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Marcinkiewicz-Zygmund Strong Laws for U -statistics of Weakly Dependent Observations

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Abstract

We prove the Marcinkiewicz-Zygmund Strong Law of Large Numbers for U -statistics of strictly stationary, absolutely regular observations $(\xi_i)_{i \geq 1}$. Under suitable moment conditions and conditions on the mixing rate, we show that

$$n^{-2+\gamma} \sum_{1 \leq i < j \leq n} (h(\xi_i, \xi_j) - Eh(\xi, \eta)) \rightarrow 0$$

for some $\gamma \geq 0$, in the non-degenerate case, and

$$n^{-1+\gamma} \sum_{1 \leq i < j \leq n} (h(\xi_i, \xi_j) - Eh(\xi, \eta)) \rightarrow 0$$

in the degenerate case.

1. Introduction

Let $(\xi_i)_{i \geq 1}$ be a strictly stationary sequence of random variables with a common distribution function $F(x)$, and let $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a symmetric function, i.e. $h(x, y) = h(y, x)$ for all $x, y \in \mathbb{R}$. We define the U -statistics with kernel h by

$$U_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} (h(\xi_i, \xi_j) - Eh(\xi, \eta)),$$

where ξ and η are independent random variables with distribution function F .

The strong law of large numbers (SLLN) for U -statistics in the case of i.i.d. observations $(\xi_i)_{i \geq 1}$ was proved independently by Hoeffding (1961) and Berk (1966). In the case of dependent observations, the SLLN was studied by Wang (1995), Aaronson et al (1996) and Arcones (1998). In the present paper, we

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investigate the Marcinkiewicz-Zygmund SLLN for U -statistics. In the case of i.i.d. observations, the Marcinkiewicz-Zygmund law has been investigated by Sen (1974), Giné and Zinn (1992) and Teicher (1998). As far as we know, there are no results on the Marcinkiewicz-Zygmund SLLN for U -statistics of dependent observations yet.

In this paper we will focus on absolutely regular processes. The absolute regularity coefficients of the sequence $(\xi_i)_{i \geq 1}$ are defined by

$$\beta(k) := \frac{1}{2} \sup \left\{ \sum_{i=1}^n \sum_{j=1}^n |P(A_i \cap B_j) - P(A_i)P(B_j)| \right\},$$

where the supremum is taken over all partitions $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_n\}$ of Ω with $A_i \in \mathcal{F}_1^l, B_j \in \mathcal{F}_{l+k}^\infty$ and all $l \geq 1$. Here \mathcal{F}_k^l denotes the σ -algebra generated by the random variables $\{\xi_i : k \leq i \leq l\}$. The process $(\xi_i)_{i \geq 1}$ is called absolutely regular if $\lim_{k \rightarrow \infty} \beta(k) = 0$.

Now we are ready to formulate our results. Our first theorem investigates U -statistics with bounded kernels.

Theorem 1. *Let $(\xi_i)_{i \geq 1}$ be a strictly stationary, absolutely regular process with mixing coefficients satisfying*

$$\sum_{k=1}^{\infty} k\beta(k) < \infty.$$

(i) *If $h(x, y)$ is a bounded and degenerate kernel, then we have for all $p \in [1, 2)$*

$$\lim_{n \rightarrow \infty} n^{2(1-\frac{1}{p})} U_n = 0 \quad a.s.$$

(ii) *If $h(x, y)$ is a bounded kernel, then we have for all $p \in [1, 2)$*

$$\lim_{n \rightarrow \infty} n^{(1-\frac{1}{p})} U_n = 0 \quad a.s.$$

In the second theorem, we allow unbounded kernels, satisfying some moment conditions.

Theorem 2. *Let $(\xi_i)_{i \geq 1}$ be a strictly stationary, absolutely regular process and suppose that for $p \in [1, 2)$ and $\delta > 0$ we have*

$$\sup_{i,j} E|h(\xi_i, \xi_j)|^{p+\delta} < \infty.$$

Moreover, assume that the mixing coefficients satisfy

$$\sum_{k=1}^{\infty} k^\gamma \beta(k) < \infty,$$

where $\gamma := \max(1, \frac{p(p-1)+\delta(p-2)}{\delta})$. Then

$$\lim_{n \rightarrow \infty} n^{(1-\frac{1}{p})} U_n = 0 \quad a.s.$$

2. Correlation and Moment Inequalities

In this section we present some moment inequalities that will be used in the proofs of our main theorems.

