Lemma 2.6. We only have the following result.

Lemma 2.18. *If $\varepsilon_{\mathcal{R},\lambda}^* < 1$, then*

$$\delta_{\mathcal{R},\lambda}^* \leq \phi_{\mathcal{R}}^* \quad \text{and} \quad \phi_{\mathcal{R},\lambda}^* \leq \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* \leq 2\varepsilon_{\mathcal{R},\lambda}^* \left(\ln \frac{\sqrt{2}}{\delta_{\mathcal{R},\lambda}^* (1 - \delta_{\mathcal{R},\lambda}^*)} \right) \left\{ 1 + \sqrt{\frac{\varepsilon_{\mathcal{R},\lambda}^*}{1 - \varepsilon_{\mathcal{R},\lambda}^*}} \right\} \quad (58)$$

Proof. The proof runs similarly to that of Lemma 2.6 and we omit it. □

Now the proofs Section 4 carry over this more general setup and we get asymptotic exponential laws for the transition time $\tau_{\mathcal{X}\setminus\mathcal{R},\lambda}$ that is the time spent in \mathcal{R} before “truly escaping” from \mathcal{R} as indicated by Theorem 2.21 below.

Theorem 2.19 (Asymptotic transition law). *For all $\lambda \geq 0$, $\phi_{\mathcal{R},\lambda} \leq \phi_{\mathcal{R}}$ and Theorem 2.7 together with its Corollary hold with an extra index λ .*

We can also give sharp estimates on the mean transition time and asymptotics of the mixing time.

Theorem 2.20 (Mean transition time estimates). *If $\varepsilon_{\mathcal{R},\lambda}^* < 1/3$, then, for all $\kappa, \lambda > 0$ and $T > \frac{1}{\kappa} + \frac{1}{\lambda}$, and setting $\phi_{\kappa}^{\lambda} = \phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})$, it holds*

$$\begin{cases} \phi_{\mathcal{R},\lambda}^* \geq \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})}{\mu(\mathcal{R})} \left\{ \frac{(\alpha_1\alpha_2) - \mu(\mathcal{R})/(\alpha_1\alpha_2)}{1 - \mu(\mathcal{R})} \alpha_4 \left(1 - \max\left(\frac{\kappa + \phi_{\kappa}^{\lambda}}{\gamma_{\mathcal{R}}}, \frac{\lambda + \phi_{\kappa}^{\lambda}}{\gamma_{\mathcal{X}\setminus\mathcal{R}}}\right) \right) \right\} \\ \phi_{\mathcal{R},\lambda}^* \leq \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X}\setminus\mathcal{R})}{\mu(\mathcal{R})} \left\{ 1 + \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* + \delta_{\mathcal{R},\lambda}^* + \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} \right\} \end{cases}, \quad (59)$$

where

$$\begin{aligned} \alpha_1 &= 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} - e^{-T\left(\frac{1}{\kappa} + \frac{1}{\lambda}\right)^{-1}} \\ \alpha_2 &= e^{-\phi_{\mathcal{R},\lambda}^* T} - \frac{1 - \alpha_1}{\alpha_1} \\ \alpha_3 &= e^{-\phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^*} - \frac{1 - \alpha_1}{\alpha_1} \\ \alpha_4 &= \alpha_3 \left(1 + \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* - \delta_{\mathcal{R},\lambda}^* + \frac{\phi_{\mathcal{R},\lambda}^*}{\lambda} \right) \end{aligned}$$

Proof. See Section 6.5. □

Remark. In the regime $\varepsilon_{\mathcal{R}}^* + \varepsilon_{\mathcal{X}\setminus\mathcal{R}}^* + \phi_{\mathcal{R}}^*/\gamma_{\mathcal{X}\setminus\mathcal{R}} \ll 1$ and assuming $\mu(\mathcal{R}) \leq \mu(\mathcal{X}\setminus\mathcal{R})$ one can choose κ and λ in such a way that $\phi_{\mathcal{R}}^* \ll \kappa \ll \gamma_{\mathcal{R}}$ and $\phi_{\mathcal{R}}^*, \phi_{\mathcal{X}\setminus\mathcal{R}}^* \ll \lambda \ll \gamma_{\mathcal{X}\setminus\mathcal{R}}$, choose T such that

$$\frac{1}{\kappa} + \frac{1}{\lambda} \ll T \ll \frac{1}{\phi_{\mathcal{R},\lambda}^*} \quad (60)$$

and then get matching bounds provided $\phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* \ll 1$. Once again, all the relevant quantities can be estimated via a two-sided variational principle.

Theorem 2.21 (Mixing time asymptotics). *If $\varepsilon_{\mathcal{X}\setminus\mathcal{R},0}^*, \varepsilon_{\mathcal{R},\lambda}^* < 1/3$, we define $\mathcal{T} = G_{\mathcal{X}\setminus\mathcal{R}}(\sigma_{\lambda})$ and, for any x in \mathcal{X} we write $\nu_x = \mathbb{P}_x(X(\mathcal{T}) = \cdot)$. Then,*

$$\|\nu_x - \mu_{\mathcal{X}\setminus\mathcal{R}}\|_{TV} \leq \lambda T_{\mathcal{X}\setminus\mathcal{R},0}^* + \frac{1}{2} \delta_{\mathcal{X}\setminus\mathcal{R},0}^*, \quad (61)$$

$$\|\nu_x - \mu\|_{TV} \leq \mu(\mathcal{R}) + \lambda T_{\mathcal{X}\setminus\mathcal{R},0}^* + \frac{1}{2} \delta_{\mathcal{X}\setminus\mathcal{R},0}^*. \quad (62)$$

In addition, if

$$\eta = \mu(\mathcal{R}) + \lambda T_{\mathcal{X}\setminus\mathcal{R},0}^* + \frac{1}{2} \left(\delta_{\mathcal{X}\setminus\mathcal{R},0}^* + \delta_{\mathcal{R},\lambda}^* + \frac{\phi_{\mathcal{R},\lambda}^*}{\lambda} + \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* \right) < 1, \quad (63)$$

then, with

$$\tau_{mix} = \max_{x \in \mathcal{X}} \inf \left\{ t \geq 0 : \|\mathbb{P}_x(X(t) = \cdot) - \mu\|_{TV} \leq \frac{1 + \eta}{2} \right\} \quad (64)$$

we have

$$\tau_{mix} \leq \frac{2}{(1 - \eta)\phi_{\mathcal{R},\lambda}^*}. \quad (65)$$

Proof. See Section 6.6. □

Remark. The theorem makes sense when $\mu(\mathcal{R}) \leq \mu(\mathcal{X} \setminus \mathcal{R})$ and in the regime $\varepsilon_{\mathcal{R}}^* + \varepsilon_{\mathcal{X} \setminus \mathcal{R}}^* + \phi_{\mathcal{R}}^* T_{\mathcal{X} \setminus \mathcal{R}, 0}^* \ll 1$. One can then choose λ such that $\phi_{\mathcal{X} \setminus \mathcal{R}}^* \ll \lambda \ll T_{\mathcal{X} \setminus \mathcal{R}, 0}^*$, and, provided $\phi_{\mathcal{R}, \lambda}^* T_{\mathcal{R}, \lambda}^* \ll 1$, our parameter η can be as close to $\mu(\mathcal{R}) \leq 1/2$ as we want. Then, since the spectral gap goes like $\phi_{\mathcal{R}, \lambda}^* / \mu(\mathcal{X} \setminus \mathcal{R})$ and $\mu(\mathcal{X} \setminus \mathcal{R}) \geq 1/2$ the theorem provides the correct order for the mixing time.

Let us finally summarize our results. To have a mathematical model of the metastability phenomenon described by properties (a)-(c), we describe metastable states by soft measures associated with a subset \mathcal{R} of a finite configuration space \mathcal{X} where runs a reversible Markov chain with respect to some probability measure μ in the regime $\varepsilon_{\mathcal{R}}^* + \varepsilon_{\mathcal{X} \setminus \mathcal{R}}^* + \phi_{\mathcal{R}}^* / \gamma_{\mathcal{X} \setminus \mathcal{R}} \ll 1$ and such that $\mu(\mathcal{R}) \leq \mu(\mathcal{X} \setminus \mathcal{R})$. In this regime all soft measures are close to the restricted ensemble (Theorem 2.16). If we choose κ and λ such that $\phi_{\mathcal{R}}^* \ll \kappa \ll \gamma_{\mathcal{R}}$ and $\phi_{\mathcal{R}}^*, \phi_{\mathcal{X} \setminus \mathcal{R}}^* \ll \lambda \ll \gamma_{\mathcal{X} \setminus \mathcal{R}}$ then we can show

- i) local thermalization towards the soft measure $\mu_{\mathcal{R}, \lambda}$ or $\mu_{\mathcal{X} \setminus \mathcal{R}, \kappa}$ starting from any configuration in \mathcal{X} and on a *short time scale* $\frac{1}{\kappa} + \frac{1}{\lambda}$ (Theorem 2.17),
- ii) exponential asymptotic transition time on a *long time scale* $\frac{1}{\phi_{\mathcal{R}, \lambda}^*} \sim \frac{\mu(\mathcal{R})}{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}$ (Theorems 2.19 and 2.20),
- iii) return time to metastable state on a *still longer time scale* $\frac{1}{\phi_{\mathcal{X} \setminus \mathcal{R}, \kappa}^*} \sim \frac{\mu(\mathcal{X} \setminus \mathcal{R})}{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}$ (Theorem 2.20 applied to $\mathcal{X} \setminus \mathcal{R}$ in place of \mathcal{R}).

In addition relaxation and mixing times are of the same order as the mean transition time (Theorems 2.12 and 2.21) - in particular the relaxation time has the same exact asymptotic up to a factor $\mu(\mathcal{X} \setminus \mathcal{R})$ - while exit times are on long, but generally shorter, time scale (Theorem 2.11). And we note once again, that all relevant quantities can be estimated via a two-sided variational principles.

3. ANALYSIS IN $\ell^2(\mu_{\mathcal{R}})$

3.1. Proof of Proposition 2.1. We recall that the dynamics on \mathcal{R} with reflecting barrier on $\partial_- \mathcal{R}$, is reversible w.r.t. $\mu_{\mathcal{R}}$ with spectral gap $\gamma_{\mathcal{R}}$. In particular, for any function $f \in \ell^2(\mu_{\mathcal{R}})$, it holds the Poincaré inequality $\text{Var}_{\mu_{\mathcal{R}}}(f) \leq \frac{1}{\gamma_{\mathcal{R}}} \mathcal{D}_{\mathcal{R}}(f)$, where $\mathcal{D}_{\mathcal{R}}(f)$ is the Dirichlet form of f given by

$$\mathcal{D}_{\mathcal{R}}(f) = \langle f, -\mathcal{L}_{\mathcal{R}} f \rangle_{\mu} = \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}(x) f(x) (\delta_x(y) - p_{\mathcal{R}}(x, y)) f(y). \quad (66)$$

Applying the Poincaré inequality to $h_{\mathcal{R}}^*$, we get

$$\begin{aligned} \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) &\leq \frac{1}{\gamma_{\mathcal{R}}} \mathcal{D}_{\mathcal{R}}(h_{\mathcal{R}}^*) = \frac{1}{\gamma_{\mathcal{R}}} \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}(x) h_{\mathcal{R}}^*(x) (\delta_x(y) - p_{\mathcal{R}}(x, y)) h_{\mathcal{R}}^*(y) \\ &= \frac{1}{\gamma_{\mathcal{R}}} \left(\mu_{\mathcal{R}}(h_{\mathcal{R}}^*)^2 - \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}^*(x) p_{\mathcal{R}}(x, y) h_{\mathcal{R}}^*(y) \right) \\ &= \frac{1}{\gamma_{\mathcal{R}}} \left(\mu_{\mathcal{R}}(h_{\mathcal{R}}^*)^2 - \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}^*(x) (p(x, y) + \delta_x(y) e_{\mathcal{R}}(x)) h_{\mathcal{R}}^*(y) \right) \\ &\leq \frac{1}{\gamma_{\mathcal{R}}} \left(\mu_{\mathcal{R}}(h_{\mathcal{R}}^*)^2 - \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}^*(x) p_{\mathcal{R}}^*(x, y) h_{\mathcal{R}}^*(y) \right) \end{aligned} \quad (67)$$

where in the first two lines we use that, by definition of $h_{\mathcal{R}}^*$,

$$\mu_{\mathcal{R}}(x) h_{\mathcal{R}}^*(x) = \mu_{\mathcal{R}}^*(x) \quad \text{and} \quad \mu_{\mathcal{R}}^*(h_{\mathcal{R}}^*) = \mu_{\mathcal{R}}(h_{\mathcal{R}}^*)^2,$$

and then we exploit the definition of $p_{\mathcal{R}}$ and $p_{\mathcal{R}}^*$. From the last line, using that $\mu_{\mathcal{R}}^*$ is a left eigenvector of $p_{\mathcal{R}}^*$ with eigenvalue $(1 - \phi_{\mathcal{R}}^*)$, we get

$$\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) \leq \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} \mu_{\mathcal{R}}(h_{\mathcal{R}}^{*2}) = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) + \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}}. \quad (68)$$

Finally, rearranging the terms in the above inequality and from the hypothesis $\varepsilon_{\mathcal{R}}^* = \frac{\phi_{\mathcal{R}}^*}{\gamma_{\mathcal{R}}} < 1$, we obtain the required upper bound.

