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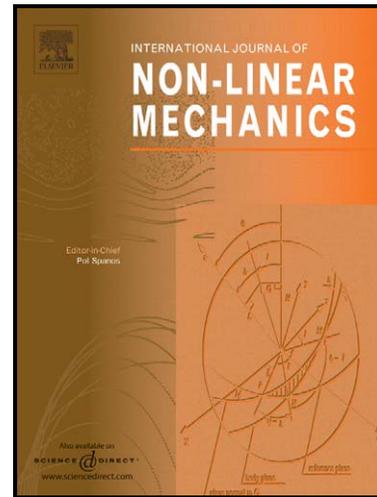
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**Micromorphic continuum.**  
**Part II: Finite deformation plasticity**  
**coupled with damage**

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**Abstract**

It is demonstrated how a micromorphic plasticity theory may be formulated on the basis of multiplicative decompositions of the macro- and microdeformation gradient tensor, respectively. The theory exhibits non-linear isotropic and non-linear kinematic hardening. The yield function is expressed in terms of Mandel stress and double stress tensors, appropriately defined for micromorphic continua. Flow rules are derived from the postulate of Il'iushin and represent generalized normality conditions. Evolution equations for isotropic and kinematic hardening are introduced as sufficient conditions for the validity of the second law of thermodynamics in every admissible process. Finally, it is sketched how isotropic damage effects may be incorporated in the theory. This is done for the concept of effective stress combined

with the hypothesis of strain equivalence.

*Key words:* micromorphic plasticity; finite deformation; isotropic and kinematic hardening rules; isotropic continuum damage mechanics; finite element implementation;

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## 1 Introduction

It has been mentioned in Part I that micropolar and micromorphic materials are continuum theories which take into account, in some sense, the microstructure of the real material (continua with microstructures). Higher order gradients of the kinematical variables are incorporated, which renders such models, among other things, to be suitable when describing localization effects. However, in opposite to micropolar continua, there are no broadly known (finite deformation) micromorphic plasticity theories. Thus the aim of Part II is to sketch how a thermodynamically consistent micromorphic plasticity theory may be formulated, by using the general framework developed in Part I.

To give an outline of the present work, we elaborate multiplicative decompositions of the macro- and microdeformation gradient tensors into elastic and plastic parts, respectively, in order to introduce a so-called plastic intermediate configuration for micromorphic continua. As in classical plasticity, this implies additive decompositions of the strain and micromorphic curvature tensors into elastic and plastic parts. It is a peculiarity of the proposed theory

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that the plastic part of the micromorphic curvature tensor is not related to some gradient operator and hence it is not subjected to some compatibility conditions. The formulation of the constitutive theory is based on three stress tensors, namely the Cauchy stress tensor, a stress tensor responsible for the microcontinuum and a double stress tensor. To these stress tensors, definitions of Mandel type stress tensors are worked out and utilized to express the intrinsic dissipation inequality, as well as to represent the yield function. Correspondingly, three back-stress tensors for modeling kinematic hardening effects are assumed. Isotropic hardening can be modeled by using three different plastic arc lengths, related respectively to a micromorphic strain, a strain for the microcontinuum and a micromorphic curvature strain, or by capturing all strains unifiedly by means of a single plastic arc length. Here, the latter course is followed, that means, a single plastic arc length approach is chosen. Similar to classical plasticity, we assume the validity of Il'iusin's postulate, appropriately generalized for micromorphic continua, and extract from this convexity of the yield surface and normality conditions for the flow rules. The evolution equations governing the response of the hardening variables are established as sufficient conditions for the validity of the dissipation inequality in every admissible process. As mentioned above, micromorphic plasticity theories may be important when localization phenomena are investigated. In such cases, the existence of some softening mechanisms inherent in the material response is significant, as e.g. damage. The simplest case is isotropic damage described by a scalar variable. We demonstrate how isotropic damage may be coupled to the micromorphic plasticity model. To this end, use is made of the effective stress concept combined with the hypothesis of strain equivalence.

Throughout the article, the notation and the assumptions made in Part I hold.

In order to make the theoretical effort as small as possible, isotropy is assumed to apply, as explained in the article.

## 2 Decompositions of deformation

### 2.1 *Multiplicative decomposition of the macro- and the microdeformation gradient tensors into elastic and plastic parts*

As in classical plasticity, it is assumed that the macrodeformation gradient tensor  $\mathbf{F}$  may be decomposed into elastic and plastic parts,

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p \quad , \quad (1)$$

where  $\det \mathbf{F}_e > 0$  is assumed, and therefore  $\det \mathbf{F}_p > 0$ , in view of  $\det \mathbf{F} > 0$ . This decomposition of  $\mathbf{F}$  has been broadly known by the works of Lee and Liu (1967) and Lee (1969). Decomposition (1) is supposed to be unique except for a rigid body rotation (see Green and Naghdi, 1971; Casey and Naghdi, 1980, 1981). In addition to (1), we assume the multiplicative decomposition of the microdeformation gradient tensor  $\mathbf{f}$  into elastic and plastic parts,

$$\mathbf{f} = \mathbf{f}_e \mathbf{f}_p \quad , \quad (2)$$

with  $\det \mathbf{f}_e > 0$ , and therefore  $\det \mathbf{f}_p > 0$  too. Decomposition (2) is supposed to be also unique except for the same rigid body rotation, which may be inserted into the decomposition (1).

It must be mentioned, that further interesting multiplicative decompositions into elastic and plastic parts have been introduced previously by Sansour (1998), and later on adopted by Forest and Sievert (2003). A generalized

deformation gradient  $\tilde{\mathbf{F}}$  is assumed by Sansour to reflect the deformation of the micro- and macrocontinuum. With respect to  $\tilde{\mathbf{F}}$ , two alternative multiplicative decompositions are proposed, which are not equivalent to our Equations (1), (2).

In opposite to  $\mathbf{F}(\mathbf{X}, t)$ ,  $\mathbf{F}_p(\mathbf{X}, t)$  (and therefore  $\mathbf{F}_e(\mathbf{X}, t)$  too) is incompatible deformation. For fixed time  $t$ ,  $\mathbf{F}_p(\mathbf{X}, t)$  induces a local configuration for the macroscopic continuum at  $\mathbf{X}$ . (We adopt the definition of local deformation and local configuration used by Noll (1967) and Truesdell and Noll (1965).) Let  $\hat{\mathbf{x}} \in \mathcal{E}$  be the position vector, in that local configuration, of the material point, which in the reference configuration possesses the position vector  $\mathbf{X}$ . Obviously, the position  $\hat{\mathbf{x}}$  can be chosen arbitrary (cf. Grammenoudis and Tsakmakis, 2008a). This fact may be visualized by imaging the local deformation  $\mathbf{F}_p(\mathbf{X}, t)$  at  $\mathbf{X}$  to map a neighborhood  $\mathcal{N}(\mathbf{X}) \in \mathcal{E}$  on a neighborhood  $\hat{\mathcal{M}}(\hat{\mathbf{x}}, t) \in \mathcal{E}$  around  $\hat{\mathbf{x}}$ , with  $\hat{\mathbf{x}}$  being arbitrary point of  $\mathcal{E}$ . (Further aspects and details may be consulted in Grammenoudis and Tsakmakis (2008a,b).) Now, as  $\hat{\mathbf{x}}$  may be chosen arbitrary, we assume in particular  $\hat{\mathbf{x}}$  to be given by an arbitrary deformation  $\hat{\chi}$ ,

$$\hat{\mathbf{x}} = \hat{\chi}(\mathbf{X}, t) \quad . \quad (3)$$

It is emphasized that  $\mathbf{F}_p(\mathbf{X}, t) \neq \frac{\partial \hat{\chi}}{\partial \mathbf{X}}$  generally. As special cases,  $\hat{\mathbf{x}} \equiv \mathbf{X}$  or  $\hat{\mathbf{x}} \equiv \mathbf{x}$  are allowed. In the following, the conceptual configuration introduced by deformation  $\hat{\chi}(\cdot, t)$  is left arbitrary. We shall write  $\hat{\mathcal{R}}_t$  for the range in  $\mathcal{E}$  occupied by the body under the configuration induced by  $\hat{\chi}$ ,  $\hat{\mathcal{R}}_t = \hat{\chi}(\mathcal{R}_{\mathcal{R}}, t)$ .

Configuration  $\hat{\chi}(\cdot, t)$ , together with a collection of local deformations for  $\mathbf{F}_p$  is referred to as plastic intermediate configuration for the macroscopic continuum. As the position vector  $\hat{\mathbf{x}}$  may be chosen arbitrary, we shall say that

the macroscopic continuum will deform in the plastic intermediate configuration locally by  $\mathbf{F}_p$ . While the macroscopic continuum deforms locally from  $\mathbf{X}$  to  $\hat{\mathbf{x}}$ , the microscopic continuum at  $\mathbf{X}$  is postulated to deform homogeneously by  $\mathbf{f}_p = \mathbf{f}_p(\mathbf{X}, t)$ , so that the position vector  $\mathbf{X}'$ , emanated from point  $\mathbf{X} \in \mathcal{R}_{\mathcal{R}}$ , will go to the position vector  $\hat{\mathbf{x}}' = \hat{\chi}'(\mathbf{X}, \mathbf{X}', t) = \mathbf{f}_p(\mathbf{X}, t)\mathbf{X}'$ , emanated from point  $\hat{\mathbf{x}} \in \hat{\mathcal{R}}_t$ . This way, the range  $\mathcal{R}'_{\mathcal{R}}(\mathbf{X})$  will be mapped to the range  $\hat{\mathcal{R}}'_t(\hat{\mathbf{x}}) = \hat{\chi}'(\mathbf{X}, \mathcal{B}')$ . For fixed  $t$ , we refer to  $\hat{\chi}'(\mathbf{X}, \cdot, t)$  as the plastic intermediate configuration of the microscopic continuum at  $\mathbf{X}$ . The plastic intermediate configuration for the macroscopic continuum together with that one for the microscopic continuum are called plastic intermediate configuration for the micromorphic continuum. Clearly,  $\mathbf{F}_p$  and  $\mathbf{f}_p$ , and therefore  $\mathbf{F}_e$  and  $\mathbf{f}_e$  too, are two-point tensor fields, satisfying polar decompositions

$$\mathbf{F}_e = \mathbf{R}_e \mathbf{U}_e = \mathbf{V}_e \mathbf{R}_e \quad , \quad \mathbf{F}_p = \mathbf{R}_p \mathbf{U}_p = \mathbf{V}_p \mathbf{R}_p \quad , \quad (4)$$

$$\mathbf{f}_e = \mathbf{r}_e \mathbf{u}_e = \mathbf{v}_e \mathbf{r}_e \quad , \quad \mathbf{f}_p = \mathbf{r}_p \mathbf{u}_p = \mathbf{v}_p \mathbf{r}_p \quad , \quad (5)$$

where  $\mathbf{U}_e, \mathbf{U}_p, \mathbf{V}_e, \mathbf{V}_p, \mathbf{u}_e, \mathbf{u}_p, \mathbf{v}_e, \mathbf{v}_p$  are symmetric, positive definite second-order tensors, and  $\mathbf{R}_e, \mathbf{R}_p, \mathbf{r}_e, \mathbf{r}_p$  are proper orthogonal second-order tensors (rotations). Aside from the velocity gradient tensors  $\mathbf{L}, \mathbf{l}$  introduced in Part I, the plastic deformation rates operating in the plastic intermediate configuration

$$\hat{\mathbf{L}}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \hat{\mathbf{D}}_p + \hat{\mathbf{W}}_p \quad , \quad \hat{\mathbf{D}}_p = \frac{1}{2}(\hat{\mathbf{L}}_p + \hat{\mathbf{L}}_p^T) \quad , \quad \hat{\mathbf{W}}_p = \frac{1}{2}(\hat{\mathbf{L}}_p - \hat{\mathbf{L}}_p^T) \quad , \quad (6)$$

$$\hat{\mathbf{l}}_p = \dot{\mathbf{f}}_p \mathbf{f}_p^{-1} = \hat{\mathbf{d}}_p + \hat{\mathbf{w}}_p \quad , \quad \hat{\mathbf{d}}_p = \frac{1}{2}(\hat{\mathbf{l}}_p + \hat{\mathbf{l}}_p^T) \quad , \quad \hat{\mathbf{w}}_p = \frac{1}{2}(\hat{\mathbf{l}}_p - \hat{\mathbf{l}}_p^T) \quad , \quad (7)$$

will be useful e.g. for defining strain and stress rates with respect to the plastic intermediate configuration.

As mentioned above, like classical plasticity (cf. Green and Naghdi, 1971; Casey and Naghdi, 1980, 1981), the plastic intermediate configuration may be determined uniquely only within an arbitrary rigid body rotation  $\mathbf{Q}_p = \mathbf{Q}_p(t)$ . Some transformation rules, which apply to both, rigid body rotations  $\mathbf{Q} = \mathbf{Q}(t)$  superposed on the actual configuration, and rigid body rotations  $\mathbf{Q}_p = \mathbf{Q}_p(t)$  superposed on the plastic intermediate configuration simultaneously, are given in Appendix A.

## 2.2 Basis systems on $\hat{\mathcal{R}}_t$

Before going any further, it is convenient to introduce some special basis systems. In conjunction with the basis systems  $\{\mathbf{g}_i\}$ ,  $\{\mathbf{E}_i\}$ , we define

$$\hat{\mathbf{g}}_i := \mathbf{F}_p \mathbf{E}_i \quad , \quad \hat{\mathbf{g}}^i = \mathbf{F}_p^{T-1} \mathbf{E}^i \quad , \quad \hat{\mathbf{g}}^i \cdot \hat{\mathbf{g}}_j = \delta_j^i \quad , \quad (8)$$

so that

$$\mathbf{g}_i = \mathbf{F}_e \hat{\mathbf{g}}_i \quad , \quad \mathbf{g}^i = \mathbf{F}_e^{T-1} \hat{\mathbf{g}}^i \quad . \quad (9)$$

Additionally, we set

$$\mathbf{F}_p = (\mathbf{F}_p)_j^i \hat{\mathbf{e}}_i \otimes \mathbf{E}^j \quad , \quad (\mathbf{F}_p)_j^i \equiv (\mathbf{F}_p)_{ij} \quad , \quad (10)$$

$$\mathbf{F}_p^{-1} = (\mathbf{F}_p^{-1})_j^i \mathbf{E}_i \otimes \hat{\mathbf{e}}^j \quad , \quad (\mathbf{F}_p^{-1})_j^i \equiv (\mathbf{F}_p^{-1})_{ij} \quad . \quad (11)$$

It follows that

$$\hat{\mathbf{g}}_i := (\mathbf{F}_p)_i^j \hat{\mathbf{e}}_j \quad , \quad \hat{\mathbf{g}}^i = (\mathbf{F}_p^{-1})_j^i \hat{\mathbf{e}}^j \quad . \quad (12)$$

Beyond  $\{\hat{\mathbf{g}}_i\}$  and  $\{\hat{\mathbf{e}}_i\}$ , one may introduce a further basis  $\{\hat{\boldsymbol{\rho}}_i\}$  at  $\hat{\mathbf{x}}$ , by

$$\hat{\boldsymbol{\rho}}_i := \mathbf{f}_p \mathbf{E}_i \quad , \quad \hat{\boldsymbol{\rho}}^i = \mathbf{f}_p^{T-1} \mathbf{E}^i \quad , \quad \hat{\boldsymbol{\rho}}^i \cdot \hat{\boldsymbol{\rho}}_j = \delta_j^i \quad . \quad (13)$$

Similar to (10), (11), we set

$$\mathbf{f}_p = (\mathbf{f}_p)^i_j \hat{\mathbf{e}}_i \otimes \mathbf{E}^j \quad , \quad (\mathbf{f}_p)^i_j \equiv (\mathbf{f}_p)_{ij} \quad , \quad (14)$$

$$\mathbf{f}_p^{-1} = (\mathbf{f}_p^{-1})^i_j \mathbf{E}_i \otimes \hat{\mathbf{e}}^j \quad , \quad (\mathbf{f}_p^{-1})^i_j \equiv (\mathbf{f}_p^{-1})_{ij} \quad , \quad (15)$$

and hence

$$\hat{\boldsymbol{\rho}}_i = (\mathbf{f}_p)^j_i \hat{\mathbf{e}}_j \quad , \quad \hat{\boldsymbol{\rho}}^i = (\mathbf{f}_p^{-1})^i_j \hat{\mathbf{e}}^j \quad . \quad (16)$$

The transformation law between  $\{\hat{\mathbf{g}}_i\}$  and  $\{\hat{\boldsymbol{\rho}}_i\}$  reads

$$\hat{\mathbf{g}}_i = A^j_i \hat{\boldsymbol{\rho}}_j \quad , \quad \hat{\mathbf{g}}^i = (A^{-1})^i_j \hat{\boldsymbol{\rho}}^j \quad , \quad (17)$$

with

$$A^i_j = (\mathbf{f}_p^{-1})^i_r (\mathbf{F}_p)^r_j \quad , \quad (A^{-1})^i_j = (\mathbf{F}_p^{-1})^i_r (\mathbf{f}_p)^r_j \quad , \quad (18)$$

$$(A^{-1})^i_r A^r_j = A^i_r (A^{-1})^r_j = \delta^i_j \quad . \quad (19)$$

### 2.3 Additive decompositions of the strain tensors

We set in Part I, Sect. 5.1,  $\mathbf{F}_a = \mathbf{F}_p$ ,  $\mathbf{f}_a = \mathbf{f}_p$ ,  $\boldsymbol{\beta}_a = \hat{\boldsymbol{\beta}}$ ,  $\boldsymbol{\epsilon}_a = \hat{\boldsymbol{\epsilon}}$  and  $\mathbf{c}_a = \hat{\mathbf{c}}$ ,  $\boldsymbol{\varphi}_a = \hat{\boldsymbol{\varphi}}$ ,  $\boldsymbol{\xi}_a = \hat{\boldsymbol{\xi}}$ ,  $\boldsymbol{\zeta}_a = \hat{\boldsymbol{\zeta}}$ , to get, on the one hand,

$$\hat{\mathbf{c}} = \mathbf{F}_p \mathbf{C} \quad , \quad \hat{\boldsymbol{\xi}} = \mathbf{F}_p^{T-1} \boldsymbol{\Xi} \quad , \quad (20)$$

$$\hat{\boldsymbol{\varphi}} = \mathbf{f}_p \boldsymbol{\Phi} \quad , \quad \hat{\boldsymbol{\zeta}} = \mathbf{f}_p^{T-1} \mathbf{Z} \quad , \quad (21)$$

and on the other hand

$$\mathbf{c} = \mathbf{F}_e \hat{\mathbf{c}} \quad , \quad \boldsymbol{\xi} = \mathbf{F}_e^{T-1} \hat{\boldsymbol{\xi}} \quad , \quad (22)$$

$$\boldsymbol{\varphi} = \mathbf{f}_e \hat{\boldsymbol{\varphi}} \quad , \quad \boldsymbol{\zeta} = \mathbf{f}_e^{T-1} \hat{\boldsymbol{\zeta}} \quad . \quad (23)$$

In addition,

$$\hat{\boldsymbol{\beta}} = \mathbf{f}_p^{T-1} \tilde{\boldsymbol{\beta}} \mathbf{f}_p^{-1} \quad , \quad \hat{\boldsymbol{\epsilon}} = \mathbf{f}_p \boldsymbol{\epsilon} \mathbf{F}_p^{-1} \quad . \quad (24)$$

These relations suggest additive decompositions of  $\Delta'_s$  and  $\Delta_s$  of the form

$$\Delta'_s = \frac{1}{2}(\boldsymbol{\varphi} \cdot \boldsymbol{\varphi} - \boldsymbol{\Phi} \cdot \boldsymbol{\Phi}) = \underbrace{\frac{1}{2}(\boldsymbol{\varphi} \cdot \boldsymbol{\varphi} - \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\varphi}})}_{=:(\Delta'_s)_e} + \underbrace{\frac{1}{2}(\hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\varphi}} - \boldsymbol{\Phi} \cdot \boldsymbol{\Phi})}_{=:(\Delta'_s)_p} = (\Delta'_s)_e + (\Delta'_s)_p \quad (25)$$

and

$$\Delta_s = \boldsymbol{\zeta} \cdot \mathbf{c} - \mathbf{Z} \cdot \mathbf{C} = \underbrace{(\boldsymbol{\zeta} \cdot \mathbf{c} - \hat{\boldsymbol{\zeta}} \cdot \hat{\mathbf{c}})}_{=:(\Delta_s)_e} + \underbrace{(\hat{\boldsymbol{\zeta}} \cdot \hat{\mathbf{c}} - \mathbf{Z} \cdot \mathbf{C})}_{=:(\Delta_s)_p} = (\Delta_s)_e + (\Delta_s)_p \quad (26)$$

On requiring from all these scalar valued differences to be form-invariant with respect to the chosen configuration, and by employing similar mathematical manipulations as in Part I, it is straightforward to deduce that (25), (26) indicate additive decompositions of the strain tensors. For example, with respect to the reference configuration, we have

$$\Delta'_s = \boldsymbol{\Phi} \cdot \tilde{\boldsymbol{\beta}} \boldsymbol{\Phi} \quad , \quad (\Delta'_s)_e = \boldsymbol{\Phi} \cdot \tilde{\boldsymbol{\beta}}_e \boldsymbol{\Phi} \quad , \quad (\Delta'_s)_p = \boldsymbol{\Phi} \cdot \tilde{\boldsymbol{\beta}}_p \boldsymbol{\Phi} \quad , \quad (27)$$

$$\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}_e + \tilde{\boldsymbol{\beta}}_p \quad , \quad (28)$$

and

$$\Delta_s = \mathbf{Z} \cdot \tilde{\boldsymbol{\epsilon}} \mathbf{C} \quad , \quad (\Delta_s)_e = \mathbf{Z} \cdot \tilde{\boldsymbol{\epsilon}}_e \mathbf{C} \quad , \quad (\Delta_s)_p = \mathbf{Z} \cdot \tilde{\boldsymbol{\epsilon}}_p \mathbf{C} \quad , \quad (29)$$

$$\tilde{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\epsilon}}_e + \tilde{\boldsymbol{\epsilon}}_p \quad . \quad (30)$$

With respect to the plastic intermediate configuration,

$$\Delta'_s = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\beta}} \hat{\boldsymbol{\varphi}} \quad , \quad (\Delta'_s)_e = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\beta}}_e \hat{\boldsymbol{\varphi}} \quad , \quad (\Delta'_s)_p = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\beta}}_p \hat{\boldsymbol{\varphi}} \quad , \quad (31)$$

$$\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}_e + \hat{\boldsymbol{\beta}}_p \quad , \quad (32)$$

$$\hat{\boldsymbol{\beta}} = \mathbf{f}_p^{T-1} \tilde{\boldsymbol{\beta}} \mathbf{f}_p^{-1} \quad , \quad \hat{\boldsymbol{\beta}}_e = \mathbf{f}_p^{T-1} \tilde{\boldsymbol{\beta}}_e \mathbf{f}_p^{-1} \quad , \quad \hat{\boldsymbol{\beta}}_p = \mathbf{f}_p^{T-1} \tilde{\boldsymbol{\beta}}_p \mathbf{f}_p^{-1} \quad , \quad (33)$$

and

$$\Delta_s = \hat{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{\epsilon}} \hat{\mathbf{c}} \quad , \quad (\Delta_s)_e = \hat{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{\epsilon}}_e \hat{\mathbf{c}} \quad , \quad (\Delta_s)_p = \hat{\boldsymbol{\zeta}} \cdot \hat{\boldsymbol{\epsilon}}_p \hat{\mathbf{c}} \quad , \quad (34)$$

$$\hat{\boldsymbol{\epsilon}} = \hat{\boldsymbol{\epsilon}}_e + \hat{\boldsymbol{\epsilon}}_p \quad , \quad (35)$$

$$\hat{\boldsymbol{\epsilon}} = \mathbf{f}_p \tilde{\boldsymbol{\epsilon}} \mathbf{F}_p^{-1} \quad , \quad \hat{\boldsymbol{\epsilon}}_e = \mathbf{f}_p \tilde{\boldsymbol{\epsilon}}_e \mathbf{F}_p^{-1} \quad , \quad \hat{\boldsymbol{\epsilon}}_p = \mathbf{f}_p \tilde{\boldsymbol{\epsilon}}_p \mathbf{F}_p^{-1} \quad . \quad (36)$$

Further details are given in Appendix C, from which it can be recognized that the tensors  $(\tilde{\boldsymbol{\beta}}, \hat{\boldsymbol{\beta}}, \boldsymbol{\beta})$ , or  $(\tilde{\boldsymbol{\beta}}_e, \hat{\boldsymbol{\beta}}_e, \boldsymbol{\beta}_e)$ , or  $\dots$ , or  $(\tilde{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\epsilon}}, \boldsymbol{\epsilon})$ , or  $(\tilde{\boldsymbol{\epsilon}}_e, \hat{\boldsymbol{\epsilon}}_e, \boldsymbol{\epsilon}_e)$ , or  $\dots$ , are respectively members of corresponding equivalence classes. Also, like  $\tilde{\boldsymbol{\beta}}$ ,  $\tilde{\boldsymbol{\epsilon}}$ , the strains  $\tilde{\boldsymbol{\beta}}_e$ ,  $\tilde{\boldsymbol{\beta}}_p$ ,  $\tilde{\boldsymbol{\epsilon}}_e$ ,  $\tilde{\boldsymbol{\epsilon}}_p$  are tensors on  $\mathcal{R}_{\mathcal{R}}$  (cf. Part I, Sect. 5.1), and so forth.

