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A note on the equation $-\Delta u + e^u - 1 = 0$

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Abstract If Ω is a bounded domain in \mathbb{R}^N , we study conditions on a Radon measure μ on $\partial\Omega$ for solving the equation $-\Delta u + e^u - 1 = 0$ in Ω with $u = \mu$ on $\partial\Omega$. The conditions are expressed in terms of nonlinear capacities.

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1 Introduction

Basic results

Consider the equation

$$-\Delta u + e^u - 1 = 0 \tag{1.1}$$

in Ω , a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega = \Sigma$. Let $\rho(x) = \text{dist}(x, \Sigma)$. We denote by $\mathbb{P}^\Omega[\mu]$ the Poisson potential of a boundary measure μ , and by $\mathbb{G}^\Omega[\phi]$ the Green potential of $\phi \in \mathfrak{M}(\Omega)$. The study is initiated by M. Grillot and L. Véron¹

Boundary measures

$$\begin{aligned} -\Delta u + e^u - 1 &= 0 && \text{in } \Omega \\ u &= \mu && \text{on } \Sigma, \end{aligned} \tag{1.2}$$

with $\mu \in \mathfrak{M}_+(\Sigma)$. If the measure μ is such that (1.24) admits a solution, always unique, is it called **admissible**. The set of admissible positive boundary measures is denoted by $\mathfrak{M}_+^{exp}(\Sigma)$. Assume $\mu = \mu_S + \mu_R$ with $\mu_S \perp dH^{N-1}$, $\mu_R \ll dH^{N-1}$ then

¹Grillot M. et Véron L., *Boundary trace of solutions of the Prescribed Gaussian curvature equation*, Proc. Roy. Soc. Edinburgh **130 A**, 1-34 (2000)

Theorem 1 *Assume*

$$\exp(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx) \quad (1.3)$$

then μ is admissible.

Proof. A weaker form is proved by Grillot and Véron. Moreover, the proof presented below is simpler. For $k > 0$, set $\mu_{R,k} = \inf\{k, \mu_R\}$ and denote by u_k the solution of

$$\begin{aligned} -\Delta u_k + e^{u_k} - 1 &= 0 & \text{in } \Omega \\ u_k &= \mu_S + \mu_{R,k} & \text{on } \Sigma. \end{aligned} \quad (1.4)$$

Such a solution exists because

$$\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) \leq e^k \exp(\mathbb{P}^\Omega[\mu_S])$$

by the maximum principle, and (1.3) implies that $\exp(\mathbb{P}^\Omega[\mu_S + \mu_{R,k}]) - 1 \in L^1(\Omega; \rho dx)$. The sequence u_k is nondecreasing. Since, for any $\zeta \in C_c^{1,1}(\bar{\Omega})$,

$$\int_{\Omega} (-u_k \Delta \zeta + (e^{u_k} - 1)\zeta) dx = \int_{\Sigma} \frac{\partial \zeta}{\partial \nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular $\zeta = \zeta_0$ as being the solution of

$$\begin{aligned} -\Delta \zeta_0 &= 1 & \text{in } \Omega \\ \zeta_0 &= 0 & \text{on } \Sigma, \end{aligned} \quad (1.5)$$

we get

$$\int_{\Omega} (u_k + (e^{u_k} - 1)\zeta_0) dx = - \int_{\Sigma} \frac{\partial \zeta_0}{\partial \nu} d(\mu_S + \mu_{R,k}) \leq c \|\mu\|_{\mathfrak{M}}. \quad (1.6)$$

Thus $u = \lim_{k \rightarrow \infty} u_k$ is integrable,

$$\int_{\Omega} (u + (e^u - 1)\zeta_0) dx \leq c \|\mu\|_{\mathfrak{M}},$$

and the convergence of u_k and e^{u_k} to u and e^u hold respectively in $L^1(\Omega)$ and $L^1(\Omega; \rho dx)$. It follows that u satisfies (1.24). \square

The proof of the next result, inspired of Brezis, Marcus and Ponce², is easy

Theorem 2 *The following properties hold:*

(i) *If $\mu \in \mathfrak{M}_+^{exp}(\Sigma)$ and $0 \leq \tilde{\mu} \leq \mu$, then $\tilde{\mu} \in \mathfrak{M}_+^{exp}(\Sigma)$.*

²Brezis H., Marcus M. and Ponce A., *Nonlinear elliptic equations with measures revisited*, preprint (2004).

