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► **To cite this version:**

Christian H. Weiss. Jumps in binomial AR(1) processes. *Statistics and Probability Letters*, 2009, 79 (19), pp.2012. 10.1016/j.spl.2009.06.010 . hal-00573469

HAL Id: hal-00573469

<https://hal.science/hal-00573469>

Submitted on 4 Mar 2011

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Accepted Manuscript

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PII: S0167-7152(09)00234-X
DOI: 10.1016/j.spl.2009.06.010
Reference: STAPRO 5450

To appear in: *Statistics and Probability Letters*

Received date: 24 April 2009

Revised date: 12 June 2009

Accepted date: 23 June 2009

Please cite this article as: Weiß, C.H., Jumps in binomial AR(1) processes. *Statistics and Probability Letters* (2009), doi:10.1016/j.spl.2009.06.010

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Jumps in Binomial AR(1) Processes

Christian H. Weiß

University of Würzburg, Inst. of Mathematics, Dep. of Statistics, Germany.

Abstract

We consider the binomial AR(1) model for serially dependent processes of binomial counts. After a review of its definition and known properties, we investigate marginal and serial properties of jumps in such processes. Based on these results, we propose the jumps control chart for monitoring a binomial AR(1) process. We show how to evaluate the performance of this control chart and give design recommendations.

Key words: Binomial AR(1) model; jumps; control chart; average run length.

1 Introduction

The field of count data time series analysis has become an area of emerging interest both in research and professional practice. Because of the broad field of potential applications, see Weiß (2008), a number of time series models for counts have been proposed in literature, which are able to describe different types of marginal distribution and autocorrelation structure. In particular, ARMA-type models based on an appropriate type of thinning operation have become quite popular in recent years, see Weiß (2008). In this article, we consider the binomial AR(1) model, which was first proposed by McKenzie (1985) and further investigated by Weiß (2009a). It is able to model processes of binomial counts with a serial dependence structure being similar to that of a usual AR(1) model. Furthermore, the binomial AR(1) model has an intuitive structure and is easy to interpret.

Definition 1 (Binomial AR(1) Model) *Let $p \in (0; 1)$ and $\rho \in \left(\max \left\{ -\frac{p}{1-p}, -\frac{1-p}{p} \right\}; 1 \right)$. Define $\beta := p \cdot (1 - \rho)$ and $\alpha := \beta + \rho$. Fix $n \in \mathbb{N}$.*

Email address: christian.weiss@mathematik.uni-wuerzburg.de (Christian H. Weiß).

The process $(X_t)_{\mathbb{N}_0}$, defined by the recursion

$$X_t = \alpha \circ X_{t-1} + \beta \circ (n - X_{t-1}) \quad \text{for } t \geq 1, \quad \text{where } X_0 \sim B(n, p),$$

where all thinnings are performed independently of each other, and where the thinnings at time t are independent of $(X_s)_{s < t}$, is referred to as the binomial AR(1) process.

The condition on ρ guarantees that $\alpha, \beta \in (0; 1)$. The following Summary 1 provides further important properties, which have been shown by McKenzie (1985); Weiß (2009a).

Summary 1 (Properties of Binomial AR(1) Model) *Let $(X_t)_{\mathbb{N}_0}$ be a process following the binomial AR(1) model of Definition 1. Then $(X_t)_{\mathbb{N}_0}$ is a stationary Markov chain with marginal distribution $B(n, p)$, i. e.,*

$$p_k := P(X_t = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

and transition probabilities

$$p_{k|l} := P(X_t = k \mid X_{t-1} = l) = \sum_{m=\max\{0, k+l-n\}}^{\min\{k, l\}} \binom{l}{m} \binom{n-l}{k-m} \alpha^m (1-\alpha)^{l-m} \beta^{k-m} (1-\beta)^{n-l+m-k}.$$

The autocorrelation function is given by $\rho(k) = \rho^k$ for $k \geq 0$. The most important regression properties are

$$E[X_t \mid X_{t-1}] = \rho \cdot X_{t-1} + n\beta,$$

$$V[X_t \mid X_{t-1}] = \rho(1-\rho)(1-2p) \cdot X_{t-1} + n\beta(1-\beta).$$

If X_1, \dots, X_T with $T \in \mathbb{N}$ is a segment from a stationary binomial AR(1) process, and if N_i is the number of X_t equal to i and $p_i := P(X_t = i)$, then Pearson's statistic

$$X_g^2 := \sum_{i=0}^n \frac{(N_i - T p_i)^2}{T p_i} \xrightarrow{D} \sum_{j=1}^n \frac{1 + \rho^j}{1 - \rho^j} \cdot Z_j^2 \quad \text{for } T \rightarrow \infty,$$

where Z_1, \dots, Z_n are independent standard normal random variables.

