A note on the equation $-\Delta u + e^u - 1 = 0$

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On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

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Abstract If Ω is a bounded domain in $\mathbb{R}^N$, we study conditions on a Radon measure μ on $\partial\Omega$ for solving the equation $-\Delta u + e^u - 1 = 0$ in Ω with $u = \mu$ on $\partial\Omega$. The conditions are expressed in terms of Orlicz capacities.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary and μ a Radon measure on $\partial\Omega$. In this paper we consider first the problem of finding a function u solution of

$$-\Delta u + e^u - 1 = 0 \tag{1.1}$$

in Ω satisfying $u = \mu$ on $\partial\Omega$. Let $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, then this problem admits a **weak formulation**: *find a function $u \in L^1(\Omega)$ such that $e^u \in L^1_\rho(\Omega)$ satisfying*

$$\int_\Omega (-u\Delta\zeta + (e^u - 1)\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \qquad \forall \zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega), \tag{1.2}$$

where ν is the unit normal outward vector. This type of problem has been initiated by Grillot and Véron [15] in 2-dim in the framework of the boundary trace theory. Much works on boundary trace problems for equation of the type

$$-\Delta u + u^q = 0 \tag{1.3}$$

with $q > 1$), have been developed by Le Gall [18], Marcus and Véron [19], [20], Dynkin and Kuznetsov [9], [10], respectively by purely probabilistic methods, by purely analytic methods or by a combination of the preceding aspects. One of the

main features of the problem with power nonlinearities is the existence of a critical exponent $q_c = \frac{N+1}{N-1}$ which is linked to the existence of boundary removable sets. Existence of boundary removable points have been discovered by Gmira and Véron [14]. Let us recall briefly the main results for (1.3):

(i) If $1 < q < q_c$, then for any $\mu \in \mathfrak{M}_+(\partial\Omega)$ there exists a unique function $u \in L^1_+(\Omega) \cap L^q_\rho(\Omega)$ which satisfies (1.3) in Ω and takes the value μ on $\partial\Omega$ in the following weak sense

$$\int_\Omega (-u\Delta\zeta + u^q\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \qquad \forall \zeta \in W^{1,\infty}_0(\Omega) \cap W^{2,\infty}(\Omega). \tag{1.4}$$

(ii) If $q \geq q_c$, the above problem can be solved if and only if μ vanishes on boundary Borel subsets with zero $C_{\frac{2}{q},q'}$-Bessel capacity. Furthermore a boundary compact set is removable if and only if it has zero $C_{\frac{2}{q},q'}$-capacity.

In this article we adapt some of the ideas used for (1.3) to problem

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 \qquad && \text{in } \Omega \\ u &= \mu \qquad && \text{on } \partial\Omega. \end{aligned} \tag{1.5}$$

Following the terminology of [5] we say that a measure $\mu \in \mathfrak{M}(\partial\Omega)$ is **good** if (1.5) admits a weak solution. Let $P^\Omega(x,y)$ (resp. $G^\Omega(x,y)$) be the Poisson kernel (resp. the Green kernel) in Ω and $\mathbb{P}^\Omega[\mu]$ the Poisson potential of a boundary mesure μ (resp. $\mathbb{G}^\Omega[\phi]$ the Green potential of a bounded measure ϕ defined in Ω). A boundary measure μ which satisfies

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx). \tag{1.6}$$

is called **admissible**. Since for $\mu \geq 0$, $\mathbb{P}^\Omega[\mu]$ is a supersolution for (1.1), an admissible measure is good (see [24]). Our first result which extends a previous one obtained in [15] is the following.

Theorem A. *Suppose $\mu \in \mathfrak{M}(\partial\Omega)$ admits Lebesgue decomposition $\mu = \mu_S + \mu_R$ where μ_S and μ_R are mutually singular and μ_R is absolutely continuous with respect to the (N-1)-dim Hausdorff measure dH^{N-1}. If*

$$\exp(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx), \tag{1.7}$$

then μ is good.

In order to go further in the study of good measures, it is necessary to introduce an *Orlicz capacity* modelized on the Legendre transform of $r \mapsto p(r) := e^r - 1$.

These capacities have been studied by Aissaoui and Benkirane [2] and they inherit most of the properties of the Bessel capacities. The capacity $C_{N^L \ln L}$ associated to the problem is constructed later and it has strong connexion with Hardy-Littlewood maximal function. In this framework we obtain the following types of results:

Theorem B. *Let $\mu \in \mathfrak{M}_+(\partial\Omega)$ be a good measure, then μ vanishes on boundary Borel subsets E with zero $C_{N^L \ln L}$-capacity.*

We also give below a result of removability of boundary singularities.

Theorem C. *Let $K \subset \partial\Omega$ be a compact subset with zero $C_{N^L \ln L}$-capacity. Suppose $u \in C(\overline{\Omega} \setminus K) \cap C^2(\Omega)$ is a positive solution of (1.1) in Ω which vanishes on K, then u is identically zero.*

In the last part of this paper we apply this approach to the problem

$$-\Delta u + e^u - 1 = \mu, \tag{1.8}$$

where μ is a bounded measure, as well as removability questions for internal singularities of solutions of (1.1). In that case the capacity associated to the problem is

$$C_{\Delta^L \ln L}(K) = \inf\{\|M[\Delta\eta]\|_{L^1} : \eta \in C_0^2(\Omega) : 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of} K\} \tag{1.9}$$

where $M[.]$ denotes Hardy-Littlewood's maximal function.

Theorem D. *Let $\mu \in \mathfrak{M}_+^b(\Omega)$ be a bounded good measure, then μ vanishes on boundary Borel subsets E with zero $C_{\Delta^L \ln L}$-capacity.*

A characterization of positive measures which have the property of vanishing on Borel subsets E with zero $C_{N^L \ln L}$-capacity is also provided. We also give below a result of removability of boundary singularities for sigma moderate solutions (see Definition 4.4).

Theorem E. *Let $K \subset \Omega$ be a compact subset with zero $C_{\Delta^L \ln L}$-capacity. Suppose $u \in C(\Omega \setminus K) \cap C^2(\Omega)$ is a positive sigma moderate solution of (1.1) in $\Omega \setminus K$ which vanishes on $\partial\Omega$, then u is identically zero.*

This note is derived from the preliminary report [25], written in 2004 and left escheated since this period. The author is gratefull to the referee for his careful verification of the manuscript which enabled several improvements.