(ii) Let $\{\mu_n\} \subset \mathfrak{M}_+^{exp}(\Sigma)$ be an increasing sequence which converges to μ in the weak sense of measures. Then $\mu \in \mathfrak{M}_+^{exp}(\Sigma)$.

(iii) $\mathfrak{M}_+^{exp}(\Sigma) + L_+^1(\Sigma) = \mathfrak{M}_+^{exp}(\Sigma)$.

Proof. I- Let $u = u_\mu$ be the solution of (1.24) and $w = \inf\{u, \mathbb{P}^\Omega[\tilde{\mu}]\}$. Since $\mathbb{P}^\Omega[\tilde{\mu}]$ is a supersolution for (1.1), w is a supersolution too. Furthermore w is nonnegative and $e^w - 1 \in L^1(\Omega; \rho dx)$. By Doob's theorem w admits a boundary trace $\mu^* \in \mathfrak{M}_+(\Sigma)$ and $\mu^* \leq \tilde{\mu} \leq \mu$. Let w^* be the solution of

$$\begin{aligned} -\Delta w^* + e^u - 1 &= 0 & \text{in } \Omega \\ w^* &= \tilde{\mu} & \text{on } \Sigma. \end{aligned}$$

then $u \geq w \geq w^*$ and ³,

$$\lim_t \int_{\Sigma_t} w^*(t, \cdot) \eta dS_t = \int_{\Sigma} \eta d\tilde{\mu} \quad \forall \eta \in C(\Sigma),$$

(here we denote by Σ_t the set of $x \in \Omega$ such that $\rho(x) = t > 0$). This implies that the boundary trace of w^* is $\tilde{\mu}$ and thus $\mu^* = \tilde{\mu}$. Set $\Omega_t = \{x \in \Omega : \rho(x) > t\}$ and let v_t be the solution of

$$\begin{aligned} -\Delta v_t + e^{v_t} - 1 &= 0 & \text{in } \Omega_t \\ v_t &= w & \text{on } \Sigma_t. \end{aligned}$$

Then $v_t \leq w$ in Ω_t . Furthermore

$$0 < t' < t \implies v_{t'} \leq v_t \quad \text{in } \Omega_{t'}.$$

Then $\tilde{u} = \lim_{t \rightarrow 0} v_t$ exists, the convergence holds in $L^1(\Omega)$ and $e^{v_t} \rightarrow e^{\tilde{u}}$ in $L^1(\Omega; \rho dx)$ (here we use the fact that $e^w \in L^1(\Omega; \rho dx)$). Because

$$\lim_{t \rightarrow 0} \int_{\Sigma_t} \tilde{w}(t, \cdot) \eta dS_t = \int_{\Sigma} \eta d\tilde{\mu} \quad \forall \eta \in C(\Sigma),$$

and $v_t = \tilde{w}$ on Σ_t , it follows that \tilde{u} admits $\tilde{\mu}$ for boundary trace and thus $\tilde{u} = u_{\tilde{\mu}}$.

II- Let $u_n = u_{\mu_n}$ be the solutions of (1.24) with boundary value μ_n . The sequence $\{u_n\}$ is increasing. Since

$$\int_{\Omega} (u_n + (e^{u_n} - 1)\zeta_0) dx = - \int_{\Sigma} \frac{\partial \zeta_0}{\partial \nu} d\mu_n \leq - \int_{\Sigma} \frac{\partial \zeta_0}{\partial \nu} d\mu, \quad (1.7)$$

we conclude as in the proof of Theorem 1, that u_n increases and converges to a solution $u = u_\mu$ of (1.24) with boundary value μ .

³Marcus M. and Véron L., *The boundary trace and generalized B.V.P. for semilinear elliptic equations with coercive absorption*, Comm. Pure Appl. Math. **56**, 689-731 (2003).