3.2. Proof of Lemma 2.2. Let us denote by $\mathcal{L}_{\mathcal{R}}^*$ the sub-Markovian generator associated to the kernel $p_{\mathcal{R}}^*$. For any function $f \in \ell^2(\mu_{\mathcal{R}})$, this is defined as

$$(\mathcal{L}_{\mathcal{R}}^* f)(x) = -f(x) + \sum_{y \in \mathcal{R}} p_{\mathcal{R}}^*(x, y) f(y). \quad (69)$$

We have

$$\phi_{\mathcal{R}}^* = \min_{\substack{f: \mathcal{R} \rightarrow \mathcal{R} \\ f \neq 0}} \frac{\langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}}}{\langle f, f \rangle_{\mathcal{R}}}. \quad (70)$$

Taking $f = \mathbb{1}_{\mathcal{R}}$ as test function, we get

$$\begin{aligned} \phi_{\mathcal{R}}^* &\leq \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(1 - \sum_{y \in \mathcal{R}} p_{\mathcal{R}}^*(x, y) \right) \\ &= \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(1 - \sum_{y \in \mathcal{R}} p(x, y) \right) \\ &= \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) e_{\mathcal{R}}(x) = \phi_{\mathcal{R}}, \end{aligned} \quad (71)$$

which concludes the proof.

3.3. Proof of Proposition 2.3. The second smallest eigenvalue of the sub-Markovian generator $\mathcal{L}_{\mathcal{R}}^*$, $\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*$, satisfies the variational formula

$$\begin{aligned} \phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^* &= \min \left\{ \frac{\langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}}}{\langle f, f \rangle_{\mathcal{R}}} : f \neq 0, \langle f, h_{\mathcal{R}}^* \rangle_{\mathcal{R}} = 0 \right\} \\ &= \min \left\{ \langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}} : \langle f, h_{\mathcal{R}}^* \rangle_{\mathcal{R}} = 0, \langle f, f \rangle_{\mathcal{R}} = 1 \right\} \end{aligned} \quad (72)$$

Let f be a function on \mathcal{R} that realizes the minimum in the above definition, with $\langle f, f \rangle_{\mathcal{R}} = 1$. Since $\langle f, h_{\mathcal{R}}^* \rangle_{\mathcal{R}} = 0$, we have

$$\langle f, h_{\mathcal{R}}^* - \mathbb{1}_{\mathcal{R}} \rangle_{\mathcal{R}} = -\langle f, \mathbb{1}_{\mathcal{R}} \rangle_{\mathcal{R}} = -\mu_{\mathcal{R}}(f)$$

and then, by the Cauchy-Schwartz inequality together with Proposition 2.1,

$$\mu_{\mathcal{R}}^2(f) \leq \|f\|_{\mathcal{R}}^2 \cdot \|h_{\mathcal{R}}^* - \mathbb{1}_{\mathcal{R}}\|_{\mathcal{R}}^2 \leq \frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*}. \quad (73)$$

Now, writing the orthogonal decomposition $f = \mu_{\mathcal{R}}(f) + g$, with $\mu_{\mathcal{R}}(g) = 0$, we have

$$1 = \|f\|_{\mathcal{R}}^2 = \mu_{\mathcal{R}}^2(f) + \|g\|_{\mathcal{R}}^2$$

and thus, from (73),

$$\|g\|_{\mathcal{R}}^2 = 1 - \mu_{\mathcal{R}}^2(f) \geq 1 - \frac{\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*} = \frac{1 - 2\varepsilon_{\mathcal{R}}^*}{1 - \varepsilon_{\mathcal{R}}^*}.$$

Using g as a test function in

$$\gamma_{\mathcal{R}} = \min \left\{ \frac{\mathcal{D}_{\mathcal{R}}(h)}{\|h\|_{\mathcal{R}}^2} : h \neq 0, \mu_{\mathcal{R}}(h) = 0 \right\}, \quad (74)$$

we get

$$\gamma_{\mathcal{R}} \leq \frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 2\varepsilon_{\mathcal{R}}^*} \mathcal{D}_{\mathcal{R}}(g) = \frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 2\varepsilon_{\mathcal{R}}^*} \mathcal{D}_{\mathcal{R}}(f). \quad (75)$$

We will conclude the proof using the following result.

Lemma 3.1. *For all $f \in \ell^2(\mu_{\mathcal{R}})$, it holds*

$$\mathcal{D}_{\mathcal{R}}(f) \leq \frac{\mathcal{D}(f)}{\mu(\mathcal{R})} = \langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}}. \quad (76)$$

Proof of Lemma 3.1. For all $x, y \in \mathcal{R}$ with $x \neq y$, $p_{\mathcal{R}}(x, y) = p(x, y)$. Then we have

$$\begin{aligned} \mathcal{D}_{\mathcal{R}}(f) &= \frac{1}{2} \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}(x) p_{\mathcal{R}}(x, y) [f(x) - f(y)]^2 \\ &= \frac{1}{2} \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}(x) p(x, y) [f(x) - f(y)]^2, \end{aligned} \quad (77)$$

since only the terms in $x \neq y$ matter in this sum. Thus

$$\mathcal{D}_{\mathcal{R}}(f) \leq \frac{1}{2} \sum_{x, y \in \mathcal{X}} \mu_{\mathcal{R}}(x) p(x, y) [f(x) - f(y)]^2 \leq \frac{\mathcal{D}(f)}{\mu(\mathcal{R})}, \quad (78)$$

and this provides the stated upper bound.

To prove the equality, we recall that the space $\ell^2(\mu_{\mathcal{R}})$ is identified with the subset of functions $f \in \ell^2(\mu)$ with $f|_{\mathcal{X} \setminus \mathcal{R}} \equiv 0$. Since, for all $x, y \in \mathcal{R}$, it holds that $\mu_{\mathcal{R}}(x) = \mu(x)/\mu(\mathcal{R})$ and $p_{\mathcal{R}}^*(x, y) = p(x, y)$, we have

$$\begin{aligned} \frac{\mathcal{D}(f)}{\mu(\mathcal{R})} &= \frac{1}{\mu(\mathcal{R})} \sum_{x, y \in \mathcal{X}} \mu(x) f(x) (\delta_x(y) - p(x, y)) f(y) \\ &= \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}(x) f(x) (\delta_x(y) - p_{\mathcal{R}}^*(x, y)) f(y) \\ &= \langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}}, \end{aligned} \quad (79)$$

that concludes the proof of the lemma. \square

Turning back to the proof of the proposition, and combining inequality (75) with Lemma 3.1, we get

$$\gamma_{\mathcal{R}} \leq \frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 2\varepsilon_{\mathcal{R}}^*} \langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}} = \frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 2\varepsilon_{\mathcal{R}}^*} (\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*), \quad (80)$$

since f was chosen in order to have $\langle f, -\mathcal{L}_{\mathcal{R}}^* f \rangle_{\mathcal{R}} = \phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*$. Setting $\phi_{\mathcal{R}}^* = \varepsilon_{\mathcal{R}}^* \gamma_{\mathcal{R}}$ and rearranging the terms in the last inequality, we get

$$\left(\frac{1 - 3\varepsilon_{\mathcal{R}}^* + \varepsilon_{\mathcal{R}}^{*2}}{1 - 2\varepsilon_{\mathcal{R}}^*} \right) \gamma_{\mathcal{R}} \leq \left(\frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 2\varepsilon_{\mathcal{R}}^*} \right) \gamma_{\mathcal{R}},$$

that under the hypothesis $\varepsilon_{\mathcal{R}}^* < 1/3$ implies

$$\frac{1}{\gamma_{\mathcal{R}}^*} \leq \frac{1}{\gamma_{\mathcal{R}}} \left\{ \frac{1 - \varepsilon_{\mathcal{R}}^*}{1 - 3\varepsilon_{\mathcal{R}}^*} \right\}.$$

3.4. Proof of Theorem 2.4. The proof is based on a classical trick to control mixing times with relaxation times. For any probability measure ν on \mathcal{R} , any $f : \mathcal{R} \rightarrow \mathbb{R}$ such that $\mu_{\mathcal{R}}^*(f) \neq 0$ and any $s, t \geq 0$, one can check that

$$\begin{aligned} & \mathbb{E}_{\nu}[f(X(s+t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > s+t\}}] - \mu_{\mathcal{R}}^*(f)\mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s+t) \\ &= \sum_{y \in \mathcal{R}} (\mathbb{P}_{\nu}(X(s) = y, \tau_{\mathcal{X}\setminus\mathcal{R}} > s) - \mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s)\mu_{\mathcal{R}}^*(y)) \\ & \quad \times \left(\mathbb{E}_y[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > t\}}] - \mathbb{P}_y(\tau_{\mathcal{X}\setminus\mathcal{R}} > t)\mu_{\mathcal{R}}^*(f) \right). \end{aligned} \quad (81)$$

Indeed, one can rewrite the right hand side of the above equality as the sum of four terms, two of which coincide with the two terms in the left hand side by the Markov property, while the other two terms cancel using the quasi-stationarity property, i.e.

$$\mathbb{E}_{\mu_{\mathcal{R}}^*} [f | \tau_{\mathcal{X}\setminus\mathcal{R}} > t] = \mu_{\mathcal{R}}^*(f). \quad (82)$$

As a consequence one gets

$$\begin{aligned} & \left| \mathbb{E}_{\nu}[f(X(s+t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > t\}}] - \mu_{\mathcal{R}}^*(f)\mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s+t) \right| \\ &= \left| \sum_{y \in \mathcal{R}} \mu_{\mathcal{R}}(y) \left(\frac{\mathbb{P}_{\nu}(X(s) = y, \tau_{\mathcal{X}\setminus\mathcal{R}} > s)}{\mu_{\mathcal{R}}(y)} - \mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s)h_{\mathcal{R}}^*(y) \right) \right. \\ & \quad \left. \times \left(\mathbb{E}_y[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > t\}}] - \mathbb{P}_y(\tau_{\mathcal{X}\setminus\mathcal{R}} > t)\mu_{\mathcal{R}}^*(f) \right) \right| \end{aligned}$$

and, by the Cauchy-Schwartz inequality

$$\begin{aligned} & \left| \mathbb{E}_{\nu}[f(X(s+t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > t\}}] - \mu_{\mathcal{R}}^*(f)\mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s+t) \right| \\ & \leq \left\| \frac{\mathbb{P}_{\nu}(X(s) = \cdot, \tau_{\mathcal{X}\setminus\mathcal{R}} > s)}{\mu_{\mathcal{R}}(\cdot)} - \mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s)h_{\mathcal{R}}^*(\cdot) \right\|_{\mathcal{R}} \\ & \quad \times \left\| \mathbb{E}_{(\cdot)}[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > t\}}] - \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\setminus\mathcal{R}} > t)\mu_{\mathcal{R}}^*(f) \right\|_{\mathcal{R}}. \end{aligned} \quad (83)$$

We now estimate these two factors. Noting that

$$\mathbb{P}_{\nu}(X(s) = \cdot, \tau_{\mathcal{X}\setminus\mathcal{R}} > s) = \nu e^{s\mathcal{L}_{\mathcal{R}}^*}(\cdot) \quad \text{and} \quad \mathbb{E}_{(\cdot)}[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > t\}}] = e^{t\mathcal{L}_{\mathcal{R}}^*} f(\cdot),$$

and diagonalizing the self-adjoint operator $\mathcal{L}_{\mathcal{R}}^*$ in orthonormal basis, one gets

$$\left\| \frac{\mathbb{P}_{\nu}(X(s) = \cdot, \tau_{\mathcal{X}\setminus\mathcal{R}} > s)}{\mu_{\mathcal{R}}(\cdot)} - \left\| \frac{\nu}{\mu_{\mathcal{R}}} \right\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_{\nu} e^{-\phi_{\mathcal{R}}^* s} \right\|_{\mathcal{R}}^2 \leq \left\| \frac{\nu}{\mu_{\mathcal{R}}} \right\|_{\mathcal{R}}^2 \sin^2 \theta_{\nu} e^{-2s(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}})}, \quad (84)$$

with $\theta_{\nu} \in [0, \pi/2[$ such that $\left\| \frac{\nu}{\mu_{\mathcal{R}}} \right\|_{\mathcal{R}} \|h_{\mathcal{R}}^*\|_{\mathcal{R}} \cos \theta_{\nu} = \langle \frac{\nu}{\mu_{\mathcal{R}}}, h_{\mathcal{R}}^* \rangle = \nu(h_{\mathcal{R}}^*)$, and

$$\left\| \mathbb{E}_{(\cdot)}[f(X(t))\mathbb{1}_{\{\tau_{\mathcal{X}\setminus\mathcal{R}} > t\}}] - \|f\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_f e^{-\phi_{\mathcal{R}}^* t} \right\|_{\mathcal{R}}^2 \leq \|f\|_{\mathcal{R}}^2 \sin^2 \theta_f e^{-2t(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}})} \quad (85)$$

with $\theta_f \in [0, \pi] \setminus \{\pi/2\}$ such that $\|f\|_{\mathcal{R}} \|h_{\mathcal{R}}^*\|_{\mathcal{R}} \cos \theta_f = \langle f, h_{\mathcal{R}}^* \rangle = \mu_{\mathcal{R}}^*(f)$.