To conclude the discussion about strain tensors, we postulate  $(\Delta'_s)_e$ ,  $(\Delta'_s)_p$ ,  $(\Delta_s)_e$  and  $(\Delta_s)_p$  to be also form-invariant with respect to the chosen configuration. This allows to define, in a natural way, associated rates for the elastic and plastic parts of the strain tensors. Clearly, the additive decomposition of the strain tensors carries over their associated rates. Appendix C summarizes formulas of this kind and illustrates how the various strain tensors, and their associated rates, are related to each other.

## 2.4 Additive decomposition of the micromorphic curvature tensors

### 2.4.1 Decomposition of $\Delta_c$

Once more, we set  $\mathbf{F}_a = \mathbf{F}_p$ ,  $\mathbf{f}_a = \mathbf{f}_p$ , as well as  $\mathcal{K}_a = \hat{\mathcal{K}}$ ,  $(\mathbf{g}_a)_k = \hat{\mathbf{g}}_k$ ,  $(\boldsymbol{\varphi}_a)_k = \hat{\boldsymbol{\varphi}}_k$ , so that (cf. Part I, Sect. 5.1.2)

$$\hat{\boldsymbol{\varphi}}_k = \mathbf{f}_p \boldsymbol{\Phi}_k \quad , \quad \hat{\boldsymbol{\varphi}}^k = \mathbf{f}_p^{T-1} \boldsymbol{\Phi}^k \quad , \quad (37)$$

and therefore

$$\boldsymbol{\varphi}_k = \mathbf{f}_e \hat{\boldsymbol{\varphi}}_k \quad , \quad \boldsymbol{\varphi}^k = \mathbf{f}_e^{T-1} \hat{\boldsymbol{\varphi}}^k \quad . \quad (38)$$

This suggests additive decomposition of  $\Delta_c$  (see Part I, Eq. (81)) into elastic  $(\Delta_c)_e$  and plastic  $(\Delta_c)_p$  parts,

$$\Delta_c = \boldsymbol{\varphi}^1 \cdot (\nabla_{\mathcal{R}_t} \boldsymbol{\varphi}_2)[\mathbf{g}_3] - \boldsymbol{\Phi}^1 \cdot (\nabla_{\mathcal{R}_\mathcal{R}} \boldsymbol{\Phi}_2)[\mathbf{E}_3] = (\Delta_c)_e + (\Delta_c)_p \quad (39)$$

with

$$(\Delta_c)_e := \boldsymbol{\varphi}^1 \cdot (\nabla_{\mathcal{R}_t} \boldsymbol{\varphi}_2)[\mathbf{g}_3] - \hat{\boldsymbol{\varphi}}^1 \cdot (\nabla_{\hat{\mathcal{R}}_t} \hat{\boldsymbol{\varphi}}_2)[\hat{\mathbf{g}}_3] \quad , \quad (40)$$

$$(\Delta_c)_p := \hat{\boldsymbol{\varphi}}^1 \cdot (\nabla_{\hat{\mathcal{R}}_t} \hat{\boldsymbol{\varphi}}_2)[\hat{\mathbf{g}}_3] - \boldsymbol{\Phi}^1 \cdot (\nabla_{\mathcal{R}_\mathcal{R}} \boldsymbol{\Phi}_2)[\mathbf{E}_3] \quad , \quad (41)$$

where, as in Part I, Eqs. (82), (83),

$$\nabla_{\mathcal{R}_t} \boldsymbol{\varphi}_2 := \text{grad} \boldsymbol{\varphi}_2 = \frac{\partial \boldsymbol{\varphi}_2}{\partial X^k} \otimes \mathbf{g}^k \quad , \quad (42)$$

$$\nabla_{\mathcal{R}_\mathcal{R}} \boldsymbol{\Phi}_2 := \text{GRAD} \boldsymbol{\Phi}_2 = \frac{\partial \boldsymbol{\Phi}_2}{\partial X^k} \otimes \mathbf{E}^k \quad . \quad (43)$$

Constitutive aspects of the underlying physic of plasticity may be addressed appropriately by using a suitable differential operator  $\nabla_{\hat{\mathcal{R}}_t}$ . In the case of micropolar plasticity, a so-called relative covariant derivative has been proposed by Grammenoudis and Tsakmakis (2008a) as a possibility. An appropriate definition for relative covariant derivative in micromorphic plasticity, has been proposed in Grammenoudis and Tsakmakis (2008b), which we shall adopt also for the present article. The most important issues of the relative covariant derivative are summerized in the next section.

#### 2.4.2 Relative covariant derivative on $\hat{\mathcal{R}}_t$

Let  $\hat{\mathbf{b}} = \hat{\mathbf{b}}(\hat{\mathbf{x}}, t)$  be a vector field on  $\hat{\mathcal{R}}_t$ ,  $\hat{\mathbf{b}}(\hat{\mathbf{x}}, t) \in T_{\hat{\mathbf{x}}} \hat{\mathcal{R}}_t$ , with  $\hat{\mathbf{b}} = b^m \hat{\boldsymbol{\rho}}_m$ . The relative covariant derivative of  $\hat{\mathbf{b}}$  is defined (relative to  $\hat{\mathcal{R}}_t$ ) by

$$\hat{\nabla} \hat{\mathbf{b}} := \left( \frac{\partial b^j}{\partial X^i} + \Lambda^j_{il} b^l \right) \hat{\boldsymbol{\rho}}_j \otimes \hat{\mathbf{g}}^i \quad . \quad (44)$$

As mentioned in Grammenoudis and Tsakmakis (2008a),  $\Lambda_{il}^j$  are symbols of connection for the space  $\mathcal{R}_{\mathcal{R}}$  but not for the space  $\hat{\mathcal{R}}_t$ , and  $\hat{\nabla}\hat{\mathbf{b}}$  does not represent a covariant derivative of  $\hat{\mathbf{b}}$  relative to  $\hat{\mathcal{R}}_t$ . Furthermore,  $\Lambda_{il}^j$  defines generally a non-torsion free connection on  $\mathcal{R}_{\mathcal{R}}$ . In addition, the space  $\mathcal{R}_{\mathcal{R}}$  may be endowed with a non-Euclidean metric

$$\tilde{G}_{ij} = (\mathbf{f}_p)_i^k \delta_{kl} (\mathbf{f}_p)_j^l = \hat{\boldsymbol{\rho}}_i \cdot \hat{\boldsymbol{\rho}}_j \quad . \quad (45)$$

This metric, together with connection  $\Lambda_{il}^j$  renders the space  $\mathcal{R}_{\mathcal{R}}$  to be a non-Euclidean and a non-Riemannian one. For the particular choice  $\Lambda_{il}^j \equiv (\Lambda_{\mathbf{f}_p})_{il}^j$ , with

$$(\Lambda_{\mathbf{f}_p})_{im}^j := (\mathbf{f}_p^{-1})_n^j \frac{\partial (\mathbf{f}_p)_m^n}{\partial X^i} \quad , \quad (46)$$

the space  $\mathcal{R}_{\mathcal{R}}$  will be flat. In this case, no constitutive laws for  $\tilde{\mathcal{K}}_p$  are necessary, provided some evolution laws for  $\mathbf{f}_p$  are available.

#### 2.4.3 Elastic and plastic parts of the curvature tensor

We turn to the scalar differences in Eqs. (40), (41), and chose the differential operator  $\nabla_{\hat{\mathcal{R}}_t}$  to be given by  $\hat{\nabla}$  (cf. Eq. (44)), so that

$$\nabla_{\hat{\mathcal{R}}_t} \hat{\boldsymbol{\varphi}}_2 = \left( \frac{\partial (\hat{\boldsymbol{\varphi}}_2)^j}{\partial X^i} + \Lambda_{im}^j (\hat{\boldsymbol{\varphi}}_2)^m \right) \hat{\boldsymbol{\rho}}_j \otimes \hat{\mathbf{g}}^i \quad , \quad (47)$$

where

$$\hat{\boldsymbol{\varphi}}_2 = (\hat{\boldsymbol{\varphi}}_2)^j \hat{\boldsymbol{\rho}}_j \quad , \quad \boldsymbol{\Phi}_2 = (\boldsymbol{\Phi}_2)^j \mathbf{E}_j \quad , \quad (\boldsymbol{\Phi}_2)^j \equiv (\boldsymbol{\varphi}_2)^j \quad . \quad (48)$$

It is readily seen that

$$\begin{aligned} \hat{\boldsymbol{\varphi}}^1 \cdot (\nabla_{\hat{\mathcal{R}}_t} \hat{\boldsymbol{\varphi}}_2)[\hat{\mathbf{g}}_3] &= \boldsymbol{\Phi}^1 \cdot \mathbf{f}_p^{-1} (\nabla_{\hat{\mathcal{R}}_t} \hat{\boldsymbol{\varphi}}_2) \mathbf{F}_p[\mathbf{E}_3] \\ &= \boldsymbol{\Phi}^1 \cdot \left\{ \left( \frac{\partial (\boldsymbol{\Phi}_2)^j}{\partial X^i} + \Lambda_{im}^j (\boldsymbol{\Phi}_2)^m \right) \mathbf{E}_j \otimes \mathbf{E}^i \right\} [\mathbf{E}_3] \quad . \quad (49) \end{aligned}$$

Also, from (43),

$$\nabla_{\mathcal{R}\mathcal{R}} \Phi_2 = \left( \frac{\partial(\Phi_2)^j}{\partial X^i} + \lambda_{im}^j (\Phi_2)^m \right) \mathbf{E}_j \otimes \mathbf{E}^i \quad , \quad (50)$$

with  $\lambda_{im}^j \equiv \lambda_{jim} = 0$  being the symbols, relative to  $\{\mathbf{E}_i\}$ , of the Levi-Civita connection in  $\mathcal{R}\mathcal{R}$ . Thus, after inserting into (41),

$$(\Delta_c)_p = \Phi^1 \cdot \{(\Lambda_{im}^j - \lambda_{im}^j)(\Phi_2 \cdot \mathbf{E}^m)\}(\mathbf{E}_j \otimes \mathbf{E}^i)[\mathbf{E}_3] \quad (51)$$

or

$$(\Delta_c)_p = \tilde{\mathcal{K}}_p[\Phi^1, \Phi_2, \mathbf{E}_3] \quad (52)$$

with

$$\tilde{\mathcal{K}}_p = (\tilde{\mathcal{K}}_p)^j_{mi} \mathbf{E}_j \otimes \mathbf{E}^m \otimes \mathbf{E}^i \quad , \quad (53)$$

$$(\tilde{\mathcal{K}}_p)^j_{mi} \equiv (\tilde{\mathcal{K}}_p)_{jmi} = \Lambda_{im}^j - \lambda_{im}^j \quad . \quad (54)$$

In addition, it can be seen that

$$(\Delta_c)_e = \tilde{\mathcal{K}}_e[\Phi^1, \Phi_2, \mathbf{E}_3] \quad (55)$$

with

$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}_e + \tilde{\mathcal{K}}_p \quad . \quad (56)$$

On requiring from the differences  $\Delta_c$ ,  $(\Delta_c)_e$ , and  $(\Delta_c)_p$  to be form-invariant with respect to the chosen configuration, and by employing similar mathematical manipulations as in Part I, it is straightforward to deduce that, e.g., relative to the plastic intermediate configuration the relations

$$\Delta_c = \hat{\mathcal{K}}[\hat{\varphi}^1, \hat{\varphi}_2, \hat{\mathbf{g}}_3] \quad , \quad \hat{\mathcal{K}} = \mathcal{L}(\mathbf{f}_p, \mathbf{f}_p^{T-1}, \mathbf{F}_p^{T-1})[\tilde{\mathcal{K}}] \quad , \quad (57)$$

$$(\Delta_c)_e = \hat{\mathcal{K}}_e[\hat{\varphi}^1, \hat{\varphi}_2, \hat{\mathbf{g}}_3] \quad , \quad \hat{\mathcal{K}}_e = \mathcal{L}(\mathbf{f}_p, \mathbf{f}_p^{T-1}, \mathbf{F}_p^{T-1})[\tilde{\mathcal{K}}_e] \quad , \quad (58)$$

$$(\Delta_c)_p = \hat{\mathcal{K}}_p[\hat{\varphi}^1, \hat{\varphi}_2, \hat{\mathbf{g}}_3] \quad , \quad \hat{\mathcal{K}}_p = \mathcal{L}(\mathbf{f}_p, \mathbf{f}_p^{T-1}, \mathbf{F}_p^{T-1})[\tilde{\mathcal{K}}_p] \quad , \quad (59)$$

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}_e + \hat{\mathcal{K}}_p \quad (60)$$

apply. Further relations are given in Appendix C. Obviously, tensors  $\tilde{\mathcal{K}}$ ,  $\hat{\mathcal{K}}$ ,  $\mathcal{K}$ , ... are members of an equivalence class.

Postulating also  $(\Delta_c)_e$ ,  $(\Delta_c)_p$ , ... to be form-invariant with respect to the chosen configuration, associated rates for the elastic and plastic parts of the micromorphic curvature tensors can be defined in a natural way. Clearly, the additive decomposition of the curvature tensors carries over their associated rates. Appendix C summarizes formulas of this kind and illustrates how the various micromorphic curvature tensors, and their associated rates, are related to each other.

*Remark*

As indicated in conjunction with Eq. (46), there are two possibilities for the curvature tensor  $\tilde{\mathcal{K}}_p$ .

- (1)  $\Lambda_{im}^j$ , and therefore  $\tilde{\mathcal{K}}_p$  too, are not subject to some compatibility conditions, so that the Riemannian curvature tensor is non-vanishing. Then, separate constitutive laws are needed for plastic strain variables and for  $\tilde{\mathcal{K}}_p$ .
- (2)  $\Lambda_{im}^j$  in Eq. (54) is assumed to be equal to  $(\Lambda_{\mathbf{f}_p})_{im}^j$  in Eq. (46). Since the right hand side of (46) is related to the gradient of  $\mathbf{f}_p$ , it is not necessary to postulate constitutive relations governing the response of  $\tilde{\mathcal{K}}_p$ , provided some evolution equations for  $\mathbf{f}_p$  are available.

In the present article we are concerned with the first possibility only. (The other case will be discussed elsewhere.)

### 2.5 Stress tensors and their associated rates

It has been shown in Part I how dual stress tensors and associated rates may be introduced with the help of the stress powers  $w'$ ,  $w$ , and  $w_c$ . By setting  $\mathbf{f}_a = \mathbf{f}_p$ ,  $\mathbf{F}_a = \mathbf{F}_p$  (see Part I, Sect. 5.3), we obtain with respect to the plastic intermediate configuration  $\hat{\mathcal{R}}_t$ ,

$$w' = \hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\beta}} \quad , \quad w = \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\epsilon}} \quad , \quad w_c = \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\kappa}} \quad , \quad (61)$$

where

$$\hat{\boldsymbol{\sigma}} := \mathbf{f}_p \tilde{\boldsymbol{\sigma}} \mathbf{f}_p^T \quad , \quad \hat{\mathbf{S}} := \mathbf{f}_p^{T-1} \tilde{\mathbf{S}} \mathbf{F}_p^T \quad , \quad \hat{\mathbf{S}} := \mathbf{f}_p^{T-1} \tilde{\mathbf{S}} \mathbf{F}_p^T \quad . \quad (62)$$

It is of interest to remark that the stress tensors  $\tilde{\boldsymbol{\sigma}}$ ,  $\hat{\boldsymbol{\sigma}}$ ,  $\boldsymbol{\sigma}$ , or  $\tilde{\mathbf{S}}$ ,  $\hat{\mathbf{S}}$ ,  $\mathbf{S}$ , or  $\tilde{\mathbf{S}}$ ,  $\hat{\mathbf{S}}$ ,  $\mathbf{S}$  are members of corresponding equivalence classes. The associated rates of the stress tensors in (62) read

$$\overset{\nabla}{\hat{\boldsymbol{\sigma}}} = \dot{\hat{\boldsymbol{\sigma}}} - \hat{\mathbf{l}}_p \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}} \hat{\mathbf{l}}_p^T \quad , \quad (63)$$

$$\overset{\nabla}{\hat{\mathbf{S}}} = \dot{\hat{\mathbf{S}}} + \hat{\mathbf{l}}_p^T \hat{\mathbf{S}} - \hat{\mathbf{S}} \hat{\mathbf{l}}_p^T \quad , \quad (64)$$

$$\overset{\nabla}{\hat{\mathbf{S}}} = \dot{\hat{\mathbf{S}}} + \hat{\mathbf{l}}_p \hat{\mathbf{S}} - \hat{\mathbf{S}} \diamond \hat{\mathbf{l}}_p^T - \hat{\mathbf{S}} \hat{\mathbf{l}}_p \quad . \quad (65)$$

Although no stress and double stress rates are needed for the purpose of the present paper, for reasons of completeness some results are given in Appendix C.

### 3 Thermodynamical framework for micromorphic plasticity

In the following all components are given with respect to the bases  $\{\mathbf{E}_i\}$ ,  $\{\hat{\mathbf{e}}_i\}$  or  $\{\mathbf{e}_i\}$ , so that no distinction between lower and upper indices is made.

We assume isothermal deformations with uniform temperature distribution. Then, the Clausius-Duhem inequality for micromorphic materials, with respect to the actual configuration, takes the form (cf. Eringen, 1999, p.50)

$$\mathbf{S} \cdot (\mathbf{L} - \mathbf{I}) + \boldsymbol{\sigma} \cdot \mathbf{d} + \boldsymbol{\mathcal{S}} \cdot \text{grad} \mathbf{l} - \varrho_{\mathcal{R}} \dot{\Psi} \equiv \mathbf{S} \cdot \hat{\boldsymbol{\epsilon}} + \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\beta}} + \boldsymbol{\mathcal{S}} \cdot \hat{\boldsymbol{\mathcal{K}}} - \varrho_{\mathcal{R}} \dot{\Psi} \geq 0 \quad , \quad (66)$$

where  $\Psi$  is the specific (per unit mass of the macroscopic continuum) free energy of the micromorphic material. As usually in classical plasticity, we assume the decomposition

$$\Psi(t) = \Psi_e(t) + \Psi_p(t) \quad . \quad (67)$$

Hence, inequality (66) is equivalent to

$$\mathbf{S} \cdot \hat{\boldsymbol{\epsilon}} + \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\beta}} + \boldsymbol{\mathcal{S}} \cdot \hat{\boldsymbol{\mathcal{K}}} - \varrho_{\mathcal{R}} \dot{\Psi}_e - \varrho_{\mathcal{R}} \dot{\Psi}_p \geq 0 \quad , \quad (68)$$

or, with respect to the plastic intermediate configuration,

$$\hat{\mathbf{S}} \cdot \hat{\boldsymbol{\epsilon}} + \hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\beta}} + \hat{\boldsymbol{\mathcal{S}}} \cdot \hat{\boldsymbol{\mathcal{K}}} - \varrho_{\mathcal{R}} \dot{\Psi}_e - \varrho_{\mathcal{R}} \dot{\Psi}_p \geq 0 \quad . \quad (69)$$

### 3.1 Elasticity laws - dissipation inequality

In analogy to the case of pure elasticity (cf. Part I, Sect. 3.4), we suppose  $\Psi_e$  to have the form

$$\Psi_e = \hat{\Psi}_e(\hat{\boldsymbol{\epsilon}}_e, \hat{\boldsymbol{\beta}}_e, \hat{\boldsymbol{\mathcal{K}}}_e) \quad . \quad (70)$$

Evidently,  $\Psi_e$  must be invariant under arbitrary rigid body rotations  $\mathbf{Q}_p$  superposed on the plastic intermediate configuration, which implies (cf. Appendix A),

$$\Psi_e = \hat{\Psi}_e(\mathbf{Q}_p \hat{\boldsymbol{\epsilon}}_e \mathbf{Q}_p^T, \mathbf{Q}_p \hat{\boldsymbol{\beta}}_e \mathbf{Q}_p^T, \mathcal{L}(\mathbf{Q}_p, \mathbf{Q}_p, \mathbf{Q}_p)[\hat{\boldsymbol{\mathcal{K}}}_e]) \quad . \quad (71)$$

But, this is exactly the condition for  $\hat{\Psi}_e$  to be an isotropic tensor function. Consequently,  $\Psi_e$  must be a function of scalar invariants of  $\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e$ , as e.g.  $(\hat{\epsilon}_e)_{ii}, (\hat{\beta}_e)_{ii}, (\hat{\beta}_e)_{ij}(\hat{\epsilon}_e + \hat{\epsilon}_e^T)_{ij}, (\hat{\mathcal{K}}_e)_{ijj}(\hat{\mathcal{K}}_e)_{imm}, \dots$ . These can be expressed in terms of  $\tilde{\epsilon}_e, \tilde{\epsilon}_p, \tilde{\beta}_e, \tilde{\beta}_p, \tilde{\mathcal{K}}_e$ ,

$$(\hat{\epsilon}_e)_{ii} = (\tilde{\epsilon}_e)_{jm} \left( (\tilde{\epsilon}_p + \mathbf{1})^{-1} \right)_{mj} , \quad (72)$$

$$(\hat{\beta}_e)_{ii} = (\tilde{\beta}_e)_{jm} \left( (2\tilde{\beta}_p + \mathbf{1})^{-1} \right)_{mj} , \quad (73)$$

$$\begin{aligned} (\hat{\beta}_e)_{ij}(\hat{\epsilon}_e + \hat{\epsilon}_e^T)_{ij} &= (\tilde{\beta}_e)_{ij} \left( (2\tilde{\beta}_p + \mathbf{1})^{-1} \right)_{im} \left( (\tilde{\epsilon}_p + \mathbf{1})^{T-1} \right)_{mk} (\tilde{\epsilon}_e)_{jk} \\ &\quad + (\tilde{\beta}_e)_{ij} (\tilde{\epsilon}_e)_{im} \left( (\tilde{\epsilon}_p + \mathbf{1})^{-1} \right)_{mk} \left( (2\tilde{\beta}_p + \mathbf{1})^{-1} \right)_{kj} , \end{aligned} \quad (74)$$

$$\begin{aligned} (\hat{\mathcal{K}}_e)_{ijj}(\hat{\mathcal{K}}_e)_{ill} &= (2\tilde{\beta}_p + \mathbf{1})_{mp} \left( (2\tilde{\beta}_p + \mathbf{1})^{-1} \right)_{nl} \left( (\tilde{\epsilon}_p + \mathbf{1})^{T-1} \right)_{ls} \\ &\quad \left( (2\tilde{\beta}_p + \mathbf{1})^{-1} \right)_{qk} \left( (\tilde{\epsilon}_p + \mathbf{1})^{T-1} \right)_{kr} (\tilde{\mathcal{K}}_e)_{mns} (\tilde{\mathcal{K}}_e)_{pqr} \end{aligned} \quad (75)$$

⋮

Therefore,  $\Psi_e$  can be represented also as a function of  $\tilde{\epsilon}, \tilde{\beta}, \tilde{\mathcal{K}}, \tilde{\epsilon}_p, \tilde{\beta}_p, \tilde{\mathcal{K}}_p$ ,

$$\Psi_e = \tilde{\Psi}_e(\tilde{\epsilon}, \tilde{\beta}, \tilde{\mathcal{K}}, \tilde{\epsilon}_p, \tilde{\beta}_p, \tilde{\mathcal{K}}_p) . \quad (76)$$