2 Good measures

Proof of Theorem A. For simplicity, we shall denote by μ_R both the regular part of μ and its density with respect to the Hausdorff measure on $\partial\Omega$. Thus for $k > 0$, we denote by $\mu_{R,k}$ the measure on $\partial\Omega$ with density $\mu_{R,k} = \inf\{k, \mu_R\}$ and denote by u_k the solution of

$$\begin{array}{ll} -\Delta u_k + e^{u_k} - 1 = 0 & \text{in } \Omega \\ u_k = \mu_S + \mu_{R,k} & \text{on } \partial\Omega. \end{array} \tag{2.1}$$

Such a solution exists because

$$\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) \leq e^k \exp(\mathbb{P}^\Omega[\mu_S])$$

by the maximum principle, and (1.7) implies that $\exp(\mathbb{P}^\Omega[\mu_S+\mu_{R,k}])-1 \in L^1(\Omega;\rho dx)$. The sequence u_k is nondecreasing. Since, for any $\zeta \in C_c^{1,1}(\bar\Omega)$,

$$\int_\Omega (-u_k\Delta\zeta + (e^{u_k} - 1)\zeta)dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular for test function ζ the solution ζ_0 of

$$\begin{array}{ll} -\Delta\zeta_0 = 1 & \text{in } \Omega \\ \zeta_0 = 0 & \text{on } \partial\Omega, \end{array} \tag{2.2}$$

we get

$$\int_\Omega (u_k + (e^{u_k} - 1)\zeta_0)dx = -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d(\mu_S + \mu_{R,k}) \leq c\,\|\mu\|_{\mathfrak{M}}\,. \tag{2.3}$$

Thus $u = \lim_{k\to\infty} u_k$ is integrable,

$$\int_\Omega (u + (e^u - 1)\zeta_0)dx \leq c\,\|\mu\|_{\mathfrak{M}}\,,$$

and the convergence of u_k and e^{u_k} to u and e^u hold respectively in $L^1(\Omega)$ and $L^1(\Omega;\rho dx)$ and u satisfies (1.2). □

The proof of the next result is directly inspired by [5] where nonlinear Poisson equations are treated.

Proposition 2.1 *The following properties hold:*
(i) If $\mu \in \mathfrak{M}_+(\partial\Omega)$ is a good measure, then any $\tilde\mu \in \mathfrak{M}_+(\partial\Omega)$ smaller than μ is good.
(ii) Let $\{\mu_n\}$ be an increasing sequence of good measures which converges to μ in the weak sense of measures. Then μ is good.
(iii) If $\mu \in \mathfrak{M}_+(\partial\Omega)$ is a good measure and $f \in L^1_+(\partial\Omega)$, then $f + \mu$ is a good measure.

Proof. We denote by $\partial\Omega_t$ the set of $x \in \Omega$ such that $\rho(x) = t > 0$. Since Ω is C^2 there exists $t_0 > 0$ such that for any $0 < t \leq t_0$, the set $\Omega \backslash \Omega_t$ is diffeomorphic to $(0, t_0] \times \partial\Omega$ by the mapping $x \mapsto (t, \sigma(x))$ where $t = \text{dist}\,(x, \partial\Omega)$ and $\sigma(x) = proj_{\partial\Omega}(x)$. Then $x = \sigma(x) - t\nu_{\sigma(x)}$ where ν_a is the outward normal unit vector to $\partial\Omega$ at a. If η is defined on $\partial\Omega$ we define a normal extension of η at $x \in \partial\Omega_t$ by assigning it the value of η at $\sigma(x)$). When there is no ambiguity, we denote this extension by the same notation.

(i) Let $u = u_\mu$ be the solution of (1.5) and $w = \inf\{u, \mathbb{P}^\Omega[\tilde{\mu}]\}$. Since $\mathbb{P}^\Omega[\tilde{\mu}]$ is a supersolution for (1.1), w is a supersolution too. Furthermore w is nonnegative and $e^w - 1 \in L^1(\Omega; \rho dx)$. By Doob's theorem w admits a boundary trace $\mu^* \in \mathfrak{M}_+(\partial\Omega)$ and $\mu^* \leq \tilde{\mu} \leq \mu$. Let w^* be the solution of

$$\begin{aligned} -\Delta w^* + e^u - 1 = 0 \qquad & \text{in } \Omega \\ w^* = \tilde{\mu} \qquad & \text{on } \partial\Omega. \end{aligned}$$

then $u \geq w \geq w^*$ and [21],

$$\lim_{t\to 0} \int_{\partial\Omega_t} w^*(t, .)\eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega).$$

This implies that the boundary trace of w^* is $\tilde{\mu}$ and thus $\mu^* = \tilde{\mu}$. Set $\Omega_t = \{x \in \Omega : \rho(x) > t\}$ and let v_t we the solution of

$$\begin{aligned} -\Delta v_t + e^{v_t} - 1 = 0 \qquad & \text{in } \Omega_t \\ v_t = w \qquad & \text{on } \partial\Omega_t. \end{aligned}$$

Then $v_t \leq w$ in Ω_t. Furthermore $0 < t' < t \Longrightarrow v_{t'} \leq v_t$ in Ω_t. Then $\tilde{u} = \lim_{t\to 0} v_t$ exists, the convergence holds in $L^1(\Omega)$ and $e^{v_t} \to e^{\tilde{u}}$ in $L^1(\Omega; \rho dx)$ (here we use the fact that $e^w \in L^1(\Omega; \rho dx)$. Because

$$\lim_{t\to 0} \int_{\partial\Omega_t} w(t, .)\eta dS_t = \int_{\partial\Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial\Omega),$$

and $v_t = w$ on $\partial\Omega_t$, is follows that $\tilde{u}$ admits $\tilde{\mu}$ for boundary trace and thus $\tilde{u} = u_{\tilde{\mu}}$.

(ii) Let $u_n = u_{\mu_n}$ be the solutions of (1.5) with boundary value μ_n. The sequence $\{u_n\}$ is increasing. Since

$$\int_\Omega (-u_n \Delta\zeta_0 + (e^{u_n} - 1)\zeta_0) dx = -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu_n \leq -\int_{\partial\Omega} \frac{\partial\zeta_0}{\partial\nu} d\mu, \tag{2.4}$$

we conclude as in the proof of Theorem 1, that u_n increases and converges to a solution $u = u_\mu$ of (1.5) with boundary value μ.

(iii) In the proof of (i) we have actually used the following result : *Let w be a nonnegative supersolution of* (1.1) *such that $e^w \in L^1(\Omega; \rho dx)$ and let $\mu \in \mathfrak{M}_+(\partial\Omega)$*

be the boundary trace of w. Then μ is good. Let $f \in L^1_+(\partial\Omega)$ and μ be an good measure. We denote by $u = u_\mu$ the solution of (1.5). For $k > 0$, set $f_k = \min\{k, f\}$. The function $w_k = u_\mu + \mathbb{P}^\Omega[f_k]$ is a nonnegative supersolution, and, since $\mathbb{P}^\Omega[f_k] \leq k$, $e^{w_k} \in L^1(\Omega; \rho dx)$. Furthermore the boundary trace of w_k is $\mu + f_k$. Therefore $\mu + f_k$ is good. We conclude by II that $\mu + f$ is good □

Remark. The assertions (i) and (ii) in Theorem 1 are still valid if we replace $r \mapsto e^r - 1$ by any continuous nondecreasing function g vanishing at 0.