Moreover, since

$$\mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s) = \mu_{\mathcal{R}} \left(\frac{\mathbb{P}_{\nu}(X(s) = \cdot, \tau_{\mathcal{X}\setminus\mathcal{R}} > s)}{\mu_{\mathcal{R}}(\cdot)} \right),$$

from inequality (84) we get

$$\begin{aligned} \left| \mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s) - \left\| \frac{\nu}{\mu_{\mathcal{R}}} \right\|_{\mathcal{R}} \frac{\cos \theta_{\nu}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-\phi_{\mathcal{R}}^* s} \right| &= \left| \mu_{\mathcal{R}} \left(\frac{\mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s)}{\mu_{\mathcal{R}}(\cdot)} - \left\| \frac{\nu}{\mu_{\mathcal{R}}} \right\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_{\nu} e^{-\phi_{\mathcal{R}}^* s} \right) \right| \\ &\leq \left\langle \mathbb{1}_{\mathcal{R}}, \left| \frac{\mathbb{P}_{\nu}(\tau_{\mathcal{X}\setminus\mathcal{R}} > s)}{\mu_{\mathcal{R}}(\cdot)} - \left\| \frac{\nu}{\mu_{\mathcal{R}}} \right\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_{\nu} e^{-\phi_{\mathcal{R}}^* s} \right| \right\rangle_{\mathcal{R}} \\ &\leq \left\| \frac{\nu}{\mu_{\mathcal{R}}} \right\|_{\mathcal{R}} \sin \theta_{\nu} e^{-s(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}})}. \end{aligned} \quad (86)$$

Using inequalities (84) and (86), we finally get

$$\begin{aligned}
& \left\| \frac{\mathbb{P}_\nu(X(s)=\cdot, \tau_{\mathcal{X}\mathcal{R}} > s)}{\mu_{\mathcal{R}}(\cdot)} - \mathbb{P}_\nu(\tau_{\mathcal{X}\mathcal{R}} > s) h_{\mathcal{R}}^*(\cdot) \right\|_{\mathcal{R}} \\
& \leq \left\| \frac{\mathbb{P}_\nu(X(s)=\cdot, \tau_{\mathcal{X}\mathcal{R}} > s)}{\mu_{\mathcal{R}}(\cdot)} - \frac{\nu}{\mu_{\mathcal{R}}} \left\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_\nu e^{-\phi_{\mathcal{R}}^* s} \right\|_{\mathcal{R}} \\
& + \left\| \left(\mathbb{P}_\nu(\tau_{\mathcal{X}\mathcal{R}} > s) - \frac{\nu}{\mu_{\mathcal{R}}} \left\|_{\mathcal{R}} \frac{\cos \theta_\nu}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-\phi_{\mathcal{R}}^* s} \right\|_{\mathcal{R}} \right) h_{\mathcal{R}}^* \right\|_{\mathcal{R}} \\
& \leq \frac{\nu}{\mu_{\mathcal{R}}} \left\|_{\mathcal{R}} (1 + \|h_{\mathcal{R}}^*\|_{\mathcal{R}}) \sin \theta_\nu e^{-s(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)}.
\end{aligned} \tag{87}$$

which provides an estimate of the first factor in (83).

To what concerns the second factor, noting that

$$\mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\mathcal{R}} > t) = \mathbb{E}_{(\cdot)}[\mathbb{1}_{\mathcal{R}}(X(t)) \mathbb{1}_{\{\tau_{\mathcal{X}\mathcal{R}} > t\}}]$$

and that, from the definition of $\cos \theta_f$ applied to $f = \mathbb{1}_{\mathcal{R}}$,

$$\cos \theta_{\mathbb{1}_{\mathcal{R}}} = \frac{1}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \quad \text{and} \quad \sin^2 \theta_{\mathbb{1}_{\mathcal{R}}} = \frac{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2 - 1}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2} = \frac{\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2},$$

we get

$$\left\| \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\mathcal{R}} > t) - \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-\phi_{\mathcal{R}}^* t} \right\|_{\mathcal{R}}^2 \leq \frac{\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2} e^{-2t(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)}, \tag{88}$$

and from inequalities (85) and (88),

$$\begin{aligned}
& \left\| \mathbb{E}_{(\cdot)}[f(X(t)) \mathbb{1}_{\{\tau_{\mathcal{X}\mathcal{R}} > t\}}] - \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\mathcal{R}} > t) \mu_{\mathcal{R}}^*(f) \right\|_{\mathcal{R}} \\
& \leq \left\| \mathbb{E}_{(\cdot)}[f(X(t)) \mathbb{1}_{\{\tau_{\mathcal{X}\mathcal{R}} > t\}}] - \|f\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_f e^{-\phi_{\mathcal{R}}^* t} \right\|_{\mathcal{R}} \\
& + \left\| \mathbb{P}_{(\cdot)}(\tau_{\mathcal{X}\mathcal{R}} > t) \|f\|_{\mathcal{R}} \|h_{\mathcal{R}}^*\|_{\mathcal{R}} \cos \theta_f - \|f\|_{\mathcal{R}} \frac{h_{\mathcal{R}}^*}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_f e^{-\phi_{\mathcal{R}}^* t} \right\|_{\mathcal{R}} \\
& \leq \|f\|_{\mathcal{R}} \sin \theta_f e^{-t(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)} + \|f\|_{\mathcal{R}} \|h_{\mathcal{R}}^*\|_{\mathcal{R}} \frac{\sqrt{\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \cos \theta_f e^{-t(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)} \\
& = \left(\|f\|_{\mathcal{R}} \sin \theta_f + \frac{\mu_{\mathcal{R}}^*(f) \sqrt{\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \right) e^{-t(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)}.
\end{aligned} \tag{89}$$

which provides an estimate of the second factor in (83).

Inserting (87) and (89) in Eq. (83), we then obtain

$$\begin{aligned}
& \left| \mathbb{E}_\nu[f(X(s+t)) \mathbb{1}_{\{\tau_{\mathcal{X}\mathcal{R}} > t\}}] - \mu_{\mathcal{R}}^*(f) \mathbb{P}_\nu(\tau_{\mathcal{X}\mathcal{R}} > s+t) \right| \\
& \leq \frac{\nu}{\mu_{\mathcal{R}}} \left\|_{\mathcal{R}} (1 + \|h_{\mathcal{R}}^*\|_{\mathcal{R}}) \sin \theta_\nu \left(\|f\|_{\mathcal{R}} \sin \theta_f + \frac{\mu_{\mathcal{R}}^*(f) \sqrt{\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \right) e^{-(s+t)(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)}.
\end{aligned} \tag{90}$$

To conclude our proof we will make two more steps. First notice that from (86) one also get that, for any $t \geq 0$,

$$\mathbb{P}_\nu(\tau_{\mathcal{X}\mathcal{R}} > t) \geq \frac{\nu}{\mu_{\mathcal{R}}} \left\|_{\mathcal{R}} \left(\frac{\cos \theta_\nu}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-t\phi_{\mathcal{R}}^*} - \sin \theta_\nu e^{-t(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)} \right).$$

In particular, as soon as the following condition is verified

$$\sin \theta_\nu e^{-t(\phi_{\mathcal{R}}^* + \gamma_{\mathcal{R}}^*)} \leq \delta \frac{\cos \theta_\nu}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-\phi_{\mathcal{R}}^* t} \iff \|h_{\mathcal{R}}^*\|_{\mathcal{R}} \tan \theta_\nu e^{-\phi_{\mathcal{R}}^* t} \leq \delta, \tag{91}$$

it holds

$$\mathbb{P}_\nu(\tau_{\mathcal{X}\mathcal{R}} > t) \geq (1 - \delta) \frac{\nu}{\mu_{\mathcal{R}}} \left\|_{\mathcal{R}} \frac{\cos \theta_\nu}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} e^{-\phi_{\mathcal{R}}^* t}. \tag{92}$$

Now, dividing both terms of (90) by $\mu_{\mathcal{R}}^*(f)\mathbb{P}_\nu(\tau_{\mathcal{X}\setminus\mathcal{R}} > s+t)$, we reach an inequality that controls the Yaglom limit and that, provided condition (91) and from the last inequality, reads as

$$\left| \frac{\mathbb{E}_\nu[f(X(t)) | \tau_{\mathcal{X}\setminus\mathcal{R}} > t]}{\mu_{\mathcal{R}}^*(f)} - 1 \right| \leq \frac{1 + \|h_{\mathcal{R}}^*\|_{\mathcal{R}}}{1-\delta} \tan \theta_\nu \left(\tan \theta_f + \sqrt{\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)} \right) e^{-\gamma_{\mathcal{R}}^* t}. \quad (93)$$

As a final step we apply this inequality to $\nu = \delta_x$ and $f = \delta_y$. For this choice of ν and f , and by definition of θ_ν and θ_f , one has

$$\tan \theta_\nu \leq \frac{1}{\cos \theta_\nu} = \frac{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}}{\sqrt{\mu_{\mathcal{R}}(x)h_{\mathcal{R}}^*(x)}} \quad \text{and} \quad \tan \theta_f \leq \frac{1}{\cos \theta_f} = \frac{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}}{\sqrt{\mu_{\mathcal{R}}(y)h_{\mathcal{R}}^*(y)}},$$

Thus, from (93), we obtain that under condition (91)

$$\begin{aligned} \left| \frac{\mathbb{P}_x((X(t)=y) | \tau_{\mathcal{X}\setminus\mathcal{R}} > t)}{\mu_{\mathcal{R}}^*(y)} - 1 \right| &\leq e^{-\gamma_{\mathcal{R}}^* t} \frac{(1 + \|h_{\mathcal{R}}^*\|_{\mathcal{R}})\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2}{(1-\delta)\sqrt{\mu_{\mathcal{R}}(x)h_{\mathcal{R}}^*(x)}} \left(\frac{1}{\sqrt{\mu_{\mathcal{R}}(y)h_{\mathcal{R}}^*(y)}} + \frac{\sqrt{\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)}}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}} \right) \\ &\leq e^{-\gamma_{\mathcal{R}}^* t} \frac{1}{1-\delta} \left(1 + \sqrt{1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right) \left(1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*} \right) \frac{1}{\sqrt{\zeta_{\mathcal{R}}^*}} \left(\frac{1}{\sqrt{\zeta_{\mathcal{R}}^*}} + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right), \end{aligned} \quad (94)$$

where in the second line we used that $\|h_{\mathcal{R}}^*\|_{\mathcal{R}} \geq 1$, the estimate given in Proposition 2.1, and we introduced the quantity $\zeta_{\mathcal{R}}^*$ defined in (19).

The right hand side of the last inequality is smaller than δ as soon as

$$t \geq \frac{1}{\gamma_{\mathcal{R}}^*} \left[\ln \frac{2}{\delta(1-\delta)\zeta_{\mathcal{R}}^*} + \ln \left(\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right) \left(1 + \frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*} \right) \left(1 + \sqrt{\frac{\zeta_{\mathcal{R}}^* \varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right) \right) \right], \quad (95)$$

which also implies (91).

Finally, from the hypothesis $\varepsilon_{\mathcal{R}}^* < 1/3$, from the concavity of the logarithm and of the square root function, and using that $\zeta_{\mathcal{R}}^* \leq 1$, then $\delta(1-\delta) \leq 1/4$ and $\ln 8 \geq 1 + 5/(4\sqrt{2})$, after some computation one obtains that the condition (95) is implied by the stronger condition

$$t \geq \frac{1}{\gamma_{\mathcal{R}}^*} \left(\ln \frac{2}{\delta(1-\delta)\zeta_{\mathcal{R}}^*} \right) \left\{ 1 + \sqrt{\frac{\varepsilon_{\mathcal{R}}^*}{1-\varepsilon_{\mathcal{R}}^*}} \right\}, \quad (96)$$

which concludes the proof.