The binomial AR(1) model of Definition 1 is easy to interpret, see Weiß (2009a). Suppose we have a system of n units, which are, independently of each other, either in state 1 or state 0. Let X_{t-1} be the number of units being in state 1 at time $t-1$. Then $\alpha \circ X_{t-1}$ is the number of units, which are still in state 1 at time t , with individual transition probability α . $\beta \circ (n - X_{t-1})$ is the

number of units, which moved from state 0 to state 1 at time t , with individual transition probability β . So for large positive values of ρ , it is much more likely that a unit remains in state 1 than to move from state 0 to state 1. Due to this simple interpretation, the binomial AR(1) model applies well to many real-world situations. Imagine, for instance, a computer pool with n machines, which are either occupied (state 1) or not (state 0). Here, X_t is the number of machines being occupied at time t , consisting of machines which have been occupied before, and machines which are newly occupied. Analogously, one might think of hotel rooms in a certain hotel being occupied at day t , clerks in a counter room serving a customer, telephones in a call center being occupied, etc.

Summary 2 (Model Estimation) *Let X_0, \dots, X_T be a segment from a binomial AR(1) process with unknown parameters p and ρ , see Definition 1. Weiß (2009a) proposes the following approaches for parameter estimation:*

- *Yule-Walker approach: Estimate p by the arithmetic mean $\frac{1}{n(T+1)} \cdot \sum_{t=0}^T X_t$, and ρ by the first order empirical autocorrelation.*
- *Conditional least squares approach: Estimate p and ρ by minimizing the conditional sum of squares*

$$CSS(p, \rho) := \sum_{t=1}^T (X_t - \rho \cdot X_{t-1} - np(1 - \rho))^2.$$

- *Maximum likelihood approach: Estimate p and ρ by numerically maximizing the logarithmic likelihood function*

$$\ell(p, \rho) := \ln p_{X_0} + \sum_{t=1}^T \ln p_{X_t|X_{t-1}}, \quad \text{see Summary 1.}$$

2 Jumps in Binomial AR(1) Processes

An interesting feature of a binomial AR(1) process are jumps $J_t := X_t - X_{t-1}$. The extent of these jumps reflects the extent of serial dependence, defined through the parameter ρ : If ρ is small, then quite large jumps can be observed.

Theorem 2 (Jumps in Binomial AR(1) Processes: Marginal Properties)

Let $(X_t)_{\mathbb{N}_0}$ be a process following the binomial AR(1) model of Definition 1. Then the moment generating function (mgf) of $(J_t)_{\mathbb{N}}$ is given by

$$\mu_J(z) = \left(1 - 2p(1-p)(1-\rho) \cdot (1 - \cosh(z)) \right)^n.$$

All odd order moments of J_t are equal to 0. Variance and excess are given by

$$2\sigma_X^2 \cdot (1 - \rho) \quad \text{and} \quad \frac{1}{2n} \cdot \left((p(1-p)(1-\rho))^{-1} - 6 \right), \quad \text{respectively.}$$

The proof of Theorem 2 is provided by Appendix A.1. Obviously, the variance of the jumps J_t increases for decreasing ρ . Hence, see Section 3, we expect a control chart monitoring the jumps J_t to be sensitive to decreases in ρ (which would imply that it becomes more likely that a unit leaves the state 1, see Section 1). Furthermore, the distribution of the jumps J_t is symmetric around 0 and the excess of J_t decreases for increasing n . So the shape of this distribution is similar to that of a normal distribution for large n , a feature that shall prove to be useful for designing a control chart for jumps J_t , see Section 3.

Theorem 3 (Jumps in Binomial AR(1) Processes: Serial Properties)

Let $(X_t)_{\mathbb{N}_0}$ be a process following the binomial AR(1) model of Definition 1. Then the autocorrelation and the partial autocorrelation function of $(J_t)_{\mathbb{N}}$ are given by

$$\rho_J(k) = -\frac{1-\rho}{2} \cdot \rho^{k-1} \quad \text{and} \quad \rho_{\text{part},J}(k) = -\frac{1-\rho}{2+(k-1) \cdot (1-\rho)} \quad \text{for } k \geq 1.$$

The proof of Theorem 3 is provided by Appendix A.2.