3 The Orlicz space framework

3.1 Orlicz capacities

The set $\mathfrak{M}^{exp}_+(\partial\Omega)$ of nonnegative measures μ on $\partial\Omega$ such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx) \tag{3.1}$$

is not a linear space, but it is a convex subset of $\mathfrak{M}_+(\partial\Omega)$. The role of this set comes from the fact that any measure in $\mathfrak{M}^{exp}(\partial\Omega)$ is good. Put

$$p(t) = \mathrm{sgn}(s)(e^s - 1),\ P(t) = e^{|t|} - 1 - |t|,$$

and

$$\bar{p}(s) = \mathrm{sgn}(s)\ln(|s| + 1),\ P^*(t) = (|t| + 1)\ln(|t| + 1) - |t|.$$

Then P and P^* are complementary functions in the sense of Legendre. Furthermore Young inequality holds

$$xy \leq P(x) + P^*(y) \qquad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if $x = \bar{p}(y)$ or $y = p(x)$. It is classical to define

$$M_P(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx)\}, \tag{3.2}$$

$$M_{P^*}(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega; \rho dx)\}. \tag{3.3}$$

The Orlicz spaces $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ are the vector spaces spanned respectively by $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$. They are endowed with the Luxemburg norms

$$\|\phi\|_{L_{P_\rho}} = \inf\left\{k > 0 : \int_\Omega P\left(\frac{\phi}{k}\right)\rho dx \leq 1\right\}. \tag{3.4}$$

and

$$\|\phi\|_{L_{P^*_\rho}} = \inf\left\{k > 0 : \int_\Omega P^*\left(\frac{\phi}{k}\right)\rho dx \leq 1\right\}. \tag{3.5}$$

Furthermore the Hölder-Young inequality asserts [16]

$$\left|\int_\Omega \phi\,\psi\,\rho\,dx\right| \le \|\phi\|_{L_{P_\rho}}\,\|\psi\|_{L_{P^*_\rho}} \quad \forall(\phi,\psi)\in L_P(\Omega;\rho dx)\times L_{P^*}(\Omega;\rho dx). \tag{3.6}$$

Since P^* satisfies the Δ_2-condition, $M_{P^*}(\Omega;\rho dx)=L_{P^*}(\Omega;\rho dx)$ and $L_P(\Omega;\rho dx)$ is the dual space of $L_{P^*}(\Omega;\rho dx)$, (see [12], [2]). Furthermore, since

$$\frac{|a|\ln(1+|a|)}{2}\le P^*(a)\le |a|\ln(1+|a|)\quad \forall a\in\mathbb{R},$$

the space $L_{P^*}(\Omega;\rho dx)$ is associated with the class $L\ln L(\Omega;\rho dx)$ and to the Hardy-Littlewood maximal function (see [12]). We recall its definition: we consider a cube Q_0 containing $\bar\Omega$, with sides parallel to the axes. If $f\in L^1(\Omega)$ we denote by $\tilde f$ its extension by 0 in $Q_0\setminus\Omega$ and put

$$M_{Q_0}[f](x)=\sup\left\{\frac{1}{|Q|}\int_Q |f|\,(y)dy : Q\in\mathcal{Q}_x\right\}$$

where $\mathcal{Q}_x$ denotes the set of all cubes containing x and contained in Q_0, with sides parallel to the axes. Thus

$$\|f\|_{L\ln L_\rho}:=\int_{Q_0} M_{Q_0}[f](x)\rho dx\approx \|f\|_{L_{P^*_\rho}}. \tag{3.7}$$

Definition 3.1 *The space of all measures on $\partial\Omega$ such that $\mathbb{P}^\Omega[\mu]\in L_P(\Omega;\rho dx)$ is denoted by $B^{\exp}(\partial\Omega)$ and endowed with the norm*

$$\|\mu\|_{B^{\exp}}=\left\|\mathbb{P}^\Omega[\mu]\right\|_{L_{P_\rho}}. \tag{3.8}$$

The set $\mathfrak{M}^{\exp}_+(\partial\Omega)$ is a subset of $B^{\exp}(\partial\Omega)$.

The following result follows from the definition of the Luxemburg norm.

Proposition 3.2 *If $\mu\in B^{\exp}_+(\partial\Omega)$ there exists $a_0>0$ such that $a\mu\in\mathfrak{M}^{\exp}_+(\partial\Omega)$ for all $0\le a<a_0$. Conversely, if $\mu\in\mathfrak{M}^{\exp}_+(\partial\Omega)$, then $a\mu\in B^{\exp}(\partial\Omega)$ for all $a>0$.*

The analytic charaterization of $B^{\exp}(\partial\Omega)$ can be done by introducing the space of normal derivatives of Green potentials of $L\ln L$ functions:

$$N^{L\ln L}(\partial\Omega)=\left\{\eta:\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta])\in L\ln L(\Omega;\rho dx)\right\}. \tag{3.9}$$

where ρ^* is a the first eigenfunction of $-\Delta$ in $H^{1,2}_0(\Omega)$ with maximum 1 (and λ is the corrresponding eigenvalue). Then $c^{-1}\rho\le\rho^*\le c\rho$ for some $c=c(\Omega)>0$, by Hopf lemma, and

$$\left|\int_{\partial\Omega}\eta d\mu\right|=\left|\int_\Omega \mathbb{P}^\Omega[\mu]\Delta(\rho^*\mathbb{P}^\Omega[\eta])\,dx\right|\le\left\|\mathbb{P}^\Omega[\mu]\right\|_{L_{P_\rho}}\left\|\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta])\right\|_{L_{P^*_\rho}}. \tag{3.10}$$

We take for norm on $N^{L\ln L}(\partial\Omega)$

$$\|\eta\|_{N^{L\ln L}} = \left\|\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta])\right\|_{L_{P^*_\rho}}, \tag{3.11}$$

and define the $C_{N^{L\ln L}}$-capacity of a compact subset K of $\partial\Omega$ by

$$C_{N^{L\ln L}}(K) = \inf\{\|\eta\|_{N^{L\ln L}} : \eta \in C^2(\partial\Omega), 0 \le \eta \le 1, \eta \ge 1 \text{ in a neighborhood of } K\}. \tag{3.12}$$

Considering the bilinear form $\mathcal{H}$ on $L_{P^*_\rho}(\partial\Omega) \times L_{P_\rho}(\partial\Omega)$

$$\mathcal{H}(\eta,\mu) := -\int_\Omega \mathbb{P}^\Omega[\mu]\Delta(\rho^*\mathbb{P}^\Omega[\eta])\,dx \tag{3.13}$$

then

$$\begin{aligned}\mathcal{H}(\eta,\mu) &= -\int_\Omega\int_{\partial\Omega} P^\Omega(x,y)d\mu(y)\Delta(\rho^*\mathbb{P}^\Omega[\eta])(x)\,dx \\ &= -\int_{\partial\Omega}\int_\Omega \Delta(\rho^*\mathbb{P}^\Omega[\eta])(x)P^\Omega(x,y)\,dx\,d\mu(y).\end{aligned} \tag{3.14}$$

It is classical to define

$$C^*_{N^{L\ln L}}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_+(\partial\Omega), \mu(K^c) = 0, \left\|\mathbb{P}^\Omega[\mu]\right\|_{L_{P_\rho}} \le 1\}. \tag{3.15}$$

The following result due to Fuglede [13] (and to Aissaoui-Benkirane in the Orlicz space framework [2]) is a consequence of the Kneser-Fan min-max theorem.