Lemma 2.1. *Let $(\xi_i)_{i \geq 1}$ be a strictly stationary, absolutely regular process, $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ a measurable kernel satisfying, for some $p > 0$,*

$$M := \sup_{i,j} E|h(\xi_i, \xi_j)|^p < \infty.$$

Then $E|h(\xi, \eta)|^p \leq M$, where ξ and η are two independent random variables with the same marginal distribution as ξ_1 .

Proof. Via standard arguments, this follows from the fact that, by definition of absolute regularity, the distribution of (ξ_1, ξ_n) converges in total variation norm to the distribution of (ξ, η) . \square

Lemma 2.2. *Let $(\xi_i)_{i \geq 1}$ be a strictly stationary, absolutely regular process, let $i_1 < \dots < i_j < i_{j+1} < \dots < i_k$ be integers and let F, G and H denote the distribution functions of $(\xi_{i_1}, \dots, \xi_{i_k})$, $(\xi_{i_1}, \dots, \xi_{i_j})$ and $(\xi_{i_{j+1}}, \dots, \xi_{i_k})$ respectively. Then the following two statements hold:*

(i) If g is a measurable function such that

$$\sup |g(x_1, \dots, x_k)| \leq M < \infty,$$

then

$$\left| \int g dF - \int \int g dG dH \right| \leq 2M\beta(i_{j+1} - i_j).$$

(ii) If g is a measurable function such that

$$M = \max \left(\int |g|^{1+\delta} dF, \int \int |g|^{1+\delta} dG dH \right) < \infty$$

for some $\delta > 0$, then

$$\left| \int g dF - \int \int g dG dH \right| \leq 3M^{\frac{1}{1+\delta}} \beta^{\delta/(1+\delta)}(i_{j+1} - i_j).$$

Part (ii) of Lemma 2.2 is proved in Yoshihara (1976) and part (i) follows from the definition of mixing coefficients $\beta(k)$, see also Arcones (1998).

Lemma 2.3. *Assume that $h(x, y)$ is degenerate and let $(\xi_i)_{i \geq 1}$ satisfy one of the following two conditions:*

(i) h is bounded and

$$\sum_{k=1}^{\infty} k\beta(k) < \infty,$$

(ii) for some $\delta > 0$ we have $\sup_{i,j} E|h(\xi_i, \xi_j)|^{2+\delta} \leq M$ and

$$\sum_{k=1}^{\infty} k(\beta(k))^{\frac{\delta}{2+\delta}} < \infty.$$

Then there exists a constant $C > 0$ such that the inequality

$$E \left(\sum_{1 \leq i < j \leq n+a, j > a} h(\xi_i, \xi_j) \right)^2 \leq C(n+a)nM^2 \quad (1)$$

holds for all integers $n \geq 1, a \geq 0$.

Proof. We will follow the proof of Lemma 2 from Yoshihara (1976). First we will prove (1) under conditions (i). We define

$$\begin{aligned} J((i_1, i_2), (j_1, j_2)) &= E(h(\xi_{i_1}, \xi_{i_2})h(\xi_{j_1}, \xi_{j_2})), \\ U_{a,n+a} &= \sum_{1 \leq i < j \leq n+a, j > a} h(\xi_i, \xi_j). \end{aligned}$$

Observe that

$$EU_{a,n+a}^2 = \sum_{1 \leq i_1 < i_2 \leq n+a, i_2 > a} \sum_{1 \leq j_1 < j_2 \leq n+a, j_2 > a} J((i_1, i_2), (j_1, j_2)). \quad (2)$$

We decompose the sum on the r.h.s. into five parts, according to the order in which the indices i_1, i_2, j_1, j_2 occur. We then apply repeatedly the correlation inequality of Lemma 2.2(i) to the function

$$g(x_1, x_2, x_3, x_4) = h(x_1, x_2)h(x_3, x_4),$$

noting that by degeneracy of h , we have

$$\int g(x_1, x_2, x_3, x_4)dF(x_j) = 0, \quad 1 \leq j \leq 4,$$

where F denotes the distribution function of ξ_1 . Using arguments similar to those used by Yoshihara (1976), we then obtain