In order to exploit inequality (69) we need  $\hat{\Psi}_e$ . After some lengthy mathematical manipulation, we deduce from (70)

$$\begin{aligned} \hat{\Psi}_e &= \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} \cdot \hat{\epsilon}_e + \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} \cdot \hat{\beta}_e + \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} \cdot \hat{\mathcal{K}}_e - \left\{ \hat{\epsilon}_e^T \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\eta} \right\} \cdot \hat{\epsilon}_p \\ &\quad - \left\{ 2\hat{\beta}_e \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} \right\}_S \cdot \hat{\beta}_p - \left\{ 2\hat{\beta}_e \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} \right\}_A \cdot \hat{\mathbf{w}}_p , \end{aligned} \quad (77)$$

where

$$\frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} := \frac{1}{\varrho_{\mathcal{R}}} \hat{\eta} - \frac{1}{\varrho_{\mathcal{R}}} \hat{\chi} + \hat{\epsilon}_e^T \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} - \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} \hat{\epsilon}_e^T , \quad (78)$$

$\{\cdot\}_S, \{\cdot\}_A$  are the symmetric and skew-symmetric parts of  $\{\cdot\}$  respectively,

and  $\hat{\eta}, \hat{\chi}$  are given by

$$\frac{1}{\varrho_{\mathcal{R}}} (\hat{\eta})_{ml} := \frac{\partial \hat{\Psi}_e}{\partial (\hat{\mathcal{K}}_e)_{rnl}} (\hat{\mathcal{K}}_e)_{rnm} , \quad (79)$$

$$\frac{1}{\varrho_{\mathcal{R}}}(\hat{\chi})_{ml} := \frac{\partial \hat{\Psi}_e}{\partial (\hat{\mathcal{K}}_e)_{mnr}}(\hat{\mathcal{K}}_e)_{lnr} - \frac{\partial \hat{\Psi}_e}{\partial (\hat{\mathcal{K}}_e)_{nlr}}(\hat{\mathcal{K}}_e)_{nmr} \quad . \quad (80)$$

Eq. (77) may be rewritten as

$$\begin{aligned} \dot{\Psi}_e &= \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} \cdot \hat{\epsilon} + \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} \cdot \hat{\beta} + \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} \cdot \hat{\mathcal{K}} - \left\{ (\mathbf{1} + \hat{\epsilon}_e^T) \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\eta} \right\} \cdot \hat{\epsilon}_p \\ &\quad - \left\{ (\mathbf{1} + 2\hat{\beta}_e) \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} \right\}_S \cdot \hat{\beta}_p - \left\{ 2\hat{\beta}_e \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} \right\}_A \cdot \hat{\mathbf{w}}_p \\ &\quad - \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} \cdot \hat{\mathcal{K}}_p \quad . \end{aligned} \quad (81)$$

Now, we shall show that  $\left\{ 2\hat{\beta}_e \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} \right\}_A \equiv 0$ . To this end, we take the material time derivative of (76),

$$\begin{aligned} \dot{\Psi}_e &= \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\epsilon}} \cdot \dot{\tilde{\epsilon}} + \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\beta}} \cdot \dot{\tilde{\beta}} + \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\mathcal{K}}} \cdot \dot{\tilde{\mathcal{K}}} + \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\epsilon}_p} \cdot \dot{\tilde{\epsilon}}_p + \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\beta}_p} \cdot \dot{\tilde{\beta}}_p + \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\mathcal{K}}_p} \cdot \dot{\tilde{\mathcal{K}}}_p \\ &= \mathbf{f}_p^{T-1} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\epsilon}} \mathbf{F}_p^T \cdot \hat{\epsilon} + \mathbf{f}_p \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\beta}} \mathbf{f}_p^T \cdot \hat{\beta} + \mathcal{L}(\mathbf{f}_p^{T-1}, \mathbf{f}_p, \mathbf{F}_p) \left[ \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\mathcal{K}}} \right] \cdot \hat{\mathcal{K}} \\ &\quad + \mathbf{f}_p^{T-1} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\epsilon}_p} \mathbf{F}_p^T \cdot \hat{\epsilon}_p + \mathbf{f}_p \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\beta}_p} \mathbf{f}_p^T \cdot \hat{\beta}_p + \mathcal{L}(\mathbf{f}_p^{T-1}, \mathbf{f}_p, \mathbf{F}_p) \left[ \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\mathcal{K}}_p} \right] \cdot \hat{\mathcal{K}}_p \quad . \end{aligned} \quad (82)$$

On comparing (81) with (82),

$$\frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} = \mathbf{f}_p^{T-1} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\epsilon}} \mathbf{F}_p^T \quad , \quad (83)$$

$$\frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} = \mathbf{f}_p \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\beta}} \mathbf{f}_p^T \quad , \quad (84)$$

$$\frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} = \mathcal{L}(\mathbf{f}_p^{T-1}, \mathbf{f}_p, \mathbf{F}_p) \left[ \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\mathcal{K}}} \right] \quad , \quad (85)$$

$$- \left\{ (\mathbf{1} + \hat{\epsilon}_e^T) \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\eta} \right\} = \mathbf{f}_p^{T-1} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\epsilon}_p} \mathbf{F}_p^T \quad , \quad (86)$$

$$- \left\{ (\mathbf{1} + 2\hat{\beta}_e) \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} \right\}_S = \mathbf{f}_p \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\beta}_p} \mathbf{f}_p^T \quad , \quad (87)$$

$$-\frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} = \mathcal{L}(\mathbf{f}_p^{T-1}, \mathbf{f}_p, \mathbf{F}_p) \left[ \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\mathcal{K}}_p} \right] , \quad (88)$$

$$\left\{ 2\hat{\beta}_e \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\Lambda} \right\}_A = 0 , \quad (89)$$

which proves the assertion.

Substituting (77) into (69),

$$\begin{aligned} & \left( \hat{\mathbf{S}} - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} \right) \cdot \hat{\epsilon} + \left\{ \varrho_{\mathcal{R}} (\mathbf{1} + \hat{\epsilon}_e^T) \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} + \hat{\eta} \right\} \cdot \hat{\epsilon}_p \\ & + \left( \hat{\sigma} - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} \right) \cdot \hat{\beta} + \left\{ \varrho_{\mathcal{R}} (\mathbf{1} + 2\hat{\beta}_e) \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} + \hat{\Lambda} \right\}_S \cdot \hat{\beta}_p \\ & + \left( \hat{\mathbf{S}} - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} \right) \cdot \hat{\mathcal{K}} + \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} \cdot \hat{\mathcal{K}}_p - \varrho_{\mathcal{R}} \dot{\Psi}_p \geq 0 , \end{aligned} \quad (90)$$

which must be satisfied for all  $\hat{\epsilon}$ ,  $\hat{\beta}$  and  $\hat{\mathcal{K}}$ .

We assume that  $\hat{\mathbf{S}}$ ,  $\hat{\sigma}$  and  $\hat{\mathbf{S}}$  are functions of  $\hat{\epsilon}_e$ ,  $\hat{\beta}_e$ ,  $\hat{\mathcal{K}}_e$ ,

$$\hat{\mathbf{S}} = \hat{\mathbf{S}}(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e) , \quad \hat{\sigma} = \hat{\sigma}(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e) , \quad \hat{\mathbf{S}} = \hat{\mathbf{S}}(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e) , \quad (91)$$

and that  $\Psi_p$  depends on internal state variables describing the hardening response of the micromorphic material. For the case of rate-dependent plasticity, referred to as viscoplasticity, we assume the evolution of the internal state variables to depend also on state variables (but not on their rates). That is, we suppose  $\hat{\epsilon}_p$ ,  $\hat{\beta}_p$ ,  $\hat{\mathcal{K}}_p$  and  $\dot{\Psi}_p$  to be functions of state variables only. Thus, using similar arguments as in Coleman and Gurtin (1967), we may conclude that the relations

$$\hat{\mathbf{S}} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\epsilon}_e} = \varrho_{\mathcal{R}} \mathbf{f}_p^{T-1} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\epsilon}} \mathbf{F}_p^T , \quad (92)$$

$$\hat{\sigma} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\beta}_e} = \varrho_{\mathcal{R}} \mathbf{f}_p \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\beta}} \mathbf{f}_p^T , \quad (93)$$

$$\hat{\mathbf{S}} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} = \varrho_{\mathcal{R}} \mathcal{L}(\mathbf{f}_p^{T-1}, \mathbf{f}_p, \mathbf{F}_p) \left[ \frac{\partial \tilde{\Psi}_e}{\partial \mathcal{K}} \right] , \quad (94)$$

$$\begin{aligned} \mathcal{D} := & \left\{ \varrho_{\mathcal{R}} (\mathbf{1} + \hat{\boldsymbol{\epsilon}}_e^T) \frac{\partial \hat{\Psi}_e}{\partial \hat{\boldsymbol{\epsilon}}_e} + \hat{\boldsymbol{\eta}} \right\} \cdot \hat{\boldsymbol{\epsilon}}_p + \left\{ \varrho_{\mathcal{R}} (\mathbf{1} + 2\hat{\boldsymbol{\beta}}_e) \frac{\partial \hat{\Psi}_e}{\partial \hat{\boldsymbol{\beta}}_e} + \hat{\boldsymbol{\Lambda}} \right\}_S \cdot \hat{\boldsymbol{\beta}}_p \\ & + \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} \cdot \hat{\mathcal{K}}_p - \varrho_{\mathcal{R}} \dot{\Psi}_p \geq 0 \end{aligned} \quad (95)$$

are necessary and sufficient conditions in order for inequality (90) to be valid in every admissible process. We call inequality (95) the internal dissipation inequality.

For rate-independent plasticity, often called plasticity, we define the evolution of internal state variables to depend on the state variables and the rates of the strain and micromorphic curvature tensors. Consequently, relations (92)-(95) are necessary and sufficient for (90) to be valid in every purely elastic admissible process, for which, by definition,  $\hat{\boldsymbol{\epsilon}}_p, \hat{\boldsymbol{\beta}}_p, \hat{\mathcal{K}}_p$  vanish. However, we assume (92)-(95) to apply also along loading paths where inelastic flow is involved, so that for (rate-independent) plasticity these relations are generally only sufficient conditions for (90).

It is convenient to introduce the stress tensors

$$\hat{\mathbf{P}} := (\mathbf{1} + \hat{\boldsymbol{\epsilon}}_e^T) \hat{\mathbf{S}} + \hat{\boldsymbol{\eta}} , \quad (96)$$

$$\hat{\boldsymbol{\Pi}} := \left\{ (\mathbf{1} + 2\hat{\boldsymbol{\beta}}_e) \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\Lambda}} \right\}_S , \quad (97)$$

where  $\hat{\boldsymbol{\Lambda}}$  reads, in terms of  $\hat{\mathbf{S}}$ ,

$$\hat{\boldsymbol{\Lambda}} = \hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\chi}} + \hat{\boldsymbol{\epsilon}}_e^T \hat{\mathbf{S}} - \hat{\mathbf{S}} \hat{\boldsymbol{\epsilon}}_e^T , \quad (98)$$

and  $\hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\eta}}$  are given by (79), (80). Then, inequality (95) becomes

$$\mathcal{D} = \hat{\mathbf{P}} \cdot \hat{\boldsymbol{\epsilon}}_p + \hat{\boldsymbol{\Pi}} \cdot \hat{\boldsymbol{\beta}}_p + \hat{\mathbf{S}} \cdot \hat{\mathcal{K}}_p - \varrho_{\mathcal{R}} \dot{\Psi}_p \geq 0 . \quad (99)$$

It is worthwhile mentioning that the plastic stress power is represented by means of the stress tensors  $\hat{\mathbf{P}}$ ,  $\hat{\mathbf{\Pi}}$ , and  $\hat{\mathbf{S}}$ , i.e. these stress tensors play a similar role as the so-called Mandel stress tensor in classical plasticity (see e.g. Lubliner, 1986; Tsakmakis, 1996). Therefore, it is meaningful to refer to these stress tensors also as Mandel stress tensors of the micromorphic material. Note in passing that Mandel stress tensors for plastically deformable micropolar materials have been introduced in Grammenoudis and Tsakmakis (2001).

### 3.2 Postulate of Il'iushin – flow rule for plasticity

The postulate of Il'iushin has been investigated in the framework of classical rate-independent plasticity, among others, by Hill (1968); Hill and Rice (1973); Dafalias (1977); Casey and Tseng (1984); Lubliner (1986); Lin and Naghdi (1989); Lucchesi and Silhavy (1991); Fosdick and Volkmann (1993); Srinivasa (1997) as well as Tsakmakis (1996, 1997, 2001). An appropriate generalization of the postulate for micropolar plasticity has been worked out in Grammenoudis and Tsakmakis (2001). Here, we shall adopt the validity of this postulate, in an appropriate fashion for micromorphic (rate independent) plasticity. Flow rules for  $\hat{\boldsymbol{\epsilon}}_p$ ,  $\hat{\boldsymbol{\beta}}_p$ , and  $\hat{\boldsymbol{\kappa}}_p$  will then be derived as sufficient conditions for the postulate.

Let

$$f(t) = \hat{f}(\hat{\mathbf{P}}, \hat{\mathbf{\Pi}}, \hat{\mathbf{S}}, \hat{\mathbf{h}}) \quad (100)$$

be a yield function with respect to the space of the stress tensors  $\hat{\mathbf{P}}$ ,  $\hat{\mathbf{\Pi}}$ ,  $\hat{\mathbf{S}}$ , with  $\hat{\mathbf{h}}$  being a set of internal state variables  $\hat{h}_i$ ,  $1 \leq i \leq M$ . The latter are scalars or components of tensors capturing hardening properties. It is assumed that (100) may be recast in a "strain-curvature space" formulation with respect to the

reference configuration in the form

$$f(t) = \tilde{g}(\tilde{\boldsymbol{\epsilon}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\mathcal{K}}}, \tilde{\boldsymbol{\epsilon}}_p, \tilde{\boldsymbol{\beta}}_p, \tilde{\boldsymbol{\mathcal{K}}}_p, \tilde{\mathbf{q}}) \quad , \quad (101)$$

where  $\tilde{\mathbf{q}}$  denotes a set of internal state variables  $\tilde{q}_j$ ,  $1 \leq j \leq N$ , associated in some way with the hardening variables  $\hat{h}_i$ .

The equation

$$f(t) = \hat{f}(\hat{\mathbf{P}}, \hat{\boldsymbol{\Pi}}, \hat{\boldsymbol{\mathcal{S}}}, \hat{\mathbf{h}}) = \tilde{g}(\tilde{\boldsymbol{\epsilon}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\mathcal{K}}}, \tilde{\boldsymbol{\epsilon}}_p, \tilde{\boldsymbol{\beta}}_p, \tilde{\boldsymbol{\mathcal{K}}}_p, \tilde{\mathbf{q}}) = 0 \quad (102)$$

is called yield condition. For fixed values of  $\hat{\mathbf{h}}$ , it describes a so-called yield surface in the space of the stress tensors  $\hat{\mathbf{P}}$ ,  $\hat{\boldsymbol{\Pi}}$ , and  $\hat{\boldsymbol{\mathcal{S}}}$ . For fixed values of  $\tilde{\boldsymbol{\epsilon}}_p$ ,  $\tilde{\boldsymbol{\beta}}_p$ ,  $\tilde{\boldsymbol{\mathcal{K}}}_p$ ,  $\tilde{\mathbf{q}}$  it describes a yield surface in the space of the strain tensors  $\tilde{\boldsymbol{\epsilon}}$ ,  $\tilde{\boldsymbol{\beta}}$  and micromorphic curvature tensors  $\tilde{\boldsymbol{\mathcal{K}}}$ . For simplicity, the yield surfaces are assumed to be smooth.

Loading processes involving plastic flow may be described by employing, instead of time  $t$ , a scalar parameter  $s$  denoting a plastic arc length. It is postulated that for  $s = \text{const.}$  all internal state variables stay constant as well. Furthermore, it is convenient to introduce a so-called loading factor  $L(t)$  (cf. Tsakmakis, 1991),

$$L := [\dot{f}]_{s=\text{const.}} \quad . \quad (103)$$

Then, the model response is characterized as follows (cf. Naghdi and Trapp, 1975; Naghdi and Casey, 1981)

$$f < 0 \Leftrightarrow \text{elastic range} \quad , \quad (104)$$

$$f = 0 \quad \& \quad L \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \Leftrightarrow \begin{cases} \text{elastic unloading} \quad , \\ \text{neutral loading} \quad , \\ \text{plastic loading} \quad . \end{cases} \quad (105)$$

Plastic flow is defined to occur only when conditions for plastic loading apply.

We remark that a cycle in the space of the tensors  $\tilde{\boldsymbol{\epsilon}}$ ,  $\tilde{\boldsymbol{\beta}}$ , and  $\tilde{\boldsymbol{\kappa}}$  is equivalent to a cycle in the space of any further strain and micromorphic curvature measure. Generalizing a proposal of Lucchesi and Silhavy (1991) (cf. also Grammenoudis and Tsakmakis, 2001, who generalize the postulate to capture micropolar material response), we denote strain-curvature cycles as small (but not necessarily infinitesimally small), if the following condition is satisfied. During the cyclic process, the initial strain-curvature state is always on or inside the yield surfaces  $\tilde{g} = 0$  corresponding to the process. In other words, the initial strain-curvature state always lies in the intersection of all the elastic ranges surrounded by the yield surfaces  $\tilde{g} = 0$  during the process.

We write  $C_s[t_0, t_e]$  for a small cycle, which begins at time  $t_0$ , and ends at time  $t_e$ . A plastically deformable micromorphic material is defined to satisfy the postulate of Il'iushin for small cycles, if for a fixed material particle

$$\begin{aligned} I(t_0, t_e) &:= \frac{1}{\varrho_{\mathcal{R}}} \int_{t_0}^{t_e} \mathbf{S} \cdot \dot{\tilde{\boldsymbol{\epsilon}}} dt + \frac{1}{\varrho_{\mathcal{R}}} \int_{t_0}^{t_e} \boldsymbol{\sigma} \cdot \dot{\tilde{\boldsymbol{\beta}}} dt + \frac{1}{\varrho_{\mathcal{R}}} \int_{t_0}^{t_e} \boldsymbol{\mathcal{S}} \cdot \dot{\tilde{\boldsymbol{\kappa}}} dt \\ &= \frac{1}{\varrho_{\mathcal{R}}} \int_{t_0}^{t_e} \tilde{\mathbf{S}} \cdot \dot{\tilde{\boldsymbol{\epsilon}}} dt + \frac{1}{\varrho_{\mathcal{R}}} \int_{t_0}^{t_e} \tilde{\boldsymbol{\sigma}} \cdot \dot{\tilde{\boldsymbol{\beta}}} dt + \frac{1}{\varrho_{\mathcal{R}}} \int_{t_0}^{t_e} \tilde{\boldsymbol{\mathcal{S}}} \cdot \dot{\tilde{\boldsymbol{\kappa}}} dt \\ &\geq 0 \quad \text{for every } C_s[t_0, t_e] \quad . \end{aligned} \quad (106)$$

In Appendix B it is proven that (106) is equivalent to

$$\hat{\mathbf{P}} \cdot \hat{\tilde{\boldsymbol{\epsilon}}}_p + \hat{\boldsymbol{\Pi}} \cdot \hat{\tilde{\boldsymbol{\beta}}}_p + \hat{\boldsymbol{\mathcal{S}}} \cdot \hat{\tilde{\boldsymbol{\kappa}}}_p \geq \hat{\mathbf{P}}^{(A)} \cdot \hat{\tilde{\boldsymbol{\epsilon}}}_p + \hat{\boldsymbol{\Pi}}^{(A)} \cdot \hat{\tilde{\boldsymbol{\beta}}}_p + \hat{\boldsymbol{\mathcal{S}}}^{(A)} \cdot \hat{\tilde{\boldsymbol{\kappa}}}_p \quad , \quad (107)$$

where  $(\hat{\mathbf{P}}, \hat{\mathbf{\Pi}}, \hat{\mathbf{S}})$  is a stress state on the yield surface, which induces the plastic strain-curvature rates  $(\hat{\boldsymbol{\epsilon}}_p, \hat{\boldsymbol{\beta}}_p, \hat{\boldsymbol{\kappa}}_p)$ . The stress state  $(\hat{\mathbf{P}}^{(A)}, \hat{\mathbf{\Pi}}^{(A)}, \hat{\mathbf{S}}^{(A)})$  is a so-called admissible stress state, i.e. a stress state which is accessible and is on or inside the yield surface,  $\hat{f}(\hat{\mathbf{P}}^{(A)}, \hat{\mathbf{\Pi}}^{(A)}, \hat{\mathbf{S}}^{(A)}, \hat{\mathbf{h}}^{(A)}) \leq 0$ .

If one introduces the notation

$$\hat{\boldsymbol{u}}_p := (\hat{\boldsymbol{\epsilon}}_p, \hat{\boldsymbol{\beta}}_p, \hat{\boldsymbol{\kappa}}_p) \quad , \quad \hat{\mathbf{s}} := (\hat{\mathbf{P}}, \hat{\mathbf{\Pi}}, \hat{\mathbf{S}}) \quad , \quad (108)$$

then (106) is equivalent to

$$\hat{\mathbf{s}} \cdot \hat{\boldsymbol{u}}_p \geq \hat{\mathbf{s}}^A \cdot \hat{\boldsymbol{u}}_p \quad . \quad (109)$$

As the plastic power of the micromorphic material may be expressed in terms of  $\hat{\mathbf{s}}, \hat{\boldsymbol{u}}_p$ ,

$$w_{pl} := \hat{\mathbf{P}} \cdot \hat{\boldsymbol{\epsilon}}_p + \hat{\mathbf{\Pi}} \cdot \hat{\boldsymbol{\beta}}_p + \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\kappa}}_p \equiv \hat{\mathbf{s}} \cdot \hat{\boldsymbol{u}}_p \quad , \quad (110)$$

inequality (109) represents a so-called principle of maximum plastic stress power, which is a natural extension of the corresponding principle of maximum plastic stress power in classical plasticity. The physical interpretation of (109) may be seen by using the definition

$$\bar{f}(\hat{\mathbf{s}}, \hat{\mathbf{h}}) := \hat{f}(\hat{\mathbf{P}}, \hat{\mathbf{\Pi}}, \hat{\mathbf{S}}, \hat{\mathbf{h}}) \quad . \quad (111)$$

Then, with respect to a pure mechanical formulation of the theory, inequality (109) states that, for a given plastic rate  $\hat{\boldsymbol{u}}_p$ , among all admissible stress states  $\hat{\mathbf{s}}^A$ , the actual stress state  $\hat{\mathbf{s}}$  maximizes the plastic power  $w_{pl}$ .

For isothermal deformations with uniform distribution, we deal with here, the internal dissipation is given by (cf. (99))

$$\mathcal{D}(\hat{\mathbf{s}}, \hat{\boldsymbol{u}}_p, \dot{\Psi}_p) = \hat{\mathbf{s}} \cdot \hat{\boldsymbol{u}}_p - \varrho_{\mathcal{R}} \dot{\Psi}_p \quad . \quad (112)$$

Keeping in mind this equation, (109) states that for given internal state variables and their rates, i.e. for given  $\hat{\mathbf{u}}_p$  and  $\dot{\Psi}_p$ , among all admissible stresses  $\hat{\mathbf{s}}^A$  the actual one  $\hat{\mathbf{s}}$  maximizes  $\mathcal{D}$ .