3 An Application: Jumps Control Chart

The discipline of *statistical process control (SPC)* provides methods for monitoring processes along time. The monitored process $(X_t)_{\mathbb{N}_0}$ is said to be *in (statistical) control* if it is stationary, following a particular process model. Otherwise, the process is said to be *out of control*. Assuming that $(X_t)_{\mathbb{N}_0}$ is usually in control, SPC aims at detecting changes in the process that may result from assignable causes at unpredictable times. The most popular tool of SPC is the *control chart*, which is applied to certain control statistics $(Z_t)_{\mathbb{N}}$ computed from the monitored process $(X_t)_{\mathbb{N}_0}$. A control chart is attached with appropriately chosen *upper and lower control limits* UCL and LCL . The observations $(Z_t)_{\mathbb{N}}$ are plotted on the control chart. If a point Z_t exceeds a control limit, i. e., if $Z_t > UCL$ or $Z_t < LCL$, then an alarm signals that the process may be out of control. A widely accepted approach to evaluate the performance of a control chart is to compute characteristics of its *run length (RL)* distribution: The *RL* of a control chart is the number of plotted points until the first alarm is triggered. In particular, the mean of the *RL* distribution, the *average run length (ARL)*, is commonly used as a quality factor for the control chart. The chart design is chosen such that the *ARL* is sufficiently large if the process is in control (the in-control *ARL* is commonly abbreviated as ARL_0), while the *ARL* should be small if the process is out-of-control. For more details on SPC and control charts, see Montgomery (2005).

As outlined in Section 1, the binomial AR(1) model has a great potential for applications in practice, also see the case study reported by Weiß (2009a). Depending on the concrete situation, the practitioner is interested in monitoring procedures, which allow to detect changes in the model parameters p and/or ρ . Concerning changes in p , especially the moving average charts investigated by Weiß (2009a) turned out to be effective. The jumps computed from a binomial AR(1) process offer the potential for creating a control chart being sensitive to a decrease in ρ , since this would increase the variance of the jumps, see Section 2. Concerning the general interpretation and the examples given in Section 1, such a decrease in ρ would imply that it becomes more likely that a unit leaves the state 1, e. g., that the users of a computer pool more often leave their machines (perhaps indicating technical problems).

3.1 Definition and Properties

According to Theorem 2, the stationary distribution of jumps J_t in a binomial AR(1) process is symmetric around 0. Hence, it is reasonable to choose symmetric control limits for a control chart monitoring the jumps, i. e., the jumps control chart requires *only one* design parameter.

Definition 4 (Jumps Control Chart) *The observed jumps $J_t := X_t - X_{t-1}$ are plotted on a chart with control limits $\mp k$ for a $k \in \mathbb{N}$, i. e., the process is considered as being in control unless $|J_t| > k$.*

The design parameter k of the jumps control chart 4 should be chosen such that a certain *average run length (ARL)* performance is reached. The following result is essential for this purpose.

Proposition 5 (Combined Jumps Process) *Let $(X_t)_{\mathbb{N}_0}$ be a binomial AR(1) process according to Definition 1. Define the process $(J_t)_{\mathbb{N}}$ of jumps by $J_t := X_t - X_{t-1}$. Since $(X_t)_{\mathbb{N}_0}$ is Markovian of order 1, also $(X_t, J_t)_{\mathbb{N}}$ is a Markov chain, with range $\{0, \dots, n\} \times \{-n, \dots, n\}$. The transition probabilities are given by*

$$\begin{aligned} p(n, j \mid m, i) &:= P(X_t = n, J_t = j \mid X_{t-1} = m, J_{t-1} = i) \\ &= P(J_t = j \mid X_t = n, X_{t-1} = m, J_{t-1} = i) \cdot P(X_t = n \mid X_{t-1} = m, J_{t-1} = i) \\ &= \delta_{j, n-m} \cdot p_{n \mid m}, \end{aligned}$$

where $\delta_{a,b}$ denotes the Kronecker delta. The marginal probabilities are

$$p(n, j) := P(X_t = n, J_t = j) = P(X_t = n, X_{t-1} = n - j) = p_{n, n-j},$$

where the $p_{k,l} := P(X_t = k, X_{t-1} = l)$ equal 0 for $l < 0$.

Since at least the bivariate process $(X_t, J_t)_\mathbb{N}$ is a Markov chain, it is possible to compute the *ARLs* of the jumps chart 4 by adapting the *Markov chain approach* of Brook & Evans (1972). For this purpose, it is necessary to identify the *reachable in-control states* of $(X_t, J_t)_\mathbb{N}$, i. e., those pairs $(x, j) \in \{0, \dots, n\} \times \{-k, \dots, k\}$, which can be reached by $(X_t, J_t)_\mathbb{N}$ such that no alarm is triggered by chart 4. It is shown in Appendix A.3 that the set $\mathcal{C}(k)$ of such reachable in-control states for given design parameter $0 < k < n$ is given by

$$\mathcal{C}(k) = \{(x, j) \in \{0, \dots, n\} \times \{-k, \dots, k\} \mid j \leq x \leq n + j\}, \quad (1)$$

$$\text{which is of size } |\mathcal{C}(k)| = (2k + 1)(n + 1) - k(k + 1).$$

Using this result and the fact that $(X_t, J_t)_\mathbb{N}$ is a discrete Markov chain according to Proposition 5, the *ARLs* of the combined jumps chart can be computed exactly by adapting the Markov chain approach of Brook & Evans (1972).