$$\begin{aligned} & \left| \sum_{1 \leq i_1 < i_2 \leq j_1 < j_2 \leq n+a, i_2 > a, j_2 > a} J((i_1, i_2), (j_1, j_2)) \right| \\ & \leq \sum_{1 \leq i_1 < i_2 \leq j_1 < j_2 \leq n+a, i_2 - i_1 \geq j_2 - j_1, i_2 > a, j_2 > a} |J((i_1, i_2), (j_1, j_2))| \\ & \quad + \sum_{1 \leq i_1 < i_2 \leq j_1 < j_2 \leq n+a, i_2 - i_1 \leq j_2 - j_1, i_2 > a, j_2 > a} |J((i_1, i_2), (j_1, j_2))| \\ & \leq Cn^2M^2, \end{aligned} \quad (3)$$

$$\begin{aligned}
& \left| \sum_{1 \leq i_1 < j_1 \leq i_2 < j_2 \leq n+a, i_2 > a, j_2 > a} J((i_1, i_2), (j_1, j_2)) \right| \\
& \leq \sum_{1 \leq i_1 < j_1 \leq i_2 < j_2 \leq n+a, j_1 - i_1 > j_2 - i_2, i_2 > a, j_2 > a} |J((i_1, i_2), (j_1, j_2))| \\
& \quad + \sum_{1 \leq i_1 < j_1 \leq i_2 < j_2 \leq n+a, j_1 - i_1 \leq j_2 - i_2, i_2 > a, j_2 > a} |J((i_1, i_2), (j_1, j_2))| \\
& \leq C(n+a)nM^2, \tag{4}
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{1 \leq i_1 < j_1 < j_2 < i_2 \leq n+a, i_2 > a, j_2 > a} J((i_1, i_2), (j_1, j_2)) \right| \\
& \leq \sum_{1 \leq i_1 < j_1 < j_2 < i_2 \leq n+a, j_1 - i_1 \geq i_2 - j_2, i_2 > a, j_2 > a} |J((i_1, i_2), (j_1, j_2))| \\
& \quad + \sum_{1 \leq i_1 < j_1 < j_2 < i_2 \leq n+a, j_1 - i_1 \leq i_2 - j_2, i_2 > a, j_2 > a} |J((i_1, i_2), (j_1, j_2))| \\
& \leq C(n+a)nM^2, \tag{5}
\end{aligned}$$

$$\begin{aligned}
& \left| \sum_{1 \leq i_1, j_1 \leq n+a} \sum_{i_2=a+1}^{n+a} J((i_1, i_2), (j_1, j_2)) \right| \\
& \leq \sum_{i_1=1}^{n+a} \sum_{i_2=a+1}^{n+a} J((i_1, i_2), (i_1, i_2)) + 2 \sum_{1 \leq i_1 < j_1 \leq n+a} \sum_{i_2=a+1}^{n+a} |J((i_1, i_2), (j_1, j_2))| \\
& \leq C(n+a)nM^2, \tag{6}
\end{aligned}$$

and finally

$$\left| \sum_{a+1 \leq i_2, j_2 \leq n+a} \sum_{i_1=1}^{n+a} J((i_1, i_2), (j_1, j_2)) \right| \leq Cn(n+a)M^2. \tag{7}$$

Inequalities (2) - (7) imply (1). Thus under conditions (i) the lemma is proved. The proof of the lemma under conditions (ii) is similar (in this case we use part (ii) of Lemma 2.2). \square

Lemma 2.4. *Let $(\xi_i)_{i \geq 1}$ satisfy the conditions of Lemma 2.3. Then*

$$E \max_{1 \leq k \leq n} \left(\sum_{1 \leq i < j \leq k+a, j > a} h(\xi_i, \xi_j) \right)^2 \leq CM^2 \left[\frac{\log 2n}{\log 2} \right] (n+a)n \quad (8)$$

for any integers $n \geq 1, a \geq 0$, and with the same C and M as in (1).

Proof. We define

$$\bar{U}_{a,n+a} = \max_{1 \leq k \leq n} \left| \sum_{1 \leq i < j \leq k+a, j > a} h(\xi_i, \xi_j) \right|^2.$$

We will follow the proof of the theorem by Serfling (1970) and give the proof for completeness.