III- In the proof of I we have actually used the following result : *Let w be a non-negative supersolution of (1.1) such that $e^w \in L^1(\Omega; \rho dx)$ and let $\mu \in \mathfrak{M}_+(\Sigma)$ be the boundary trace of w . Then μ is admissible. Let $f \in L^1_+(\Sigma)$ and μ be an admissible measure. We denote by $u = u_\mu$ the solution of (1.24). For $k > 0$, set $f_k = \min\{k, f\}$. The function $w_k = u_\mu + \mathbb{P}^\Omega[f_k]$ is a nonnegative supersolution, and, since $\mathbb{P}^\Omega[f_k] \leq k$, $e^{w_k} \in L^1(\Omega; \rho dx)$. Furthermore the boundary trace of w_k is $\mu + f_k$. Therefore $\mu + f_k$ is admissible. We conclude by II that $\mu + f$ is admissible \square*
Remark. The assertions I and II in Theorem 1 are still valid if we replace $r \mapsto e^r - 1$ by any continuous nondecreasing function f vanishing at 0.

The Orlicz space framework

Let $M_{exp}(\Sigma)$ be the set of nonnegative boundary measures μ such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx). \quad (1.8)$$

The set $M_{exp}(\Sigma)$ is not a linear space, but it is a convex subset of $\mathfrak{M}_+(\Sigma)$. Put

$$p(t) = \operatorname{sgn}(s)(e^s - 1), \quad P(t) = e^{|t|} - 1 - |t|,$$

and

$$\bar{p}(s) = \operatorname{sgn}(s) \ln(|s| + 1), \quad P^*(t) = (|t| + 1) \ln(|t| + 1) - |t|.$$

Then P and P^* are complementary functions in the sense that Young inequality holds

$$xy \leq P(x) + P^*(y) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if $x = \bar{p}(y)$ or $y = p(x)$. It is classical to define

$$M_P(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx)\}, \quad (1.9)$$

$$M_{P^*}(\Omega; \rho dx) = \{\phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega; \rho dx)\}. \quad (1.10)$$

The Orlicz spaces $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ are the vector spaces spanned respectively by $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$. They are endowed with the Luxemburg norms

$$\|\phi\|_{L_{P\rho}} = \inf \left\{ k > 0 : \int_{\Omega} P\left(\frac{f}{k}\right) \rho dx \leq 1 \right\}. \quad (1.11)$$

and

$$\|\phi\|_{L_{P^*\rho}} = \inf \left\{ k > 0 : \int_{\Omega} P^*\left(\frac{f}{k}\right) \rho dx \leq 1 \right\}. \quad (1.12)$$

Furthermore the Hölder-Young inequality holds ⁴

$$\left| \int_{\Omega} \phi \psi \rho dx \right| \leq \|\phi\|_{L_{P\rho}} \|\psi\|_{L_{P^*}} \quad \forall (\phi, \psi) \in L_P(\Omega; \rho dx) \times L_{P^*}(\Omega; \rho dx). \quad (1.13)$$

Since P^* satisfies the Δ_2 -condition, $M_{P^*}(\Omega; \rho dx) = L_{P^*}(\Omega; \rho dx)$. Furthermore, since

$$\frac{|a| \ln(1 + |a|)}{2} \leq P^*(a) \leq |a| \ln(1 + |a|) \quad \forall a \in \mathbb{R},$$

the space $L_{P^*}(\Omega; \rho dx)$ is associated with the class $L \ln L(\Omega; \rho dx)$ and to the Hardy-Littlewood maximal function. We recall its definition: we consider a cube Q_0 containing $\bar{\Omega}$, with sides parallel to the axes. If $f \in L^1(\Omega)$ we denote by \tilde{f} its extension by 0 in $Q_0 \setminus \Omega$ and put

$$M_{Q_0}[f](x) = \sup \left\{ \frac{1}{|Q|} \int_Q |f|(y) dy : Q \in \mathcal{Q}_x \right\}$$

where \mathcal{Q}_x denotes the set of all cubes containing x and contained in Q_0 , with sides parallel to the axes. Thus

$$\|f\|_{L \ln L_{\rho}} = \int_{Q_0} M_{Q_0}[f](x) \rho dx.$$

and

$$f \in L \ln L(\Omega; \rho dx) \iff \int_{\Omega} |f| \ln(1 + |f|) \rho dx$$

Furthermore, $L_P(\Omega; \rho dx)$ is the dual space of $L_{P^*}(\Omega; \rho dx)$, ⁵ if ℓ is a continuous linear functional on $L \ln L(\Omega; \rho dx)$, there exist a measurable function g_{ℓ} and some $\theta > 0$ such that

$$\begin{cases} \ell(f) = \int_{\Omega} g_{\ell} f dx \quad \forall f \in L \ln L(\Omega; \rho dx), \\ \int_{\Omega} e^{\theta |g_{\ell}|} \rho dx < \infty. \end{cases}$$