Proposition 3.3 *For any compact set $K \subset \partial\Omega$, there holds*

$$C^*_{N^{L\ln L}}(K) = C_{N^{L\ln L}}(K). \tag{3.16}$$

As a direct consequence of (3.10), we have the following

Proposition 3.4 *If $\mu \in B^{\exp}_+(\partial\Omega)$, it does not charge Borel subsets with $C_{N^{L\ln L}}$-capacity zero.*

3.2 Good measures and removable sets

Proof of Theorem B. If K is compact and $C_{N^{L\ln L}}(K) = 0$, there exist a sequence $\{\eta_n\} \subset C^2(\partial\Omega)$ such that $0 \le \eta_n \le 1$, $\eta_n = 1$ in a neighborhood of K and

$$\lim_{n\to\infty} \|\eta_n\|_{N^{L\ln L}} = \left\|\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta_n])\right\|_{L_{P^*_\rho}} = 0. \tag{3.17}$$

Take $\rho^*\mathbb{P}^\Omega[\eta_n])$ as a test function, then

$$\int_\Omega \left(-u\Delta(\rho^*\mathbb{P}^\Omega[\eta_n]) + (e^u-1)\rho^*\mathbb{P}^\Omega[\eta_n])\right)dx = -\int_{\partial\Omega}\frac{\partial(\rho^*\mathbb{P}^\Omega[\eta_n]))}{\partial\nu}d\mu$$

Since $-\dfrac{\partial(\rho^*\mathbb{P}^\Omega[\eta_n]))}{\partial\nu} = \eta_n$ and $\mu > 0$, there holds $-\displaystyle\int_{\partial\Omega}\frac{\partial(\rho^*\mathbb{P}^\Omega[\eta_n]))}{\partial\nu}d\mu \geq \mu(K)$. Furthermore

$$\left|\int_\Omega u\Delta(\rho^*\mathbb{P}^\Omega[\eta_n])dx\right| \leq \|u\|_{L_{P_\rho}}\left\|\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta_n])\right\|_{L_{P^*_\rho}}. \tag{3.18}$$

Then

$$\mu(E) \leq \int_\Omega (e^u-1)\rho^*\mathbb{P}^\Omega[\eta_n])dx + \|u\|_{L_{P_\rho}}\left\|\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta_n])\right\|_{L_{P^*_\rho}}.$$

By the same argument as in [5], $\lim_{n\to\infty}\rho^*\mathbb{P}^\Omega[\eta_n] = 0$, a.e. in Ω, and there exists a nonnegative L^1_ρ-function Φ such that $0 \leq \rho^*\mathbb{P}^\Omega[\eta_n] \leq \Phi$. By (3.17), (3.18) and Lebesgue's theorem, $\mu(K) = 0$. □

Definition 3.5 *A subset $E \subset \partial\Omega$ is said removable for equation* (1.1)*, if any positive solution $u \in C^2(\Omega)$ of* (1.1) *in Ω, which is continuous in $\overline{\Omega}\backslash E$ and vanishes on $\partial\Omega\backslash E$, is identically zero.*

Proof of Theorem C. Let $u \in C(\overline{\Omega}\setminus K)$ be a solution of (1.1) which is zero on $\partial\Omega\setminus K$. As a consequence of Keller-Osserman estimate (see e.g. [23]), there holds

$$u(x) \leq 2\ln\left(\frac{1}{\rho(x)}\right) + D, \tag{3.19}$$

but since u vanishes on $\partial\Omega\setminus K$, we can extend it by 0 in $\overline{\Omega}^c$ in order it becomes a subsolution and obtain, always by Keller-Osserman method, that $\rho(x)$ can be replaced by $\rho_K(x) := \operatorname{dist}(x, K)$ in (3.19). Furthermore, for any open subset containing K, there exists a constant c_G such that $u(x) \leq c_G\rho(x)$ for all $x \in \overline{\Omega}\setminus G$.

Let $\{\eta_n\} \subset C^2(\partial\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a relative neighborhood $\mathcal{V} = G \cap \partial\Omega$ of K, where G is open. Put $\theta_n = 1-\eta_n$. The function $\zeta_n = \rho^*\mathbb{P}^\Omega[\theta_n]$ satisfies $\Delta\zeta_n = -\lambda\zeta_n + 2\nabla\rho^*.\nabla\mathbb{P}^\Omega[\theta_n]$. Therefore $|\Delta\zeta_n|$ remains bounded in $G\cap\Omega$ where there also holds $\zeta_n(x) \leq c\rho^2(x)$. Using (3.19) and an easy approximation argument, we can take ζ_n as a test function and obtain

$$\int_\Omega (-u\Delta\zeta_n + (e^u-1)\zeta_n)\,dx = 0.$$

We derive

$$\begin{aligned}-\int_\Omega u\Delta\zeta_n\,dx &= -\int_\Omega \zeta_n^{-1}\Delta\zeta_n\,u\zeta_n dx\\ &\geq -2^{-1}\int_\Omega (e^u-1-u)\zeta_n\,dx - \int_\Omega Q(\zeta_n^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta_n]))\,\zeta_n dx,\end{aligned}$$

where

$$Q(r) = (|r| + 2^{-1}) \ln(2|r| + 1) - |r| \leq C\,|r| \ln(|r| + 1) \quad \forall r \in \mathbb{R}. \tag{3.20}$$

Therefore

$$\int_\Omega (e^u - 1 - u)\zeta_n\, dx \leq 2C \int_\Omega \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln(1 + \rho^{-2} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|) dx, \tag{3.21}$$

since $\zeta_n^{-1} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \leq \rho^{-2} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|$. Furthermore

$$\begin{aligned} \ln(1 + \rho^{-2} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|)) &= -\ln \rho + \ln(\rho + \rho^{-1} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|) \\ &\leq -\ln \rho + \ln(1 + \rho^{-1} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|) \end{aligned}$$