4. AROUND THE EXPONENTIAL LAW

4.1. Proof of Lemma 2.6. The proof uses the super-harmonicity of $h_{\mathcal{R}}^*$ on \mathcal{R} , where $h_{\mathcal{R}}^*$ is thought as a function in $\ell^2(\mu)$ with $h_{\mathcal{R}}^*(x) = 0$ for $x \neq \mathcal{R}$. To prove that $h_{\mathcal{R}}^*$ is a super-harmonic function, notice that, for all $x, z \in \mathcal{R}$,

$$\begin{aligned} (\mathcal{L}h_{\mathcal{R}}^*)(x) &= -h_{\mathcal{R}}^*(x) + \sum_{y \in \mathcal{X}} p(x, y) h_{\mathcal{R}}^*(y) \\ &= -h_{\mathcal{R}}^*(x) + \sum_{y \in \mathcal{X}} p(x, y) \frac{\mu_{\mathcal{R}}^*(y)}{\mu_{\mathcal{R}}(y)} \\ &= -h_{\mathcal{R}}^*(x) + \lim_{t \rightarrow \infty} \sum_{y \in \mathcal{X}} \frac{p(y, x)}{\mu_{\mathcal{R}}(x)} \frac{\mathbb{P}_z(X(t) = y, \tau_{\mathcal{X}\setminus\mathcal{R}} > t)}{\mathbb{P}_z(\tau_{\mathcal{X}\setminus\mathcal{R}} > t)} \\ &= -h_{\mathcal{R}}^*(x) + \lim_{t \rightarrow \infty} \sum_{y \in \mathcal{X}} \frac{\mathbb{P}_z(X(t+\tau) = x, \tau_{\mathcal{X}\setminus\mathcal{R}} > t+\tau)}{\mu_{\mathcal{R}}(x) \mathbb{P}_z(\tau_{\mathcal{X}\setminus\mathcal{R}} > t)}, \end{aligned} \quad (97)$$

where $t + \tau$ is the time of the first clock ring after t . Then

$$(\mathcal{L}h_{\mathcal{R}}^*)(x) \leq -h_{\mathcal{R}}^*(x) + \lim_{t \rightarrow \infty} \sum_{y \in \mathcal{X}} \frac{\mathbb{P}_z(X(t+\tau) = x | \tau_{\mathcal{X}\setminus\mathcal{R}} > t+\tau)}{\mu_{\mathcal{R}}(x)} = -h_{\mathcal{R}}^*(x) + h_{\mathcal{R}}^*(x) = 0.$$

This implies that $h_{\mathcal{R}}^*$ reaches its minimum in \mathcal{R} on $\partial_- \mathcal{R}$. As a consequence, being $\mu_{\mathcal{R}}(h_{\mathcal{R}}^*) = 1$, there exists $x_0 \in \partial_- \mathcal{R}$ such that $h_{\mathcal{R}}^*(x_0) \leq 1$. Setting $\underline{e}_{\mathcal{R}} = \min_{x \in \partial_- \mathcal{R}} e_{\mathcal{R}}(x)$ and recalling that $\zeta_{\mathcal{R}}^* = \min_{x \in \mathcal{R}} \mu_{\mathcal{R}}^*(x) h_{\mathcal{R}}^*(x)$, we then have

$$\delta_{\mathcal{R}}^* = \underline{e}_{\mathcal{R}} \zeta_{\mathcal{R}}^* \leq \underline{e}_{\mathcal{R}} \mu_{\mathcal{R}}^*(x_0) h_{\mathcal{R}}^*(x_0) \leq e_{\mathcal{R}}(x_0) \mu_{\mathcal{R}}^*(x_0) \leq \phi_{\mathcal{R}}^*, \quad (98)$$

which proves the first inequality.

Now, since $\gamma_{\mathcal{R}}^* \leq 2$, $\delta_{\mathcal{R}}^*(1 - \delta_{\mathcal{R}}^*) \zeta_{\mathcal{R}}^* \leq \delta_{\mathcal{R}}^*(1 - \delta_{\mathcal{R}}^*) \leq 1/4$ and $(\ln 8)/2 \geq 1$, from the definition of $T_{\mathcal{R}}^*$ we get that $T_{\mathcal{R}}^* \geq 1$ and prove the second inequality.

The last inequality is equivalent to

$$\ln \frac{2}{\delta_{\mathcal{R}}^*(1 - \delta_{\mathcal{R}}^*) \zeta_{\mathcal{R}}^*} = \ln \frac{\sqrt{2}}{\delta_{\mathcal{R}}^*(1 - \delta_{\mathcal{R}}^*)} + \ln \frac{\sqrt{2}}{\zeta_{\mathcal{R}}^*} \leq 2 \ln \frac{\sqrt{2}}{\delta_{\mathcal{R}}^*(1 - \delta_{\mathcal{R}}^*)}$$

and it follows from $\delta_{\mathcal{R}}^*(1 - \delta_{\mathcal{R}}^*) \leq \delta_{\mathcal{R}}^* \leq \zeta_{\mathcal{R}}^*$.

4.2. Proof of Theorem 2.7. We write

$$\mathbb{E}_{\nu}[\tau_{\mathcal{X}\mathcal{R}}] = \pi_{\mathcal{R}}(\nu) \mathbb{E}_{\nu}[\tau_{\mathcal{X}\mathcal{R}} | \tau_{\mathcal{X}\mathcal{R}} < T_{\mathcal{R}}^*] + (1 - \pi_{\mathcal{R}}(\nu)) \mathbb{E}_{\nu}[\tau_{\mathcal{X}\mathcal{R}} | \tau_{\mathcal{X}\mathcal{R}} > T_{\mathcal{R}}^*],$$

and notice that trivially

$$0 \leq \mathbb{E}_{\nu}[\tau_{\mathcal{X}\mathcal{R}} | \tau_{\mathcal{X}\mathcal{R}} < T_{\mathcal{R}}^*] \leq T_{\mathcal{R}}^*.$$

Moreover, by Theorem 2.4, one gets

$$T_{\mathcal{R}}^* + \frac{1}{\phi_{\mathcal{R}}^*} \{1 - \delta_{\mathcal{R}}^*\} \leq \mathbb{E}_{\nu}[\tau_{\mathcal{X}\mathcal{R}} | \tau_{\mathcal{X}\mathcal{R}} > T_{\mathcal{R}}^*] \leq T_{\mathcal{R}}^* + \frac{1}{\phi_{\mathcal{R}}^*} \{1 + \delta_{\mathcal{R}}^*\}.$$

Altogether this proves statement (i).

In the same way, we can write

$$\begin{aligned} \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*}) &= \pi_{\mathcal{R}}(\nu) \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*} | \tau_{\mathcal{X}\mathcal{R}} < T_{\mathcal{R}}^*) \\ &\quad + (1 - \pi_{\mathcal{R}}(\nu)) \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*} | \tau_{\mathcal{X}\mathcal{R}} > T_{\mathcal{R}}^*), \end{aligned}$$

If $t > \phi_{\mathcal{R}}^* T_{\mathcal{R}}^*$, the first term in the r.h.s cancel and we get

$$\mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*}) = (1 - \pi_{\mathcal{R}}(\nu)) \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*} | \tau_{\mathcal{X}\mathcal{R}} > T_{\mathcal{R}}^*).$$

By Theorem 2.4, we also have

$$\left| \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R}} > \frac{t}{\phi_{\mathcal{R}}^*} | \tau_{\mathcal{X}\mathcal{R}} > T_{\mathcal{R}}^*) - e^{-\phi_{\mathcal{R}}^*(\frac{t}{\phi_{\mathcal{R}}^*} - T_{\mathcal{R}}^*)} \right| \leq \delta_{\mathcal{R}}^* e^{-\phi_{\mathcal{R}}^*(\frac{t}{\phi_{\mathcal{R}}^*} - T_{\mathcal{R}}^*)},$$

that, together with the previous equality, provides the relation (ii).

4.3. Proof of Lemma 2.8. Let \hat{X} denote the discrete version of X , such that X follows \hat{X} at each ring of a Poissonian clock of intensity 1, and let $N(t)$ denote the number of rings up to time t . By the natural coupling between the dynamics $X_{\mathcal{R}}$ and its discrete version $\hat{X}_{\mathcal{R}}$ we have

$$\begin{aligned} \mathbb{P}_{\mu_{\mathcal{R}}}(\tau_{\mathcal{X}\mathcal{R}} \leq T_{\mathcal{R}}^*) &= \sum_{k \geq 1} \mathbb{P}_{\mu_{\mathcal{R}}}(\hat{\tau}_{\mathcal{X}\mathcal{R}} = k) \mathbb{P}(N(T_{\mathcal{R}}^*) \geq k) \\ &\leq \sum_{k \geq 1} \sum_{x \in \mathcal{R}} \mathbb{P}_{\mu_{\mathcal{R}}}(\hat{X}_{\mathcal{R}}(k-1) = x) e_{\mathcal{R}}(x) \mathbb{P}(N(T_{\mathcal{R}}^*) \geq k) \\ &= \phi_{\mathcal{R}} \mathbb{E}[N(T_{\mathcal{R}}^*)] = \phi_{\mathcal{R}} T_{\mathcal{R}}^*. \end{aligned} \quad (99)$$

5. WORKING WITH (κ, λ) -CAPACITIES

5.1. Proof of the upper bound in Theorem 2.11. Let \tilde{X} denote the continuous time Markov chain on $\tilde{\mathcal{X}}$ defined, for $\tilde{\kappa} > 0$, by the generator

$$(\tilde{\mathcal{L}}f)(\tilde{x}) = \begin{cases} \tilde{\kappa}(f(x) - f(\tilde{x})) & \text{if } \tilde{x} = \bar{x} \in \bar{\mathcal{R}} \\ (\mathcal{L}f)(x) + \tilde{\kappa}(f(\bar{x}) - f(x)) & \text{if } \tilde{x} = x \in \mathcal{R} \\ (\mathcal{L}f)(x) & \text{if } \tilde{x} = x \in \mathcal{X} \setminus \mathcal{R} \end{cases}. \quad (100)$$

This is a reversible process with respect to a measure $\tilde{\mu}$ defined as

$$\tilde{\mu}(\tilde{x}) = \begin{cases} \mu(x) & \text{if } \tilde{x} = x \in \mathcal{X} \\ \frac{\tilde{\kappa}}{\kappa}\mu(x) & \text{if } \tilde{x} = \bar{x} \in \bar{\mathcal{R}} \end{cases}. \quad (101)$$

Note that $\tilde{\mu}$ is not a probability measure. Let us denote by $\tilde{\nu}_{\bar{\mathcal{R}}}$ the harmonic measure on $\bar{\mathcal{R}}$ associated with $\mathcal{X} \setminus \mathcal{R}$, i.e., the probability measure on $\bar{\mathcal{R}}$ defined by

$$\tilde{\nu}_{\bar{\mathcal{R}}}(\bar{x}) = \frac{-\tilde{\mu}(\bar{x})(\tilde{\mathcal{L}}\tilde{V}_{\tilde{\kappa}})(\bar{x})}{C_{\tilde{\kappa}}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} \quad (102)$$

and with

$$\tilde{V}_{\tilde{\kappa}}(\tilde{x}) = \begin{cases} V_{\tilde{\kappa}}(x) = \mathbb{P}_x(\sigma_{\tilde{\kappa}} < \tau_{\mathcal{X} \setminus \mathcal{R}}) & \text{if } \tilde{x} = x \in \mathcal{R} \\ 1 & \text{if } \tilde{x} = \bar{x} \in \bar{\mathcal{R}} \\ 0 & \text{if } \tilde{x} = x \in \mathcal{X} \setminus \mathcal{R} \end{cases}.$$

With obvious notation, we then have

$$\mathbb{E}_{\tilde{\nu}_{\bar{\mathcal{R}}}}[\tilde{\tau}_{\mathcal{X} \setminus \mathcal{R}}] = \frac{\tilde{\mu}(\tilde{V}_{\tilde{\kappa}})}{C_{\tilde{\kappa}}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}. \quad (103)$$

Such kind of formula was introduced for the study of metastability in [16]. We refer to lecture notes [31] for a derivation.

Now setting $\nu(x) = \tilde{\nu}_{\bar{\mathcal{R}}}(\bar{x})$ for all $x \in \mathcal{R}$, we can write

$$\mathbb{E}_{\tilde{\nu}_{\bar{\mathcal{R}}}}[\tilde{\tau}_{\mathcal{X} \setminus \mathcal{R}}] = \frac{1}{\tilde{\kappa}} + \mathbb{E}_{\nu}[\tau_{\mathcal{X} \setminus \mathcal{R}}] + \mathbb{E}_{\nu}[\tau_{\mathcal{X} \setminus \mathcal{R}}] \cdot \tilde{\kappa} \cdot \frac{1}{\tilde{\kappa}} = \frac{1}{\tilde{\kappa}} + \mathbb{E}_{\nu}[\tau_{\mathcal{X} \setminus \mathcal{R}}](1 + \frac{\tilde{\kappa}}{\kappa}).$$

and

$$\frac{\tilde{\mu}(\tilde{V}_{\tilde{\kappa}})}{C_{\tilde{\kappa}}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} = \frac{\mu(V_{\tilde{\kappa}}) + \sum_{\bar{x} \in \bar{\mathcal{R}}} \tilde{\mu}(\bar{x})}{C_{\tilde{\kappa}}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} = \frac{\mu(V_{\tilde{\kappa}}) + \frac{\tilde{\kappa}}{\kappa}\mu(\mathcal{R})}{C_{\tilde{\kappa}}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}.$$

Inserting the above equalities in (103) and multiplying by $\tilde{\kappa}$, we then get

$$1 + \mathbb{E}_{\nu}[\tau_{\mathcal{X} \setminus \mathcal{R}}](\tilde{\kappa} + \kappa) = \frac{\tilde{\kappa}\mu(V_{\tilde{\kappa}}) + \kappa\mu(\mathcal{R})}{C_{\tilde{\kappa}}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}. \quad (104)$$

Note that $\mu(\mathcal{R})$, $\mu(V_{\tilde{\kappa}})$, $C_{\tilde{\kappa}}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})$ and $\mathbb{E}_{\nu}[\tau_{\mathcal{X} \setminus \mathcal{R}}]$ do not depend on $\tilde{\kappa}$. Then, in the limit of a vanishing $\tilde{\kappa}$, it holds

$$1 + \kappa\mathbb{E}_{\nu}[\tau_{\mathcal{X} \setminus \mathcal{R}}] = \frac{\kappa\mu(\mathcal{R})}{C_{\kappa}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}. \quad (105)$$

By Theorem 2.7, this implies

$$\frac{\kappa}{\phi_{\mathcal{R}}^*} \{1 + \phi_{\mathcal{R}}^* T_{\mathcal{R}}^* + \delta_{\mathcal{R}}^*\} + 1 \geq \frac{\kappa\mu(\mathcal{R})}{C_{\kappa}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}$$

that is equivalent to the upper bound in (35).