This can be seen as a consequence of Young's inequality.

Definition 1.1 *The space of all distributions on Σ such that*

$$\mathbb{P}^{\Omega}[\mu] \in L_P(\Omega; \rho dx)$$

⁴M.A. Krasnosel'skii and Y. B. Rutickii *Convex functions and Orlicz spaces* P. Noordhoff Ltd, Groningen (1961).

⁵M. Fuchs and G. Seregin *A regularity theory for variational integrals with $L \ln L$ -growth* **Calc. Var.** **6** (1998), 171-187.

is denoted by $B^{\text{exp}}(\Sigma)$, with norm

$$\|\mu\|_{B^{\text{exp}}} = \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}}. \quad (1.14)$$

The subset of distributions such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx)$$

is denoted by $M^{\text{exp}}(\Sigma)$.

The analytic characterization of $B^{\text{exp}}(\Sigma)$ can be done in introducing the space of normal derivatives of Green potentials of $L \ln L$ functions:

$$N^{L \ln L}(\Sigma) = \{\eta : \rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho dx)\}. \quad (1.15)$$

where ρ^* is a smooth positive function with value ρ in a neighborhood of Σ . Then

$$\left| \int_\Sigma \eta d\mu \right| = \left| \int_\Omega \mathbb{P}^\Omega[\mu] \Delta(\rho^* \mathbb{P}^\Omega[\eta]) dx \right| \leq \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P_\rho^*}}. \quad (1.16)$$

Notice that the actual regularity of the η function is not clear, although

$$\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho dx) \implies \Delta(\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega).$$

If we take, as a norm on $N^{L \ln L}(\Sigma)$

$$\|\eta\|_{N^{L \ln L}} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta])\|_{L_{P_\rho^*}}, \quad (1.17)$$

and define the $C_{N^{L \ln L}}$ -capacity of a compact subset K of Σ by

$$C_{N^{L \ln L}}(K) = \inf\{\|\eta\|_{N^{L \ln L}} : \eta \in C^2(\Sigma), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}, \quad (1.18)$$

the following result follows from (1.16):

Proposition 1 *If $\mu \in B_+^{\text{exp}}(\Sigma)$, it does not charge Borel subsets with $C_{N^{L \ln L}}$ -capacity zero.*

Because P^* satisfies the Δ_2 -condition, $L_P(\Omega; \rho dx)$ is the dual space of $L_{P^*}(\Omega; \rho dx)$. Now we admit provisionally the following result of Feyel-de la Pradelle type ⁶

Theorem 3 *Let $\mu \in \mathfrak{M}_+(\Sigma)$, which does not charge Borel sets with $C_{N^{L \ln L}}$ -capacity zero. Then there exists an increasing sequence $\{\mu_n\} \subset B^{\text{exp}}(\Sigma)$ which converges*

⁶D. Feyel and A. de la Pradelle *Topologies fines et compactification associées à certains espaces de Dirichlet* **Ann. Inst. Fourier** **27** (1977), 121-146.

weakly to μ .

The proof could be a parallel copy of the original proof. It may have been proved for general capacities in duality (we shall see later on). A variant of this result is the Dal Maso form ⁷

Theorem 4 *Let $\mu \in \mathfrak{M}_+(\Sigma)$, which does not charge Borel sets with $C_{N^L \ln L}$ -capacity zero. Then there exists $\nu \in N_+^{L \ln L}(\Sigma)$ and a nonnegative Borel function h such that*

$$\mu = h d\nu.$$

The role of Orlicz capacities: admissible measures and removable sets

As we have already seen it, a measure in $B_+^{exp}(\Sigma)$ is admissible, and does not charge Borel subsets of $C_{N^L \ln L}$ -capacity zero. The following result is a slight extension of a result of Grillot-Véron, with a proof which inherits some observations of Brezis-Marcus-Ponce.