But (we can assume $\rho \leq 1$)

$$\begin{aligned} &\int_\Omega \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln(1 + \rho^{-2} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|) dx \\ &\qquad \leq -\int_\Omega \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln \rho dx + \int_\Omega \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln(1 + \rho^{-1} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|) dx, \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln \rho^{-1} dx \\ &\quad = \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq 1\}} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln \rho^{-1} dx + \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| > 1\}} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln \rho^{-1} dx \\ &\quad \leq \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq 1\}} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln \rho^{-1} dx + \int_\Omega \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln(1 + \rho^{-1} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|) dx \end{aligned}$$

By assumption $C_{N^L \ln L}(K) = 0$, then we take $\{\eta_n\}$ such that $\|\eta_n\|_{N^L \ln L} \to 0$ and

$$\lim_{n \to \infty} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| = 0 \quad \text{a. e. in } \Omega,$$

at least up to some subsequence. Thus

$$\lim_{n \to \infty} \int_\Omega \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right| \ln(1 + \rho^{-2} \left|\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\right|) dx = 0. \tag{3.22}$$

Using (3.21), we derive $u = 0$.

Conversely, assume that $C_{N^L \ln L}(K) > 0$. By Proposition 3.3 there exists a non negative non-zero measure $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that $\mu(K^c) = 0$ in the space $B_+^{exp}(\partial\Omega)$. This means that $\theta\mu \in M_+^{exp}(\partial\Omega)$ for some $\theta > 0$. Thus problem (1.5) admits a

solution. □

Several open problems can be posed

1- If a measure μ is good, does there exist an increasing sequence of measures $\{\mu_n\}$ which converges to μ such that $\theta_n\mu_n$ is admissible for some $\theta_n > 0$?

2- If a measure μ, singular with respect to $\mathcal{H}^{N-1}$ is good does, it exist an increasing sequence of admissible measures $\{\mu_n\}$ converging to μ ?

3- If a measure μ does not charge Borel sets with $C^{L\ln L}$-capacity zero, doest it exist $\theta > 0$ such that $\theta\mu$ is admissible ?

4- If a singular measure μ is good, is $(1-\delta)\mu$ admissible for any $\delta \in (0,1)$?

3.3 More general nonlinearities

In the section we consider the problem

$$\begin{array}{rll} -\Delta u + P(u) = 0 & & \text{in } \Omega \\ u = \mu & & \text{on } \partial\Omega, \end{array} \tag{3.23}$$

where P is a convex increasing function vanishing at 0 and such that $\lim_{r\to\infty} P(r)/r = \infty$: In Theorem A-$P$, (1.7) should be replaced by

$$P(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega;\rho\,dx). \tag{3.24}$$

In Proposition 2.1-P, (i), (ii) and (iii) still hold. For simplicity we assume that P is a N-function in the sense of Orlicz spaces i.e.

$$P(r) = \int_0^r p(s)ds$$

where p is increasing, vanishes at 0 and tends to infinity at infinity. Let P^* be the conjugate N-function, $L_P(\Omega;\rho\,dx)$ and $L_{P^*}(\Omega;\rho\,dx)$ the corresponding Orlicz spaces endowed with the Luxemburg norms. Then Proposition 3.4-P is valid, provided the space

$$B^P(\partial\Omega) := \{\mu \in \mathfrak{M}(\partial\Omega) : \mathbb{P}^\Omega[\mu] \in L_P(\Omega;\rho\,dx)\}$$

endowed with its natural norm replaces $B^{\exp}(\partial\Omega)$ with the norm (4.10). We set

$$N^{P^*}(\partial\Omega) = \{\eta : \rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta]) \in L_{P^*}(\Omega;\rho\,dx)\}$$

with corresponding norm

$$\|\eta\|_{N^{P^*}} = \left\|\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta])\right\|_{L_{P^*_\rho}}$$

and the corresponding capacity $C_{N^{P^*}}$. The proof of Proposition 3.4-P, consequence of Young inequality between Orlicz space is valid without modification. **However**, it appears that the full characterization of removable sets cannot be adapted without further properties of the function P^* like the Δ_2-condition. Some results in this directions have been obtained in [17] where a necessary and sufficient condition for removability of boundary set is given, under a very restrictive growth condition on P which reduces the nonlinearity to power-like with limited exponent.

4 Internal measures

Several above techniques can be extended to the following types of problem in which $\mu \in \mathfrak{M}^b_+(\Omega)$:

$$\begin{array}{rll} -\Delta u + e^u - 1 = \mu & & \text{in } \Omega \\ u = 0 & & \text{on } \partial\Omega. \end{array} \tag{4.1}$$

For this specific problem many interesting results can be found in [3] where the analysis of μ is made by comparison with the Hausdorff measure in dimension $N-2$, $\mathcal{H}^{N-2}$. It is proved in particular that if a measure μ satisfies $\mu \leq 4\pi\mathcal{H}^{N-2}$, then problem (4.1) admits a solution, while if μ charges some Borel set A with Hausdorff dimension less than $N-2$, no solution exists. The results we provide are different and in the Orlicz capacities framework.

We define the classes $M_P(\Omega)$ and $M_{P^*}(\Omega)$ similarly to $M_P(\Omega;\rho dx)$ and $M_{P^*}(\Omega;\rho dx)$ except that the measure ρdx is replaced by the Lebesgue measure dx. The Orlicz spaces $L_P(\Omega)$ and $L_{P^*}(\Omega)$ are defined from $M_P(\Omega)$ and $M_{P^*}(\Omega)$ and endowed with the respective Luxemburg norms $\| \ \|_P$ and $\| \ \|_{P^*}$. We put

$$\Delta^{L\ln L}(\Omega) := \{\eta \in W^{1,1}_0(\Omega) : \Delta\eta \in L_{P^*}(\Omega)\}, \tag{4.2}$$

with natural norm

$$\|\eta\|_{\Delta^{L\ln L}} := \|\eta\|_{L^1} + \|\Delta\eta\|_{L_{P^*}}. \tag{4.3}$$

The norm in $M_{P^*}(\Omega)$ can be characterized using the Hardy-Littlewood maximal function $f \mapsto M_{Q_0}[f]$ since

$$\|f\|_{L\ln L} := \int_{Q_0} M_{Q_0}[f](x)dx \approx \|f\|_{L_{P^*}}. \tag{4.4}$$

Since P^* satisfies the Δ_2-condition, $C^\infty_0(\Omega)$ is dense in $\Delta^{L\ln L}(\Omega)$. Inequality (3.10) becomes

$$\left|\int_\Omega \eta d\mu\right| = \left|\int_\Omega \eta\Delta\mathbb{G}^\Omega[\mu]\,dx\right| = \left|\int_\Omega \mathbb{G}^\Omega[\mu]\Delta\eta\,dx\right| \leq \left\|\mathbb{G}^\Omega[\mu]\right\|_{L_P} \|\Delta\eta\|_{L_{P^*}}, \tag{4.5}$$

for $\eta \in C_c^{1,1}(\bar{\Omega})$. We define the $C_{\Delta^L \ln L}$-capacity of a compact subset K of $\partial\Omega$ by

$$C_{\Delta^L \ln L}(K) = \inf\{\|\Delta\eta\|_{L^{P^*}} : \eta \in C_c^2(\Omega), 0 \le \eta \le 1, \eta = 1 \text{ in a neighborhood of } K\}, \tag{4.6}$$