We will use the induction method. For $n = 2$, (8) follows from (1). Assume (8) holds for all $n < N$ and all $a \geq 0$ taking N to be even. We will bound $U_{a,n+a}^2$ for each $n \leq N$. There are two cases: either $1 \leq n \leq \frac{N}{2}$ or $\frac{N}{2} < n \leq N$. If $1 \leq n \leq \frac{N}{2}$

$$U_{a,n+a}^2 \leq \bar{U}_{a, \frac{N}{a}}^2.$$

If $\frac{N}{2} < n \leq N$

$$\begin{aligned} U_{a,n+a}^2 &= (U_{a, \frac{N}{2}+a} + U_{\frac{N}{2}+a, n+a})^2 \\ &= U_{a, \frac{N}{2}+a}^2 + 2U_{a, \frac{N}{2}+a} \cdot U_{\frac{N}{2}+a, n+a} + U_{\frac{N}{2}+a, n+a}^2 \\ &\leq \bar{U}_{a, \frac{N}{2}+a} + 2|U_{a, \frac{N}{2}+a}| \cdot \bar{U}_{\frac{N}{2}+a, \frac{N}{2}+a + \frac{N}{2}} + U_{\frac{N}{2}+a, \frac{N}{2}+a + \frac{N}{2}}^2. \end{aligned}$$

Hence

$$\bar{U}_{a, N+a}^2 \leq U_{a, \frac{N}{2}+a} + 2|U_{a, \frac{N}{2}+a}| \bar{U}_{\frac{N}{2}+a, \frac{N}{2}+a + \frac{N}{2}} + \bar{U}_{\frac{N}{2}+a, \frac{N}{2}+a + \frac{N}{2}}^2.$$

Taking expectations on both sides and applying the induction hypothesis we obtain

$$\begin{aligned} E \bar{U}_{a, N+a}^2 &\leq \left(\frac{\log N}{\log 2} \right)^2 CM^2 \left(\frac{N}{2} + a \right) \frac{N}{2} + 2E \left(|U_{a, \frac{N}{2}+a}| \bar{U}_{\frac{N}{2}+a, \frac{N}{2}+a + \frac{N}{2}} \right) \\ &\quad + \left(\frac{\log N}{\log 2} \right)^2 CM^2 \left(\frac{N}{2} + \frac{N}{2} + a \right) \frac{N}{2}, \end{aligned} \quad (9)$$

and moreover

$$\begin{aligned} &2E \left(|U_{a, \frac{N}{2}+a}| \bar{U}_{\frac{N}{2}+a, \frac{N}{2}+a + \frac{N}{2}} \right) \\ &\leq 2E^{1/2} |U_{a, \frac{N}{2}+a}|^2 \cdot E^{1/2} |\bar{U}_{\frac{N}{2}+a, \frac{N}{2}+a + \frac{N}{2}}|^2 \\ &\leq 2 \left(\frac{\log N}{\log 2} \right) \left(CM^2 \left(\frac{N}{2} + a \right) \frac{N}{2} \right)^{1/2} \cdot \left(CM^2 (N+a) \frac{N}{2} \right)^{1/2} \\ &\leq \left(\frac{\log N}{\log 2} \right) \left[CM^2 \left(\frac{N}{2} + a \right) \frac{N}{2} + CM^2 (N+a) \frac{N}{2} \right]. \end{aligned} \quad (10)$$

Then, (10) together with (9) implies

$$\begin{aligned} E\bar{U}_{a,N+a}^2 &\leq \left[\frac{\log N}{\log 2} + \left(\frac{\log N}{\log 2} \right)^2 \right] \left(CM^2 \left(\frac{N}{2} + a \right) CM^2(N+a) \frac{N}{2} \right) \\ &\leq \left[\frac{\log 2N}{\log 2} \right]^2 CM^2(N+a)N. \end{aligned} \quad (11)$$

For N odd (8) can be proved in the same way considering the two cases $1 \leq n \leq \frac{N+1}{2}$ and $\frac{N+1}{2} < n \leq N$. \square

In the next lemma we formulate strong laws of large numbers for $(\xi_i)_{i \geq 1}$ under additional conditions.

Lemma 2.5. *Let $(\xi_i)_{i \geq 1}$ be a strictly stationary, absolutely regular process satisfying $E|\xi_1| < \infty$. Then*

$$\frac{1}{n}(\xi_1 + \dots + \xi_n) \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty.$$

Moreover, for $p \in [1, 2)$,

$$\frac{1}{n^{1/p}}(\xi_1 + \dots + \xi_n) \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty,$$

provided one of the following two sets of conditions is satisfied,

(i) The random variables ξ_i are almost surely bounded and

$$\sum_{n=1}^{\infty} n^{p-2} \beta(n) < \infty.$$

(ii) For some $\delta > 0$, the random variables satisfy $E|\xi_i|^{p+\delta} < \infty$ and

$$\sum_{k=1}^{\infty} k^\nu \beta(k) < \infty,$$

where $\nu = \frac{p(p-1)+\delta(p-2)}{\delta}$.