Theorem 5 *Let μ be an admissible measure. Then μ does not charge Borel subsets of $C_{N^L \ln L}$ -capacity zero.*

Proof. Let K be a compact subset with $C_{N^L \ln L}$ -capacity zero. There exist a sequence $\{\eta_n\} \subset C^2(\Sigma)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of K and

$$\lim_{n \rightarrow \infty} \|\eta_n\|_{N^L \ln L} = \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}} = 0. \quad (1.19)$$

Take $\rho^* \mathbb{P}^\Omega[\eta_n]$ as a test function, then

$$\int_{\Omega} (-u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) + (e^u - 1) \rho^* \mathbb{P}^\Omega[\eta_n]) dx = - \int_{\Sigma} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial \nu} d\mu$$

Since $\frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial \nu} = \eta_n$ and $\mu > 0$, there holds $-\int_{\Sigma} \frac{\partial(\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial \nu} d\mu \geq \mu(K)$.

Furthermore

$$\left| \int_{\Omega} u \Delta(\rho^* \mathbb{P}^\Omega[\eta_n]) dx \right| \leq \|u\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}}. \quad (1.20)$$

Then

$$\mu(E) \leq \int_{\Omega} (e^u - 1) \rho^* \mathbb{P}^\Omega[\eta_n] dx + \|u\|_{L_{P_\rho}} \|\rho^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L_{P_\rho^*}}.$$

⁷G. Dal Maso *On the integral representation of certain local functionals* **Recherche Math.** **32** (1983), 85-113.

By the same argument as in Brezis-Marcus-Ponce, $\lim_n \rightarrow \infty \rho^* \mathbb{P}^\Omega[\eta_n] = 0$, a.e. in Ω , and there exists a nonnegative L^1_ρ -function Φ such that $0 \leq \rho^* \mathbb{P}^\Omega[\eta_n] \leq \Phi$. By (1.19), (1.20) and Lebesgue's theorem, $\mu(E) = 0$. \square

Definition 1.2 *A subset $E \subset \Sigma$ is said removable, if and only if any positive solution u of (1.1) in Ω , which is continuous in $\bar{\Omega} \setminus E$ and vanishes on $\Sigma \setminus E$, is identically zero.*

Theorem 6 *A compact subset $K \subset \Sigma$ is removable if and only if $C_{NL \ln L}(K) = 0$.*

Proof. Let $u \in C(\bar{\Omega} \setminus K)$ be a solution of (1.1) which is zero on $\Sigma \setminus K$. Let $\{\eta_n\} \subset C^2(\Sigma)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood \mathcal{V} of K and (1.19) holds. Put $\theta_n = 1 - \eta_n$. Put $\rho_K(x) = \text{dist}(x, K)$. Then, as a consequence of Keller-Osserman estimate and the fact that u vanishes on K^c , there holds

$$u(x) \leq C \frac{\rho(x) \ln(2/\rho_K(x))}{\rho_K(x)} + D.$$

Thus the function $\zeta_n = \rho^* \mathbb{P}^\Omega[\theta_n]$ is an admissible test function for u , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

Clearly $\mathbb{P}^\Omega[\theta_n] = 1 - \mathbb{P}^\Omega[\eta_n]$ and

$$\Delta \zeta_n = \Delta \rho^* - \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])$$

Inasmuch we can modify ρ^* in order to have $-\Delta \rho^* \geq 0$, in which case $\rho^* = \rho$ near Σ is replaced by $\rho^* \approx \rho$, we derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u \zeta_n dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) \zeta_n dx - \int_{\Omega} Q(\zeta_n^{-1} \Delta(\rho^* \mathbb{P}^\Omega[\eta_n])) \zeta_n dx, \end{aligned}$$

where

$$Q(r) = (|r| + 2^{-1}) \ln(2|r| + 1) - |r| \leq C |r| \ln(|r| + 1) \quad \forall r \in \mathbb{R}.$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) dx, \quad (1.21)$$

since $\zeta_n^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \leq \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|$. Furthermore

$$\begin{aligned} \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|) &= -\ln \rho + \ln(\rho + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|) \\ &\leq -\ln \rho + \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|) \end{aligned}$$