Proposition 6 (ARL Computation of Combined Jumps Chart) *Let $(X_t, J_t)_\mathbb{N}$ be the combined jumps process of Proposition 5. Let $\mathcal{C} := \mathcal{C}(k)$ be the control region of Formula (1). Define $p_{m,i}(r)$ for $r \geq 1$ as the probability that the $(r+1)^{st}$ pair leads to the first out-of-control signal, given that $(X_1, J_1) = (m, i) \in \mathcal{C}$, i. e.,*

$$p_{m,i}(r) := P\left((X_{r+1}, J_{r+1}) \notin \mathcal{C}, (X_r, J_r), \dots, (X_2, J_2) \in \mathcal{C} \mid (X_1, J_1) = (m, i)\right),$$

let $\mu_{m,i} := \sum_{r=1}^{\infty} r \cdot p_{m,i}(r)$. Assume that \mathcal{C} is ordered in a certain manner, define

$$\boldsymbol{\mu} := (\dots, \mu_{n,j}, \dots)^\top, \quad \mathbf{Q}^\top := \left(p(n, j \mid m, i)\right)_{(n,j), (m,i) \in \mathcal{C}}.$$

Then $\boldsymbol{\mu}$ is computed as the solution of the linear equation $(\mathbf{I} - \mathbf{Q}) \cdot \boldsymbol{\mu} = \mathbf{1}$. The *ARL* is obtained as $ARL = 1 + \sum_{(m,i) \in \mathcal{C}} \mu_{m,i} \cdot p_{m,m-i}$.

The proof of Proposition 6 is done in complete analogy to Weiß (2009b).

3.2 Design and Performance

Assume that the parameter values p_0 and ρ_0 of the in-control model of the monitored binomial AR(1) process are available; in practice, these parameters are usually estimated from given in-control data using, e. g., one of the approaches of Summary 2. To design an appropriate jumps chart for the given in-control situation, we have to find a value of k such that a satisfactory ARL_0 is reached. In practice, the desired ARL_0 level is defined by considering aspects like the process speed or the costs related to a false alarm, but in literature, it is common to choose $ARL_0 \approx 370$ to make results comparable (The value

370 is motivated by the in-control performance of a standard Shewhart chart with $3\text{-}\sigma$ limits applied to i.i.d. normal observations.).

To find an appropriate value of the design parameter k , we apply Proposition 6 for the in-control model $(p, \rho) = (p_0, \rho_0)$ and for varying values of k . Based on the computed values $ARL_0(k)$, we choose k such that $ARL_0(k)$ is close to the desired in-control performance. The search for an appropriate parameter k can be accelerated by considering that the distribution of jumps has a shape similar to that of a normal distribution, see the discussion after Theorem 2. As an example, if one is interested in reaching an in-control ARL of about 370, then it is known from normal theory ($3\text{-}\sigma$ limits) that the design parameter k should be close to

$$3 \cdot \sqrt{2 \cdot np_0(1 - p_0) \cdot (1 - \rho_0)}.$$

So compared to the conditional and residuals control chart for a binomial AR(1) process proposed by Weiß (2009a), which can also be used for detecting a decrease in ρ , the new jumps control chart is more easy to design and allows to compute ARL s exactly, without the need for simulations.

Next, we study the ARL performance of the jumps chart. Using Proposition 6 for the ARL computations, the performance of the jumps control chart 4 concerning a decrease in ρ compared to the in-control value ρ_0 is investigated. Some of the results are summarized in Figure 1, where the respective in-control situations (n, p_0, ρ_0) and chart designs k can be read from the legend. The charts considered there are designed in such a way that their in-control performance is similar, reaching an ARL_0 between 300 and 400. It becomes clear that the jumps control chart is sensitive to such a decrease in ρ , with the best sensitivity shown for large values of ρ_0 . Given a particular value of ρ_0 , the sensitivity is better in situations, where n is large and where p_0 has a moderate value (around 0.5).

3.3 An Illustrative Example

To illustrate the application of the jumps chart in practice, let us assume that the monitored binomial AR(1) process $(X_t)_{\mathbb{N}_0}$ has the model parameters $(n, p, \rho) = (n, p_0, \rho_0) := (25, 0.2, 0.75)$ in its in-control state. Furthermore, assume that it is desirable to reach an in-control ARL of about 370, which would correspond to $3\text{-}\sigma$ limits of a standard Shewhart chart applied to i.i.d. normal observations. Adapted to the standard deviation of the jumps, we obtain