As in classical plasticity (see e.g. Lubliner, 1990, Sect. 3.2.2), the convexity of the yield surface  $\bar{f} = 0$ , and the normality rule for  $\hat{\mathbf{u}}_p$ , are sufficient conditions for inequality (109) to hold. This means that (109) is always satisfied, if  $\hat{\mathbf{u}}_p$  is directed along the outward normal on the yield surface  $\bar{f} = 0$ , which has been assumed to be smooth,

$$\hat{\mathbf{u}}_p = \dot{s} \frac{\frac{\partial \bar{f}}{\partial \hat{\mathbf{s}}}}{\left\| \frac{\partial \bar{f}}{\partial \hat{\mathbf{s}}} \right\|} , \quad (113)$$

or, equivalently

$$\hat{\boldsymbol{\epsilon}}_p = \dot{s} \frac{\frac{\partial \hat{f}}{\partial \hat{\mathbf{P}}}}{\left\| \frac{\partial \hat{f}}{\partial \hat{\mathbf{P}}} \right\|} , \quad \hat{\boldsymbol{\beta}}_p = \dot{s} \frac{\frac{\partial \hat{f}}{\partial \hat{\boldsymbol{\Pi}}}}{\left\| \frac{\partial \hat{f}}{\partial \hat{\boldsymbol{\Pi}}} \right\|} , \quad \hat{\boldsymbol{\kappa}}_p = \dot{s} \frac{\frac{\partial \hat{f}}{\partial \hat{\boldsymbol{S}}}}{\left\| \frac{\partial \hat{f}}{\partial \hat{\boldsymbol{S}}} \right\|} , \quad (114)$$

with

$$\left\| \frac{\partial \bar{f}}{\partial \hat{\mathbf{s}}} \right\| := \sqrt{\frac{\partial \hat{f}}{\partial \hat{\mathbf{P}}} \cdot \frac{\partial \hat{f}}{\partial \hat{\mathbf{P}}} + \frac{\partial \hat{f}}{\partial \hat{\boldsymbol{\Pi}}} \cdot \frac{\partial \hat{f}}{\partial \hat{\boldsymbol{\Pi}}} + \frac{\partial \hat{f}}{\partial \hat{\boldsymbol{S}}} \cdot \frac{\partial \hat{f}}{\partial \hat{\boldsymbol{S}}}} . \quad (115)$$

$\dot{s}$  is a positive scalar for plastic loading, which has to be determined from the so-called consistency condition  $\dot{f} = 0$ . We see from (113)-(115) that

$$\dot{s} = \sqrt{\hat{\mathbf{u}}_p \cdot \hat{\mathbf{u}}_p} := \sqrt{\hat{\boldsymbol{\epsilon}}_p \cdot \hat{\boldsymbol{\epsilon}}_p + \hat{\boldsymbol{\beta}}_p \cdot \hat{\boldsymbol{\beta}}_p + \hat{\boldsymbol{\kappa}}_p \cdot \hat{\boldsymbol{\kappa}}_p} . \quad (116)$$

Clearly, convexity of  $\bar{f}(\hat{\mathbf{s}}, \hat{\mathbf{h}}) = 0$  with respect to  $\hat{\mathbf{s}}$  is equivalent to convexity of  $\hat{f}(\hat{\mathbf{P}}, \hat{\boldsymbol{\Pi}}, \hat{\boldsymbol{S}}, \hat{\mathbf{h}}) = 0$  with respect to  $\hat{\mathbf{P}}, \hat{\boldsymbol{\Pi}}, \hat{\boldsymbol{S}}$ .

Plastic incompressibility is defined by the constraints

$$\det \mathbf{F}_p = \det \mathbf{f}_p = 1 \quad \Leftrightarrow \quad \text{tr} \hat{\mathbf{L}}_p = \text{tr} \hat{\mathbf{l}}_p = \text{tr} \hat{\boldsymbol{\epsilon}}_p = \text{tr} \hat{\boldsymbol{\beta}}_p = 0 . \quad (117)$$

If this is assumed, then the yield function must have such a form that  $\frac{\partial \hat{f}}{\partial \hat{\mathbf{P}}}$  and  $\frac{\partial \hat{f}}{\partial \hat{\mathbf{\Pi}}}$  are deviatoric.

Note in passing, that in contrast to the very interesting approach of Ehlers (see e.g. Ehlers and Volk, 1997b,a, 1998),  $\hat{\mathcal{K}}_p$  does not satisfy any compatibility conditions. This is the reason why evolution equations for  $\hat{\mathcal{K}}_p$  are necessary. Also several yield functions, corresponding to multi-mechanism plasticity have been suggested by Forest and Sievert (2003). This could be an alternative approach, which may be more proper when discussing practical problems.

### 3.3 Flow rule for viscoplasticity

Consider micromorphic viscoplasticity models which arise from those of micromorphic plasticity by adopting all the constitutive equations except for the evolution equation for  $s$ . This is now defined in a quite similar way as in classical viscoplasticity in terms of a so-called over-stress. Thus, whereas for rate-independent micromorphic plasticity the yield function is subject to the constraint  $f = \hat{f}(\hat{\mathbf{P}}, \hat{\mathbf{\Pi}}, \hat{\mathbf{S}}, \hat{\mathbf{h}}) \leq 0$ , in the case of micromorphic viscoplasticity no such restrictions on  $f$  are imposed. We call a positive value of  $f$  an over-stress, so that  $\dot{s}$  is supposed to be given as a function of  $\langle f \rangle$ . As an example, we propose the evolution equation (cf. Chaboche, 1989)

$$\dot{s} = \frac{\langle f \rangle^m}{\eta} \geq 0 \quad , \quad (118)$$

with  $m$  and  $\eta$  being positive material parameters.

### 3.4 Hardening rules

We suppose the micromorphic material to exhibit isotropic and kinematic hardening. Let  $r$  be a scalar valued internal state variable responsible for isotropic hardening. With respect to the plastic intermediate configuration, we introduce strain and micromorphic curvature tensors  $\hat{\boldsymbol{\epsilon}}_k$ ,  $\hat{\boldsymbol{\beta}}_k$ ,  $\hat{\boldsymbol{\kappa}}_k$ , responsible for kinematic hardening, so that the additive decompositions

$$\hat{\boldsymbol{\epsilon}}_p = \hat{\boldsymbol{\epsilon}}_k + \hat{\boldsymbol{\epsilon}}_d \quad , \quad \hat{\boldsymbol{\beta}}_p = \hat{\boldsymbol{\beta}}_k + \hat{\boldsymbol{\beta}}_d \quad , \quad \hat{\boldsymbol{\kappa}}_p = \hat{\boldsymbol{\kappa}}_k + \hat{\boldsymbol{\kappa}}_d \quad , \quad (119)$$

apply. The index  $d$  indicates that the corresponding variables are related with the work dissipated as heat. We think of the additive decompositions (119) to be induced by multiplicative decompositions of  $\mathbf{F}_p$ ,  $\mathbf{f}_p$ ,

$$\mathbf{F}_p = \mathbf{F}_k \mathbf{F}_d \quad , \quad \mathbf{f}_p = \mathbf{f}_k \mathbf{f}_d \quad , \quad (120)$$

with  $\det \mathbf{F}_k > 0$ ,  $\det \mathbf{f}_k > 0$ .  $\mathbf{F}_k$ ,  $\mathbf{f}_k$  introduce a new intermediate configuration  $\check{\mathcal{R}}_t$ , characterized by the property that the stress and back-stress tensors are vanishing there. According to our work until now, it is a straightforward matter to establish the kinematical relations given in Appendix D, which are similar to those in Appendix C.

Following classical proposals (see e.g. Diegele et al., 2000), we assume the additive decomposition for  $\Psi_p$

$$\Psi_p(t) = \Psi_{is}(t) + \Psi_k(t) \quad (121)$$

with

$$\Psi_{is} = \bar{\Psi}_{is}(r) \quad , \quad \Psi_k = \hat{\Psi}_k(\hat{\boldsymbol{\epsilon}}_k, \hat{\boldsymbol{\beta}}_k, \hat{\boldsymbol{\kappa}}_k) \quad . \quad (122)$$

It is convenient to introduce the stresses

$$R := \varrho_{\mathcal{R}} \frac{\partial \bar{\Psi}_{is}}{\partial r} \quad , \quad \hat{\mathbf{S}}_k := \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\epsilon}}_k} \quad , \quad \hat{\boldsymbol{\sigma}}_k := \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\beta}}_k} \quad , \quad \hat{\mathbf{S}}_k := \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_k}{\partial \hat{\mathcal{K}}_k} \quad . \quad (123)$$

$R$  denotes the scalar valued stress modeling isotropic hardening, so that the yield stress  $k$  is given by

$$k := R + \bar{k}_0 \quad , \quad \bar{k}_0 = \text{const.} \geq 0 \quad . \quad (124)$$

Kinematic hardening is modeled by the back-stress tensor  $\hat{\mathbf{s}}_k$ ,

$$\hat{\mathbf{s}}_k := (\hat{\mathbf{P}}_k, \hat{\mathbf{\Pi}}_k, \hat{\mathbf{S}}_k) \quad , \quad (125)$$

where the tensors  $\hat{\mathbf{P}}_k, \hat{\mathbf{\Pi}}_k$  are defined in the following.

Assume that  $\Psi_k$  in (122) may be represented also in the form (cf. Eqs. (70), (76))

$$\Psi_k = \tilde{\Psi}_k(\tilde{\boldsymbol{\epsilon}}_p, \tilde{\boldsymbol{\beta}}_p, \tilde{\mathcal{K}}_p, \tilde{\boldsymbol{\epsilon}}_d, \tilde{\boldsymbol{\beta}}_d, \tilde{\mathcal{K}}_d) \quad . \quad (126)$$

Performing mathematical manipulations similar to those in Sect. 3.1, we arrive at the results

$$\begin{aligned} \dot{\Psi}_k &= \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\epsilon}}_k} \cdot \overset{\Delta}{\hat{\boldsymbol{\epsilon}}_k} + \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\beta}}_k} \cdot \overset{\Delta}{\hat{\boldsymbol{\beta}}_k} + \frac{\partial \hat{\Psi}_k}{\partial \hat{\mathcal{K}}_k} \cdot \overset{\Delta}{\hat{\mathcal{K}}_k} - \left\{ \hat{\boldsymbol{\epsilon}}_k^T \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\epsilon}}_k} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\boldsymbol{\eta}}_k \right\} \cdot \overset{\Delta}{\hat{\boldsymbol{\epsilon}}_p} \\ &\quad - \left\{ 2\hat{\boldsymbol{\beta}}_k \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\beta}}_k} + \hat{\boldsymbol{\Lambda}}_k \right\}_S \cdot \overset{\Delta}{\hat{\boldsymbol{\beta}}_p} - \left\{ 2\hat{\boldsymbol{\beta}}_k \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\beta}}_k} + \frac{1}{\varrho_{\mathcal{R}}} \hat{\boldsymbol{\Lambda}}_k \right\}_A \cdot \hat{\mathbf{w}}_p \\ &= \frac{\partial \tilde{\Psi}_k}{\partial \tilde{\boldsymbol{\epsilon}}_d} \cdot \dot{\tilde{\boldsymbol{\epsilon}}_d} + \frac{\partial \tilde{\Psi}_k}{\partial \tilde{\boldsymbol{\beta}}_d} \cdot \dot{\tilde{\boldsymbol{\beta}}_d} + \frac{\partial \tilde{\Psi}_k}{\partial \tilde{\mathcal{K}}_d} \cdot \dot{\tilde{\mathcal{K}}_d} \\ &\quad + \frac{\partial \tilde{\Psi}_k}{\partial \tilde{\boldsymbol{\epsilon}}_p} \cdot \dot{\tilde{\boldsymbol{\epsilon}}_p} + \frac{\partial \tilde{\Psi}_k}{\partial \tilde{\boldsymbol{\beta}}_p} \cdot \dot{\tilde{\boldsymbol{\beta}}_p} + \frac{\partial \tilde{\Psi}_k}{\partial \tilde{\mathcal{K}}_p} \cdot \dot{\tilde{\mathcal{K}}_p} \quad , \quad (127) \end{aligned}$$

from which we deduce

$$-\frac{1}{\varrho_{\mathcal{R}}} \hat{\mathbf{S}}_k = -\frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\epsilon}}_k} = \mathbf{f}_p^{T-1} \frac{\partial \tilde{\Psi}_k}{\partial \tilde{\boldsymbol{\epsilon}}_d} \mathbf{F}_p^T \quad , \quad (128)$$

$$-\frac{1}{\varrho_{\mathcal{R}}}\hat{\boldsymbol{\sigma}}_k = -\frac{\partial\hat{\Psi}_k}{\partial\hat{\boldsymbol{\beta}}_k} = \mathbf{f}_p \frac{\partial\tilde{\Psi}_k}{\partial\tilde{\boldsymbol{\beta}}_d} \mathbf{f}_p^T \quad , \quad (129)$$

$$-\frac{1}{\varrho_{\mathcal{R}}}\hat{\mathbf{S}}_k = -\frac{\partial\hat{\Psi}_k}{\partial\hat{\mathcal{K}}_k} = \mathcal{L}(\mathbf{f}_p^{T-1}, \mathbf{f}_p, \mathbf{F}_p) \left[ \frac{\partial\tilde{\Psi}_k}{\partial\tilde{\mathcal{K}}_d} \right] \quad , \quad (130)$$

$$\left\{ (\mathbf{1} - \hat{\boldsymbol{\epsilon}}_k^T) \frac{\partial\hat{\Psi}_k}{\partial\hat{\boldsymbol{\epsilon}}_k} - \frac{1}{\varrho_{\mathcal{R}}}\hat{\boldsymbol{\eta}}_k \right\} = \mathbf{f}_p^{T-1} \frac{\partial\tilde{\Psi}_k}{\partial\tilde{\boldsymbol{\epsilon}}_p} \mathbf{F}_p^T \quad , \quad (131)$$

$$\left\{ (\mathbf{1} - 2\hat{\boldsymbol{\beta}}_k) \frac{\partial\hat{\Psi}_k}{\partial\hat{\boldsymbol{\beta}}_k} - \hat{\boldsymbol{\Lambda}}_k \right\}_S = \mathbf{f}_p \frac{\partial\tilde{\Psi}_k}{\partial\tilde{\boldsymbol{\beta}}_p} \mathbf{f}_p^T \quad , \quad (132)$$

$$\frac{\partial\hat{\Psi}_k}{\partial\hat{\mathcal{K}}_k} = \mathcal{L}(\mathbf{f}_p^{T-1}, \mathbf{f}_p, \mathbf{F}_p) \left[ \frac{\partial\tilde{\Psi}_k}{\partial\tilde{\mathcal{K}}_p} \right] \quad , \quad (133)$$

$$\left\{ 2\hat{\boldsymbol{\beta}}_k \frac{\partial\hat{\Psi}_k}{\partial\hat{\boldsymbol{\beta}}_k} + \frac{1}{\varrho_{\mathcal{R}}}\hat{\boldsymbol{\Lambda}}_k \right\}_A = 0 \quad , \quad (134)$$

with

$$\frac{1}{\varrho_{\mathcal{R}}}\hat{\boldsymbol{\Lambda}}_k := \frac{1}{\varrho_{\mathcal{R}}}\hat{\boldsymbol{\eta}}_k - \frac{1}{\varrho_{\mathcal{R}}}\hat{\boldsymbol{\chi}}_k + \frac{1}{\varrho_{\mathcal{R}}}\{\hat{\boldsymbol{\epsilon}}_k^T \hat{\mathbf{S}}_k - \hat{\mathbf{S}}_k \hat{\boldsymbol{\epsilon}}_k^T\} \quad , \quad (135)$$

$$\frac{1}{\varrho_{\mathcal{R}}}(\hat{\boldsymbol{\eta}}_k)_{ml} := \frac{\partial\hat{\Psi}_k}{\partial(\hat{\mathcal{K}}_k)_{rnl}} (\hat{\mathcal{K}}_k)_{rnm} \quad , \quad (136)$$

$$\frac{1}{\varrho_{\mathcal{R}}}(\hat{\boldsymbol{\chi}}_k)_{ml} := \frac{\partial\hat{\Psi}_k}{\partial(\hat{\mathcal{K}}_k)_{mnr}} (\hat{\mathcal{K}}_k)_{lnr} - \frac{\partial\hat{\Psi}_k}{\partial(\hat{\mathcal{K}}_k)_{nlr}} (\hat{\mathcal{K}}_k)_{nmr} \quad . \quad (137)$$

These relations suggest to define the back-stress tensors  $\hat{\mathbf{P}}_k, \hat{\boldsymbol{\Pi}}_k$  by

$$\hat{\mathbf{P}}_k := (\mathbf{1} - \hat{\boldsymbol{\epsilon}}_k^T) \hat{\mathbf{S}}_k - \hat{\boldsymbol{\eta}}_k \quad , \quad \hat{\boldsymbol{\Pi}}_k := \left\{ (\mathbf{1} - 2\hat{\boldsymbol{\beta}}_k) \hat{\boldsymbol{\sigma}}_k - \hat{\boldsymbol{\Lambda}}_k \right\}_S \quad . \quad (138)$$

This way,  $\varrho_{\mathcal{R}}\hat{\Psi}_p$  becomes

$$\varrho_{\mathcal{R}}\hat{\Psi}_p = -\hat{\mathbf{S}}_k \cdot \hat{\boldsymbol{\epsilon}}_d - \hat{\boldsymbol{\sigma}}_k \cdot \hat{\boldsymbol{\beta}}_d - \hat{\mathbf{S}}_k \cdot \hat{\mathcal{K}}_d + \hat{\mathbf{P}}_k \cdot \hat{\boldsymbol{\epsilon}}_p + \hat{\boldsymbol{\Pi}}_k \cdot \hat{\boldsymbol{\beta}}_p + \hat{\mathbf{S}}_k \cdot \hat{\mathcal{K}}_p + R\dot{r} \quad . \quad (139)$$

After inserting in the dissipation inequality (99),

$$\mathcal{D} = (\hat{\mathbf{P}} - \hat{\mathbf{P}}_k) \cdot \hat{\boldsymbol{\epsilon}}_p + (\hat{\boldsymbol{\Pi}} - \hat{\boldsymbol{\Pi}}_k) \cdot \hat{\boldsymbol{\beta}}_p + (\hat{\mathbf{S}} - \hat{\mathbf{S}}_k) \cdot \hat{\mathcal{K}}_p - R\dot{r} + \hat{\mathbf{S}}_k \cdot \hat{\boldsymbol{\epsilon}}_d + \hat{\boldsymbol{\sigma}}_k \cdot \hat{\boldsymbol{\beta}}_d + \hat{\mathbf{S}}_k \cdot \hat{\mathcal{K}}_d \geq 0 \quad , \quad (140)$$

or

$$\mathcal{D} = (\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) \cdot \hat{\boldsymbol{u}}_p - R\dot{r} + \hat{\mathbf{s}}_d \cdot \hat{\boldsymbol{u}}_d \geq 0 \quad , \quad (141)$$

where

$$\hat{\mathbf{s}}_d := (\hat{\mathbf{S}}_k, \hat{\boldsymbol{\sigma}}_k, \hat{\mathbf{S}}_k) \quad , \quad \hat{\boldsymbol{\mathcal{U}}}_d := (\hat{\boldsymbol{\epsilon}}_d, \hat{\boldsymbol{\beta}}_d, \hat{\boldsymbol{\mathcal{K}}}_d) \quad . \quad (142)$$

We separate effects due to isotropic hardening from those due to kinematic hardening by requiring the two inequalities

$$\mathcal{D}_{is} := (\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) \cdot \hat{\boldsymbol{\mathcal{U}}}_p - R\dot{r} \geq 0 \quad , \quad (143)$$

$$\mathcal{D}_k := \hat{\mathbf{s}}_d \cdot \hat{\boldsymbol{\mathcal{U}}}_d \geq 0 \quad , \quad (144)$$

which are sufficient conditions for (141).

### 3.4.1 Isotropic hardening

Let the yield function in (111) obey the representation

$$f(t) = \bar{f}(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) - k \equiv \bar{f}(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) - R - \bar{k}_0 \quad , \quad (145)$$

with  $\bar{f}$  being a homogeneous function of degree one, so that, according to Euler's theorem,

$$\frac{\partial \bar{f}}{\partial(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k)} \cdot (\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) = \bar{f} \quad . \quad (146)$$

We recall from the normality rule (113) that (143) is equivalent to

$$\mathcal{D}_{is} = (\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) \cdot \dot{s} \frac{\frac{\partial \bar{f}}{\partial(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k)}}{\left\| \frac{\partial \bar{f}}{\partial(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k)} \right\|} - R\dot{r} \geq 0 \quad , \quad (147)$$

or, by virtue of (146),

$$\mathcal{D}_{is} = \dot{s} \frac{\bar{f}}{\zeta} - R\dot{r} \geq 0 \quad , \quad \zeta := \left\| \frac{\partial \bar{f}}{\partial(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k)} \right\| \quad . \quad (148)$$

When inelastic flow is involved,  $f = 0 \Leftrightarrow \bar{f} = k$  for plasticity, and  $f \geq 0 \Leftrightarrow$

$\bar{f} \geq k$  for viscoplasticity. Hence, we conclude that

$$\dot{s} \frac{k}{\zeta} - R\dot{r} \geq 0 \quad (149)$$

is a sufficient condition for (148). Keeping in mind (124), it follows that (149) is equivalent to

$$R \left( \frac{\dot{s}}{\zeta} - \dot{r} \right) + \bar{k}_0 \frac{\dot{s}}{\zeta} \geq 0 \quad . \quad (150)$$

Since  $\bar{k}_0 \frac{\dot{s}}{\zeta} \geq 0$ , it suffices to require

$$R \left( \frac{\dot{s}}{\zeta} - \dot{r} \right) \geq 0 \quad , \quad (151)$$

in order to satisfy (143) always. A sufficient condition for the validity of the latter reads

$$\frac{\dot{s}}{\zeta} - \dot{r} = \frac{\beta \dot{s}}{\gamma \zeta} (R - R_0) \quad \Leftrightarrow \quad \dot{r} = \left( 1 - \frac{\beta}{\gamma} (R - R_0) \right) \frac{\dot{s}}{\zeta} \quad , \quad (152)$$

$$R_0 := R|_{r=0} \quad , \quad (153)$$

where  $\beta \geq 0$ ,  $\gamma \geq 0$  are material parameters subject to the condition  $\frac{\beta}{\gamma} \geq 0$ .

As a particular example consider the case

$$\Psi_{is} = \bar{\Psi}_{is}(r) = \frac{\gamma}{2\varrho_{\mathcal{R}}} (r^2 + 2r_0 r) \quad \Rightarrow \quad R = \varrho_{\mathcal{R}} \frac{\partial \bar{\Psi}_{is}}{\partial r} = \gamma(r + r_0) \quad . \quad (154)$$

Thus,

$$k = R + \bar{k}_0 \quad \Rightarrow \quad k_0 := k|_{r=0} = R_0 + \bar{k}_0 \quad , \quad R_0 = \gamma r_0 \quad , \quad (155)$$

and (152) is equivalent to

$$\dot{r} = (1 - \beta r) \frac{\dot{s}}{\zeta} \quad , \quad (156)$$

or

$$\dot{R} = \{ \gamma - \beta(R - R_0) \} \frac{\dot{s}}{\zeta} \quad (157)$$

or

$$\dot{k} = \{\gamma - \beta(k - k_0)\} \frac{\dot{s}}{\zeta} . \quad (158)$$

In essence, these results for isotropic hardening are similar to those in classical plasticity established by Chaboche (see e.g. Chaboche, 1993).

### 3.4.2 Kinematic hardening

In order to satisfy (144),

$$\mathcal{D}_k = \hat{s}_d \cdot \hat{\mathbf{U}}_d \equiv \hat{\mathbf{S}}_k \cdot \hat{\boldsymbol{\epsilon}}_d + \hat{\boldsymbol{\sigma}}_k \cdot \hat{\boldsymbol{\beta}}_d + \hat{\mathbf{S}}_k \cdot \hat{\boldsymbol{\kappa}}_d \geq 0 , \quad (159)$$

it suffices to assume

$$\hat{\boldsymbol{\epsilon}}_d = \hat{s} \hat{\mathbf{M}}_k[\hat{\mathbf{S}}_k] , \quad \hat{\boldsymbol{\beta}}_d = \hat{s} \hat{\mathbf{N}}_k[\hat{\boldsymbol{\sigma}}_k] , \quad \hat{\boldsymbol{\kappa}}_d = \hat{s} \hat{\mathcal{P}}_k[\hat{\mathbf{S}}_k] , \quad (160)$$

where  $\hat{\mathbf{M}}_k$ ,  $\hat{\mathbf{N}}_k$  are respectively semi-definite isotropic fourth-order tensors, and  $\hat{\mathcal{P}}_k$  is a semi-definite isotropic sixth-order tensor. Clearly, the evolution equations (160) may be rewritten in the form

$$\hat{\boldsymbol{\epsilon}}_k = \hat{\boldsymbol{\epsilon}}_p - \hat{s} \hat{\mathbf{M}}_k[\hat{\mathbf{S}}_k] , \quad (161)$$

$$\hat{\boldsymbol{\beta}}_k = \hat{\boldsymbol{\beta}}_p - \hat{s} \hat{\mathbf{N}}_k[\hat{\boldsymbol{\sigma}}_k] , \quad (162)$$

$$\hat{\boldsymbol{\kappa}}_k = \hat{\boldsymbol{\kappa}}_p - \hat{s} \hat{\mathcal{P}}_k[\hat{\mathbf{S}}_k] . \quad (163)$$

These evolution equations represent generalized Armstrong-Frederick rules (cf. Armstrong and Frederick, 1966) for the micromorphic material as adopted here.