For future use, we note that dividing Eq. (104) by $\tilde{\kappa}$, and then sending $\tilde{\kappa}$ to $+\infty$, we get

$$\mathbb{E}_{\nu}[\tau_{\mathcal{X} \setminus \mathcal{R}}] = \frac{\mu(V_{\kappa})}{C_{\kappa}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}. \quad (106)$$

Together with (105), this implies $\frac{\mu(V_{\kappa})}{C_{\kappa}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} = \frac{\mu(\mathcal{R})}{C_{\kappa}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} - \frac{1}{\kappa}$ or equivalently

Lemma 5.1.

$$\mu_{\mathcal{R}}(V_{\kappa}) = 1 - \frac{C_{\kappa}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\kappa \mu(\mathcal{R})} \quad (107)$$

5.2. Proof of the upper bound in Theorem 2.12. For any $f \in \ell^2(\mu)$, we have

$$\begin{aligned} \text{Var}_{\mu}(f) &= \mu(\text{Var}_{\mu}(f|_{\mathbb{1}_{\mathcal{R}}})) + \text{Var}_{\mu}(\mu(f|_{\mathbb{1}_{\mathcal{R}}})) \\ &= \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \mu(\mathcal{X} \setminus \mathcal{R})\text{Var}_{\mu_{\mathcal{X} \setminus \mathcal{R}}}(f|_{\mathcal{X} \setminus \mathcal{R}}) \\ &\quad + \mu(\mathcal{R})\mu(\mathcal{X} \setminus \mathcal{R}) \left(\mu_{\mathcal{R}}(f|_{\mathcal{R}}) - \mu_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{X} \setminus \mathcal{R}}) \right)^2. \end{aligned} \quad (108)$$

Now, using the test function

$$\tilde{f} = \frac{f - \mu_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{X} \setminus \mathcal{R}})}{\mu_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{R}}) - \mu_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{X} \setminus \mathcal{R}})}$$

in the definition (30) of (κ, λ) -capacity, we get

$$\begin{aligned} C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R}) &\leq \mathcal{D}(\tilde{f}) + \kappa \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(\tilde{f}|_{\mathcal{R}}) + \lambda \mu(\mathcal{X} \setminus \mathcal{R})\text{Var}_{\mu_{\mathcal{X} \setminus \mathcal{R}}}(\tilde{f}|_{\mathcal{X} \setminus \mathcal{R}}) \\ &= \left(\mu_{\mathcal{R}}(f|_{\mathcal{R}}) - \mu_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{X} \setminus \mathcal{R}}) \right)^{-2} \left(\mathcal{D}(f) + \kappa \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \lambda \mu(\mathcal{X} \setminus \mathcal{R})\text{Var}_{\mu_{\mathcal{X} \setminus \mathcal{R}}}(f|_{\mathcal{X} \setminus \mathcal{R}}) \right), \end{aligned}$$

which provides an upper bound over $\left(\mu_{\mathcal{R}}(f|_{\mathcal{R}}) - \mu_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{X} \setminus \mathcal{R}}) \right)^2$. Applying that bound in Eq. (108), and from the definition of $\phi_{\kappa}^{\lambda}(A, B)$, we get

$$\begin{aligned} \text{Var}_{\mu}(f) &\leq \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \mu(\mathcal{X} \setminus \mathcal{R})\text{Var}_{\mu_{\mathcal{X} \setminus \mathcal{R}}}(f|_{\mathcal{X} \setminus \mathcal{R}}) \\ &\quad + \left(\mathcal{D}(f) + \kappa \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(f|_{\mathcal{R}}) + \lambda \mu(\mathcal{X} \setminus \mathcal{R})\text{Var}_{\mu_{\mathcal{X} \setminus \mathcal{R}}}(f|_{\mathcal{X} \setminus \mathcal{R}}) \right) \phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})^{-1} \\ &\leq \frac{\mathcal{D}(f)}{\phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} + \frac{\mu(\mathcal{R})\mathcal{D}_{\mathcal{R}}(f|_{\mathcal{R}})}{\gamma_{\mathcal{R}}} \left(1 + \frac{\kappa}{\phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} \right) + \frac{\mu(\mathcal{X} \setminus \mathcal{R})\mathcal{D}_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{X} \setminus \mathcal{R}})}{\gamma_{\mathcal{X} \setminus \mathcal{R}}} \left(1 + \frac{\kappa}{\phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} \right) \\ &\leq \frac{\mathcal{D}(f)}{\phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} \left\{ 1 + \max \left(\frac{\phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R}) + \kappa}{\gamma_{\mathcal{R}}}; \frac{\phi_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R}) + \lambda}{\gamma_{\mathcal{X} \setminus \mathcal{R}}} \right) \right\}, \end{aligned} \quad (109)$$

where in the last step we used that

$$\mathcal{D}(f) \leq \mu(\mathcal{R})\mathcal{D}_{\mathcal{R}}(f|_{\mathcal{R}}) + \mu(\mathcal{X} \setminus \mathcal{R})\mathcal{D}_{\mathcal{X} \setminus \mathcal{R}}(f|_{\mathcal{X} \setminus \mathcal{R}}).$$

The upper bound in (37) follows directly.

5.3. Proof of the lower bound of Theorem 2.11. From inequality (109) applied to $f = h_{\mathcal{R}}^*$ and $\lambda = +\infty$, and since $h_{\mathcal{R} \setminus \mathcal{X} \setminus \mathcal{R}}^* = 0$, we get

$$\text{Var}_{\mu}(h_{\mathcal{R}}^*) \leq \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) + \frac{\mathcal{D}(h_{\mathcal{R}}^*)}{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} \mu(\mathcal{R})(1 - \mu(\mathcal{R})) \left\{ 1 + \frac{\kappa}{\gamma_{\mathcal{R}}} \right\}. \quad (110)$$

On the other and, by (108),

$$\text{Var}_{\mu}(h_{\mathcal{R}}^*) = \mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) + \mu(\mathcal{R})(1 - \mu(\mathcal{R})).$$

Inserting this formula in (110), the term $\mu(\mathcal{R})\text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*)$ cancels and then, dividing by $\mu(\mathcal{R})$, we have

$$1 - \mu(\mathcal{R}) \leq \frac{\mathcal{D}(h_{\mathcal{R}}^*)}{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})} (1 - \mu(\mathcal{R})) \left\{ 1 + \frac{\kappa}{\gamma_{\mathcal{R}}} \right\}. \quad (111)$$

Now, modifying the transition probabilities along the edges that connect $\mathcal{X} \setminus \mathcal{R}$ to \mathcal{R} without changing the (non-diagonal) ratios $\frac{c(x, y)}{c(x', y')}$ but modifying also μ , in order to keep a probability measure on \mathcal{X} , we can send $\mu(\mathcal{R})$ to 0 without changing $\mu_{\mathcal{R}}$, $h_{\mathcal{R}}^*$ and the ratio $\mathcal{D}(h_{\mathcal{R}}^*)/C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})$. Then we obtain, for the modified system as well as for the original system,

$$C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R}) \left\{ 1 + \frac{\kappa}{\gamma_{\mathcal{R}}} \right\}^{-1} \leq \mathcal{D}(h_{\mathcal{R}}^*). \quad (112)$$

Now, dividing by $\mu(\mathcal{R})\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2$ and using that, by Prop. 2.1, $\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2 = \text{Var}_{\mu_{\mathcal{R}}}(h_{\mathcal{R}}^*) + 1 \leq 1/(1 - \varepsilon_{\mathcal{R}}^*)$, we get

$$\frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})} \left\{ \frac{1 - \varepsilon_{\mathcal{R}}}{1 + \frac{\kappa}{\gamma_{\mathcal{R}}}} \right\} \leq \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2} \left\{ 1 + \frac{\kappa}{\gamma_{\mathcal{R}}} \right\}^{-1} \leq \frac{\mathcal{D}(h_{\mathcal{R}}^*)}{\|h_{\mathcal{R}}^*\|_{\mathcal{R}}^2 \mu(\mathcal{R})} = \phi_{\mathcal{R}}^*, \quad (113)$$

where the last equality comes from Lemma 3.1 and the fact that

$$\langle h_{\mathcal{R}}^*, -\mathcal{L}_{\mathcal{R}}^* h_{\mathcal{R}}^* \rangle_{\mathcal{R}} = \phi_{\mathcal{R}}^*.$$

Finally, using the convexity of the function $x \mapsto \frac{1}{1+x}$, we obtain

$$\frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})} \left\{ 1 - \varepsilon_{\mathcal{R}} - \frac{\kappa}{\gamma_{\mathcal{R}}} \right\} \leq \phi_{\mathcal{R}}^*, \quad (114)$$

which concludes the proof of the lower bound in (35).

5.4. Proof of the lower bound in Theorem 2.12. We use the test function V_{κ}^{λ} , for which we know that (see (30))

$$\mathcal{D}(V_{\kappa}^{\lambda}) + \kappa\mu(\mathcal{R})\mathbb{E}_{\mu_{\mathcal{R}}} \left[\left(V_{\kappa}^{\lambda} |_{\mathcal{R}} - 1 \right)^2 \right] + \lambda\mu(\mathcal{X} \setminus \mathcal{R})\mathbb{E}_{\mu_{\mathcal{X} \setminus \mathcal{R}}} \left[\left(V_{\kappa}^{\lambda} |_{\mathcal{X} \setminus \mathcal{R}} - 0 \right)^2 \right] = C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})$$

so that

$$\mathcal{D}(V_{\kappa}^{\lambda}) \leq C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R}). \quad (115)$$

We then look for a lower bound on $\text{Var}_{\mu}(V_{\kappa}^{\lambda})$. From (108) we have

$$\text{Var}_{\mu}(V_{\kappa}^{\lambda}) \geq \mu(\mathcal{R})\mu(\mathcal{X} \setminus \mathcal{R}) \left(\mu_{\mathcal{R}}(V_{\kappa}^{\lambda} |_{\mathcal{R}}) - \mu_{\mathcal{X} \setminus \mathcal{R}}(V_{\kappa}^{\lambda} |_{\mathcal{X} \setminus \mathcal{R}}) \right)^2,$$

thus we need to estimate $\mu_{\mathcal{R}}(V_{\kappa}^{\lambda} |_{\mathcal{R}})$ and $\mu_{\mathcal{X} \setminus \mathcal{R}}(V_{\kappa}^{\lambda} |_{\mathcal{X} \setminus \mathcal{R}})$.

By the monotonicity in λ , for all $x \in \mathcal{R}$ we get

$$V_{\kappa}^{\lambda}(x) = \mathbb{P}_x(G_{\mathcal{R}}(\sigma_{\kappa}) < G_{\mathcal{X} \setminus \mathcal{R}}(\sigma_{\lambda})) \geq \mathbb{P}_x(\sigma_{\kappa} < \tau_{\mathcal{X} \setminus \mathcal{R}}) = V_{\kappa}(x),$$

which implies, together with Lemma 5.1,

$$\mu_{\mathcal{R}}(V_{\kappa}^{\lambda} |_{\mathcal{R}}) \geq 1 - \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\kappa\mu(\mathcal{R})}.$$

In the same way we have

$$\mu_{\mathcal{X} \setminus \mathcal{R}}(V_{\kappa}^{\lambda} |_{\mathcal{X} \setminus \mathcal{R}}) \leq \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\lambda\mu(\mathcal{X} \setminus \mathcal{R})}.$$

Altogether, we finally get

$$\gamma \leq \frac{\mathcal{D}(V_{\kappa}^{\lambda})}{\text{Var}_{\mu}(V_{\kappa}^{\lambda})} \leq \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})\mu(\mathcal{X} \setminus \mathcal{R})} \left(1 - \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\kappa\mu(\mathcal{R})} - \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\lambda\mu(\mathcal{X} \setminus \mathcal{R})} \right)^{-2}. \quad (116)$$

6. WORKING WITH SOFT MEASURES

6.1. Proof of Lemma 2.13. If $\lambda = 0$ the first statement holds trivially since, in that case, $\phi_{\mathcal{R}, \lambda}^* = 0 = \mu_{\mathcal{R}, \lambda}^*(e_{\mathcal{R}, \lambda})$. If $\lambda > 0$, we can write

$$\mathbb{P}_{\mu_{\mathcal{R}, \lambda}^*}(\tau_{\mathcal{X} \setminus \mathcal{R}, \lambda} \leq t) = \sum_{k \geq 1} \mathbb{P}_{\mu_{\mathcal{R}, \lambda}^*}(N_{\mathcal{R}}(t) \geq k)(1 - \phi_{\mathcal{R}, \lambda}^*)^{k-1} \mu_{\mathcal{R}, \lambda}^*(e_{\mathcal{R}, \lambda}),$$

where $N(t)$ is the number of clock rings inside \mathcal{R} for the Poissonian clock associated to X . Taking the limit as $t \rightarrow \infty$ in the above equation, we get that

$$1 = \mu_{\mathcal{R}, \lambda}^*(e_{\mathcal{R}, \lambda}) / \phi_{\mathcal{R}, \lambda}^* \iff \phi_{\mathcal{R}, \lambda}^* = \mu_{\mathcal{R}, \lambda}^*(e_{\mathcal{R}, \lambda}),$$

which provides identity i).