$$3 \cdot \sqrt{2 \cdot np_0(1 - p_0) \cdot (1 - \rho_0)} \approx 4.24.$$

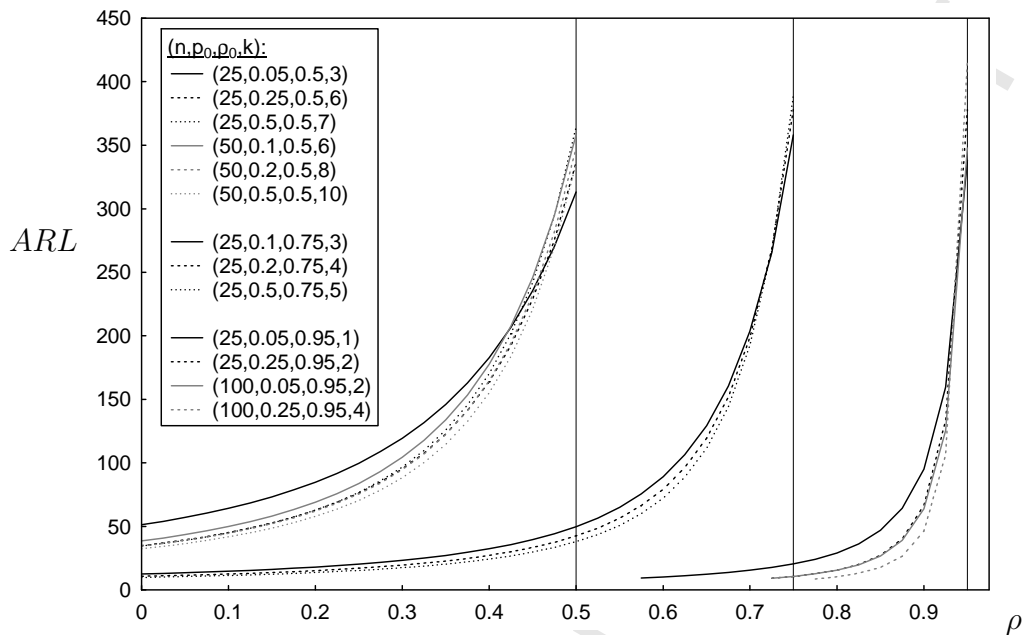


Fig. 1. ARL performance of jumps charts for different in-control models.

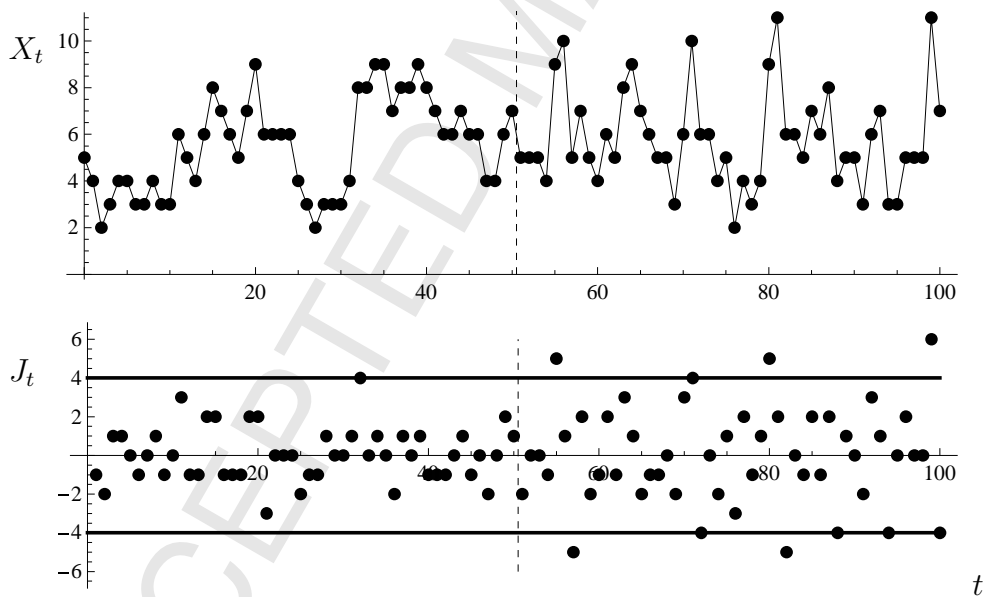


Fig. 2. Data example, see Section 3.3: Counts and jumps against time t .

So we expect that an appropriate value of the design parameter k of the jumps chart will be close to 4. And in fact, we obtain by applying Proposition 6 that

$$\dots, ARL_0(3) \approx 64.9961, ARL_0(4) \approx 378.012, ARL_0(5) \approx 2731.2, \dots$$

so we choose the design parameter $k := 4$. The theoretical performance of this particular jumps chart concerning a decrease in ρ can be read from Figure 1.

To get an impression of the performance in practice, a segment X_0, \dots, X_{100} from a binomial AR(1) process was simulated in the following way: The observations X_0, \dots, X_{50} stem from the in-control model, but then the parameter ρ is shifted to a value of 0.5. The simulated data is shown in the upper part of Figure 2. The lower part shows the computed jumps J_t on a jumps chart with design parameter $k = 4$ (having the out-of-control $ARL \approx 42.4975 \ll 378.012$, also see Figure 1). While the jumps chart does not signal an alarm during the in-control phase (before the vertical line), the out-of-control situation is detected quickly, with the first alarm being triggered at time $t = 55$.