### 3.5 Coupling with damage

Micromorphic plasticity models have considerable influence whenever localization phenomena are studied. Such phenomena can result as a consequence of some softening mechanisms inherent in the model response. Damage models induce softening and are employed to describe the progressive material degradation due to the loading process. A simple damage model arises if one assumes the concept of effective stresses combined with the principle of strain equivalence. This approach has been initiated and intensively investigated by Lemaitre and Chaboche (1990) (see e.g. Chaboche, 1999; Lemaitre, 1985) (A comprehensive study is also given in Reckwerth and Tsakmakis (2003).) We shall now apply this approach in our micropolar plasticity to capture damage effects.

We start from the second law of thermodynamics (69) and assume again the additive decomposition (67). But now,  $\Psi_e$  and  $\Psi_p$  depend also on the scalar damage variable  $D$ :

$$\Psi(t) = \hat{\Psi}(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e, \hat{\epsilon}_k, \hat{\beta}_k, \hat{\mathcal{K}}_k, r, D) = \Psi_e(t) + \Psi_p(t) \quad , \quad \Psi_p = \Psi_{is} + \Psi_k \quad , \quad (164)$$

$$\Psi_e = \hat{\Psi}_e(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e, D) \quad , \quad \Psi_{is} = \bar{\Psi}_{is}(r, D) \quad , \quad \Psi_k = \hat{\Psi}_k(\hat{\epsilon}_k, \hat{\beta}_k, \hat{\mathcal{K}}_k, D) \quad . \quad (165)$$

This is the simplest possibility to describe isotropic damage. It is assumed that  $D \in [0, 1]$ . The values  $D = 0$  and  $D = 1$  correspond to the undamaged state and the complete local rupture, respectively, while  $D \in (0, 1)$  reflects a partially damaged state.

Quite similar to the approach until now, we establish the relations

$$\hat{\mathbf{S}} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\boldsymbol{\epsilon}}_e} \quad , \quad \hat{\boldsymbol{\sigma}} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\boldsymbol{\beta}}_e} \quad , \quad \hat{\mathbf{S}} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e}{\partial \hat{\mathcal{K}}_e} \quad , \quad (166)$$

$$R := \varrho_{\mathcal{R}} \frac{\partial \bar{\Psi}_{is}}{\partial r} \quad , \quad \hat{\mathbf{S}}_k := \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\epsilon}}_k} \quad , \quad \hat{\boldsymbol{\sigma}}_k := \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_k}{\partial \hat{\boldsymbol{\beta}}_k} \quad , \quad \hat{\mathbf{S}}_k := \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_k}{\partial \hat{\mathcal{K}}_k} \quad , \quad (167)$$

$$\begin{aligned} \mathcal{D} = & (\hat{\mathbf{P}} - \hat{\mathbf{P}}_k) \cdot \hat{\boldsymbol{\epsilon}}_p + (\hat{\boldsymbol{\Pi}} - \hat{\boldsymbol{\Pi}}_k) \cdot \hat{\boldsymbol{\beta}}_p + (\hat{\mathbf{S}} - \hat{\mathbf{S}}_k) \cdot \hat{\mathcal{K}}_p - R \dot{r} \\ & + \hat{\mathbf{S}}_k \cdot \hat{\boldsymbol{\epsilon}}_d + \hat{\boldsymbol{\sigma}}_k \cdot \hat{\boldsymbol{\beta}}_d + \hat{\mathbf{S}}_k \cdot \hat{\mathcal{K}}_d - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}}{\partial D} \dot{D} \geq 0 \quad , \end{aligned} \quad (168)$$

where the stresses  $\hat{\mathbf{P}}$ ,  $\hat{\boldsymbol{\Pi}}$ ,  $\hat{\mathbf{S}}$ ,  $\hat{\mathbf{S}}_k$ ,  $\hat{\boldsymbol{\sigma}}$ ,  $\hat{\boldsymbol{\sigma}}_k$ ,  $\hat{\mathbf{S}}_k$ ,  $\hat{\mathbf{P}}_k$ ,  $\hat{\boldsymbol{\Pi}}_k$ ,  $R$  are defined as above, but with  $\Psi$  given by (164), (165).

According to the version of the principle of strain equivalence as adopted here, the constitutive equations governing the response of the real, damaged material may be gained as follows. At every material point, we assign to the real material a fictitious, undamaged material which obeys the constitutive laws established in Sect. 3.3, 3.4, but with the variables of stress replaced by so-called effective stresses. The strains for the real and the fictitious material are assumed to be equal (strain equivalence).

To elaborate, let  $\mathbf{X}$  be any one of the stress variables  $\hat{\mathbf{S}}$ ,  $\hat{\boldsymbol{\sigma}}$ ,  $\hat{\mathbf{S}}$ ,  $R$ ,  $\hat{\mathbf{S}}_k$ ,  $\hat{\boldsymbol{\sigma}}_k$ ,  $\hat{\mathbf{S}}_k$ ,  $\hat{\mathbf{P}}$ ,  $\hat{\boldsymbol{\Pi}}$ ,  $\hat{\mathbf{P}}_k$ ,  $\hat{\boldsymbol{\Pi}}_k$ . The corresponding effective stress  $\mathbf{X}^{(eff)}$  is defined by

$$\mathbf{X}^{(eff)} := \frac{\mathbf{X}}{1 - D} \quad . \quad (169)$$

Then, from (92)-(94) we obtain

$$\hat{\mathbf{S}}^{(eff)} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e^{(f)}}{\partial \hat{\boldsymbol{\epsilon}}_e} \quad , \quad \hat{\boldsymbol{\sigma}}^{(eff)} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e^{(f)}}{\partial \hat{\boldsymbol{\beta}}_e} \quad , \quad \hat{\mathbf{S}}^{(eff)} = \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}_e^{(f)}}{\partial \hat{\mathcal{K}}_e} \quad , \quad (170)$$

where

$$\Psi_e^{(f)} = \hat{\Psi}_e^{(f)}(\hat{\boldsymbol{\epsilon}}_e, \hat{\boldsymbol{\beta}}_e, \hat{\mathcal{K}}_e) \quad (171)$$

is the specific free energy for the fictitious materials. Here and in the sequel, the superfix  $f$  denotes the fictitious material. Eqs. (166), (169), and (170) imply, after integration,

$$\hat{\Psi}_e(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e, D) = (1 - D)\hat{\Psi}_e^{(f)}(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e) \quad . \quad (172)$$

Similarly, we have

$$\bar{\Psi}_{is}(r, D) = (1 - D)\bar{\Psi}_{is}^{(f)}(r) \quad , \quad \hat{\Psi}_k(\hat{\epsilon}_k, \hat{\beta}_k, \hat{\mathcal{K}}_k, D) = (1 - D)\hat{\Psi}_k^{(f)}(\hat{\epsilon}_k, \hat{\beta}_k, \hat{\mathcal{K}}_k) \quad . \quad (173)$$

Since only isotropic and kinematic hardening are assumed to be present, the yield function reads

$$f = \bar{F}(\hat{s} - \hat{s}_k, R, D) = \bar{f}^{(f)}(\hat{s}^{(eff)} - \hat{s}_k^{(eff)}) - R^{(eff)} - \bar{k}_0 \quad , \quad (174)$$

in view of (145), and the flow rule (114) becomes

$$\hat{\epsilon}_p^\Delta = \dot{s} \frac{\partial \bar{F}}{\zeta} \quad , \quad \hat{\beta}_p^\Delta = \dot{s} \frac{\partial \bar{\Pi}}{\zeta} \quad , \quad \hat{\mathcal{K}}_p^\Delta = \dot{s} \frac{\partial \bar{\mathcal{S}}}{\zeta} \quad , \quad (175)$$

$$\zeta := \left\| \frac{\partial \bar{F}}{\partial \hat{s}} \right\| = \frac{1}{1 - D} \left\| \frac{\partial \bar{f}^{(f)}(\hat{s}^{(eff)} - \hat{s}_k^{(eff)})}{\partial \hat{s}^{(eff)}} \right\| \quad , \quad (176)$$

$$\dot{s} = \sqrt{\hat{\epsilon}_p^\Delta \cdot \hat{\epsilon}_p^\Delta + \hat{\beta}_p^\Delta \cdot \hat{\beta}_p^\Delta + \hat{\mathcal{K}}_p^\Delta \cdot \hat{\mathcal{K}}_p^\Delta} \quad . \quad (177)$$

The isotropic hardening rule in Sect. 3.4.1 suggests

$$\dot{\Psi}_{is} = \frac{\gamma(1 - D)}{2\varrho\mathcal{R}}(r^2 + 2r_0r) \quad , \quad \frac{R}{1 - D} = \gamma(r + r_0) \quad , \quad (178)$$

$$\dot{r} = (1 - \beta r) \frac{\dot{s}}{\zeta} = \left( 1 - \frac{\beta}{\gamma} \left( \frac{R}{1 - D} - \gamma r_0 \right) \right) \frac{\dot{s}}{\zeta} \quad , \quad (179)$$

with  $\zeta$ ,  $R_0$  being defined as in (176) and (153), respectively. From the kinematic hardening law in (161)- (163), we get

$$\hat{\epsilon}_k^\Delta = \hat{\epsilon}_p^\Delta - \dot{s} \hat{\mathbf{M}}_k [\hat{\mathbf{S}}_k^{(eff)}] = \hat{\epsilon}_p^\Delta - \frac{\dot{s}}{(1 - D)} \hat{\mathbf{M}}_k [\hat{\mathbf{S}}_k] \quad , \quad (180)$$

$$\hat{\beta}_k = \hat{\beta}_p - s \hat{\mathbf{N}}_k [\hat{\sigma}_k^{(eff)}] = \hat{\beta}_p - \frac{\dot{s}}{(1-D)} \hat{\mathbf{N}}_k [\hat{\sigma}_k] \quad , \quad (181)$$

$$\hat{\mathcal{K}}_k = \hat{\mathcal{K}}_p - s \hat{\mathcal{P}}_k [\hat{\mathcal{S}}_k^{(eff)}] = \hat{\mathcal{K}}_p - \frac{\dot{s}}{(1-D)} \hat{\mathcal{P}}_k [\hat{\mathcal{S}}_k] \quad . \quad (182)$$

It remains to verify whether the dissipation inequality is satisfied. To this end, we insert into (168) to obtain

$$\begin{aligned} \mathcal{D} &= (\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) \cdot \dot{s} \frac{\partial \bar{f}^{(f)}(\hat{\mathbf{s}}^{(eff)} - \hat{\mathbf{s}}_k^{(eff)})}{\partial (\hat{\mathbf{s}} - \hat{\mathbf{s}}_k)} - R\dot{r} \\ &\quad + \frac{\dot{s}}{1-D} \left\{ \hat{\mathbf{S}}_k \cdot \hat{\mathbf{M}}_k [\hat{\mathbf{S}}_k] + \hat{\sigma}_k \cdot \hat{\mathbf{N}}_k [\hat{\sigma}_k] + \hat{\mathcal{S}}_k \cdot \hat{\mathcal{P}}_k [\hat{\mathcal{S}}_k] \right\} - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}}{\partial D} \dot{D} \geq 0 \quad . \end{aligned} \quad (183)$$

Since the term in curls is always nonnegative, inequality (183) will be satisfied whenever

$$\begin{aligned} &(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k) \cdot \frac{\dot{s}}{\zeta} \frac{\partial \bar{f}^{(f)}}{\partial (\hat{\mathbf{s}} - \hat{\mathbf{s}}_k)} - R\dot{r} - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}}{\partial D} \dot{D} \\ &= (\hat{\mathbf{s}}^{(eff)} - \hat{\mathbf{s}}_k^{(eff)}) \cdot \frac{\dot{s}}{\zeta} \frac{\partial \bar{f}^{(f)}}{\partial (\hat{\mathbf{s}}^{(eff)} - \hat{\mathbf{s}}_k^{(eff)})} - R\dot{r} - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}}{\partial D} \dot{D} \\ &\geq \underbrace{R^{(eff)} \frac{\dot{s}}{\zeta}}_{\geq 0} - \underbrace{R \frac{\dot{s}}{\zeta}}_{\geq 0} + \underbrace{\beta R r \frac{\dot{s}}{\zeta}}_{\geq 0} - \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}}{\partial D} \dot{D} \geq 0 \quad . \end{aligned} \quad (184)$$

Thus, the dissipation inequality will always be satisfied, provided

$$-\varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}}{\partial D} \dot{D} \geq 0 \quad . \quad (185)$$

A sufficient condition for this inequality reads (cf. Lemaitre, 1987a,b; Lämmer and Tsakmakis, 2000)

$$\dot{D} = -\alpha_1 \dot{s} \left( \varrho_{\mathcal{R}} \frac{\partial \hat{\Psi}}{\partial D} \right) \quad , \quad (186)$$

where  $\alpha_1 \geq 0$  denotes a material parameter.

### 3.6 Simple constitutive relations

Various constitutive functions, like free energy or yield function, indicate a simple form if linear, isotropic behavior is assumed to be present. This is expressed in terms of isotropic tensors  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$ ,  $\mathbf{C}$ ,

$$\mathbb{A}_{ijpq} = \mathbb{A}_{pqij} = A_1 \delta_{ij} \delta_{pq} + A_2 \delta_{ip} \delta_{jq} + A_3 \delta_{iq} \delta_{jp} \quad , \quad (187)$$

$$\mathbb{B}_{ijpq} = \mathbb{B}_{pqij} = \mathbb{B}_{ijqp} = B_1 \delta_{ij} \delta_{pq} + B_2 (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) \quad , \quad (188)$$

$$\mathbb{D}_{ijpq} = \mathbb{D}_{ijqp} = D_1 \delta_{ij} \delta_{pq} + D_2 (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) \quad , \quad (189)$$

$$\begin{aligned} \mathbb{C}_{ijkpqr} = \mathbb{C}_{pqrjik} = & C_1 (\delta_{ij} \delta_{kp} \delta_{qr} + \delta_{jk} \delta_{ir} \delta_{pq}) + C_2 (\delta_{ij} \delta_{kq} \delta_{rp} + \delta_{ki} \delta_{jr} \delta_{pq}) \\ & + C_3 \delta_{ij} \delta_{kr} \delta_{pq} + C_4 \delta_{jk} \delta_{ip} \delta_{qr} + C_5 (\delta_{jk} \delta_{iq} \delta_{pr} + \delta_{ki} \delta_{jp} \delta_{qr}) \\ & + C_6 \delta_{ki} \delta_{jq} \delta_{rp} + C_7 \delta_{ip} \delta_{jq} \delta_{kr} + C_8 (\delta_{jp} \delta_{kq} \delta_{ir} + \delta_{kp} \delta_{iq} \delta_{jr}) \\ & + C_9 \delta_{ip} \delta_{jr} \delta_{kq} + C_{10} \delta_{jp} \delta_{kr} \delta_{iq} + C_{11} \delta_{kp} \delta_{ir} \delta_{jq} \quad . \quad (190) \end{aligned}$$

#### 3.6.1 Elasticity laws

Following Mindlin (1964) and Eringen (1999), we assume  $\Psi_e$  to be given by

$$\begin{aligned} \varrho_{\mathcal{R}} \Psi_e = & (1 - D) \left\{ \frac{1}{2} (\mathbf{A}_e)_{ijpq} (\hat{\boldsymbol{\epsilon}}_e)_{ij} (\hat{\boldsymbol{\epsilon}}_e)_{pq} + \frac{1}{2} (\mathbf{B}_e)_{ijpq} (\hat{\boldsymbol{\beta}}_e)_{ij} (\hat{\boldsymbol{\beta}}_e)_{pq} \right. \\ & \left. + (\mathbf{D}_e)_{ijpq} (\hat{\boldsymbol{\epsilon}}_e)_{ij} (\hat{\boldsymbol{\beta}}_e)_{pq} + \frac{1}{2} (\mathbf{C}_e)_{ijkpqr} (\hat{\boldsymbol{\kappa}}_e)_{ijk} (\hat{\boldsymbol{\kappa}}_e)_{pqr} \right\} \quad , \quad (191) \end{aligned}$$

where  $\mathbf{A}_e$ ,  $\mathbf{B}_e$ ,  $\mathbf{D}_e$ ,  $\mathbf{C}_e$  indicate the same form as  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$ ,  $\mathbf{C}$ , respectively. The elements of these tensors are denoted respectively by  $A_1^e$ ,  $A_2^e$ ,  $A_3^e$ ,  $B_1^e$ ,  $B_2^e$ ,  $D_1^e$ ,  $D_2^e$ ,  $C_1^e$ ,  $C_2^e$ ,  $\dots$ ,  $C_{11}^e$ . In particular we set

$$A_1^e = \lambda \quad , \quad A_2^e = \mu + \alpha \quad , \quad A_3^e = \mu - \alpha \quad . \quad (192)$$

Generally, there are involved 18 material parameters, which have to satisfy some conditions in order for  $\Psi_e$  to be always non-negative. Such conditions

have been worked out by Eringen (1999) and Smith (1968). From (166), we deduce

$$\hat{\mathbf{S}} = (1 - D) \{ A_1^e (\text{tr} \hat{\boldsymbol{\epsilon}}_e) \mathbf{1} + A_2^e \hat{\boldsymbol{\epsilon}}_e + A_3^e \hat{\boldsymbol{\epsilon}}_e^T + D_1^e (\text{tr} \hat{\boldsymbol{\beta}}_e) \mathbf{1} + 2D_2^e \hat{\boldsymbol{\beta}}_e \} \quad , \quad (193)$$

$$\hat{\boldsymbol{\sigma}} = (1 - D) \{ B_1^e (\text{tr} \hat{\boldsymbol{\beta}}_e) \mathbf{1} + 2B_2^e \hat{\boldsymbol{\beta}}_e + D_1^e (\text{tr} \hat{\boldsymbol{\epsilon}}_e) \mathbf{1} + D_2^e (\hat{\boldsymbol{\epsilon}}_e + \hat{\boldsymbol{\epsilon}}_e^T) \} \quad , \quad (194)$$

$$\begin{aligned} \hat{S}_{ijk} = & (1 - D) [\delta_{ij} \{ C_1^e (\hat{\mathcal{K}}_e)_{krr} + C_2^e (\hat{\mathcal{K}}_e)_{rkr} + C_3^e (\hat{\mathcal{K}}_e)_{rrk} \} \\ & + \delta_{jk} \{ C_1^e (\hat{\mathcal{K}}_e)_{rri} + C_4^e (\hat{\mathcal{K}}_e)_{irr} + C_5^e (\hat{\mathcal{K}}_e)_{rir} \} \\ & + \delta_{ki} \{ C_2^e (\hat{\mathcal{K}}_e)_{rrj} + C_5^e (\hat{\mathcal{K}}_e)_{jrr} + C_6^e (\hat{\mathcal{K}}_e)_{rjr} \} \\ & + C_7^e (\hat{\mathcal{K}}_e)_{ijk} + C_9^e (\hat{\mathcal{K}}_e)_{ikj} + C_{10}^e (\hat{\mathcal{K}}_e)_{jik} \\ & + C_{11}^e (\hat{\mathcal{K}}_e)_{kji} + C_8^e \{ (\hat{\mathcal{K}}_e)_{jki} + (\hat{\mathcal{K}}_e)_{kij} \}] \quad . \quad (195) \end{aligned}$$

### 3.6.2 Kinematic hardening

Intending to obtain, at the end, a theory for small deformations, we set, in analogy to (191),

$$\begin{aligned} \varrho_{\mathcal{R}} \Psi_k = & (1 - D) \left\{ \frac{1}{2} (\mathbf{A}_k)_{ijpq} (\hat{\boldsymbol{\epsilon}}_k)_{ij} (\hat{\boldsymbol{\epsilon}}_k)_{pq} + \frac{1}{2} (\mathbf{B}_k)_{ijpq} (\hat{\boldsymbol{\beta}}_k)_{ij} (\hat{\boldsymbol{\beta}}_k)_{pq} \right. \\ & \left. + (\mathbf{D}_k)_{ijpq} (\hat{\boldsymbol{\epsilon}}_k)_{ij} (\hat{\boldsymbol{\beta}}_k)_{pq} + \frac{1}{2} (\mathbf{C}_k)_{ijkpqr} (\hat{\mathcal{K}}_k)_{ijk} (\hat{\mathcal{K}}_k)_{pqr} \right\} \quad , \quad (196) \end{aligned}$$

where  $\mathbf{A}_k$ ,  $\mathbf{B}_k$ ,  $\mathbf{D}_k$ ,  $\mathbf{C}_k$  exhibit the same form as  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$ ,  $\mathbf{C}$ . Their parameters are  $A_1^k$ ,  $A_2^k$ ,  $A_3^k$ ,  $B_1^k$ ,  $B_2^k$ ,  $D_1^k$ ,  $D_2^k$ ,  $C_1^k$ ,  $C_2^k$ ,  $\dots$ ,  $C_{11}^k$ , respectively. It is readily seen, by substituting in (167), that  $\hat{\mathbf{S}}_k$ ,  $\hat{\boldsymbol{\sigma}}_k$ ,  $\hat{\mathcal{S}}_k$  are related to  $\hat{\boldsymbol{\epsilon}}_k$ ,  $\hat{\boldsymbol{\beta}}_k$  and  $\hat{\mathcal{K}}_k$  in a way similar to that in Eqs. (193)–(195). Moreover, we suppose  $\hat{\mathbf{M}}_k$ ,  $\hat{\mathbf{N}}_k$ ,  $\hat{\mathcal{P}}_k$  in (180)–(182) to exhibit the same form as  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  with material parameters  $M_1^k$ ,  $M_2^k$ ,  $M_3^k$ ,  $N_1^k$ ,  $N_2^k$ ,  $P_1^k$ ,  $P_2^k$ ,  $\dots$ ,  $P_{11}^k$ , respectively.

Hence, Eqs. (180)–(182) yield

$$\hat{\boldsymbol{\epsilon}}_k^\Delta = \hat{\boldsymbol{\epsilon}}_p^\Delta - \frac{\dot{s}}{(1 - D)} \{ M_1^k (\text{tr} \hat{\mathbf{S}}_k) \mathbf{1} + M_2^k \hat{\mathbf{S}}_k + M_3^k \hat{\mathbf{S}}_k^T \} \quad , \quad (197)$$

$$\hat{\beta}_k^\Delta = \hat{\beta}_p^\Delta - \frac{\dot{s}}{(1-D)} \{N_1^k(\text{tr} \hat{\sigma}_k) \mathbf{1} + 2N_2^k \hat{\sigma}_k\} \quad , \quad (198)$$

$$\hat{\mathcal{K}}_k^\Delta = \hat{\mathcal{K}}_p^\Delta - \frac{\dot{s}}{(1-D)} \hat{\mathcal{P}}_k[\hat{\mathcal{S}}_k] \quad . \quad (199)$$

### 3.6.3 Yield function – Flow rule

Assuming plastic incompressibility to apply for both, the micro- and the macrocontinuum, we postulate for the yield function in (174) the form

$$\begin{aligned} f &= \bar{F}(\hat{\mathbf{s}} - \hat{\mathbf{s}}_k, R, D) = \bar{f}^{(f)}(\hat{\mathbf{s}}^{(eff)} - \hat{\mathbf{s}}_k^{(eff)}) - R^{(eff)} - \bar{k}_0 \\ &= \left( \frac{(\hat{\mathbf{P}} - \hat{\mathbf{P}}_k)^D}{1-D} \cdot \mathbf{A}_y \left[ \frac{(\hat{\mathbf{P}} - \hat{\mathbf{P}}_k)^D}{1-D} \right] + \frac{(\hat{\mathbf{\Pi}} - \hat{\mathbf{\Pi}}_k)^D}{1-D} \cdot \mathbf{B}_y \left[ \frac{(\hat{\mathbf{\Pi}} - \hat{\mathbf{\Pi}}_k)^D}{1-D} \right] \right. \\ &\quad \left. + \frac{\hat{\mathcal{S}} - \hat{\mathcal{S}}_k}{1-D} \cdot \mathbf{C}_y \left[ \frac{\hat{\mathcal{S}} - \hat{\mathcal{S}}_k}{1-D} \right] \right)^{\frac{1}{2}} - \frac{R}{1-D} - \bar{k}_0 \quad . \quad (200) \end{aligned}$$

$\mathbf{A}_y, \mathbf{B}_y, \mathbf{C}_y$  indicate the same form as  $\mathbf{A}, \mathbf{B}, \mathbf{C}$ , with material parameters  $A_1^y = 0$ ,  $A_2^y, A_3^y, B_1^y = 0, B_2^y, C_1^y, \dots, C_{11}^y$ .