By the min-max theorem there holds

$$C_{\Delta^L \ln L}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_+^b(\Omega), \mu(K^c) = 0, \left\|\mathbb{G}^\Omega[\mu]\right\|_{L_P} \le 1\}. \tag{4.7}$$

Remark. The characterization of the $C_{\Delta^L \ln L}$-capacity is not simple, however, by a result of [7, Th1], there holds

$$\left\|D^2\eta\right\|_{L^{1,\infty}} \le C \|\Delta\eta\|_{L \ln L} \qquad \forall \eta \in C_c^{1,1}(\overline{\Omega}) \tag{4.8}$$

where $L^{1,\infty}(\Omega)$ denotes the weak L^1-space, that is the space of all measurable functions f defined in Ω satisfying

$$meas\left(\{x \in \Omega : |f(x)| > t\}\right) \le \frac{c}{t}, \qquad \forall t > 0 \tag{4.9}$$

and $\|f\|_{L^{1,\infty}}$ is the smallest constant such that (4.9) holds.

Definition 4.1 *The space of all bounded measures in Ω such that $\mathbb{G}^\Omega[\mu] \in L_P(\Omega)$ is denoted by $B^{\exp}(\Omega)$, with norm*

$$\|\mu\|_{B^{\exp}} = \left\|\mathbb{G}^\Omega[\mu]\right\|_{L_P}. \tag{4.10}$$

The subset of nonnegative measures μ in Ω such that $\exp(\mathbb{G}^\Omega[\mu]) \in L^1(\Omega)$ is denoted by $\mathfrak{M}_+^{\exp}(\Omega)$.

Proposition 3.4 and Theorem B admit the following counterparts

Proposition 4.2 *If $\mu \in B_+^{\exp}(\Omega)$, it does not charge Borel subsets with $C_{\Delta^L \ln L}$-capacity zero.*

Theorem 4.3 *Let $\mu \in \mathfrak{M}_+(\Omega)$ be a good measure, then μ vanishes on Borel subsets E with zero $C_{\Delta^L \ln L}$-capacity.*

Proof. The proof of Proposition 4.2 is straightforward from the definition. For Theorem 4.3 we consider a solution u of (4.1) and $K \subset \Omega$ a compact set. Then there exists a sequence $\{\eta_n\} \subset C_0^2(\Omega)$ satisfying $0 \le \eta_n \le 1$, $\eta_n = 1$ in a neighborhood $\mathcal{V}$ of K such that $\lim_{n\to\infty} \|\Delta\eta_n\|_{L^{P^*}} = 0$. Then

$$\int_\Omega \left(-u\Delta\eta_n^3 + (e^u - 1)\eta_n\right) dx = \int_\Omega \eta_n^3 d\mu \ge \mu(K).$$

Since u is positive and $-u\Delta\eta_n^3 \leq -u\Delta\eta_n$ we derive by Hölder-Young inequality (3.6)

$$3\left\|u\right\|_{L_P}\left\|\Delta\eta_n\right\|_{L_{P^*}} + \int_\Omega (e^u - 1)\,\eta_n dx \geq \mu(K). \tag{4.11}$$

Notice that $u \in L_P(\Omega; dx)$ since $e^u \in L^1(\Omega)$. If $C_{\Delta^L \ln L}(K) = 0$, the sequence $\{\eta_n\}$ can be taken such that $\left\|\Delta\eta_n\right\|_{L_{P^*}} + \left\|\eta_n\right\|_{L^1} \to 0$. Therefore $\mu(K) = 0$. □

Following Dynkin [10] (although in a slightly different context) it is natural to introduce the notions of moderate and sigma-moderate solutions.

Definition 4.4 *et $K \subset \Omega$ be compact. A positive solution u of (1.1) in $\Omega \setminus K$ is called moderate if $e^u \in L^1(\Omega \setminus K)$. It is sigma-moderate if there exists an increasing sequence $\{u_n\}$ of moderate solutions in $\Omega \setminus K$ which converges to u in $\Omega \setminus K$.*

Theorem 4.5 *Let $K \subset \Omega$ be compact. A sigma-moderate solution of (1.1) in $\Omega \setminus K$ is a solution in Ω if and only if $C_{\Delta^L \ln L}(K) = 0$.*

Proof. We first assume that u is a moderate solution. Let $\{\eta_n\} \subset C_0^2(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood $\mathcal{V}$ of K and $\left\|\Delta\eta_n\right\|_{L_{P^*}} + \left\|\eta_n\right\|_{L^1} \to 0$ when $n \to \infty$. If $\zeta \in C_0^2(\Omega)$, we set $\zeta_n = (1 - \eta_n)\zeta$. Then

$$\int_\Omega (-u\Delta\zeta_n + (e^u - 1)\zeta_n)\, dx = 0.$$

Therefore

$$\int_\Omega (-u(1 - \eta_n)\Delta\zeta + (e^u - 1)\zeta_n)\, dx = -\int_\Omega (\zeta\Delta\eta_n + 2\nabla\zeta.\nabla\eta_n)\, u dx. \tag{4.12}$$

Since $e^u - 1 \in L^1(\Omega \setminus K)$ and $|K| = 0$, $e^u - 1 \in L^1(\Omega)$. But $0 \leq u \leq e^u - 1$, therefore $u \in L^1(\Omega)$. By Lebesgue's theorem

$$\lim_{n\to\infty} \int_\Omega (-u(1 - \eta_n)\Delta\zeta + (e^u - 1)\zeta_n)\, dx = \int_\Omega (-u\Delta\zeta + (e^u - 1)\zeta)\, dx.$$

Furthermore

$$\left|\int_\Omega (\zeta\Delta\eta_n + 2\nabla\zeta.\nabla\eta_n)\, u dx\right| \leq \left(\left\|\zeta\right\|_{L^\infty}\left\|\Delta\eta_n\right\|_{L_{P^*}} + 2\left\|\nabla\zeta\right\|_{L^\infty}\left\|\nabla\eta_n\right\|_{L_{P^*}}\right)\left\|u\right\|_{L_P}.$$

By standard regularity $\left\|\nabla\eta_n\right\|_{L^r} \leq \left\|\Delta\eta_n\right\|_{L^1}$ for any $r \in (1, \frac{N}{N-1})$. Since

$$\int_\Omega |\nabla\eta_n| \ln(1 + |\nabla\eta_n|) dx \leq C\int_\Omega (|\nabla\eta_n|^r + |\nabla\eta_n|)\, dx,$$

the right-hand side of (4.12) tends to zero as $n \to \infty$ which implies that u is a solution in whole Ω. If u is a sigma-moderate solution in $\Omega \setminus K$, it is the limit of an increasing sequence $\{u_n\}$ of positive moderate solutions in $\Omega \setminus K$. These solutions are solutions in whole Ω, so is u. Finally, if $C_{\Delta^L \ln L}(K) > 0$, by the dual definition (4.7) there exists a positive bounded measure μ with support in K such that $\mu(K) > 0$ and $\left\|\mathbb{G}^{\Omega}[\mu]\right\|_{L_P} \leq 1$. For this measure problem (4.1) admits a solution and this solution is not a solution of (1.1) in whole Ω. □

When the solution is not sigma-moderate we have a weaker result.