4 Conclusions

We discussed the binomial AR(1) model for serially dependent processes of binomial counts. After having reviewed its definition and known properties, we investigated marginal and serial properties of jumps in such processes. It turned out that the shape of the marginal distribution is similar to that of a normal distribution. This fact is useful for designing the newly proposed jumps control chart for monitoring a binomial AR(1) process. We showed how to compute the ARL s of this control chart and investigated its sensitivity concerning a decrease in ρ compared to the in-control value ρ_0 . Especially for large values of ρ_0 , a decrease in ρ is detected effectively.

Acknowledgements. The author thanks the referee for useful comments on an earlier draft of this article.

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Appears in Applied Stochastic Models in Business and Industry 25.

A Proofs

A.1 Proof of Theorem 2

Let $(X_t)_{\mathbb{N}_0}$ be a process following the binomial AR(1) model of Definition 1 with parameters n, ρ, p . Then the bivariate probability generating function (pgf) $p_{X_t, X_{t-1}}(z_0, z_1)$ of X_t and X_{t-1} is obtained as

$$p_{X_t, X_{t-1}}(z_0, z_1) = \left((1-p)(1-\beta+\beta z_0) + pz_1 \cdot (1-\alpha+\alpha z_0) \right)^n. \quad (\text{A.1})$$

Since $p_X(z) = (1-p+pz)^n$ according to Summary 1, this follows from

$$\begin{aligned} p_{X_t, X_{t-1}}(z_0, z_1) &= E[z_0^{\alpha X_{t-1} + \beta(n-X_{t-1})} \cdot z_1^{X_{t-1}}] \\ &= E[z_1^{X_{t-1}} \cdot E[z_0^{\alpha X_{t-1} + \beta(n-X_{t-1})} \mid X_{t-1}]] \\ &= (1-\beta+\beta z_0)^n \cdot E\left[\left(\frac{1-\alpha+\alpha z_0}{1-\beta+\beta z_0}\right)^{X_{t-1}} \cdot z_1^{X_{t-1}}\right] \\ &= (1-\beta+\beta z_0)^n \cdot p_X\left(\frac{1-\alpha+\alpha z_0}{1-\beta+\beta z_0} \cdot z_1\right) \\ &= (1-\beta+\beta z_0)^n \cdot \left(1-p + pz_1 \cdot \frac{1-\alpha+\alpha z_0}{1-\beta+\beta z_0}\right)^n. \end{aligned}$$

Using the expression (A.1) for the bivariate pgf, we obtain the mgf of $(J_t)_{\mathbb{N}}$. Here, we use the relations $\alpha = \beta + \rho$, $\beta = p(1-\rho)$, $1-\alpha = (1-p)(1-\rho)$, see Definition 1:

$$\begin{aligned} \mu_{J_t}(z) &= p_{X_t, X_{t-1}}(e^z, e^{-z}) = \left((1-p)(1-\beta+\beta e^z) + pe^{-z} \cdot (1-\alpha+\alpha e^z) \right)^n \\ &= \left((1-p)(1-\beta) + \alpha p + (1-p)\beta \cdot e^z + p(1-\alpha) \cdot e^{-z} \right)^n \\ &= \left(1-2p(1-p)(1-\rho) + (1-p)p(1-\rho) \cdot e^z + p(1-p)(1-\rho) \cdot e^{-z} \right)^n \\ &= \left(1-2p(1-p)(1-\rho) \cdot \left(1 - \frac{1}{2}(e^z + e^{-z})\right) \right)^n. \end{aligned}$$

Since the series expansion of $\cosh(z)$ contains only even terms, the same is true for $\mu_{J_t}(z)$, i. e., all odd order moments of J_t are equal to 0. Even order moments are obtained from the relation $\mu_{J_t}^{(k)}(0) = E[J_t^k] = E[(J_t - E[J_t])^k]$. Defining $f(z)$ via

$\mu_J(z) := f(z)^n$ and $n_{(k)} := n \cdots (n - k + 1)$, one computes the derivatives

$$\mu_J'(z) = n \cdot f(z)^{n-1} \cdot f'(z),$$

$$\mu_J''(z) = n_{(2)} \cdot f'(z)^2 \cdot f(z)^{n-2} + n \cdot f''(z) \cdot f(z)^{n-1},$$

$$\mu_J^{(3)}(z) = n_{(3)} \cdot f'(z)^3 \cdot f(z)^{n-3} + 3n_{(2)} \cdot f'(z) \cdot f''(z) \cdot f(z)^{n-2} + n \cdot f^{(3)}(z) \cdot f(z)^{n-1},$$

$$\begin{aligned} \mu_J^{(4)}(z) &= n_{(4)} \cdot f'(z)^4 \cdot f(z)^{n-4} + 6n_{(3)} \cdot f'(z)^2 \cdot f''(z) \cdot f(z)^{n-3} \\ &\quad + 3n_{(2)} \cdot f''(z)^2 \cdot f(z)^{n-2} + 4n_{(2)} \cdot f'(z) \cdot f^{(3)}(z) \cdot f(z)^{n-2} + n \cdot f^{(4)}(z) \cdot f(z)^{n-1}. \end{aligned}$$