Let us now define the operator $\mathcal{L}_{\mathcal{R}, \lambda}^*$ on $\ell^2(\mu_{\mathcal{R}})$ as

$$(\mathcal{L}_{\mathcal{R}, \lambda}^* f)(x) = -f(x) + \sum_{y \in \mathcal{R}} p_{\mathcal{R}, \lambda}^*(x, y) f(y) \quad \forall x \in \mathcal{R}, f \in \ell^2(\mu_{\mathcal{R}}) \quad (117)$$

and notice that, for any probability measure ν on \mathcal{X} , it holds

$$\mathbb{E}_\nu \left[f(X \circ G_{\mathcal{R}}(t) \mathbb{1}_{\{\tau_{\mathcal{X}|\mathcal{R},\lambda} > t\}} \right] = \nu \left(e^{t\mathcal{L}_{\mathcal{R},\lambda}^*} f \right). \quad (118)$$

The exponential law given in ii) follows from the above identity applied to $\nu = \mu_{\mathcal{R},\lambda}^*$ and $f = \mathbb{1}_{\mathcal{R}}$.

Finally, since $1 - \phi_{\mathcal{R},\lambda}^*$ is a simple eigenvalue equal to the spectral radius of $p_{\mathcal{R},\lambda}^*$, for any $x, y \in \mathcal{R}$ and in the large t regime, we have

$$\mathbb{P}_x(X \circ G_{\mathcal{R}}(t = y), \tau_{\mathcal{X}|\mathcal{R},\lambda} > t) \sim c_x \mu_{\mathcal{R},\lambda}^*(y) e^{-t\phi_{\mathcal{R},\lambda}^*}, \quad (119)$$

where $c_x \mu_{\mathcal{R},\lambda}^*$ is the canonical projection of δ_x on the one-dimensional eigenspace associated with $\mu_{\mathcal{R},\lambda}^*$. Note that c_x is positive, otherwise we would have $\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_{\mathcal{X}|\mathcal{R},\lambda} > t) \leq (1 - \mu_{\mathcal{R},\lambda}^*(x)) e^{-t\phi_{\mathcal{R},\lambda}^*} + o(e^{-t\phi_{\mathcal{R},\lambda}^*})$ while $\mu_{\mathcal{R},\lambda}^* > 0$ and this would contradict the exponential law. From (119), and taking the limit of $t \rightarrow \infty$, it follows

$$\lim_{t \rightarrow \infty} \mathbb{P}_x(X \circ G_{\mathcal{R}}(t = y) | \tau_{\mathcal{X}|\mathcal{R},\lambda} > t) = \mu_{\mathcal{R},\lambda}^*.$$

6.2. Proof of Lemma 2.14. The result is once again a consequence of Perron-Frobenius theorem. Let $\chi_\lambda(y)$ denote the characteristic polynomial of $\mathcal{L}_{\mathcal{R},\lambda}^*$, that can be written as $\chi_\lambda(y) = (y - \phi_{\mathcal{R},\lambda}^*)a(y)$. If $a(y) = (y - \phi_{\mathcal{R},\lambda}^*)q(y) + r(y)$ is the Euclidian division of $a(y)$ by $(y - \phi_{\mathcal{R},\lambda}^*)$, then $r(y) \equiv r(0) \neq 0$, and we have the Bézout identity

$$\frac{1}{r(0)}a(y) - \frac{1}{r(0)}q(y)(y - \phi_{\mathcal{R},\lambda}^*) = 1. \quad (120)$$

In particular, for any $x \in \mathcal{R}$, $\frac{1}{r(0)}\delta_x a(\mathcal{L}_{\mathcal{R},\lambda}^*) = c_x \mu_{\mathcal{R},\lambda}^*$ is the canonical projection of δ_x on the eigenspace associated to $\mu_{\mathcal{R},\lambda}^*$, and since $c_x > 0$ as previously noticed, we have

$$\mu_{\mathcal{R},\lambda}^* = \frac{\delta_x a(\mathcal{L}_{\mathcal{R},\lambda}^*)}{\sum_{y \in \mathcal{R}} \delta_x a(\mathcal{L}_{\mathcal{R},\lambda}^*) \delta_y}. \quad (121)$$

Since $a(y) = \frac{\chi_\lambda(y)}{(y - \phi_{\mathcal{R},\lambda}^*)}$, the above equation expresses the map $\lambda \mapsto \mu_{\mathcal{R},\lambda}^*$ as a composition of continuous functions of λ .

6.3. Proof of Proposition 2.15. As far as $\phi_{\mathcal{R},\lambda}$ is concerned, continuity and monotonicity derive from continuity and monotonicity of $e_{\mathcal{R},\lambda}(x)$ for any $x \in \mathcal{R}$. We then consider the other parameters. The continuity follows from the continuity of the eigenvalues as root of the characteristic polynomial. To prove the monotonicity, we notice that when λ decreases to zero, $p_{\mathcal{R},\lambda}^*(x, y)$ grows for for all x and y in \mathcal{R} as well as $c_{\mathcal{R},\lambda}(x, y)$ for any distinct $x, y \in \mathcal{R}$. From the variational characterization of $\phi_{\mathcal{R},\lambda}^*$, i.e.

$$\phi_{\mathcal{R},\lambda}^* = \min \left\{ \langle f, -\mathcal{L}_{\mathcal{R},\lambda}^* f \rangle_{\mathcal{R}} : \langle f, f \rangle_{\mathcal{R}} = 1, f > 0 \right\} \quad (122)$$

$$= \min_{\substack{\langle f, f \rangle_{\mathcal{R}} = 1 \\ f > 0}} \sum_{x, y \in \mathcal{R}} \mu_{\mathcal{R}}(x) f(x) \left(f(x) - \sum_{y \in \mathcal{R}} p_{\mathcal{R},\lambda}^*(x, y) f(y) \right) \quad (123)$$

where the restriction $f > 0$ comes from the fact that, by the Perron-Frobenius theorem, the right eigenvector has positive coordinates, we see that $\phi_{\mathcal{R},\lambda}^*$ is decreasing in λ . Similarly, using

$$\gamma_{\mathcal{R},\lambda} = \min \left\{ \frac{1}{2} \sum_{x, t \in \mathcal{R}} c_{\mathcal{R},\lambda}(x, y) (f(x) - f(y))^2 : \text{Var}_{\mu_{\mathcal{R}}}(f) = 1 \right\}, \quad (124)$$

we see that $\gamma_{\mathcal{R},\lambda}$ is increasing in λ . As a consequence $\varepsilon_{\mathcal{R},\lambda}^*$ is decreasing in λ , and we have

$$\varepsilon_{\mathcal{R},0} = \frac{\phi_{\mathcal{R},0}^*}{\gamma_{\mathcal{R},0}} = \frac{\mu_{\mathcal{R},0}^*(e_{\mathcal{R},0})}{\gamma_{\mathcal{R},0}} = 0. \quad (125)$$

6.4. Proof of Theorem 2.17. Proof of (55): We first write

$$\begin{aligned} \mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{R}) &= \frac{1}{\mathbb{P}_\nu(X(T_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \\ &\times \mathbb{P}_{x_i}(X \circ G_{\mathcal{R}}(\sigma_\kappa) = x, G_{\mathcal{R}}(\sigma_\kappa) < G_{\mathcal{X} \setminus \mathcal{R}}(\sigma_\lambda), \sigma_\kappa > T_{\delta, \mathcal{R}, \lambda}^*) \end{aligned} \quad (126)$$

Now, conditioning on σ_κ and setting $\mathbb{P}_{x_i}^{\sigma_\kappa} = \mathbb{P}_{x_i}(\cdot \mid \sigma_\kappa)$, we get

$$\begin{aligned} \mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{R}) &= \frac{1}{\mathbb{P}_\nu(X(T_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \\ &\times \mathbb{E} \left[\mathbb{P}_{x_i}^{\sigma_\kappa}(X \circ G_{\mathcal{R}}(\sigma_\kappa) = x, G_{\mathcal{R}}(\sigma_\kappa) < G_{\mathcal{X} \setminus \mathcal{R}}(\sigma_\lambda), \sigma_\kappa > T_{\delta, \mathcal{R}, \lambda}^*) \right] \\ &= \frac{1}{\mathbb{P}_\nu(X(T_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \\ &\times \mathbb{E} \left[\mathbb{1}_{\{\sigma_\kappa > T_{\delta, \mathcal{R}, \lambda}^*\}} \mathbb{P}_{x_i}^{\sigma_\kappa}(X \circ G_{\mathcal{R}}(\sigma_\kappa) = x \mid \sigma_\kappa < \tau_{\mathcal{X} \setminus \mathcal{R}, \lambda}) \mathbb{P}_{x_i}^{\sigma_\kappa}(\sigma_\kappa < \tau_{\mathcal{X} \setminus \mathcal{R}, \lambda}) \right], \end{aligned} \quad (127)$$

where the second equality comes from the independence between X , σ_κ and σ_λ . Since

$$\mathbb{P}_\nu(X(T_\delta) \in \mathcal{R}) = \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \mathbb{E} \left[\mathbb{1}_{\{\sigma_\kappa > T_{\delta, \mathcal{R}, \lambda}^*\}} \mathbb{P}_{x_i}^{\sigma_\kappa}(\sigma_\kappa < \tau_{\mathcal{X} \setminus \mathcal{R}, \lambda}) \right], \quad (128)$$

from (127) we get

$$\begin{aligned} \frac{\mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{R})}{\mu_{\mathcal{R}, \lambda}^*} - 1 &= \frac{1}{\mathbb{P}_\nu(X(T_\delta) \in \mathcal{R})} \sum_{i \geq 0} \sum_{x_i \in \mathcal{X}} \mathbb{P}_\nu(i_0 > i, X(\tau_i) = x_i) \\ &\times \mathbb{E} \left[\mathbb{1}_{\{\sigma_\kappa > T_{\delta, \mathcal{R}, \lambda}^*\}} \mathbb{P}_{x_i}^{\sigma_\kappa}(\sigma_\kappa < \tau_{\mathcal{X} \setminus \mathcal{R}, \lambda}) \left(\frac{\mathbb{P}_{x_i}^{\sigma_\kappa}(X \circ G_{\mathcal{R}}(\sigma_\kappa) = x \mid \sigma_\kappa < \tau_{\mathcal{X} \setminus \mathcal{R}, \lambda})}{\mu_{\mathcal{R}, \lambda}^*} - 1 \right) \right]. \end{aligned} \quad (129)$$

An analogous expression can be found for $\frac{\mathbb{P}_\nu(X(\tau_\delta) = x \mid X(\tau_\delta) \in \mathcal{X} \setminus \mathcal{R})}{\mu_{\mathcal{X} \setminus \mathcal{R}, \kappa}^*} - 1$. The result then follows from Th.2.16, and in particular for the equivalent of Th. 2.4.

To prove inequality (56), we first state the following lemma.