Theorem 4.6 *Let $K \subset \Omega$ be compact such that*

$$\inf\left\{\int_{\Omega} |\Delta\eta| + |\nabla\eta|^2)dx : \eta \in C_c^2(\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\right\} = 0. \tag{4.13}$$

If u is a positive solution of (1.1) *in $\Omega \setminus K$, it can be extended as a solution in Ω.*

Proof. If $\psi \in C_c^{\infty}(\Omega)$ is nonnegative, there holds

$$\begin{aligned}\int_{\Omega}(e^u - 1)\psi dx &= \int_{\Omega} u\Delta\psi dx = \int_{\Omega} u(\psi^{-1}\Delta\psi)\psi dx \\ &\leq \frac{1}{2}\int_{\Omega}(e^u - 1 - u)\psi dx + c\int_{\Omega} Q(\psi^{-1}|\Delta\psi|)\psi dx,\end{aligned}$$

where Q is defined in (3.20). Consider $\phi \in C_c^{\infty}(\Omega)$, $0 \leq \phi \leq 1$, $\phi = 1$ in a neighborhood G of K and a sequence of functions $\{\eta_n\} \subset C_c^{\infty}(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in some neighborhood of K We set $\psi = \psi_n^3 = \phi^3(1 - \eta_n)^3$ and derive

$$\begin{aligned}Q(\psi^{-1}|\Delta\psi|) &\leq (3\psi_n^{-1}|\Delta\psi_n| + 6\psi_n^{-2}|\nabla\psi_n|^2)\ln(1 + 3\psi_n^{-1}|\Delta\psi_n| + 6\psi_n^{-2}|\nabla\psi_n|^2) \\ &\leq 6\psi_n^{-1}|\Delta\psi_n|\ln(1 + 3\psi_n^{-1}|\Delta\psi_n|) + 12\psi_n^{-2}|\nabla\psi_n|^2\ln(1 + 6\psi_n^{-2}|\nabla\psi_n|^2)\end{aligned}$$

It follows from the Keller-Osserman estimate for this type of nonlinearity (see e.g. [23]) that u is bounded on each compact subset of $\Omega \setminus K$; it is in particular the case of on $H := supp(\phi) \setminus G$. Using the fact that ϕ is constant on G, which implies $|\Delta\psi_n| \leq |\Delta\eta_n| + c_1$, we derive

$$\begin{aligned}6\psi_n^2|\Delta\psi_n|\ln(1 + 3\psi_n^{-1}|\Delta\psi_n|) &\leq 6\psi_n^2|\Delta\psi_n|\left(\ln(\psi_n + 3|\Delta\psi_n|) - \ln\psi_n\right) \\ &\leq 6|\Delta\psi_n|\ln(1 + |\Delta\psi_n|) + c_2|\Delta\psi_n| + c_3.\end{aligned}$$

Similarly

$$\begin{aligned}12\psi_n|\nabla\psi_n|^2\ln(1 + 6\psi_n^{-2}|\nabla\psi_n|^2) &\leq 12\psi_n|\nabla\psi_n|^2\left(\ln(\psi_n^2 + 6|\nabla\psi_n|^2) - 2\ln\psi_n\right) \\ &\leq 12|\nabla\psi_n|^2(\ln(1 + |\nabla\psi_n|) + c_4|\nabla\psi_n|^2 + c_5,\end{aligned}$$

where the c_j do not depend on n. Since there always hold (as $0 \leq \eta_n \leq 1$ and Ω is bounded)

$$c\int_\Omega \eta_n^2 dx \leq \int_\Omega |\nabla \eta_n|^2 dx \leq \int_\Omega |\Delta \eta_n| dx,$$

we derive

$$\begin{aligned}\int_G (e^u - 1 - u)dx &\leq \limsup_{n\to\infty} \int_\Omega (e^u - 1 - u)\psi_n^3 dx \\ &\leq 2 \limsup \int_\Omega Q(\psi_n^{-3}|\Delta \psi_n^3|)\psi_n^3 dx \leq |H|(c_3 + c_5).\end{aligned}$$

Therefore u is moderate and the conclusion follows from Theorem 4.5. □

Remark. It is an open question wether all positive solutions of (1.1) in $\Omega \setminus K$ are sigma-moderate.

4.1 More on good measures

The main characterization of good measures is the following

Theorem 4.7 *Assume μ is a positive good measure, then there exists an increasing sequence $\{\mu_n\} \subset B_+^{exp}(\Omega)$ which converges weakly to μ.*

The proof will necessitate several intermediate results which are classical in the framework of Lebesgue measure or Bessel capacities, but appear to be new for Orlicz capacities.

Lemma 4.8 *Let $K \subset \Omega$, then $C_{\Delta^{L\ln L}}(K) = 0$ if and only if there exists $\eta \in \Delta^{L\ln L}(\Omega)$ such that $\eta \geq 0$ and $K \subset \{y \in \Omega : \eta(y) = \infty\}$.*

Proof. By the definition of the capacity, for any $\lambda > 0$ and $\eta \in \Delta^{L\ln L}(\Omega)$, $\eta \geq 0$,

$$C_{\Delta^{L\ln L}}\left(\{y \in \Omega : \eta(y) \geq \lambda\}\right) \leq \frac{1}{\lambda}\left\|\eta\right\|_{\Delta^{L\ln L}}. \tag{4.14}$$

This implies

$$C_{\Delta^{L\ln L}}\left(\{y \in \Omega : \eta(y) = \infty\}\right) = 0.$$

□

Lemma 4.9 *Suppose $\{\eta_j\}$ is a Cauchy sequence in $\Delta^{L\ln L}(\Omega)$. Then there exist a subsequence $\{\eta_{j_\ell}\}$ and $\eta \in \Delta^{L\ln L}(\Omega)$ such that*

$$\lim_{i_\ell\to\infty} \eta_{j_\ell} = \eta,$$

uniformly outside an open subset of arbitrary small $C_{\Delta^{L\ln L}}$-capacity.