The first part of the lemma follows from Birkhoff's ergodic theorem. The second part was proved by Berbee (1987) under conditions (i). Under (ii), it is a special case of Theorem 1 by Rio (1995).

3. Proofs of the Theorems

Proof of Theorem 1. First we prove part (i). It suffices to prove the following relations

$$\frac{S_{2^k}}{2^{2k/p}} \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty, \quad (12)$$

$$\frac{\max_{2^k < i \leq 2^{k+1}} |S_i|}{2^{2k/p}} \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty. \quad (13)$$

where

$$S_n = \sum_{1 \leq i < j \leq n} h(\xi_i, \xi_j).$$

Using Chebyshev's inequality and Lemma 2.3 we have for any $\varepsilon > 0$

$$\sum_{k=1}^{\infty} P\left(\frac{|S_{2^k}|}{2^{2k/p}} > \varepsilon\right) \leq \sum_{k=1}^{\infty} \frac{CE|S_{2^k}|^2}{\varepsilon 2^{4k/p}} \leq \sum_{k=1}^{\infty} \frac{C2^{2k}M^2}{\varepsilon 2^{4k/p}} < \infty.$$

Now the Borel-Cantelli lemma implies (12). Again using Chebyshev's inequality and Lemma 2.4 we obtain for any $\varepsilon > 0$

$$\begin{aligned} \sum_{k=1}^{\infty} P\left(\frac{1}{2^{2k/p}} \max_{2^k < i \leq 2^{k+1}} |S_i| > \varepsilon\right) &\leq \sum_{k=1}^{\infty} C \frac{E(\max_{2^k < i \leq 2^{k+1}} |S_i|^2)}{\varepsilon 2^{4k/p}} \\ &\leq \sum_{k=1}^{\infty} \frac{C \cdot 2^{2k} \cdot (\log 2^k)^2}{\varepsilon 2^{4k/p}} < \infty. \end{aligned}$$

The latter together with the Borel-Cantelli lemma implies (13). Thus part (i) is proved. In order to prove part (ii), we use Hoeffding's decomposition, according to which we can write

$$U_n = \frac{2}{n} \sum_{i=1}^n (h_1(\xi_i) - \theta(F)) + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} h_2(\xi_i, \xi_j), \quad (14)$$

where

$$\begin{aligned} h_1(x) &= \int h(x, y) dF(y), \\ \theta(F) &= \int \int h(x, y) dF(x) dF(y), \\ h_2(x, y) &= h(x, y) - h_1(x) - h_1(y) + \theta(F). \end{aligned}$$

Now part (ii) of the theorem easily follows from part (i) and Lemma 2.5. \square

Proof of Theorem 2. Again, by Hoeffding's decomposition we have

$$U_n = \frac{2}{n} \sum_{i=1}^n (h_1(\xi_i) - \theta(F)) + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} h_2(\xi_i, \xi_j),$$

where $h_1(x)$, $\theta(F)$ and $h_2(x, y)$ are defined as above. Thus it suffices to show

$$\frac{1}{n^{1/p}} \sum_{i=1}^n (h_1(\xi_i) - \theta(F)) \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty, \quad (15)$$

$$\frac{2n^{1-1/p}}{n(n-1)} \sum_{1 \leq i < j \leq n} h_2(\xi_i, \xi_j) \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty. \quad (16)$$