Lemma 6.1. *Let $T > 0$ and $\{\sigma_i : i \geq 1\}$ be a sequence of independent exponential random variables of rate κ such that $e^{\kappa T} - 1 < 1$. If $N = \min\{i \geq 1 : \sigma_i > T\}$, then*

$$\mathbb{P} \left(\sum_{j=1}^N \sigma_j > \frac{t}{\kappa} \right) \leq \frac{e^{-t}}{1 - (e^{\kappa T} - 1)} \quad (130)$$

Proof of Lemma 6.1. Using the property of the exponential distribution, we have

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^N \sigma_j > \frac{t}{\kappa}\right) &= \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{P}\left(\sum_{j=1}^N \sigma_j > \frac{t}{\kappa} \mid \sigma_1 < T, \dots, \sigma_{n-1} < T, \sigma_n > T\right) \\
&\leq \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{P}\left(\sigma_n > \frac{t}{\kappa} - (n-1)T \mid \sigma_n > T\right) \\
&= \sum_{n \geq 1} \mathbb{P}(N = n) \mathbb{P}\left(\sigma_n > \frac{t}{\kappa} - nT\right) \\
&= \sum_{n \geq 1} (1 - e^{-\kappa T})^{n-1} e^{-\kappa T} e^{-t+n\kappa T} \\
&= e^{-t} \sum_{n \geq 1} (e^{\kappa T} - 1)^{n-1} = \frac{e^{-t}}{1 - (e^{\kappa T} - 1)},
\end{aligned} \tag{131}$$

which concludes the proof. \square

Coming back to the proof of Th. 2.17, we first notice that if $\tau_\delta > t(\frac{1}{\kappa} + \frac{1}{\lambda})$, then $L_{\mathcal{R}}(\tau_\delta) > \frac{1}{\kappa}$ or $L_{\mathcal{X} \setminus \mathcal{R}}(\tau_\delta) > \frac{1}{\lambda}$. As a consequence, defining

$$\begin{aligned}
A_{\mathcal{R}} &= \{\kappa L_{\mathcal{R}}(\tau_\delta) \vee \lambda L_{\mathcal{X} \setminus \mathcal{R}}(\tau_\delta) = \kappa L_{\mathcal{R}}(\tau_\delta) > t\} \\
A_{\mathcal{X} \setminus \mathcal{R}} &= \{\kappa L_{\mathcal{R}}(\tau_\delta) \vee \lambda L_{\mathcal{X} \setminus \mathcal{R}}(\tau_\delta) = \lambda L_{\mathcal{X} \setminus \mathcal{R}}(\tau_\delta) > t\}
\end{aligned} \tag{132}$$

so that $\mathbb{P}(A_{\mathcal{R}}) + \mathbb{P}(A_{\mathcal{X} \setminus \mathcal{R}}) \leq 1$, we have

$$\mathbb{P}_\nu(\tau_\delta > t(\frac{1}{\kappa} + \frac{1}{\lambda})) = \mathbb{P}_\nu(\tau_\delta > t(\frac{1}{\kappa} + \frac{1}{\lambda}) \mid A_{\mathcal{R}}) \mathbb{P}_\nu(A_{\mathcal{R}}) + \mathbb{P}_\nu(\tau_\delta > t(\frac{1}{\kappa} + \frac{1}{\lambda}) \mid A_{\mathcal{X} \setminus \mathcal{R}}) \mathbb{P}_\nu(A_{\mathcal{X} \setminus \mathcal{R}}).$$

Using the independence between σ_κ , σ_λ and X , together with the previous lemma, we finally get

$$\mathbb{P}_\nu(T - \delta > t(\frac{1}{\kappa} + \frac{1}{\lambda})) \leq e^{-t} \left\{ \frac{1}{1-\xi} \right\} (\mathbb{P}_\nu(A_{\mathcal{R}}) + \mathbb{P}_\nu(A_{\mathcal{X} \setminus \mathcal{R}})) \leq e^{-t} \left\{ \frac{1}{1-\xi} \right\}.$$

6.5. Proof of Theorem 2.20. Considering the extended electrical network associated with $C_\kappa^\lambda(A, B)$, the proof of the upper bound is similar to that of Theorem 2.11 and we omit it.

As far as the lower bound is concerned, the situation is much more delicate. In the proof of the analog Theorem 2.11 we used a partial Poincaré inequality to control the mean exit time from \mathcal{R} . Here we will do the same, with the difference that we will have to work on the whole space \mathcal{X} and not only on \mathcal{R} . Since $\mu_{\mathcal{R}, \lambda}^*$ is concentrated on \mathcal{R} , we will first compare it with another quasi-stationary measure, $\tilde{\mu}_{\mathcal{X}}^*$, that spreads on the whole space \mathcal{X} . Then we will control the escape rate of $\tilde{\mu}_{\mathcal{X}}^*$ from \mathcal{X} with the spectral gap estimated in Theorem 2.12. This second step will be the easier one.

Let $\tilde{\mu}_{\mathcal{X}}^*$ be the quasi-stationary measure on \mathcal{X} associated with the Markovian process \tilde{X} on $\tilde{\mathcal{X}} = \mathcal{X} \cup \overline{\mathcal{X} \setminus \mathcal{R}}$ with generator $\tilde{\mathcal{L}}$ defined, for some $\tilde{\lambda} > 0$, by

$$(\tilde{\mathcal{L}}f)(\tilde{x}) = \begin{cases} (\mathcal{L}f)(x) & \text{if } \tilde{x} = x \in \mathcal{R} \\ (\mathcal{L}f)(x) + \lambda(f(\tilde{x}) - f(x)) & \text{if } \tilde{x} = x \in \mathcal{X} \setminus \mathcal{R} \\ \tilde{\lambda}(f(\tilde{x}) - f(x)) & \text{if } \tilde{x} = \bar{x} \in \overline{\mathcal{X} \setminus \mathcal{R}} \end{cases}. \tag{133}$$

With obvious notation, for any $x \in \mathcal{X}$ and any probability measure ν on \mathcal{X} , we have

$$\tilde{\mu}_{\mathcal{X}}^*(x) = \lim_{t \rightarrow \infty} \mathbb{P}_\nu(\tilde{X}(t) = x \mid \tilde{\tau}_{\mathcal{X} \setminus \mathcal{R}} > t). \tag{134}$$

We want to show that $\tilde{\mu}_{\mathcal{X}}^*$ is in some sense close to $\mu_{\mathcal{R}, \lambda}^*$. The difficulty comes from the fact that while $\tilde{\mu}_{\mathcal{X}}^*$ is defined through the global time of the dynamics, $\mu_{\mathcal{R}, \lambda}^*$ is defined through the local time in \mathcal{R} . In particular, it is not true that $\mu_{\mathcal{R}, \lambda}^* = \tilde{\mu}_{\mathcal{X}}^*(\cdot \mid \mathcal{R})$.

For any $a > 1$ and $T > 0$, there is a large enough $t > T$ such that, for all $x \in \mathcal{R}$,

$$\tilde{\mu}_{\mathcal{X}}^*(x) \geq \frac{1}{a} \frac{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(t) = x, \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t)}{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tilde{\tau}_{\mathcal{X}\mathcal{R}} > t)}. \quad (135)$$

Recalling the definition of the times τ_i given at page 12, we set

$$\mathcal{T} = \min\{\tau_i \geq 0 : \tau_i > t - T\}.$$

The numerator in (135) then satisfies

$$\begin{aligned} \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(t) = x, \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t) &\geq \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(t) = x, \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t > \mathcal{T}) \\ &= \mathbb{E}_{\mu_{\mathcal{R},\lambda}^*} \left[\mathbb{E}_{\mu_{\mathcal{R},\lambda}^*} \left[\mathbb{1}_{\{X(t)=x, \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t\}} | \mathcal{F}_{\mathcal{T}} \right] \mathbb{1}_{\{t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}} > \mathcal{T}\}} \right] \\ &= \mathbb{E}_{\mu_{\mathcal{R},\lambda}^*} \left[\mathbb{P}_{X(\mathcal{T})}(X(t - \mathcal{T}) = x, \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t - \mathcal{T}) \mathbb{1}_{\{t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}} > \mathcal{T}\}} \right] \end{aligned}$$

To what concerns the denominator, we have

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tilde{\tau}_{\mathcal{X}\mathcal{R}} > t) \leq \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tilde{\tau}_{\mathcal{X}\mathcal{R}} > t \wedge \mathcal{T}) = \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}}) + \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t). \quad (136)$$

Now, since $\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(\tau_1) = \cdot | \tilde{\tau}_{\mathcal{X}\mathcal{R}} > \tau_1) = \mu_{\mathcal{R},\lambda}^*$, we get

$$\frac{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t)}{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}})} \leq \frac{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 > T)}{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 < T, X(\tau_1) \in \mathcal{R})}.$$

Now, similarly to the proof of Theorem (2.17) but, since we do not care about conditions (53-54), without the correcting factor $\frac{1}{1-\xi}$, we have, on one hand,

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 > T) \leq e^{-T\left(\frac{1}{\kappa} + \frac{1}{\lambda}\right)^{-1}} \quad (137)$$

and, on the other hand,

$$\begin{aligned} \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 < T, X(\tau_1) \in \mathcal{R}) &= \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(\tau_1) \in \mathcal{R}) - \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 > \mathcal{T}, X(\tau_1) \in \mathcal{R}) \\ &= 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\phi_{\mathcal{R},\lambda}^* + \kappa} - \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\sigma_{\kappa} + \tau_{\mathcal{R},\kappa} > T, \tau_{\mathcal{R},\kappa} < \sigma_{\lambda}) \\ &\geq 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} - \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\sigma_{\kappa} + \sigma_{\lambda} > T) \\ &\geq 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} e^{-T\left(\frac{1}{\kappa} + \frac{1}{\lambda}\right)^{-1}} \end{aligned} \quad (138)$$

where the last inequality comes from the following direct computation for $\kappa \neq \lambda$ and $\alpha = T/\left(\frac{1}{\kappa} + \frac{1}{\lambda}\right)$

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\sigma_{\kappa} + \sigma_{\lambda} > T) = e^{-\alpha} \left(\frac{\lambda e^{-\kappa\alpha/\lambda} - \kappa e^{-\lambda\alpha/\kappa}}{\lambda - \kappa} \right) \leq e^{-\alpha} = e^{-T\left(\frac{1}{\kappa} + \frac{1}{\lambda}\right)^{-1}},$$

and a continuity argument for the case $\kappa = \lambda$.

We conclude that

$$\begin{aligned}
\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tilde{\tau}_{\mathcal{X}\mathcal{R}} > t) &\leq \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}}) \left(1 + \frac{e^{-T(\frac{1}{\kappa} + \frac{1}{\lambda})^{-1}}}{1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} - e^{-T(\frac{1}{\kappa} + \frac{1}{\lambda})^{-1}}} \right) \\
&\leq \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}}) \frac{1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa}}{1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} - e^{-T(\frac{1}{\kappa} + \frac{1}{\lambda})^{-1}}} \\
&\leq \frac{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}})}{1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} - e^{-T(\frac{1}{\kappa} + \frac{1}{\lambda})^{-1}}},
\end{aligned} \tag{139}$$

and coming back to (135), with $\alpha_1 = 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} - e^{-T(\frac{1}{\kappa} + \frac{1}{\lambda})^{-1}}$,

$$\tilde{\mu}_{\mathcal{X}}^*(x) \geq \frac{\alpha_1}{a} \frac{\mathbb{E}_{\mu_{\mathcal{R},\lambda}^*} \left[\mathbb{P}_{X(\mathcal{T})}(X(t - \mathcal{T}) = x, \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t - \mathcal{T}) \mathbb{1}_{\{t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}} > \mathcal{T}\}} \right]}{\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}})}. \tag{140}$$

Now we note that starting from $\mu_{\mathcal{R},\lambda}^*$, and conditionally on $\{\mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}}\}$, the law of $X(\mathcal{T})$ is close to $\mu_{\mathcal{R}}^*$ in total variation. Indeed,

$$\begin{aligned}
\mu_{\mathcal{R},\lambda}^* &= \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(\tau_1) = \cdot | x(\tau_1 \in \mathcal{R})) \\
&= \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(\tau_1) = \cdot | x(\tau_1 \in \mathcal{R}), \tau_1 < T) \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 < T | x(\tau_1 \in \mathcal{R})) \\
&\quad + \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(\tau_1) = \cdot | x(\tau_1 \in \mathcal{R}), \tau_1 > T) \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 > T | x(\tau_1 \in \mathcal{R}))
\end{aligned} \tag{141}$$

and, by (138)

$$\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 < T | X(\tau_1) \in \mathcal{R}) \geq \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_1 < T, X(\tau_1) \in \mathcal{R}) \geq 1 - \frac{\phi_{\mathcal{R},\lambda}^*}{\kappa} e^{-T(\frac{1}{\kappa} + \frac{1}{\lambda})^{-1}} = \alpha_1.$$

As a consequence

$$\left\| \mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(X(\mathcal{T}) = \cdot | \mathcal{T} < t \wedge \tilde{\tau}_{\mathcal{X}\mathcal{R}}) - \mu_{\mathcal{R}}^* \right\|_{\text{TV}} \leq \frac{1 - \alpha_1}{\alpha_1}. \tag{142}$$