But (we can assume $\rho \leq 1$)

$$\begin{aligned} &\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|) dx \\ &\leq -\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln \rho dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|) dx, \end{aligned}$$

and

$$\begin{aligned} &\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln \rho^{-1} dx \\ &= \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln \rho^{-1} dx + \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| > 1\}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln \rho^{-1} dx \\ &\leq \int_{\{|\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \leq 1\}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln \rho^{-1} dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|) dx \end{aligned}$$

But

$$\lim_{n \rightarrow \infty} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| = 0 \quad \text{a. e. in } \Omega,$$

at least up to some subsequence. Thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_m])|) dx = 0 \quad (1.22)$$

Using (1.30), we derive $u = 0$.

Conversely, assume that $C_{NL \ln L}(K) > 0$. Since the dual definition of $C_{NL \ln L}(K)$ appears to be⁸

$$C_{NL \ln L}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}_+(\Sigma), \mu(K^c) = 0, \|\mathbb{P}^\Omega[\mu]\|_{L_{P_\rho}} \leq 1\},$$

there exists a nonzero measure $\mu \in \mathfrak{M}_+(\Sigma)$ such that $\mu(K^c) = 0$ in the space $B_+^{exp}(\Sigma)$. This means that $\theta\mu \in M_+^{exp}(\Sigma)$ for some $\theta > 0$. Thus problem (1.24) admits a solution. \square

⁸S. E. Kuznetsov *Removable singularities for $Lu = \Psi(u)$ and Orlicz capacities* **J. Funct. Anal.** **170** (2000), 428-449.

By Theorems 1 and 3 we have a partial characterization of measures for which problems (1.24) admits a solution and K is not removable.

Theorem 7 *If a measure μ is admissible there exists an increasing sequence of $\{\mu_n\}$ satisfying*

$$\int_{\Omega} \exp(\theta_n \mathbb{P}^{\Omega}[\mu_n]) \rho dx < \infty \quad (1.23)$$

for some $\theta_n > 0$ which converges to μ .

Several questions can be adressed

1- If a singular measure μ is admissible does it exist an increasing sequence of $\{\mu_n\}$ converging to μ such that (1.23) holds with $\theta_n = 1$?

2- If a measure μ does not charge Borel sets with $C^{L \ln L}$ -capacity zero, does it exist $\theta > 0$ such that $\theta \mu \in M^{exp}(\Sigma)$?

3- If a singular measure μ is admissible, then $(1 - \delta)\mu \in M^{exp}(\Sigma)$ for any $\delta > 0$?

Further extensions

A part of the above construction could be extended to problems with more general nonlinearity such as

Boundary measures

$$\begin{aligned} -\Delta u + P(u) &= 0 \quad \text{in } \Omega \\ u &= \mu \quad \text{on } \Sigma, \end{aligned} \quad (1.24)$$

where P is a convex increasing function vanishing at 0 and such that $\lim_{r \rightarrow \infty} P(r)/r = \infty$: In Theorem 1- P , (1.3) should be replaced by

$$P(\mathbb{P}^{\Omega}[\mu_n]) \in L^1(\Omega; \rho dx). \quad (1.25)$$

In Theorem 2- P , (i) and (ii) still hold. For simplicity we assume that P is a N -function in the sense of Orlicz spaces

$$P(r) = \int_0^r p(s) ds$$

where p is increasing and vanishes at 0. Let P^* be the conjugate N -function, $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ the corresponding Orlicz spaces endowed with the

Luxemburg norms. Then Proposition 1- P is valid, provided the space $B^P(\Sigma)$ and $M^P(\Sigma)$ are accordingly defined with the following notations:

$$N^{P^*}(\Sigma) = \{\eta : \rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta]) \in L_{P^*}(\Omega; \rho dx)\}$$

with corresponding norm

$$\|\eta\|_{N^{P^*}} = \|\rho^{-1}\Delta(\rho^*\mathbb{P}^\Omega[\eta])\|_{L_{P^*}}$$

and the corresponding capacity $C_{N^{P^*}}$. It is still likely that Theorem 3- P , 4- P hold. The proof of Theorem 5- P should be valid without any major modification. **However**, it appears that the characterization of removable sets cannot be adapted without further properties of the function P^* like the Δ_2 -condition. Such a condition holds usually when P has a power-like growth (> 1) and a logarithmic type growth.