As (15) follows from Lemma 2.5, it remains to show (16), which again follows from

$$\frac{2 \cdot 2^{(1-1/p)k}}{2^k(2^k - 1)} S_{2^k} \longrightarrow 0 \quad \text{a.s. as } k \rightarrow \infty, \quad (17)$$

$$\frac{2 \cdot 2^{(1-1/p)k}}{2^k(2^k - 1)} \max_{2^k < n \leq 2^{k+1}} |S_n - S_{2^k}| \longrightarrow 0 \quad \text{a.s. as } k \rightarrow \infty, \quad (18)$$

where

$$S_n = \sum_{1 \leq i < j \leq n} h_2(\xi_i, \xi_j).$$

Let ξ, η be two independent random variables with the same distribution as ξ_i . As $Eh_2(\xi, \eta) = 0$, we get

$$\begin{aligned} h_2(\xi_i, \xi_j) &= h_2(\xi_i, \xi_j)I(|h_2(\xi_i, \xi_j)| \leq 2^{\alpha k}) - Eh_2(\xi, \eta)I(|h_2(\xi, \eta)| \leq 2^{\alpha k}) \\ &\quad + h_2(\xi_i, \xi_j)I(|h_2(\xi_i, \xi_j)| > 2^{\alpha k}) - Eh_2(\xi, \eta)I(|h_2(\xi, \eta)| > 2^{\alpha k}). \end{aligned}$$

We define the kernels $\bar{h}_2(\xi_i, \xi_j) = \bar{h}_{2,k}(\xi_i, \xi_j)$ and $\check{h}_2(\xi_i, \xi_j) = \check{h}_{2,k}(\xi_i, \xi_j)$ by

$$\begin{aligned} \bar{h}_{2,k}(\xi_i, \xi_j) &= h_2(\xi_i, \xi_j)I(|h_2(\xi_i, \xi_j)| \leq 2^{\alpha k}) - Eh_2(\xi, \eta)I(|h_2(\xi, \eta)| \leq 2^{\alpha k}), \\ \check{h}_{2,k}(\xi_i, \xi_j) &= h_2(\xi_i, \xi_j)I(|h_2(\xi_i, \xi_j)| > 2^{\alpha k}) - Eh_2(\xi, \eta)I(|h_2(\xi, \eta)| > 2^{\alpha k}), \end{aligned}$$

and the associated non-normalized U -statistics by

$$\begin{aligned} \bar{S}_{2^k} &= \sum_{1 \leq i < j \leq 2^k} \bar{h}_2(\xi_i, \xi_j), \\ \check{S}_{2^k} &= \sum_{1 \leq i < j \leq 2^k} \check{h}_2(\xi_i, \xi_j). \end{aligned}$$

We can take $\alpha = \frac{1}{p+\delta/2}$. First we prove (17). Using the triangle, Hölder and Markov inequalities we obtain for any $\epsilon > 0$

$$\begin{aligned} &\sum_{k=1}^{\infty} P\left(\frac{1}{2^{(1+1/p)k}} |\check{S}_{2^k}| > \epsilon\right) \\ &\leq C \sum_{k=1}^{\infty} \frac{2^{2k} (\sup_{i,j} E|\check{h}_2(\xi_i, \xi_j)|I(|h_2(\xi_i, \xi_j)| > 2^{\alpha k}) + E|h_2(\xi, \eta)|I(|h_2(\xi, \eta)| \geq 2^{\alpha k}))}{\epsilon 2^{(1/p+1)k}} \\ &\leq C \sum_{k=1}^{\infty} \frac{\sup_{i,j} (E|h_2(\xi_i, \xi_j)|^{p+\delta})^{1/(p+\delta)} \sup_{i,j} (P(|h_2(\xi_i, \xi_j)| > 2^{\alpha k}))^{(p+\delta-1)/(p+\delta)}}{\epsilon 2^{(1/p-1)k}} \\ &\quad + C \sum_{k=1}^{\infty} \frac{(E|h_2(\xi, \eta)|^{p+\delta})^{1/(p+\delta)} (P(|h_2(\xi, \eta)| > 2^{\alpha k}))^{(p+\delta-1)/(p+\delta)}}{\epsilon 2^{(1/p-1)k}} \\ &\leq \frac{C}{\epsilon} \sum_{k=1}^{\infty} \frac{1}{(2^{(1/p-1)2^{\alpha(p+\delta-1)}})^k} < \infty. \end{aligned} \quad (19)$$

Using the Borel-Cantelli lemma, we obtain

$$\frac{1}{2^{(1+1/p)k}} \check{S}_{2^k} \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty.$$