Since $\cosh'(z) = \sinh(z)$ and $\sinh'(z) = \cosh(z)$, we have

$$f^{(2k-1)}(z) = 2p(1-p)(1-\rho) \cdot \sinh(z), \quad f^{(2k)}(z) = 2p(1-p)(1-\rho) \cdot \cosh(z), \quad \text{for } k \in \mathbb{N}.$$

Since $\cosh(0) = 1$ and $\sinh(0) = 0$, it follows that

$$f(0) = 1, \quad f^{(2k-1)}(0) = 0, \quad f^{(2k)}(0) = 2p(1-p)(1-\rho), \quad \text{for } k \in \mathbb{N}.$$

Hence, $V[J_t] = \mu_J''(0) = 0 + n \cdot 2p(1-p)(1-\rho) \cdot 1$, and

$$\mu_J^{(4)}(0) = 0 + 0 + 3n_{(2)} \cdot (2p(1-p)(1-\rho))^2 \cdot 1 + 0 + n \cdot 2p(1-p)(1-\rho) \cdot 1.$$

So the excess $E[(J_t - E[J_t])^4]/V[J_t]^2 - 3$ of J_t equals

$$\frac{3(n-1)}{n} + (2np(1-p)(1-\rho))^{-1} - 3 = \frac{1}{2n} \cdot \left((p(1-p)(1-\rho))^{-1} - 6 \right).$$

A.2 Proof of Theorem 3

Since $V[J_t] = 2\sigma_X^2 \cdot (1-\rho)$ and $\sigma_X^2 := V[X_t] = np(1-p)$, we obtain that

$$\begin{aligned} \rho_J(k) &= \frac{1}{2\sigma_X^2 \cdot (1-\rho)} \cdot \text{Cov}[X_t - X_{t-1}, X_{t-k} - X_{t-k-1}] \\ &= \frac{1}{2(1-\rho)} \cdot (2\rho(k) - \rho(k+1) - \rho(k-1)) = \frac{\rho^{k-1}}{2(1-\rho)} \cdot (2\rho - \rho^2 - 1), \end{aligned}$$

which proves the expression for $\rho_J(k)$. So it remains to prove the expression for the partial autocorrelation function. For this purpose, we adopt the notations of the following Definition 7:

Definition 7 (Partial Autocorrelation) Let $(X_t)_{\mathbb{Z}}$ be a weakly stationary process with $0 < \gamma(0) < \infty$, denote $\rho(k) := \text{Corr}[X_t, X_{t-k}]$ for $k \in \mathbb{N}_0$, with $\rho(0) = 1$. For $k \in \mathbb{N}$, define the matrix

$$\mathbf{R}_k := (\rho(|i-j|))_{1 \leq i, j \leq k}.$$

For each $k \in \mathbb{N}$, we define $(a_{k1}, \dots, a_{kk})^\top$ as the solution of the equation

$$\mathbf{R}_k (a_{k1}, \dots, a_{kk})^\top = (\rho(1), \dots, \rho(k))^\top.$$

Then $\rho_{part}(k) := a_{kk}$ is referred to as the k^{th} order partial autocorrelation.

According to Box et al. (1994), the a_{kj} of Definition 7 follow the recursive scheme

$$a_{k+1,k+1} = \frac{\rho(k+1) - \sum_{i=1}^k a_{ki} \cdot \rho(k+1-i)}{1 - \sum_{i=1}^k a_{ki} \cdot \rho(i)}, \quad (\text{A.2})$$

$$a_{k+1,j} = a_{kj} - a_{k+1,k+1} \cdot a_{k,k-j+1} \quad \text{for } j = 1, \dots, k.$$