### 3.6.4 Small deformations

Intrinsic model properties may be discussed appropriately by confining to small deformations, excluding thus effects due to geometrical non-linearities.

Let  $\mathbf{H}, \mathbf{h}$  be the displacement gradients for the macro- and the microcontinuum,

$$\mathbf{H} := \mathbf{F} - \mathbf{1} \quad , \quad \mathbf{h} := \mathbf{f} - \mathbf{1} \quad . \quad (201)$$

Consider the set  $\mathcal{F}$ , elements of which are the tensors

$$\mathbf{H}, \mathbf{U}_e - \mathbf{1}, \mathbf{U}_p - \mathbf{1}, \mathbf{h}, \mathbf{u}_e - \mathbf{1}, \mathbf{u}_p - \mathbf{1}, \mathbf{R}_p^T \mathbf{r}_p - \mathbf{1}, \mathbf{R}_e \mathbf{r}_e^T - \mathbf{1}, \tilde{\mathcal{K}}, \tilde{\mathcal{K}}_e, \tilde{\mathcal{K}}_p, \tilde{\beta}_k, \tilde{\epsilon}_k, \tilde{\mathcal{K}}_k, \quad (202)$$

as well as their time and spatial (with respect to  $X^i$ ) derivatives. Let  $\varepsilon := \max\{\sup \|A\|/A \in \mathcal{F}\}$  be a measure of smallness, where  $\|\cdot\|$  is the Euclidean norm and  $\sup$  stands for supremum over the region  $\mathcal{R}_{\mathcal{R}}$ . Assume relations of the form

$$\mathbf{H} = \mathbf{O}(\varepsilon) \quad , \quad \mathbf{F} = \mathbf{1} + \mathbf{O}(\varepsilon) \quad , \quad (203)$$

$$\mathbf{U}_e = \mathbf{1} + \mathbf{O}(\varepsilon) \quad , \quad \mathbf{U}_p = \mathbf{1} + \mathbf{O}(\varepsilon) \quad , \quad (204)$$

$$\mathbf{F}_e = \mathbf{R}_e + \mathbf{O}(\varepsilon) \quad , \quad \mathbf{F}_p = \mathbf{R}_p + \mathbf{O}(\varepsilon) \quad , \quad (205)$$

$$\mathbf{h} = \mathbf{O}(\varepsilon) \quad , \quad \mathbf{f} = \mathbf{1} + \mathbf{O}(\varepsilon) \quad , \quad (206)$$

$$\mathbf{u}_e = \mathbf{1} + \mathbf{O}(\varepsilon) \quad , \quad \mathbf{u}_p = \mathbf{1} + \mathbf{O}(\varepsilon) \quad , \quad (207)$$

$$\mathbf{f}_e = \mathbf{r}_e + \mathbf{O}(\varepsilon) \quad , \quad \mathbf{f}_p = \mathbf{r}_p + \mathbf{O}(\varepsilon) \quad , \quad (208)$$

$$\tilde{\boldsymbol{\beta}} = \frac{1}{2}(\mathbf{h} + \mathbf{h}^T) + \mathbf{O}(\varepsilon^2) \quad , \quad \tilde{\boldsymbol{\beta}}_e = \mathbf{O}(\varepsilon) \quad , \quad \tilde{\boldsymbol{\beta}}_p = \mathbf{O}(\varepsilon) \quad , \quad (209)$$

$$\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\beta}} + \mathbf{O}(\varepsilon^2) \quad , \quad \boldsymbol{\beta}_e = \tilde{\boldsymbol{\beta}}_e + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\beta}}_e + \mathbf{O}(\varepsilon^2) \quad , \quad (210)$$

$$\boldsymbol{\beta}_p = \tilde{\boldsymbol{\beta}}_p + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\beta}}_p + \mathbf{O}(\varepsilon^2) \quad , \quad (211)$$

$$\tilde{\boldsymbol{\epsilon}} = \mathbf{H} - \mathbf{h} + \mathbf{O}(\varepsilon^2) \quad , \quad \tilde{\boldsymbol{\epsilon}}_e = \mathbf{O}(\varepsilon) \quad , \quad \tilde{\boldsymbol{\epsilon}}_p = \mathbf{O}(\varepsilon) \quad , \quad (212)$$

$$\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}} + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\epsilon}} + \mathbf{O}(\varepsilon^2) \quad , \quad \boldsymbol{\epsilon}_e = \tilde{\boldsymbol{\epsilon}}_e + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\epsilon}}_e + \mathbf{O}(\varepsilon^2) \quad , \quad (213)$$

$$\boldsymbol{\epsilon}_p = \tilde{\boldsymbol{\epsilon}}_p + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\epsilon}}_p + \mathbf{O}(\varepsilon^2) \quad , \quad (214)$$

$$\tilde{\boldsymbol{\kappa}} = \text{GRAD}\mathbf{h} + \mathbf{O}(\varepsilon^2) \quad , \quad \tilde{\boldsymbol{\kappa}}_e = \mathbf{O}(\varepsilon) \quad , \quad \tilde{\boldsymbol{\kappa}}_p = \mathbf{O}(\varepsilon) \quad , \quad (215)$$

$$\boldsymbol{\kappa} = \tilde{\boldsymbol{\kappa}} + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\kappa}} + \mathbf{O}(\varepsilon^2) \quad , \quad (216)$$

$$\boldsymbol{\kappa}_e = \tilde{\boldsymbol{\kappa}}_e + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\kappa}}_e + \mathbf{O}(\varepsilon^2) \quad , \quad (217)$$

$$\boldsymbol{\kappa}_p = \tilde{\boldsymbol{\kappa}}_p + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\kappa}}_p + \mathbf{O}(\varepsilon^2) \quad , \quad (218)$$

$$\hat{\mathbf{S}} = \mathbf{O}(\varepsilon) \quad , \quad \mathbf{S} = \hat{\mathbf{S}} + \mathbf{O}(\varepsilon^2) = \tilde{\mathbf{S}} + \mathbf{O}(\varepsilon^2) = \mathbf{T} + \mathbf{O}(\varepsilon^2) = \hat{\mathbf{P}} + \mathbf{O}(\varepsilon^2) \quad , \quad (219)$$

$$\hat{\boldsymbol{\sigma}} = \mathbf{O}(\varepsilon) \quad , \quad \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}} + \mathbf{O}(\varepsilon^2) = \tilde{\boldsymbol{\sigma}} + \mathbf{O}(\varepsilon^2) = \boldsymbol{\Sigma} + \mathbf{O}(\varepsilon^2) = \hat{\boldsymbol{\Pi}} + \mathbf{O}(\varepsilon^2) \quad , \quad (220)$$

$$\hat{\mathbf{S}} = \mathcal{O}(\varepsilon) \quad , \quad \mathbf{S} = \hat{\mathbf{S}} + \mathcal{O}(\varepsilon^2) = \tilde{\mathbf{S}} + \mathcal{O}(\varepsilon^2) = \boldsymbol{\tau} + \mathcal{O}(\varepsilon^2) \quad , \quad (221)$$

$$\boldsymbol{\beta}_k = \tilde{\boldsymbol{\beta}}_k + \mathcal{O}(\varepsilon^2) = \hat{\boldsymbol{\beta}}_k + \mathcal{O}(\varepsilon^2) \quad , \quad (222)$$

$$\vdots \quad (223)$$

$$\dot{\boldsymbol{\beta}}_p = \overset{\Delta}{\boldsymbol{\beta}}_p + \mathcal{O}(\varepsilon^2) = \dot{\boldsymbol{\beta}} + \mathcal{O}(\varepsilon^2) = \overset{\Delta}{\hat{\boldsymbol{\beta}}} + \mathcal{O}(\varepsilon^2) \quad , \quad (224)$$

$$\vdots \quad (225)$$

$$\varrho = \varrho_R + \mathcal{O}(\varepsilon) \quad , \quad (226)$$

$$\text{DIV}\mathbf{T} = \text{div}\mathbf{T} + \mathcal{O}(\varepsilon^2) \quad , \quad (227)$$

$$\vdots \quad (228)$$

to hold. Whenever terms only up to order  $\mathcal{O}(\varepsilon)$  are explicitly retained, the resulting theory is said to be of small deformations.

### A Transformations under rigid body rotations superposed on both the current and the plastic intermediate configuration

It can be seen (for some of the subsequent relations cf. Green and Naghdi, 1971; Casey and Naghdi, 1980, 1981) that under rigid body rotations  $\mathbf{Q} = \mathbf{Q}(t)$  superposed on the current configuration, and rigid body rotations  $\mathbf{Q}_p = \mathbf{Q}_p(t)$  superposed on the plastic intermediate configuration simultaneously, the transformation rules for the macroscopic continuum

$$\mathbf{F} \rightarrow \mathbf{F}^* = \mathbf{Q}\mathbf{F} = \mathbf{Q}\mathbf{F}_e\mathbf{Q}_p^T\mathbf{Q}_p\mathbf{F}_p \quad , \quad (A.1)$$

$$\mathbf{F}_e \rightarrow \mathbf{F}_e^* = \mathbf{Q}\mathbf{F}_e\mathbf{Q}_p^T \quad , \quad \mathbf{F}_p \rightarrow \mathbf{F}_p^* = \mathbf{Q}_p\mathbf{F}_p \quad , \quad (A.2)$$

$$\mathbf{R}_e \rightarrow \mathbf{R}_e^* = \mathbf{Q}\mathbf{R}_e\mathbf{Q}_p^T \quad , \quad \mathbf{R}_p \rightarrow \mathbf{R}_p^* = \mathbf{Q}_p\mathbf{R}_p \quad , \quad (A.3)$$

$$\mathbf{U}_e \rightarrow \mathbf{U}_e^* = \mathbf{Q}_p\mathbf{U}_e\mathbf{Q}_p^T \quad , \quad \mathbf{U}_p \rightarrow \mathbf{U}_p^* = \mathbf{U}_p \quad , \quad (A.4)$$

$$\mathbf{V}_e \rightarrow \mathbf{V}_e^* = \mathbf{Q}\mathbf{V}_e\mathbf{Q}^T \quad , \quad \mathbf{V}_p \rightarrow \mathbf{V}_p^* = \mathbf{Q}_p\mathbf{V}_p\mathbf{Q}_p^T \quad , \quad (A.5)$$

$$\hat{\mathbf{L}}_p \rightarrow \hat{\mathbf{L}}_p^* = \mathbf{Q}_p\hat{\mathbf{L}}_p\mathbf{Q}_p^T + \dot{\mathbf{Q}}_p\mathbf{Q}_p^T \quad , \quad (A.6)$$

$$\hat{\mathbf{D}}_p \rightarrow \hat{\mathbf{D}}_p^* = \mathbf{Q}_p \hat{\mathbf{D}}_p \mathbf{Q}_p^T \quad , \quad \hat{\mathbf{W}}_p \rightarrow \hat{\mathbf{W}}_p^* = \mathbf{Q}_p \hat{\mathbf{W}}_p \mathbf{Q}_p^T + \dot{\mathbf{Q}}_p \mathbf{Q}_p^T \quad , \quad (\text{A.7})$$

and for the microcontinuum

$$\mathbf{f} \rightarrow \mathbf{f}^* = \mathbf{Q}\mathbf{f} = \mathbf{Q}\mathbf{f}_e \mathbf{Q}_p^T \mathbf{Q}_p \mathbf{f}_p \quad , \quad (\text{A.8})$$

$$\mathbf{f}_e \rightarrow \mathbf{f}_e^* = \mathbf{Q}\mathbf{f}_e \mathbf{Q}_p^T \quad , \quad \mathbf{f}_p \rightarrow \mathbf{f}_p^* = \mathbf{Q}_p \mathbf{f}_p \quad , \quad (\text{A.9})$$

$$\mathbf{r}_e \rightarrow \mathbf{r}_e^* = \mathbf{Q}\mathbf{r}_e \mathbf{Q}_p^T \quad , \quad \mathbf{r}_p \rightarrow \mathbf{r}_p^* = \mathbf{Q}_p \mathbf{r}_p \quad , \quad (\text{A.10})$$

$$\mathbf{u}_e \rightarrow \mathbf{u}_e^* = \mathbf{Q}_p \mathbf{u}_e \mathbf{Q}_p^T \quad , \quad \mathbf{u}_p \rightarrow \mathbf{u}_p^* = \mathbf{u}_p \quad , \quad (\text{A.11})$$

$$\mathbf{v}_e \rightarrow \mathbf{v}_e^* = \mathbf{Q}\mathbf{v}_e \mathbf{Q}^T \quad , \quad \mathbf{v}_p \rightarrow \mathbf{v}_p^* = \mathbf{Q}_p \mathbf{v}_p \mathbf{Q}_p^T \quad , \quad (\text{A.12})$$

apply. Let  $\hat{\mathbf{X}}$  denote any one of the tensors  $\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\beta}}_e, \hat{\boldsymbol{\beta}}_p, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\beta}}_e, \hat{\boldsymbol{\beta}}_p, \hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\epsilon}}_e, \hat{\boldsymbol{\epsilon}}_p, \hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\epsilon}}_e, \hat{\boldsymbol{\epsilon}}_p, \hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{\Gamma}}_e, \hat{\boldsymbol{\Gamma}}_p, \hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{\Gamma}}_e, \hat{\boldsymbol{\Gamma}}_p$ . Then

$$\hat{\mathbf{X}} \rightarrow \hat{\mathbf{X}}^* = \mathbf{Q}_p \hat{\mathbf{X}} \mathbf{Q}_p^T \quad . \quad (\text{A.13})$$

For the micromorphic curvature tensors  $\tilde{\boldsymbol{\mathcal{K}}}, \hat{\boldsymbol{\mathcal{K}}}$  we have

$$\tilde{\boldsymbol{\mathcal{K}}}^* = \tilde{\boldsymbol{\mathcal{K}}} \quad , \quad (\text{A.14})$$

$$\begin{aligned} \hat{\boldsymbol{\mathcal{K}}}^* &= \mathcal{L}(\mathbf{f}_p^*, (\mathbf{f}_p^*)^{T-1}, (\mathbf{f}^*)^{T-1})[\tilde{\boldsymbol{\mathcal{K}}}^*] \\ &= \mathcal{L}(\mathbf{Q}_p, \mathbf{Q}_p, \mathbf{Q}_p) \mathcal{L}(\mathbf{f}_p, \mathbf{f}_p^{T-1}, \mathbf{f}_p^{T-1})[\tilde{\boldsymbol{\mathcal{K}}}] \\ &= \mathcal{L}(\mathbf{Q}_p, \mathbf{Q}_p, \mathbf{Q}_p)[\hat{\boldsymbol{\mathcal{K}}}] \quad . \end{aligned} \quad (\text{A.15})$$

In a similar manner, if  $\hat{\boldsymbol{\mathcal{P}}}$  represents any one of the tensors  $\hat{\boldsymbol{\mathcal{K}}}_e, \hat{\boldsymbol{\mathcal{K}}}_p, \hat{\boldsymbol{\mathcal{K}}}, \hat{\boldsymbol{\mathcal{K}}}_e, \hat{\boldsymbol{\mathcal{K}}}_p$ , then

$$\hat{\boldsymbol{\mathcal{P}}}^* = \mathcal{L}(\mathbf{Q}_p, \mathbf{Q}_p, \mathbf{Q}_p)[\hat{\boldsymbol{\mathcal{P}}}] \quad . \quad (\text{A.16})$$

## B Conditions for the validity of Il'iushin's postulate

We recall from (92)- (94) that

$$\tilde{\mathbf{S}} = \varrho_{\mathcal{R}} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\boldsymbol{\epsilon}}} \quad , \quad \tilde{\boldsymbol{\sigma}} = \varrho_{\mathcal{R}} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\boldsymbol{\beta}}} \quad , \quad \tilde{\mathbf{S}} = \varrho_{\mathcal{R}} \frac{\partial \tilde{\Psi}_e}{\partial \tilde{\boldsymbol{\mathcal{K}}}} \quad . \quad (\text{B.1})$$

Assume (106) to apply and consider a small strain-curvature cycle  $ABCD$  (see Fig. B.1), which is parameterized by time  $t$ . Denote by  $M^{(X)}$  the value of some quantity  $M$  at point  $X$ . Then, the times connected with points  $A, B, C, D$  are  $t^{(A)}, t^{(B)}, t^{(C)}, t^{(D)}$ , respectively, ( $t^{(A)} < t^{(B)} < t^{(C)} < t^{(D)}$ ). The strain-curvature cycle begins and ends at  $\tilde{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\epsilon}}^{(A)} = \tilde{\boldsymbol{\epsilon}}^{(D)}$ ,  $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}^{(A)} = \tilde{\boldsymbol{\beta}}^{(D)}$ ,  $\tilde{\boldsymbol{\kappa}} = \tilde{\boldsymbol{\kappa}}^{(A)} = \tilde{\boldsymbol{\kappa}}^{(D)}$ , while plastic flow occurs only between  $B$  and  $C$ . In analogy to Lin and Naghdi (1989), we use the notation

$$\mathbf{u}(t) := (\tilde{\boldsymbol{\epsilon}}(t), \tilde{\boldsymbol{\beta}}(t), \tilde{\boldsymbol{\kappa}}(t)) \quad , \quad \mathbf{u}_p(t) := (\tilde{\boldsymbol{\epsilon}}_p(t), \tilde{\boldsymbol{\beta}}_p(t), \tilde{\boldsymbol{\kappa}}_p(t)) \quad , \quad (\text{B.2})$$

$$\mathbf{u}^{(X)} := (\tilde{\boldsymbol{\epsilon}}^{(X)}, \tilde{\boldsymbol{\beta}}^{(X)}, \tilde{\boldsymbol{\kappa}}^{(X)}) \quad , \quad \mathbf{u}_p^{(X)} := (\tilde{\boldsymbol{\epsilon}}_p^{(X)}, \tilde{\boldsymbol{\beta}}_p^{(X)}, \tilde{\boldsymbol{\kappa}}_p^{(X)}) \quad . \quad (\text{B.3})$$

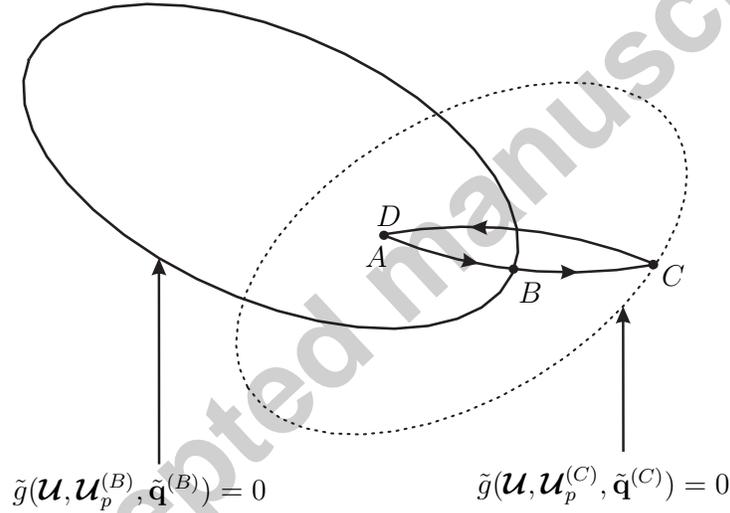


Fig. B.1. A small strain-curvature cycle with plastic flow occurring between  $B$  and  $C$  only.

Since (92)- (94), and therefore also (B.1), are assumed to hold during plastic loading as well, we have

$$\begin{aligned} I(t^{(A)}, t^{(D)}) &= \frac{1}{\varrho_{\mathcal{R}}} \int_{t^{(A)}}^{t^{(D)}} \left\{ \tilde{\mathbf{S}} \cdot \dot{\tilde{\boldsymbol{\epsilon}}} + \tilde{\boldsymbol{\sigma}} \cdot \dot{\tilde{\boldsymbol{\beta}}} + \tilde{\mathbf{S}} \cdot \dot{\tilde{\boldsymbol{\kappa}}} \right\} dt \\ &= \int_{t^{(A)}}^{t^{(D)}} \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\epsilon}}(t)} \cdot \dot{\tilde{\boldsymbol{\epsilon}}}(t) + \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\beta}}(t)} \cdot \dot{\tilde{\boldsymbol{\beta}}}(t) \right\} dt \end{aligned}$$

$$\begin{aligned}
& + \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\mathcal{K}}(t)} \cdot \dot{\tilde{\mathcal{K}}}(t) \Big\} dt \\
= & \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p^{(C)}) - \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p^{(B)}) \\
& - \int_{t^{(B)}}^{t^{(C)}} \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\epsilon}_p(t)} \cdot \dot{\tilde{\epsilon}}_p(t) dt \\
& - \int_{t^{(B)}}^{t^{(C)}} \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\beta}_p(t)} \cdot \dot{\tilde{\beta}}_p(t) dt \\
& - \int_{t^{(B)}}^{t^{(C)}} \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\mathcal{K}}_p(t)} \cdot \dot{\tilde{\mathcal{K}}}_p(t) dt \quad . \tag{B.4}
\end{aligned}$$

We make note of the identity

$$\begin{aligned}
\tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}^{(C)}) - \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}^{(B)}) = & \int_{t^{(B)}}^{t^{(C)}} \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\epsilon}_p(t)} \cdot \dot{\tilde{\epsilon}}_p(t) \right. \\
& + \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\beta}_p(t)} \cdot \dot{\tilde{\beta}}_p(t) \\
& \left. + \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\mathcal{K}}_p(t)} \cdot \dot{\tilde{\mathcal{K}}}_p(t) \right\} dt \quad , \tag{B.5}
\end{aligned}$$

and conclude from (B.4), (106)

$$\begin{aligned}
I(t^{(A)}, t^{(D)}) = & \int_{t^{(B)}}^{t^{(C)}} \left[ \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\epsilon}_p(t)} - \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\epsilon}_p(t)} \right\} \cdot \dot{\tilde{\epsilon}}_p(t) \right. \\
& + \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\beta}_p(t)} - \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\beta}_p(t)} \right\} \cdot \dot{\tilde{\beta}}_p(t) \\
& \left. + \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\mathcal{K}}_p(t)} - \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\mathcal{K}}_p(t)} \right\} \cdot \dot{\tilde{\mathcal{K}}}_p(t) \right] dt \geq 0 \quad . \tag{B.6}
\end{aligned}$$

By using Taylor's theorem

$$\lim_{t^{(C)} \rightarrow t^{(B)}} \frac{I(t^{(A)}, t^{(D)})}{t^{(C)} - t^{(B)}}$$

$$\begin{aligned}
&= \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\epsilon}}_p(t)} \cdot \dot{\tilde{\boldsymbol{\epsilon}}}_p(t) - \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\epsilon}}_p(t)} \cdot \dot{\tilde{\boldsymbol{\epsilon}}}_p(t) \right\}_{t=t^{(B)}} \\
&+ \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\beta}}_p(t)} \cdot \dot{\tilde{\boldsymbol{\beta}}}_p(t) - \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\beta}}_p(t)} \cdot \dot{\tilde{\boldsymbol{\beta}}}_p(t) \right\}_{t=t^{(B)}} \\
&+ \left\{ \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\kappa}}_p(t)} \cdot \dot{\tilde{\boldsymbol{\kappa}}}_p(t) - \frac{\partial \tilde{\Psi}_e(\mathbf{u}(t), \mathbf{u}_p(t))}{\partial \tilde{\boldsymbol{\kappa}}_p(t)} \cdot \dot{\tilde{\boldsymbol{\kappa}}}_p(t) \right\}_{t=t^{(B)}} \geq 0 \quad .
\end{aligned} \tag{B.7}$$

Since the point  $B$  can be chosen arbitrarily on the yield surface, we may drop the index  $t^{(B)}$  in the last relation to get, as a necessary condition for (106), the inequality

$$\begin{aligned}
&-\frac{\partial \tilde{\Psi}_e(\mathbf{u}, \mathbf{u}_p)}{\partial \tilde{\boldsymbol{\epsilon}}_p} \cdot \dot{\tilde{\boldsymbol{\epsilon}}}_p - \frac{\partial \tilde{\Psi}_e(\mathbf{u}, \mathbf{u}_p)}{\partial \tilde{\boldsymbol{\beta}}_p} \cdot \dot{\tilde{\boldsymbol{\beta}}}_p - \frac{\partial \tilde{\Psi}_e(\mathbf{u}, \mathbf{u}_p)}{\partial \tilde{\boldsymbol{\kappa}}_p} \cdot \dot{\tilde{\boldsymbol{\kappa}}}_p \\
&\geq -\frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p)}{\partial \tilde{\boldsymbol{\epsilon}}_p} \cdot \dot{\tilde{\boldsymbol{\epsilon}}}_p - \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p)}{\partial \tilde{\boldsymbol{\beta}}_p} \cdot \dot{\tilde{\boldsymbol{\beta}}}_p - \frac{\partial \tilde{\Psi}_e(\mathbf{u}^{(A)}, \mathbf{u}_p)}{\partial \tilde{\boldsymbol{\kappa}}_p} \cdot \dot{\tilde{\boldsymbol{\kappa}}}_p \quad ,
\end{aligned} \tag{B.8}$$

where  $\mathbf{u} = (\tilde{\boldsymbol{\epsilon}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\kappa}})$  denotes a strain-curvature state on the yield surface and  $\mathbf{u}_p = (\tilde{\boldsymbol{\epsilon}}_p, \tilde{\boldsymbol{\beta}}_p, \tilde{\boldsymbol{\kappa}}_p)$  are the plastic strain and plastic micromorphic curvature tensors associated with this state.  $\mathbf{u}^{(A)} = (\tilde{\boldsymbol{\epsilon}}^{(A)}, \tilde{\boldsymbol{\beta}}^{(A)}, \tilde{\boldsymbol{\kappa}}^{(A)})$  is a strain-curvature state on or inside the yield surface, i.e.  $\tilde{\mathbf{g}}(\mathbf{u}^{(A)}, \mathbf{u}_p, \tilde{\mathbf{q}}) \leq 0$ , with the internal state variables  $\tilde{\mathbf{q}}$  being associated with the strain-curvature state  $\mathbf{u}$ .