Proof. By Lemma 4.8, η_j and η are finite outside a set F with zero $C_{\Delta^{L\ln L}}$-capacity. There exists a subsequence $\{\eta_{j_\ell}\}$ such that

$$\|\eta_{j_\ell} - \eta\|_{\Delta^{L\ln L}} \leq 2^{-2\ell}.$$

Put $E_\ell = \{y \in \Omega : \eta_{j_\ell}(y) - \eta(y) \geq 2^{-\ell}\}$. By (4.14) $C_{\Delta^{L\ln L}}(E_\ell) \leq 2^{-\ell}$, and if $G_m = \cup_{\ell \geq m} E_\ell$, there holds $C_{\Delta^{L\ln L}}(G_m) \leq 2^{1-m}$. Therefore

$$C_{\Delta^{L\ln L}}(\cap_{m\geq 1} G_m) = 0.$$

Since for any $y \notin G_m \cup F$, there holds

$$|(\eta_{j_\ell} - \eta)(y)| \leq 2^{-\ell},$$

the claim follows. □

Lemma 4.10 *If $\eta \in \Delta^{L\ln L}(\partial\Omega)$ it has a unique quasi-continuous representative with respect to the capacity $C_{\Delta^{L\ln L}}$.*

Proof. Uniqueness is clear as in the Bessel capacity case [1, Chap 6]. Let $\{\eta_j\} \subset C_0^2(\Omega)$ be a sequence which converges to η in $\Delta^{L\ln L}(\Omega)$. Then there exists a subsequence $\{\eta_{j_\ell}\}$ such that η_{j_ℓ} converges to η uniformly on the complement of an open set of arbitrarily small $C_{\Delta^{L\ln L}}$-capacity. This is the claim. □

Proof of Theorem 4.7. The method is adapted from [11, Th 8], [4, Lemma 4.2]. By Lemma 4.10 we can define the functional h on $\Delta^{L\ln L}(\Omega)$ by

$$h(\eta) = \int_\Omega \overline{\eta}_+ d\mu \qquad \forall \eta \in \Delta^{L\ln L}(\Omega),$$

where $\overline{\eta}$ stands for the $C_{\Delta^{L\ln L}}$-quasi-continuous representative of η. Notice that we can write

$$h(\eta) = -\int_\Omega \Delta \mathbb{G}^\Omega[\mu]\eta dx = -\int_\Omega \mathbb{G}^\Omega[\mu]\Delta\eta dx$$

The following steps are similar to the previous proofs:

Step 1- The functional h is convex, positively homogeneous and l.s.c. The convexity and the homogeneity are clear. If $\eta_n \to \eta$ in $\Delta^{L\ln L}(\partial\Omega)$, then by Lemma 4.10 we can extract a subsequence which is converging everywhere except for a set with zero capacity. The conclusion follows from Fatou's lemma.

Step 2- Since $L_P(\Omega)$ is the dual space of $L_{P^*}(\Omega)$, for any continuous linear form α on $\Delta^{L\ln L}(\Omega)$ there exists $\beta \in L_P(\Omega)$ such that

$$\alpha(\eta) = -\int_\Omega \beta\Delta\eta dx \qquad \forall \eta \in \Delta^{L\ln L}(\Omega).$$

Therefore, in the sense of distributions there holds

$$\alpha(\eta) = -\langle \Delta\beta, \eta \rangle \qquad \forall \eta \in C_0^\infty(\Omega).$$

Step 3- By the geometric Hahn-Banch theorem, h is the upper convex hull of the continuous linear functionals on $\Delta^{L\ln L}(\partial\Omega)$ it dominates. Fix a function $\eta_0 \in C_0^\infty(\Omega)$ and $\epsilon > 0$, there exists a continuous linear form α on $\Delta^{L\ln L}(\Omega)$ and constants a, b such that

$$a + bt + \alpha(\eta) \leq 0 \qquad \forall (\eta, t) \in \mathcal{E} := \{(\eta, t) \in \Delta^{L\ln L}(\Omega) \times \mathbb{R} : h(\eta) \leq t\},$$

and

$$a + b(h(\eta_0) - \epsilon) + \alpha(\eta_0) > 0.$$

The same ideas as in [4, Lemma 4.2] yields successively to $a = 0$ and $b < 0$. If we put $\sigma(\eta) = -b^{-1}\alpha(\eta)$ we derive $\sigma(\eta) \leq h(\eta)$ for all $\eta \in \Delta^{L\ln L}(\Omega)$. This implies in particular that $\sigma(\eta) \leq 0$ if $\eta \leq 0$, thus σ is a positive linear form on $\Delta^{L\ln L}(\Omega)$. Therefore there exist a Radon measure ν on Ω and $\beta \in L_P(\Omega)$ such that $-\Delta\beta = \nu$, $0 \leq \nu \leq \mu$ and

$$\int_\Omega \eta_0 d\mu \leq \epsilon + \int_\Omega \eta_0 d\nu.$$

Step 4- Considering an increasing sequence of compact sets K_j such that $K_j \subset \overset{o}{K}_{j+1}$ and $\cup_j K_j = \Omega$, we construct for each $j \in \mathbb{N}^*$ a Radon measure ν_j and $\beta_j \in L_P(\Omega)$ such that $-\Delta\beta_j = \nu_j$, $0 \leq \nu_j \leq \mu$ and

$$\int_{K_j} d\mu \leq j^{-1} + \int_{K_j} d\nu_j.$$

At last we can assume that the sequence $\{\nu_j\}$ is increasing since if $-\Delta\beta_j = \nu_j$ for $j = 1, 2$, then

$$-\Delta\beta_{1,2} = \sup\{\nu_1, \nu_2\} \leq \nu_1 + \nu_2 = -\Delta\beta_1 - \Delta\beta_2$$

thus $\beta_{1,2} \in L_P(\Omega)$. Iterating this process, we can replace the sequence $\{\nu_j\}$ by $\{\nu'_j\} := \{\nu_1, \sup\{\nu_1, \nu_2\}, \sup\{\nu_3, \sup\{\nu_1, \nu_2\}\}...\}$. The sequence $\{\nu'_j\}$ is increasing, converges to μ and since $\beta_j = \mathbb{G}^\Omega[\nu'_j]$ with $\beta_j \in L_P(\Omega)$, ν'_j belongs to $B^{exp}(\Omega)$. □

As a consequence of this result and the characterization of linear functionals over $L\ln L(\Omega)$, the following result holds.

Corollary 4.11 *Assume μ is a bounded positive good measure in Ω, then there exist an increasing sequence of positive measures ν_j in Ω and positive real numbers θ_j such that $\nu_j \to \mu$ in the weak sense of measures and* $\exp\left(\theta_j \mathbb{G}^\Omega[\nu_j]\right) \in L^1(\Omega)$.

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