This is useful because, from the estimate

$$\begin{aligned}
\mathbb{P}_{X(\mathcal{T})}(X(t - \mathcal{T}) = x, \tilde{\tau}_{\mathcal{X}\mathcal{R}} > t - \mathcal{T}) &\geq \mathbb{P}_{X(\mathcal{T})}(X(t - \mathcal{T}) = x, \tau_{\mathcal{X}\mathcal{R},\lambda} > t - \mathcal{T}) \\
&\geq \mathbb{P}_{X(\mathcal{T})}(X(t - \mathcal{T}) = x | \tau_{\mathcal{X}\mathcal{R},\lambda} > t - \mathcal{T}) \mathbb{P}_{X(\mathcal{T})}(\tau_{\mathcal{X}\mathcal{R},\lambda} > T)
\end{aligned}$$

and since $\mathbb{P}_{\mu_{\mathcal{R},\lambda}^*}(\tau_{\mathcal{X}\mathcal{R},\lambda} > T) = e^{-\phi_{\mathcal{R},\lambda}^* T}$, we get that for all $a > 1$,

$$\tilde{\mu}_{\mathcal{X}}^*(x) \geq \frac{\alpha_1}{a} \left(e^{-\phi_{\mathcal{R},\lambda}^* T - \frac{1 - \alpha_1}{\alpha_1}} \right) \nu(x) = \frac{\alpha_1 \alpha_2}{a} \nu(x), \tag{143}$$

where $\nu(x)$ is a probability measure such that $\|\nu - \mu_{\mathcal{R},\lambda}^*\|_{\text{TV}} \leq \frac{1 - \alpha_1}{\alpha_1}$. In particular,

$$\pi_{\mathcal{R},\lambda}(\nu) \leq \pi_{\mathcal{R},\lambda}(\mu_{\mathcal{R},\lambda}^*) + \frac{1 - \alpha_1}{\alpha_1} = 1 - e^{-\phi_{\mathcal{R},\lambda}^* T} + \frac{1 - \alpha_1}{\alpha_1} = 1 - \alpha_3,$$

where, we recall, $\pi_{\mathcal{R},\lambda}(\nu) = \mathbb{P}_{\nu}(\tau_{\mathcal{X}\mathcal{R},\lambda} < T_{\mathcal{R}}^*)$, and by Theorem (2.19) we get

$$\mathbb{E}_{\nu}[\tilde{\tau}_{\mathcal{X}\mathcal{R}}] = \mathbb{E}_{\nu}[\tilde{\tau}_{\mathcal{X}\mathcal{R}} + \sigma_{\lambda}] \geq \frac{\alpha_3}{\phi_{\mathcal{R},\lambda}^*} \{1 + \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R}}^* - \delta_{\mathcal{R},\lambda}^*\} + \frac{1}{\lambda} = \frac{\alpha_4}{\phi_{\mathcal{R},\lambda}^*} \tag{144}$$

Finally, dropping the coefficient $1/a$ in (143) since the inequality holds for all $a > 1$, we get

$$\frac{1}{\phi_{\mathcal{X}}^*} = \mathbb{E}_{\tilde{\mu}_{\mathcal{X}}^*}[\tilde{\tau}_{\mathcal{X}\mathcal{R}}] \geq \sum_{x \in \mathcal{R}} \tilde{\mu}_{\mathcal{X}}^*(x) \mathbb{E}_x[\tilde{\tau}_{\mathcal{X}\mathcal{R}}] \geq \alpha_1 \alpha_2 \mathbb{E}_{\nu}[\tilde{\tau}_{\mathcal{X}\mathcal{R}}] \geq \frac{\alpha_1 \alpha_2 \alpha_4}{\phi_{\mathcal{R},\lambda}^*}. \tag{145}$$

This gives a lower bound on $\phi_{\mathcal{R},\lambda}^*$, provided we have an estimate on $\tilde{\phi}_{\mathcal{X}}^*$. We get such an estimate by comparison with the spectral gap. By Lemma 3.1 applied with $\mathcal{R} = \mathcal{X}$ and, with obvious notation, $f = \tilde{h}_{\mathcal{X}}^*$, we have

$$\tilde{\phi}_{\mathcal{X}}^* \geq \frac{\mathcal{D}(\tilde{h}_{\mathcal{X}}^*)}{\|\tilde{h}_{\mathcal{X}}^*\|^2} = \frac{\|\tilde{h}_{\mathcal{X}}^*\|^2 - 1}{\|\tilde{h}_{\mathcal{X}}^*\|^2} \frac{\mathcal{D}(\tilde{h}_{\mathcal{X}}^*)}{\text{Var}_{\mu}(\tilde{h}_{\mathcal{X}}^*)} \geq \left(1 - \frac{1}{\|\tilde{h}_{\mathcal{X}}^*\|^2}\right) \gamma. \quad (146)$$

Now,

$$\|\tilde{h}_{\mathcal{X}}^*\|^2 \geq \sum_{x \in \mathcal{R}} \mu(x) \left(\frac{\tilde{\mu}_{\mathcal{X}}^*(x)}{\mu(x)} \right)^2 \quad (147)$$

$$= \mu(\mathcal{R}) \sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \left(\frac{\tilde{\mu}_{\mathcal{X}}^*(x)}{\mu(x)} \right)^2 \quad (148)$$

$$\geq \mu(\mathcal{R}) \left(\sum_{x \in \mathcal{R}} \mu_{\mathcal{R}}(x) \frac{\tilde{\mu}_{\mathcal{X}}^*(x)}{\mu(x)} \right)^2 \quad (149)$$

$$= \frac{1}{\mu(\mathcal{R})} (\tilde{\mu}_{\mathcal{X}}^*(\mathcal{R}))^2 \geq \frac{(\alpha_1 \alpha_2)^2}{\mu(\mathcal{R})} \nu^2(\mathcal{R}) = \frac{(\alpha_1 \alpha_2)^2}{\mu(\mathcal{R})}. \quad (150)$$

so that, by Theorem 2.12,

$$\tilde{\phi}_{\mathcal{X}}^* \geq \left(1 - \frac{\mu(\mathcal{R})}{(\alpha_1 \alpha_2)^2}\right) \frac{C_{\kappa}^{\lambda}(\mathcal{R}, \mathcal{X} \setminus \mathcal{R})}{\mu(\mathcal{R})(1 - \mu(\mathcal{R}))} \left\{ \frac{1}{1 + \max\left(\frac{\kappa + \phi_{\kappa}^{\lambda}}{\gamma_{\mathcal{R}}}, \frac{\lambda + \phi_{\lambda}^{\kappa}}{\gamma_{\mathcal{X} \setminus \mathcal{R}}}\right)} \right\}. \quad (151)$$

Plugging this last estimate in (145) and using the convexity of $x \mapsto 1/(1+x)$, we get the desired result.

6.6. Proof of Theorem 2.21. By Theorem 2.16, for all x in \mathcal{X} ,

$$\max_{y \in \mathcal{R}} \left| \frac{\mathbb{P}_x(X(\mathcal{T}) = y | \sigma_{\lambda} > T_{\mathcal{X} \setminus \mathcal{R}, 0}^*)}{\mu_{\mathcal{X} \setminus \mathcal{R}}(y)} - 1 \right| \leq \delta_{\mathcal{X} \setminus \mathcal{R}, 0}^*. \quad (152)$$

Then

$$\|\nu_x - \mu_{\mathcal{X} \setminus \mathcal{R}}\|_{\text{TV}} \leq \frac{1}{2} \delta_{\mathcal{X} \setminus \mathcal{R}, 0}^* + \mathbb{P}(\sigma_{\lambda} < T_{\mathcal{X} \setminus \mathcal{R}, 0}^*) \quad (153)$$

$$= \frac{1}{2} \delta_{\mathcal{X} \setminus \mathcal{R}, 0}^* + 1 - e^{-\lambda T_{\mathcal{X} \setminus \mathcal{R}, 0}^*} \quad (154)$$

$$\leq \frac{1}{2} \delta_{\mathcal{X} \setminus \mathcal{R}, 0}^* + \lambda T_{\mathcal{X} \setminus \mathcal{R}, 0}^*. \quad (155)$$

Also $\|\mu_{\mathcal{X} \setminus \mathcal{R}} - \mu\| = \mu(\mathcal{R})$ and the upper on $\|\nu_x - \mu\|_{\text{TV}}$ follows from the triangular inequality.

Now, for all $t > 0$,

$$\|\mathbb{P}_x(X(t) = \cdot) - \mu\|_{\text{TV}} \leq \|\nu_x - \mu\|_{\text{TV}} + \mathbb{P}_x(\mathcal{T} > t) \quad (156)$$

and to prove our mixing time estimate, it is sufficient to show

$$\mathbb{P}_x\left(\mathcal{T} > \frac{2}{(1-\eta)\phi_{\mathcal{R},\lambda}^*}\right) \leq \frac{1+\eta}{2} - \mu(\mathcal{R}) - \lambda T_{\mathcal{X} \setminus \mathcal{R}, 0}^* - \frac{1}{2} \delta_{\mathcal{X} \setminus \mathcal{R}, 0}^*. \quad (157)$$

To obtain such an estimate we give an upper bound on the mean value of \mathcal{T} and use Markov inequality.

With $\mathcal{T}' = \sigma_{\lambda} \wedge \tau_{\mathcal{R}}$, conditionally to $X(\mathcal{T}') \in \mathcal{R}$, we have

$$\mathbb{E}_{X(\mathcal{T}')}[\mathcal{T}_{\mathcal{R},\lambda}] \leq \frac{1}{\phi_{\mathcal{R},\lambda}^*} \{1 + \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* + \delta_{\mathcal{R},\lambda}^*\} \quad (158)$$

Then

$$\mathbb{E}_x[\mathcal{T}] \leq \mathbb{E}[\sigma_\lambda] + \mathbb{E}_x \left[\mathbb{E}_{X(\mathcal{T}')} [\tau_{\mathcal{R},\lambda}] \mid X(\mathcal{T}') \in \mathcal{R} \right] \quad (159)$$

$$\leq \frac{1}{\lambda} + \frac{1}{\phi_{\mathcal{R},\lambda}^*} \{1 + \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* + \delta_{\mathcal{R},\lambda}^*\} \quad (160)$$

and

$$\mathbb{P}_x \left(\mathcal{T} > \frac{2}{(1-\eta)\phi_{\mathcal{R},\lambda}^*} \right) \leq \frac{1-\eta}{2} \frac{\phi_{\mathcal{R},\lambda}^*}{\lambda} + \frac{1-\eta}{2} + \frac{1-\eta}{2} (\phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* + \delta_{\mathcal{R},\lambda}^*) \quad (161)$$

$$\leq \frac{1+\eta}{2} - \eta + \frac{1}{2} \left(\frac{\phi_{\mathcal{R},\lambda}^*}{\lambda} + \phi_{\mathcal{R},\lambda}^* T_{\mathcal{R},\lambda}^* + \delta_{\mathcal{R},\lambda}^* \right) \quad (162)$$

$$= \frac{1+\eta}{2} - \mu(\mathcal{R}) - \lambda T_{\mathcal{X}\setminus\mathcal{R},0}^* - \frac{1}{2} \delta_{\mathcal{X}\setminus\mathcal{R},0}^*. \quad (163)$$

APPENDIX A. A VERY CRUDE ESTIMATE

We prove here lemma 2.5. The first inequality is obvious from the definition of $\zeta_{\mathcal{R}}^*$. Let \hat{X} denote the discrete version of X , such that X follows \hat{X} at each ring of a Poissonian clock of intensity 1, and let $N(t)$ denote the number of rings up to time t . Then, for $z \in \mathcal{R}$, we have

$$\begin{aligned} \mu_{\mathcal{R}}^*(z) &= \lim_{t \rightarrow \infty} \mathbb{P}_x(X(t) = z \mid \tau_{\mathcal{X}\setminus\mathcal{R}} > t) \\ &= \lim_{t \rightarrow \infty} \sum_{k \geq 0} \mathbb{P}_x(\hat{X}(k) = z \mid \hat{\tau}_{\mathcal{X}\setminus\mathcal{R}} > k) \mathbb{P}(N(t) = k) \\ &\geq \lim_{t \rightarrow \infty} \sum_{k \geq 0} \mathbb{P}_x(\hat{X}(k + D_{\mathcal{R}}) = z \mid \hat{\tau}_{\mathcal{X}\setminus\mathcal{R}} > k + D_{\mathcal{R}}) \mathbb{P}(N(t) = k + D_{\mathcal{R}}) \\ &= \lim_{t \rightarrow \infty} \sum_{k \geq 0} \sum_{y \in \mathcal{R}} \mathbb{P}_x(\hat{X}(k) = y \mid \hat{\tau}_{\mathcal{X}\setminus\mathcal{R}} > k) \mathbb{P}_y(\hat{X}(D_{\mathcal{R}}) = z \mid \hat{\tau}_{\mathcal{X}\setminus\mathcal{R}} > D_{\mathcal{R}}) \\ &\quad \times \mathbb{P}(N(t) = k + D_{\mathcal{R}}), \end{aligned} \quad (164)$$

where we used the notation $\hat{\tau}_{\mathcal{X}\setminus\mathcal{R}}$ for the hitting time of the chain \hat{X} on $\mathcal{X}\setminus\mathcal{R}$. Since for all $y \in \mathcal{R}$ we have

$$\mathbb{P}_y(\hat{X}(D_{\mathcal{R}}) = z \mid \hat{\tau}_{\mathcal{X}\setminus\mathcal{R}} > D_{\mathcal{R}}) \geq \mathbb{P}_y(\hat{X}(D_{\mathcal{R}}) = z, \hat{\tau}_{\mathcal{X}\setminus\mathcal{R}} > D_{\mathcal{R}}) \geq e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}},$$

we get

$$\begin{aligned} \mu_{\mathcal{R}}^*(z) &\geq e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}} \lim_{t \rightarrow \infty} \sum_{k \geq 0} \sum_{y \in \mathcal{R}} \mathbb{P}_x(\hat{X}(k) = y \mid \hat{\tau}_{\mathcal{X}\setminus\mathcal{R}} > k) \mathbb{P}(N(t) = k + D_{\mathcal{R}}) \\ &= e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}} \lim_{t \rightarrow \infty} \mathbb{P}(N(t) \geq D_{\mathcal{R}}) = e^{-\Delta_{\mathcal{R}} D_{\mathcal{R}}}. \end{aligned} \quad (165)$$

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