Conversely, (B.8) is a sufficient condition for (106) to hold. This can be verified by taking the integral of (B.8) along a strain-curvature cycle as shown in Fig. B.1 For (B.8) to remain valid during this strain-curvature cycle,  $\mathbf{u}^{(A)}$  must always lie in the intersection of all elastic ranges, which in turn implies that the cycle  $ABCD$  is small. Then, following the same steps as in (B.4)-

(B.6), but in the inverse direction, it is a straightforward matter to arrive at (106).

In view of (86)–(88), one obtains from (B.8)

$$\begin{aligned}
& \left\{ \left(1 + \hat{\epsilon}_e^T\right) \frac{\partial \hat{\Psi}_e \left(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e\right)}{\partial \hat{\epsilon}_e} + \frac{1}{\varrho \mathcal{R}} \hat{\eta} \right\} \cdot \hat{\epsilon}_p \\
& + \left\{ \left(1 + 2\hat{\beta}_e\right) \frac{\partial \hat{\Psi}_e \left(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e\right)}{\partial \hat{\beta}_e} + \frac{1}{\varrho \mathcal{R}} \hat{\Lambda} \right\}_S \cdot \hat{\beta}_p + \frac{\partial \hat{\Psi}_e \left(\hat{\epsilon}_e, \hat{\beta}_e, \hat{\mathcal{K}}_e\right)}{\partial \hat{\mathcal{K}}_e} \cdot \hat{\mathcal{K}}_p \\
\geq & \left\{ \left(1 + \left(\hat{\epsilon}_e^{(A)}\right)^T\right) \frac{\partial \hat{\Psi}_e \left(\hat{\epsilon}_e^{(A)}, \hat{\beta}_e^{(A)}, \hat{\mathcal{K}}_e^{(A)}\right)}{\partial \hat{\epsilon}_e} + \frac{1}{\varrho \mathcal{R}} \hat{\eta}^{(A)} \right\} \cdot \hat{\epsilon}_p \\
& + \left\{ \left(1 + 2\hat{\beta}_e^{(A)}\right) \frac{\partial \hat{\Psi}_e \left(\hat{\epsilon}_e^{(A)}, \hat{\beta}_e^{(A)}, \hat{\mathcal{K}}_e^{(A)}\right)}{\partial \hat{\beta}_e} + \frac{1}{\varrho \mathcal{R}} \hat{\Lambda}^{(A)} \right\}_S \cdot \hat{\beta}_p \\
& + \frac{\partial \hat{\Psi}_e \left(\hat{\epsilon}_e^{(A)}, \hat{\beta}_e^{(A)}, \hat{\mathcal{K}}_e^{(A)}\right)}{\partial \hat{\mathcal{K}}_e} \cdot \hat{\mathcal{K}}_p, \tag{B.9}
\end{aligned}$$

or, by virtue of (96), (97), and (92)–(94),

$$\hat{\mathbf{P}} \cdot \hat{\epsilon}_p + \hat{\Pi} \cdot \hat{\beta}_p + \hat{\mathbf{S}} \cdot \hat{\mathcal{K}}_p \geq \hat{\mathbf{P}}^{(A)} \cdot \hat{\epsilon}_p + \hat{\Pi}^{(A)} \cdot \hat{\beta}_p + \hat{\mathbf{S}}^{(A)} \cdot \hat{\mathcal{K}}_p. \tag{B.10}$$

Inequality (B.10) is equivalent to (B.9) and therefore equivalent to (106).

## C Decompositions of strain and micromorphic curvature tensors – dual stress and couple stress tensors

### C.1 Decomposition of the strain tensors for the microstructure

$\mathcal{R}'_{\mathcal{R}}$		$\mathcal{R}'_t$
$\begin{aligned}\tilde{\boldsymbol{\beta}} &= \frac{1}{2}(\mathbf{u}^2 - \mathbf{1}) \\ \tilde{\boldsymbol{\beta}}_p &= \frac{1}{2}(\mathbf{u}_p^2 - \mathbf{1}) \\ \tilde{\boldsymbol{\beta}}_e &= \frac{1}{2}(\mathbf{u}^2 - \mathbf{u}_p^2) \\ \tilde{\boldsymbol{\beta}} &= \tilde{\boldsymbol{\beta}}_e + \tilde{\boldsymbol{\beta}}_p\end{aligned}$	$\xrightarrow{\mathbf{f}^{T-1}(\cdot)\mathbf{f}^{-1}}$	$\begin{aligned}\boldsymbol{\beta} &= \frac{1}{2}(\mathbf{1} - \mathbf{v}^{-2}) \\ \boldsymbol{\beta}_p &= \frac{1}{2}(\mathbf{v}_e^{-2} - \mathbf{v}^{-2}) \\ \boldsymbol{\beta}_e &= \frac{1}{2}(\mathbf{1} - \mathbf{v}_e^{-2}) \\ \boldsymbol{\beta} &= \boldsymbol{\beta}_e + \boldsymbol{\beta}_p\end{aligned}$
$\downarrow \mathbf{f}_p^{T-1}(\cdot)\mathbf{f}_p^{-1}$		$\uparrow \mathbf{f}_e^{T-1}(\cdot)\mathbf{f}_e^{-1}$
	$\hat{\mathcal{R}}'_t$	$\begin{aligned}\hat{\boldsymbol{\beta}} &= \frac{1}{2}(\mathbf{u}_e^2 - \mathbf{v}_p^{-2}) \\ \hat{\boldsymbol{\beta}}_p &= \frac{1}{2}(\mathbf{1} - \mathbf{v}_p^{-2}) \\ \hat{\boldsymbol{\beta}}_e &= \frac{1}{2}(\mathbf{u}_e^2 - \mathbf{1}) \\ \hat{\boldsymbol{\beta}} &= \hat{\boldsymbol{\beta}}_e + \hat{\boldsymbol{\beta}}_p\end{aligned}$

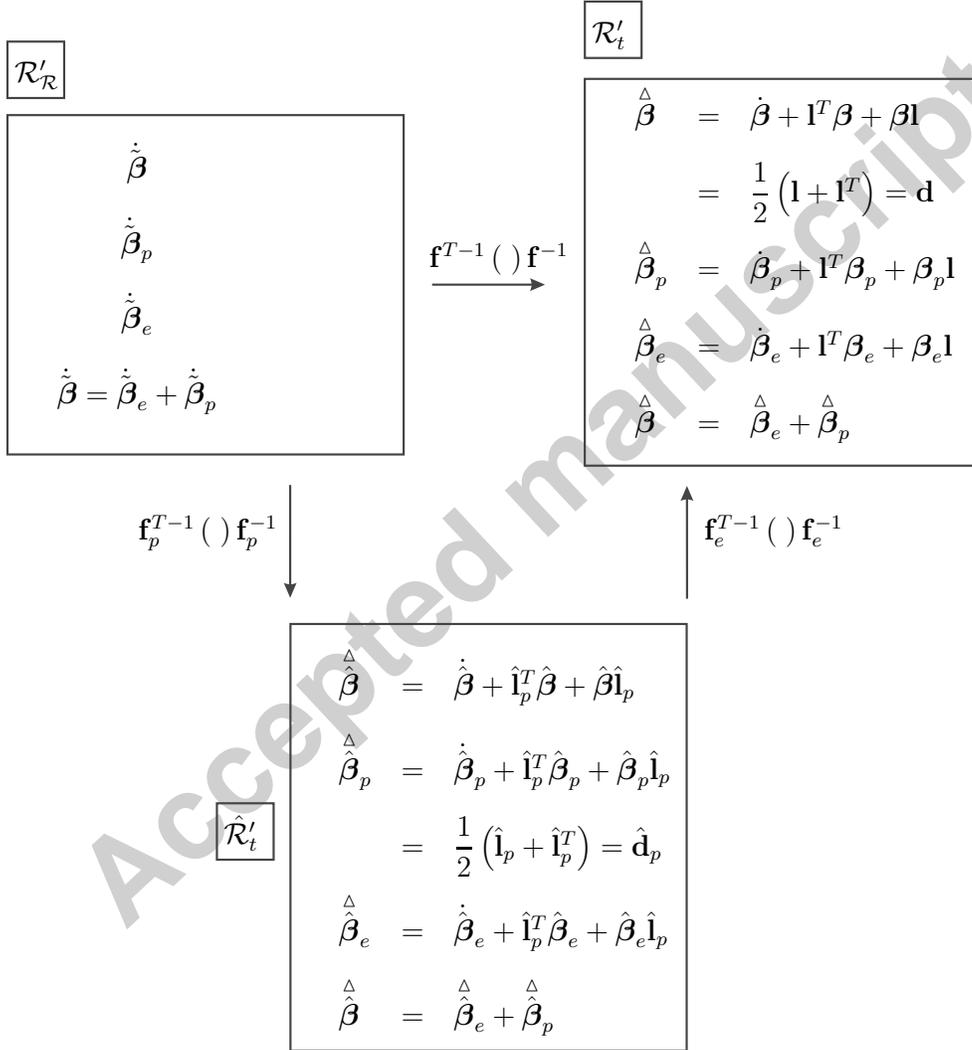
## C.2 Decomposition of the strain rate tensors for the microstructure

$$\hat{\mathbf{l}}_p = \dot{\mathbf{f}}_p \mathbf{f}_p^{-1} = \hat{\mathbf{d}}_p + \hat{\mathbf{w}}_p, \quad \mathbf{l} = \dot{\mathbf{f}} \mathbf{f}^{-1} = \mathbf{d} + \mathbf{w}$$

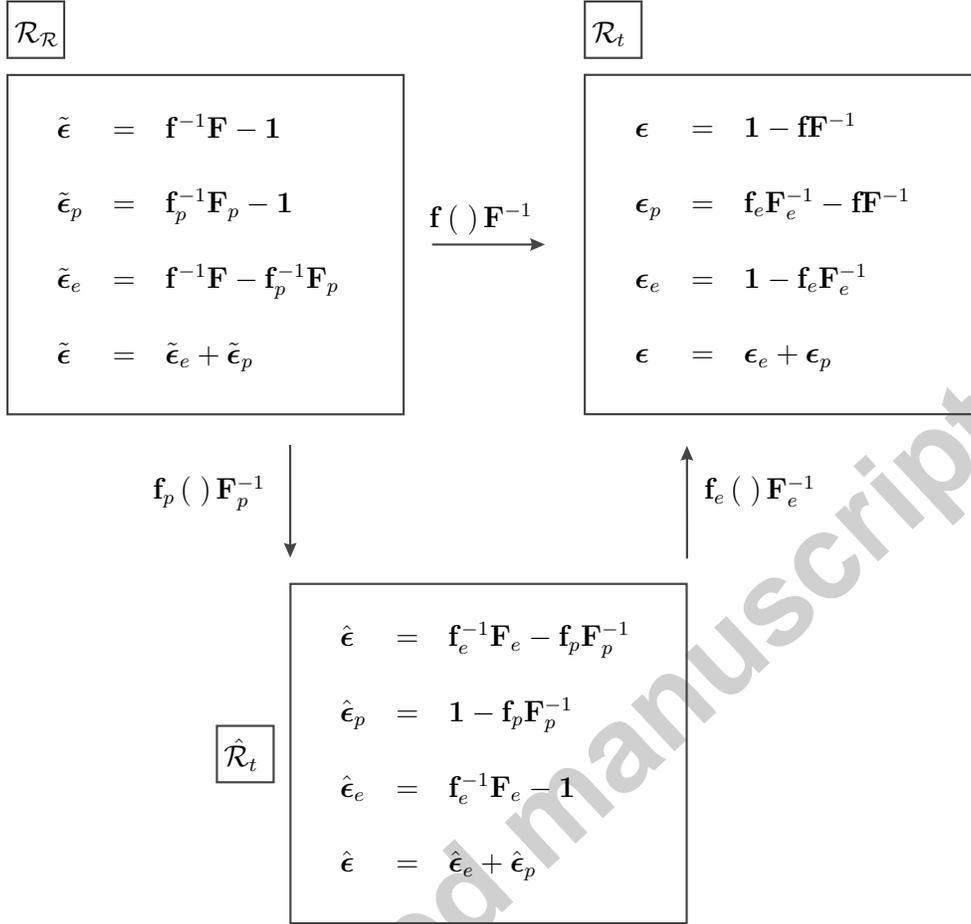
( $\cdot$ )': relative to  $\mathcal{R}'_{\mathcal{R}}$

( $\Delta$ ) = ( $\cdot$ )' +  $\hat{\mathbf{l}}_p^T$  ( $\cdot$ ) + ( $\cdot$ )  $\hat{\mathbf{l}}_p$ : relative to  $\hat{\mathcal{R}}'_t$

( $\Delta$ ) = ( $\cdot$ )' +  $\mathbf{l}^T$  ( $\cdot$ ) + ( $\cdot$ )  $\mathbf{l}$ : relative to  $\mathcal{R}'_t$



## C.3 Decomposition of the micromorphic strain tensors



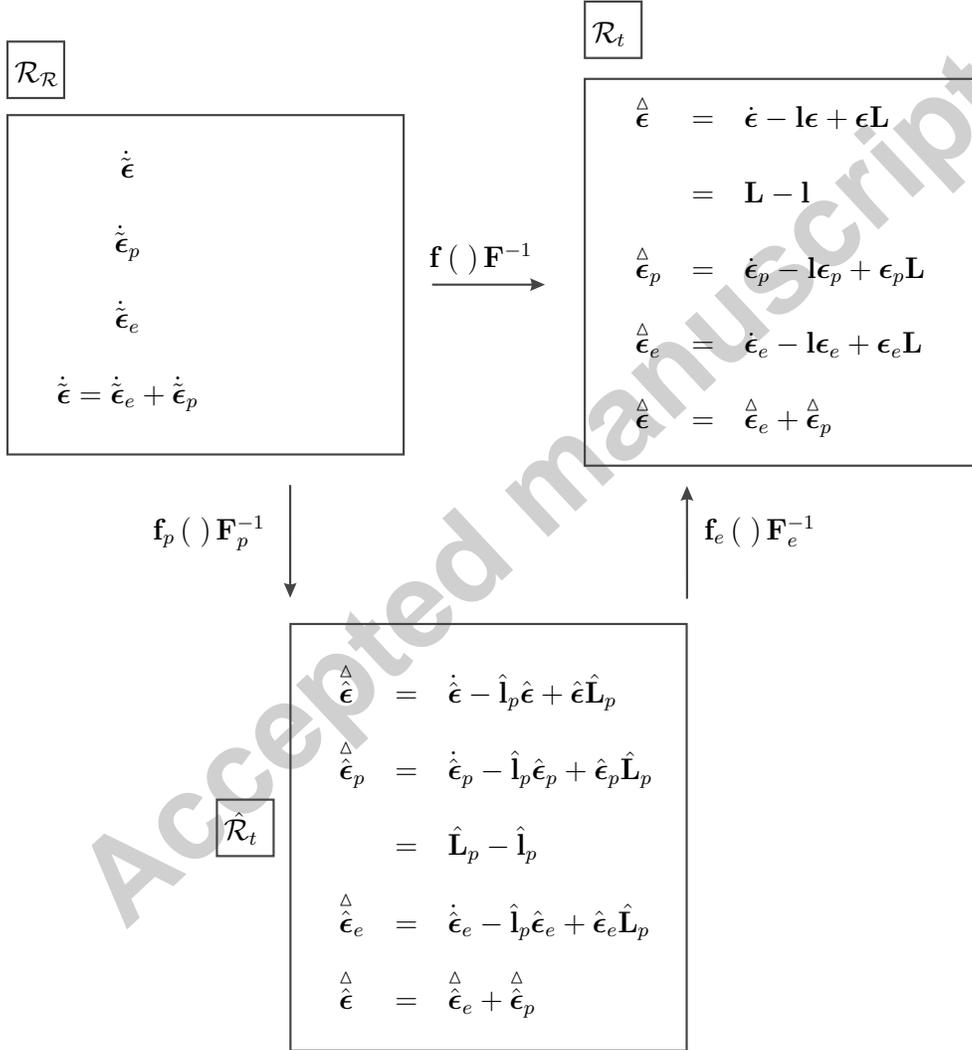
## C.4 Decomposition of the micromorphic strain rates

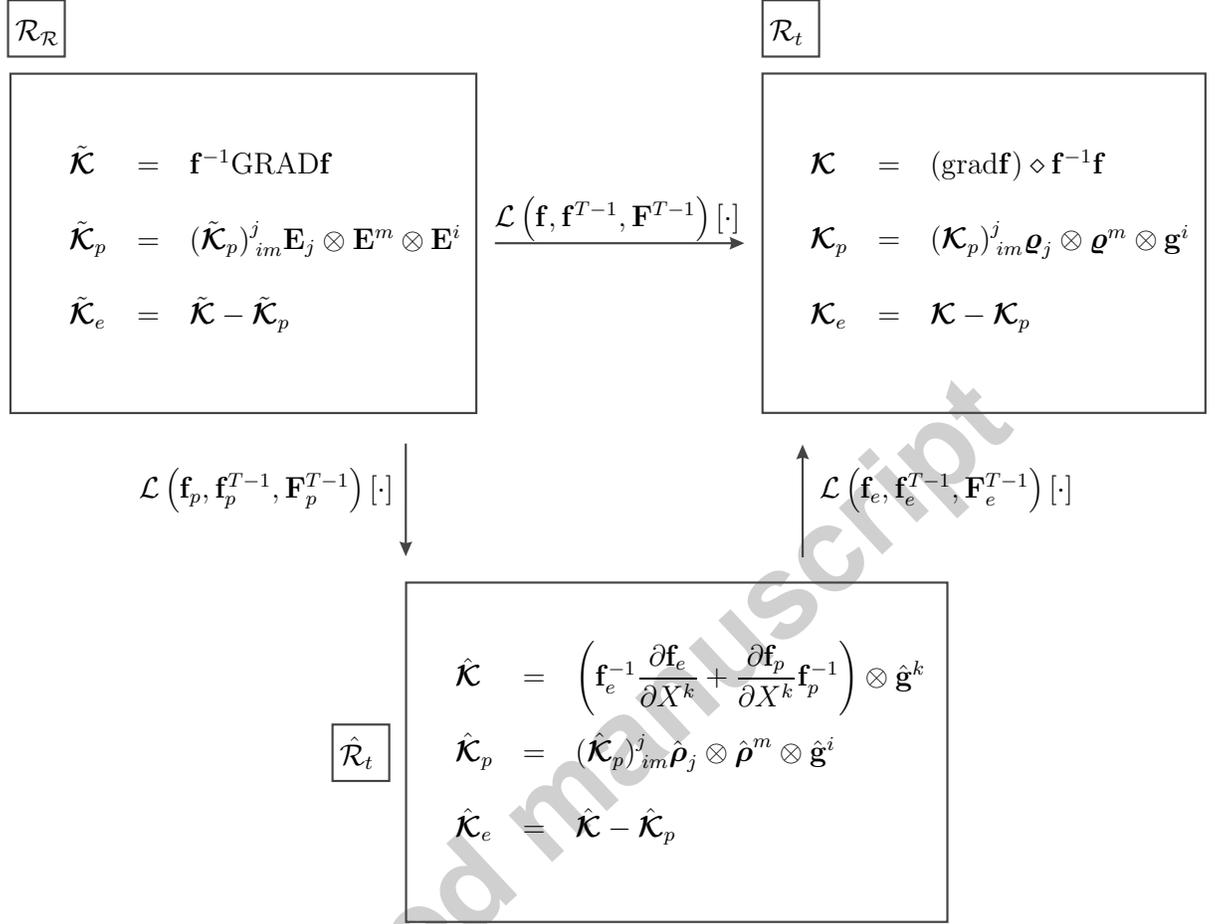
$$\hat{\mathbf{L}}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1} = \hat{\mathbf{D}}_p + \hat{\mathbf{W}}_p, \quad \mathbf{L} = \dot{\mathbf{F}} \mathbf{F}^{-1} = \mathbf{D} + \mathbf{W}$$

$(\cdot)'$ : relative to  $\mathcal{R}_{\mathcal{R}}$

$(\cdot)^\Delta = (\cdot)' - \hat{\mathbf{l}}_p(\cdot) + (\cdot)\hat{\mathbf{L}}_p$ : relative to  $\hat{\mathcal{R}}_t$

$(\cdot)^\Delta = (\cdot)' - \mathbf{l}(\cdot) + (\cdot)\mathbf{L}$ : relative to  $\mathcal{R}_t$