Since $h_2(x, y)$ will generally be non-degenerate, we use Hoeffding's decomposition to get

$$\begin{aligned} & \frac{2^{(1-1/p)k} \cdot 2}{2^k(2^k - 1)} \check{S}_{2^k} \\ &= 2^{k(1-1/p)} \left[\bar{\theta}(F) + \frac{2}{2^k} \sum_{i=1}^{2^k} (h_3(\xi_i) - \bar{\theta}(F)) + \frac{2}{2^k(2^k - 1)} \sum_{1 \leq i < j \leq 2^k} h_4(\xi_i, \xi_j) \right], \end{aligned} \quad (20)$$

where

$$\begin{aligned} h_3(x) &= \int \bar{h}_2(x, y) dF(y), \\ \bar{\theta}(F) &= \int \bar{h}_2(x, y) dF(x) dF(y) = 0, \\ h_4(x, y) &= \bar{h}_2(x, y) - h_3(x) - h_3(y) + \bar{\theta}(F). \end{aligned}$$

Now we prove that

$$\frac{1}{2^{(1+1/p)k}} \sum_{1 \leq i < j \leq 2^k} h_4(\xi_i, \xi_j) \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty. \quad (21)$$

Since $h_4(x, y)$ is degenerate, we obtain, using Chebyshev's inequality and Lemma 2.3, for any $\varepsilon > 0$

$$\begin{aligned} \sum_{k=1}^{\infty} P \left(\frac{1}{2^{(1+1/p)k}} \left| \sum_{1 \leq i < j \leq 2^k} h_4(\xi_i, \xi_j) \right| > \varepsilon \right) &\leq C \sum_{k=1}^{\infty} \frac{E \left| \sum_{1 \leq i < j \leq 2^k} h_4(\xi_i, \xi_j) \right|^2}{\varepsilon^2 2^{2(1+1/p)k}} \\ &\leq \frac{C}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{2^{2k} 2^{2\alpha k}}{2^{2(1+1/p)k}} < \infty. \end{aligned}$$

The latter together with the Borel-Cantelli lemma implies (21). Next we show that

$$\frac{1}{2^{k/p}} \sum_{i=1}^{2^k} h_3(\xi_i) \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty. \quad (22)$$

Note that

$$h_3(x) = \int h_2(x, y) dF(y) - \int \check{h}_2(x, y) dF(y) := h_5(x) - \check{h}_6(x).$$

From Lemma 2.5 we have

$$\frac{1}{2^{k/p}} \sum_{i=1}^{2^k} h_5(\xi_i) \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty. \quad (23)$$

Using the triangle and the Markov inequality we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}
 & \sum_{k=1}^{\infty} P \left(\frac{1}{2^{k/p}} \left| \sum_{i=1}^{2^k} \check{h}_6(\xi_i) \right| > \varepsilon \right) \\
 & \leq C \sum_{k=1}^{\infty} \frac{2^k E |\check{h}_6(\xi_1)|}{2^{k/p} \varepsilon} \\
 & \leq C \sum_{k=1}^{\infty} \frac{E |\check{h}_2(\xi, \eta)|}{2^{(1/p-1)k} \varepsilon} \\
 & = C \sum_{k=1}^{\infty} \frac{E |h_2(\xi, \eta) I(|h_2(\xi, \eta)| > 2^{\alpha k}) - E h_2(\xi, \eta) I(|h_2(\xi, \eta)| > 2^{\alpha k})|}{2^{(1/p-1)k} \varepsilon} \\
 & \leq C \sum_{k=1}^{\infty} \frac{E |h_2(\xi, \eta) I(|h_2(\xi, \eta)| > 2^{\alpha k})|}{2^{(1/p-1)k} \varepsilon} \\
 & \leq \frac{C}{\varepsilon} \sum_{k=1}^{\infty} \frac{(E |h_2(\xi, \eta)|^{p+\delta})}{2^{(1/p-1)k} 2^{\alpha(p+\delta-1)k}} < \infty. \tag{24}
 \end{aligned}$$

Hence using the Borel-Cantelli lemma we have

$$\frac{1}{2^{k/p}} \sum_{i=1}^{2^k} \check{h}_6(\xi_i) \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty. \tag{25}$$

(23) and (25) imply (22). Now, (20), (21) and (22) imply (17). Thus, in order to finish the proof of the theorem, it remains to prove (18). Again we use a truncation method. We note that (18) holds if

$$\frac{2 \cdot 2^{(1-1/p)k}}{2^k(2^k - 1)} \max_{2^k < n \leq 2^{k+1}} |\check{S}_n - \check{S}_{2^k}| \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty \tag{26}$$

and

$$\frac{2 \cdot 2^{(1-1/p)k}}{2^k(2^k - 1)} \max_{2^k < n \leq 2^{k+1}} |\bar{S}_n - \bar{S}_{2^k}| \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty, \tag{27}$$

where

$$\begin{aligned}
 \check{S}_n &= \sum_{1 \leq i < j \leq n} \check{h}_2(\xi_i, \xi_j), \\
 \bar{S}_n &= \sum_{1 \leq i < j \leq n} \bar{h}_2(\xi_i, \xi_j).
 \end{aligned}$$