We shall use the recursive scheme (A.2) and prove the following result (by induction):

$$a_{kj} = -\frac{(1-\rho) \cdot (1+(k-j)(1-\rho))}{2+(k-1)(1-\rho)} \quad \text{for } 1 \leq j \leq k \text{ and } k \in \mathbb{N}, \quad (\text{A.3})$$

with $\rho_{\text{part},J}(k) = a_{kk}$. For $k = 1$, we simply obtain $\rho_{\text{part},J}(1) = \rho_J(1) = -\frac{1-\rho}{2}$, which coincides with (A.3) in this case. So assume that (A.3) has been shown for all $k \leq n$ with some $n \in \mathbb{N}$, consider $k = n + 1$. We shall use the following recursive scheme, see formula (A.2):

$$\rho_{\text{part},J}(n+1) = a_{n+1,n+1} = \frac{\rho_J(n+1) - \sum_{i=1}^n a_{ni} \cdot \rho_J(n+1-i)}{1 - \sum_{i=1}^n a_{ni} \cdot \rho_J(i)},$$

$$a_{n+1,j} = a_{nj} - a_{n+1,n+1} \cdot a_{n,n+1-j} \quad \text{for } j = 1, \dots, n.$$

Applying the geometric sum formulae, one obtains the following results:

$$\begin{aligned} \sum_{i=1}^n a_{ni} \cdot \rho_J(n+1-i) &= \frac{(1-\rho)^2}{2(2+(n-1)(1-\rho))} \cdot \sum_{i=1}^n (1+(n-i)(1-\rho)) \cdot \rho^{n-i} \\ &= \frac{1-\rho}{2(2+(n-1)(1-\rho))} \cdot (1+\rho - (n+1) \cdot \rho^n + (n-1) \cdot \rho^{n+1}), \\ \sum_{i=1}^n a_{ni} \cdot \rho_J(i) &= \frac{(1-\rho)^2}{2(2+(n-1)(1-\rho))} \cdot \sum_{i=1}^n (1+(n-i)(1-\rho)) \cdot \rho^{i-1} \\ &= \frac{n(1-\rho)^2}{2(2+(n-1)(1-\rho))}. \end{aligned}$$

Therefore, it results that

$$\begin{aligned} \rho_J(n+1) - \sum_{i=1}^n a_{ni} \cdot \rho_J(n+1-i) &= \frac{-(1-\rho^2)}{2(2+(n-1)(1-\rho))}, \quad \text{and} \\ 1 - \sum_{i=1}^n a_{ni} \cdot \rho_J(i) &= \frac{(1+\rho) \cdot (2+n(1-\rho))}{2(2+(n-1)(1-\rho))}, \end{aligned}$$

from which the expression for $a_{n+1,n+1} = -(1-\rho)/(2+n(1-\rho))$ follows. Next, one shows the identity

$$\begin{aligned} &2 + (1-\rho) \cdot (2(n-j) + n+1) + (1-\rho)^2 \cdot (n(n-j) + j-1) \\ &= (2+(n-1) \cdot (1-\rho)) \cdot (1+(n+1-j) \cdot (1-\rho)) \end{aligned}$$

by simply multiplying out the product on the right-hand side and collecting the resulting terms. Using this identity, one obtains

$$\begin{aligned}
a_{n+1,j} &= a_{nj} - a_{n+1,n+1} \cdot a_{n,n+1-j} \\
\stackrel{(A.3)}{\text{with } k=n} &= \frac{-(1-\rho)}{(2+(n-1)(1-\rho)) \cdot (2+n(1-\rho))} \\
&\quad \cdot \left((1+(n-j)(1-\rho)) \cdot (2+n(1-\rho)) + (1-\rho) \cdot (1+(j-1)(1-\rho)) \right) \\
&= \frac{-(1-\rho)}{(2+(n-1)(1-\rho)) \cdot (2+n(1-\rho))} \\
&\quad \cdot \left(2+(1-\rho) \cdot (2(n-j)+n+1) + (1-\rho)^2 \cdot (n(n-j)+j-1) \right) \\
\text{identity} &= \frac{-(1-\rho)}{(2+(n-1)(1-\rho)) \cdot (2+n(1-\rho))} \cdot (2+(n-1)(1-\rho)) \cdot (1+(n+1-j)(1-\rho)),
\end{aligned}$$

which completes the proof.

A.3 Proof of Formula (1)

Since $J_t = X_t - X_{t-1}$, we have $J_t = j$ iff $X_t = j + X_{t-1}$. Since $0 \leq X_{t-1} \leq n$, the restriction $j \leq X_t \leq n + j$ follows. The size $|\mathcal{C}(k)|$ follows from evaluating the sum

$$\begin{aligned}
|\mathcal{C}(k)| &= \sum_{j=-k}^k (\min\{n, n+j\} - \max\{0, j\} + 1) \\
&= \left(\sum_{j=-k}^{-1} (n+j) + \sum_{j=0}^n n \right) - \left(\sum_{j=-k}^0 0 + \sum_{j=1}^k j \right) + (2k+1) \\
&= \left(k \cdot n - \frac{1}{2}k(k+1) \right) + (k+1) \cdot n - \left(0 + \frac{1}{2}k(k+1) \right) + (2k+1) \\
&= (2k+1) \cdot n - k(k+1) + (2k+1) = (2k+1)(n+1) - k(k+1).
\end{aligned}$$