Internal measures

Most of the above techniques can be extended to problem of the types

$$\begin{aligned} -\Delta u + e^u - 1 &= \mu & \text{in } \Omega \\ u &= 0 & \text{on } \Sigma, \end{aligned} \quad (1.26)$$

and

$$\begin{aligned} -\Delta u + P(u) &= \mu & \text{in } \Omega \\ u &= 0 & \text{on } \Sigma. \end{aligned} \quad (1.27)$$

where $\mu \in \mathfrak{M}_+^b(\Omega)$. If we denote by $\mathfrak{M}_+^{exp}(\Omega)$ the set of admissible measures for (1.26), then the internal outer part of Theorems 2 and 5 are already known (BMP). Inequality (1.16) becomes

$$\left| \int_\Omega \eta d\mu \right| = \left| \int_\Omega \eta \Delta \mathbb{G}^\Omega[\mu] dx \right| = \left| \int_\Omega \mathbb{G}^\Omega[\mu] \Delta \eta dx \right| \leq \|\mathbb{G}^\Omega[\mu]\|_{L_P} \|\Delta \eta\|_{L_{P^*}}. \quad (1.28)$$

for $\eta \in C_c^{1,1}(\bar{\Omega})$. Define the $C_{\Delta L \ln L}$ -capacity of a compact subset K of Σ by

$$C_{\Delta L \ln L}(K) = \inf\{\|\Delta \eta\|_{L_{P^*}} : \eta \in C_c^2(\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}, \quad (1.29)$$

Thus Proposition 1 is valid under the form

Proposition 1 bis *If $\mu \in B_+^{\text{exp}}(\Omega)$, it does not charge Borel subsets with $C_{\Delta L \ln L}$ -capacity zero.*

Theorem 6 becomes

Theorem 6 bis *A compact subset $K \subset \Omega$ is removable if and only if $C_{\Delta L \ln L}(K) = 0$.*

Proof. Let $u \in C(\Omega \setminus K)$ be a solution of (1.1) which is zero on Σ . Let $\{\eta_n\} \subset C^2(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood \mathcal{V} of K and (1.19) holds. Put $\rho_K(x) = \text{dist}(x, K)$. Then, as a consequence of Keller-Osserman estimate, there holds

$$u(x) \leq C \ln(2/\rho_K(x)) + D.$$

Put $\theta_n = 1 - \eta_n$. Then the function $\zeta_n = \phi_1 \theta_n$ (ϕ_1 being the first eigenfunction of $-\Delta$) is an admissible test function for u , and

$$\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) dx = 0.$$

We derive

$$\begin{aligned} - \int_{\Omega} u \Delta \zeta_n dx &= - \int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u dx \\ &\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) dx - \int_{\Omega} Q(\Delta(\zeta_n)) dx. \end{aligned}$$

Therefore

$$\int_{\Omega} (e^u - 1 - u) \zeta_n dx \leq 2C \int_{\Omega} |\Delta \zeta_n| \ln(1 + |\Delta \zeta_n|) dx, \quad (1.30)$$

The conclusion is as in BMP. \square

Reduced measures

What are the reduced measures both for the boundary and internal problems (1.24) (resp. 1.27)? A projection onto the closure of the sets $M^{\text{exp}}(\Sigma)$ (resp. $M^{\text{exp}}(\Omega)$) of positive measures in Σ (resp. Ω) satisfying

$$\int_{\Omega} \exp(\mathbb{P}^{\Omega}[\mu]) \rho dx < \infty$$

(resp.

$$\int_{\Omega} \exp(\mathbb{G}^{\Omega}[\mu]) dx < \infty.)$$

The definition of the projection is not clear, although an important fact is that it ensures uniqueness. This conjectures could be extended to problems involving operator $u \mapsto -\Delta u + P(u)$.