Using the triangle, Hölder and Markov inequalities we obtain, for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} P \left(\frac{1}{2^{(1+1/p)k}} \max_{2^k < n \leq 2^{k+1}} |\check{S}_n - \check{S}_{2^k}| > \varepsilon \right) \leq C \sum_{k=1}^{\infty} \frac{2^{2k} \sup_{i,j} E |\check{h}_2(\xi_i, \xi_j)|}{2^{(1+1/p)k} \varepsilon}.$$

The last series converges by the same arguments which were used in (19) and Borel-Cantelli lemma implies (26).

In order to prove (27) we need to prove that, as $k \rightarrow \infty$

$$\frac{1}{2^{(1+1/p)k}} \max_{1 \leq l \leq 2^k} \left| \sum_{1 \leq i < j \leq 2^k + l, j > 2^k} h_4(\xi_i, \xi_j) \right| \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty, \quad (28)$$

$$\frac{1}{2^{k/p}} \max_{2^k < l \leq 2^{k+1}} \left| \sum_{i=2^k}^l h_4(\xi_i) \right| \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty. \quad (29)$$

Using Chebyshev's inequality and Lemma 2.4 we obtain, for any $\varepsilon > 0$

$$\begin{aligned} & \sum_{k=1}^{\infty} P \left(\frac{1}{2^{(1+1/p)k}} \max_{1 \leq l \leq 2^k} \left| \sum_{1 \leq i < j \leq 2^k + l, j > 2^k} h_4(\xi_i, \xi_j) \right| > \varepsilon \right) \\ & \leq C \sum_{k=1}^{\infty} \frac{E \left(\max_{1 \leq l \leq 2^k} \left| \sum_{1 \leq i < j \leq 2^k + l, j > 2^k} h_4(\xi_i, \xi_j) \right|^2 \right)}{2^{2(1+1/p)k} \varepsilon^2} \\ & \leq \frac{C}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{2^{2k} 2^{2\alpha k} k^2}{2^{2(1+1/p)k}} < \infty. \end{aligned}$$

The latter together with the Borel-Cantelli lemma implies (28).

Now we will prove that, as $k \rightarrow \infty$,

$$\frac{1}{2^{k/p}} \max_{1 \leq l \leq 2^k} \left| \sum_{i=2^k+1}^{2^k+l} h_5(\xi_i) \right| \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty \quad (30)$$

and

$$\frac{1}{2^{k/p}} \max_{1 \leq l \leq 2^k} \left| \sum_{i=2^k+1}^{2^k+l} \check{h}_6(\xi_i) \right| \rightarrow 0 \quad \text{a.s. as } k \rightarrow \infty. \quad (31)$$

(30) follows from Lemma 2.5 and the inequality

$$\frac{1}{2^{k/p}} \max_{1 \leq l \leq 2^k} \left| \sum_{i=2^k+1}^{2^k+l} h_5(\xi_i) \right| \leq \frac{C}{2^{k/p}} \left| \sum_{i=1}^{2^k} h_5(\xi_i) \right| + \frac{C}{2^{k/p}} \max_{1 \leq l \leq 2^{k+1}} \left| \sum_{i=1}^l h_5(\xi_i) \right|.$$

Using the inequality

$$\max_{1 \leq l \leq 2^k} E \left| \sum_{i=2^k+1}^{2^k+l} \check{h}_6(\xi_i) \right| \leq 2^k E |\check{h}_6(\xi_1)|,$$

and arguments which were used in (24) one can prove that, for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} P \left(\frac{1}{2^{k/p}} \max_{1 \leq l \leq 2^k} \left| \sum_{i=2^k+1}^{2^k+l} \check{h}_6(\xi_i) \right| > \varepsilon \right) < \infty.$$

The latter and the Borel-Cantelli lemma imply (31). Next, (23) and (31) imply (29), which together with (28) implies (27). Thus the theorem is proved. \square

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