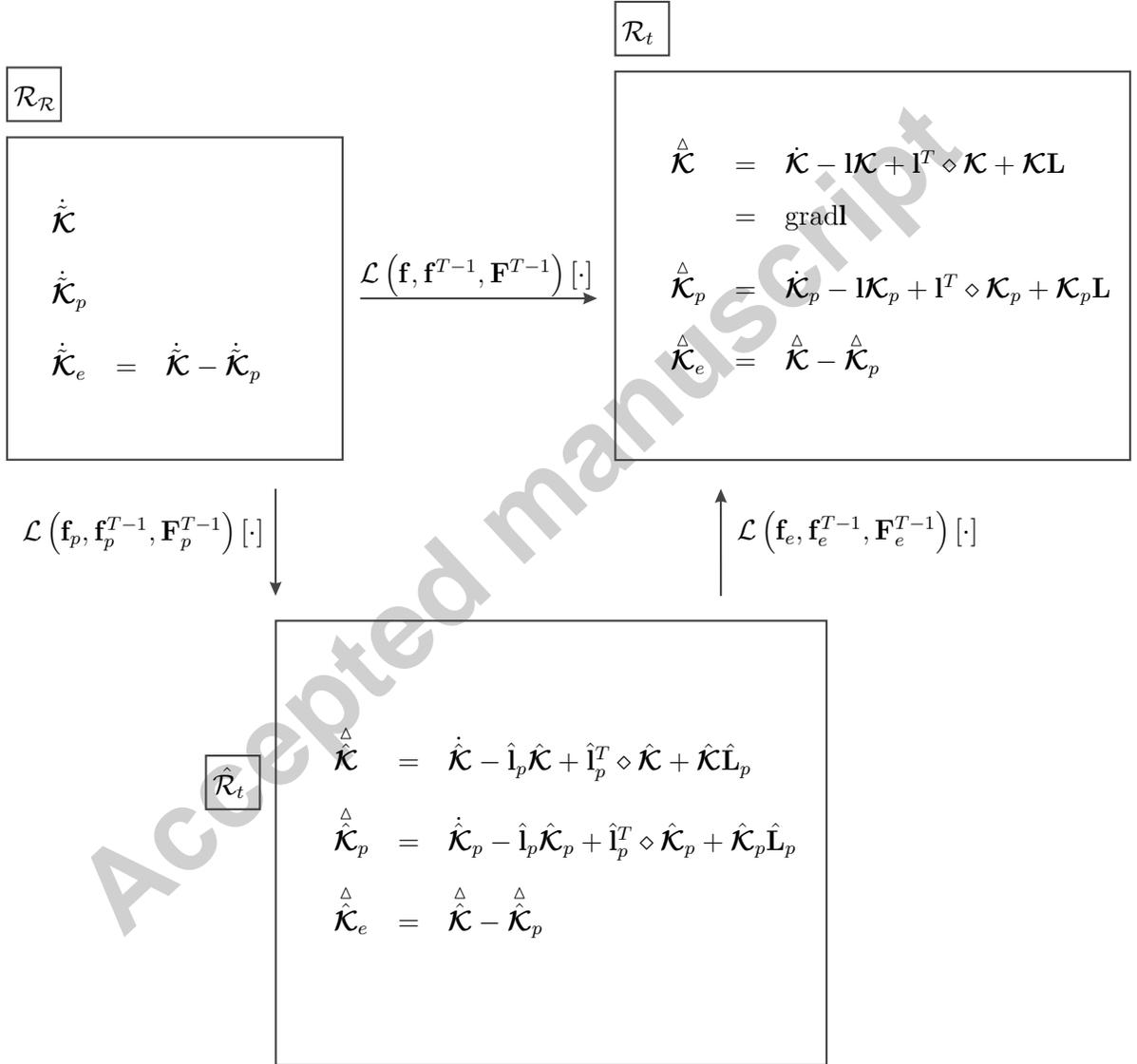
C.5 Decomposition of the micromorphic curvature tensors  $\tilde{\mathcal{K}}$ ,  $\hat{\mathcal{K}}$ ,  $\mathcal{K}$ 

C.6 Decomposition of the associated rates for  $\tilde{\mathcal{K}}$ ,  $\hat{\mathcal{K}}$ ,  $\mathcal{K}$

$(\cdot)^\cdot$ : relative to  $\mathcal{R}_{\mathcal{R}}$

$(\cdot)^\Delta = (\cdot)^\cdot - \hat{\mathbf{l}}_p(\cdot) + \hat{\mathbf{l}}_p^T \diamond (\cdot) + (\cdot) \hat{\mathbf{L}}_p$ : relative to  $\hat{\mathcal{R}}_t$

$(\cdot)^\Delta = (\cdot)^\cdot - \mathbf{l}(\cdot) + \mathbf{l}^T \diamond (\cdot) + (\cdot) \mathbf{L}$ : relative to  $\mathcal{R}_t$

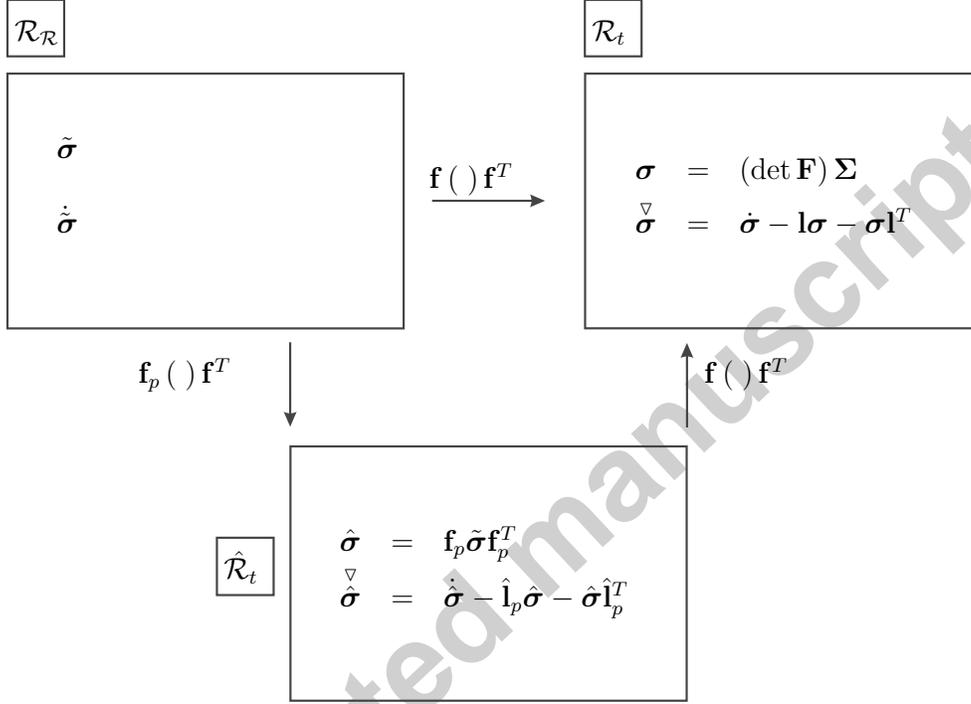


## C.7 Stress tensors related to the microcontinuum and associated rates

$(\cdot)^\cdot$ : relative to  $\mathcal{R}_{\mathcal{R}}$

$(\cdot)^\nabla = (\cdot)^\cdot - \hat{\mathbf{l}}_p(\cdot) - (\cdot)\hat{\mathbf{l}}_p^T$ : relative to  $\hat{\mathcal{R}}_t$

$(\cdot)^\nabla = (\cdot)^\cdot - \mathbf{l}(\cdot) - (\cdot)\mathbf{l}^T$ : relative to  $\mathcal{R}_t$

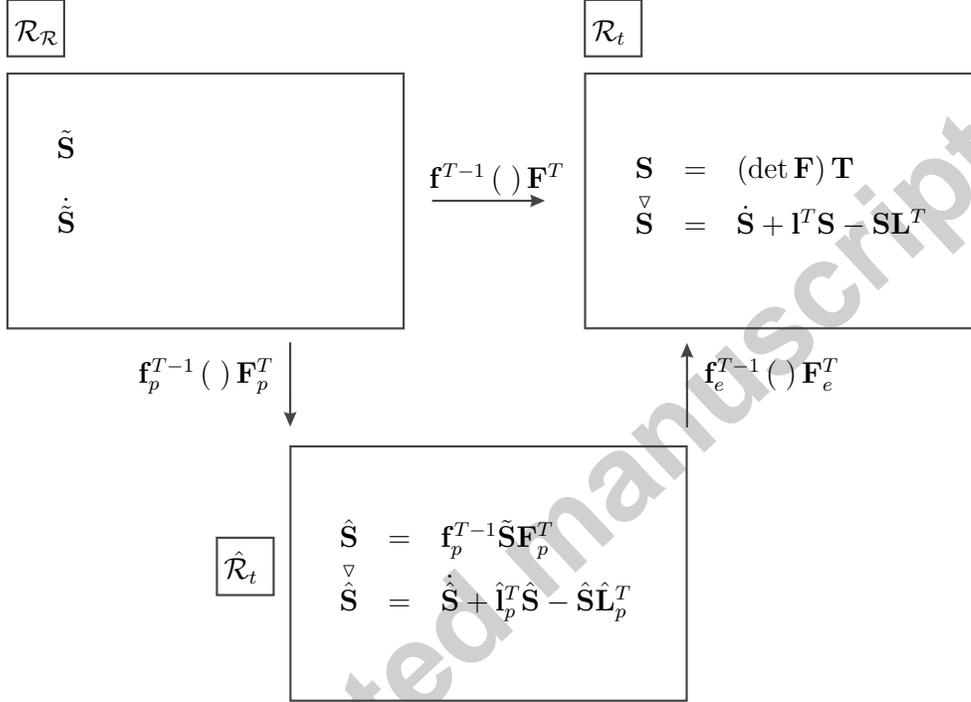


## C.8 Stress tensors related to the macroscopic continuum and associated rates

$(\cdot)^\cdot$ : relative to  $\mathcal{R}_{\mathcal{R}}$

$(\cdot)^\nabla = (\cdot)^\cdot + \hat{\mathbf{L}}_p^T (\cdot) - (\cdot) \hat{\mathbf{L}}_p^T$ : relative to  $\hat{\mathcal{R}}_t$

$(\cdot)^\nabla = (\cdot)^\cdot + \mathbf{L}^T (\cdot) - (\cdot) \mathbf{L}^T$ : relative to  $\mathcal{R}_t$

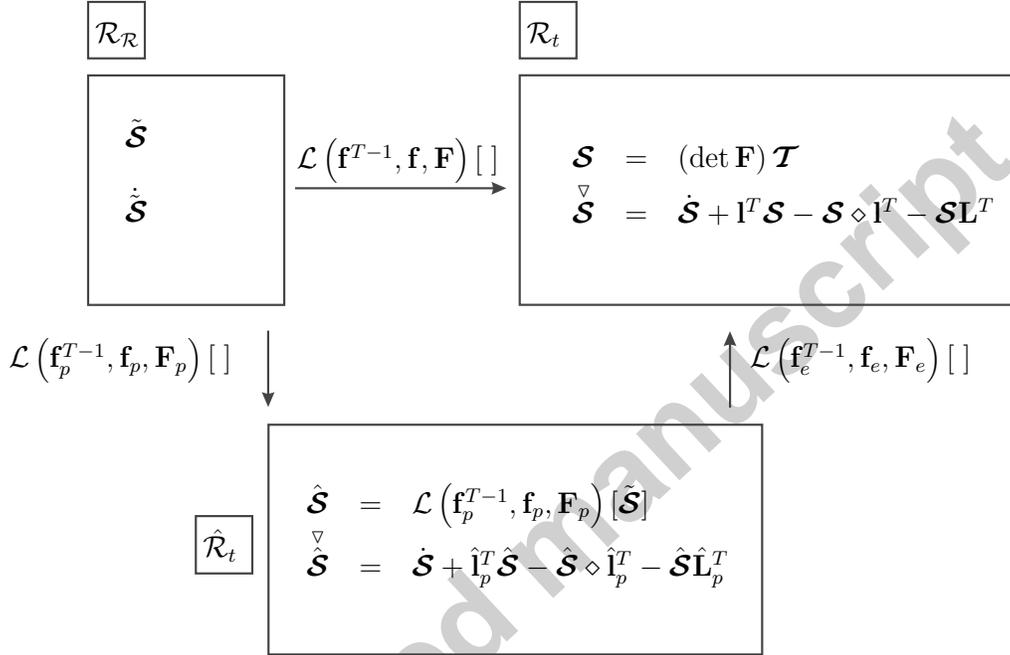


## C.9 Double stress tensors and associated rates

$(\cdot)^\cdot$ : relative to  $\mathcal{R}_{\mathcal{R}}$

$(\cdot)^\nabla = (\cdot)^\cdot + \hat{\mathbf{I}}_p^T (\cdot) - (\cdot) \diamond \hat{\mathbf{I}}_p^T - (\cdot) \hat{\mathbf{L}}_p^T$ : relative to  $\hat{\mathcal{R}}_t$

$(\cdot)^\nabla = (\cdot)^\cdot + \mathbf{I}^T (\cdot) - (\cdot) \diamond \mathbf{I}^T - (\cdot) \mathbf{L}^T$ : relative to  $\mathcal{R}_t$



## D Kinematic hardening – Decompositions of plastic strain and micromorphic curvature tensors

### D.1 Kinematic hardening – decomposition of plastic strain tensors for the microstructure

$$\mathbf{f}_p = \mathbf{f}_k \mathbf{f}_d$$

$\mathcal{R}'_{\mathcal{R}}$		$\hat{\mathcal{R}}'_t$
$\begin{aligned}\tilde{\boldsymbol{\beta}} &= \frac{1}{2}(\mathbf{u}_p^2 - \mathbf{1}) \\ \dot{\tilde{\boldsymbol{\beta}}} &= \frac{1}{2}(\mathbf{u}_d^2 - \mathbf{1}) \\ \tilde{\boldsymbol{\beta}}_k &= \frac{1}{2}(\mathbf{u}_p^2 - \mathbf{u}_d^2) \\ \tilde{\boldsymbol{\beta}}_p &= \tilde{\boldsymbol{\beta}}_k + \tilde{\boldsymbol{\beta}}_d\end{aligned}$	$\xrightarrow{\mathbf{f}_p^{T-1}(\cdot)\mathbf{f}_p^{-1}}$	$\begin{aligned}\hat{\boldsymbol{\beta}}_p &= \frac{1}{2}(\mathbf{1} - \mathbf{v}_p^{-2}) \\ \hat{\boldsymbol{\beta}}_d &= \frac{1}{2}(\mathbf{v}_k^{-2} - \mathbf{v}_p^{-2}) \\ \hat{\boldsymbol{\beta}}_k &= \frac{1}{2}(\mathbf{1} - \mathbf{v}_k^{-2}) \\ \hat{\boldsymbol{\beta}}_p &= \hat{\boldsymbol{\beta}}_k + \hat{\boldsymbol{\beta}}_d\end{aligned}$
$\downarrow \mathbf{f}_d^{T-1}(\cdot)\mathbf{f}_d^{-1}$		$\uparrow \mathbf{f}_k^{T-1}(\cdot)\mathbf{f}_k^{-1}$
	$\mathcal{R}'_t$	
	$\begin{aligned}\check{\boldsymbol{\beta}}_p &= \frac{1}{2}(\mathbf{u}_k^2 - \mathbf{v}_d^{-2}) \\ \check{\boldsymbol{\beta}}_d &= \frac{1}{2}(\mathbf{1} - \mathbf{v}_d^{-2}) \\ \check{\boldsymbol{\beta}}_k &= \frac{1}{2}(\mathbf{u}_k^2 - \mathbf{1}) \\ \check{\boldsymbol{\beta}}_p &= \check{\boldsymbol{\beta}}_k + \check{\boldsymbol{\beta}}_d\end{aligned}$	

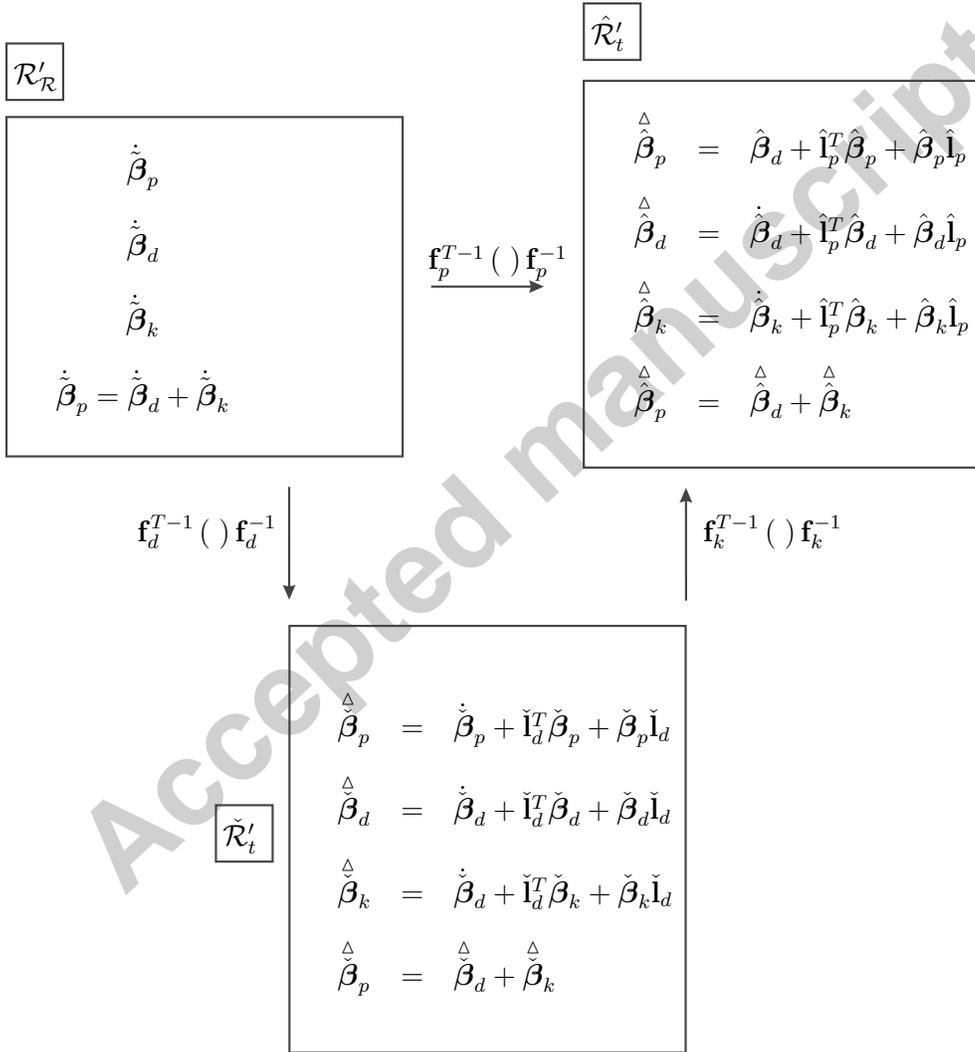
D.2 Kinematic hardening – decomposition of plastic strain rate tensors for the microstructure

$$\check{\mathbf{l}}_d = \dot{\mathbf{f}}_d \mathbf{f}_d^{-1}, \quad \hat{\mathbf{l}}_p = \dot{\mathbf{f}}_p \mathbf{f}_p^{-1}$$

$(\cdot)^\cdot$ : relative to  $\mathcal{R}'_{\mathcal{R}}$

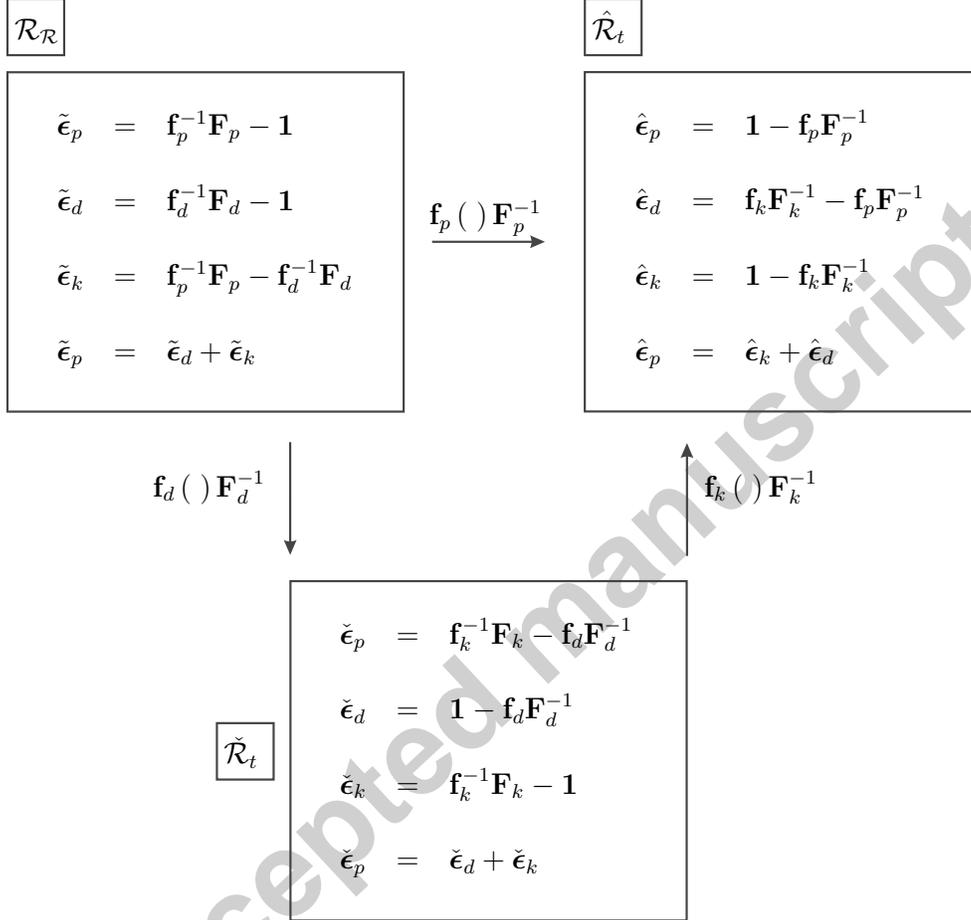
$(\cdot)^\Delta = (\cdot)^\cdot + \check{\mathbf{l}}_d^T (\cdot) + (\cdot) \check{\mathbf{l}}_d$ : relative to  $\check{\mathcal{R}}'_t$

$(\cdot)^\Delta = (\cdot)^\cdot + \hat{\mathbf{l}}_p^T (\cdot) + (\cdot) \hat{\mathbf{l}}_p$ : relative to  $\hat{\mathcal{R}}'_t$



D.3 Kinematic hardening – decomposition of plastic strain for the micromorphic continuum

$$\mathbf{F}_p = \mathbf{F}_k \mathbf{F}_d$$



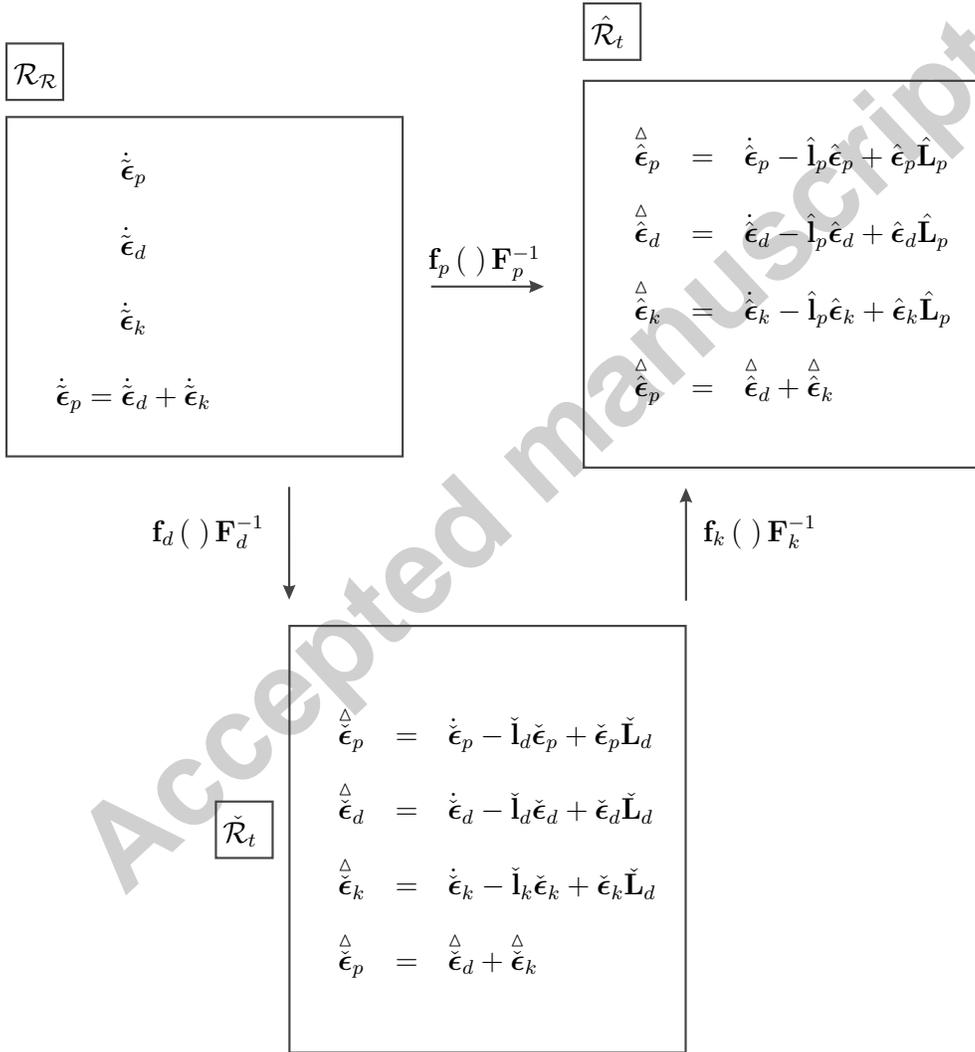
D.4 Kinematic hardening – decomposition of plastic strain rates for micro-morphic continuum

$$\check{\mathbf{L}}_d = \dot{\mathbf{F}}_d \mathbf{F}_d^{-1}, \quad \hat{\mathbf{L}}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$$

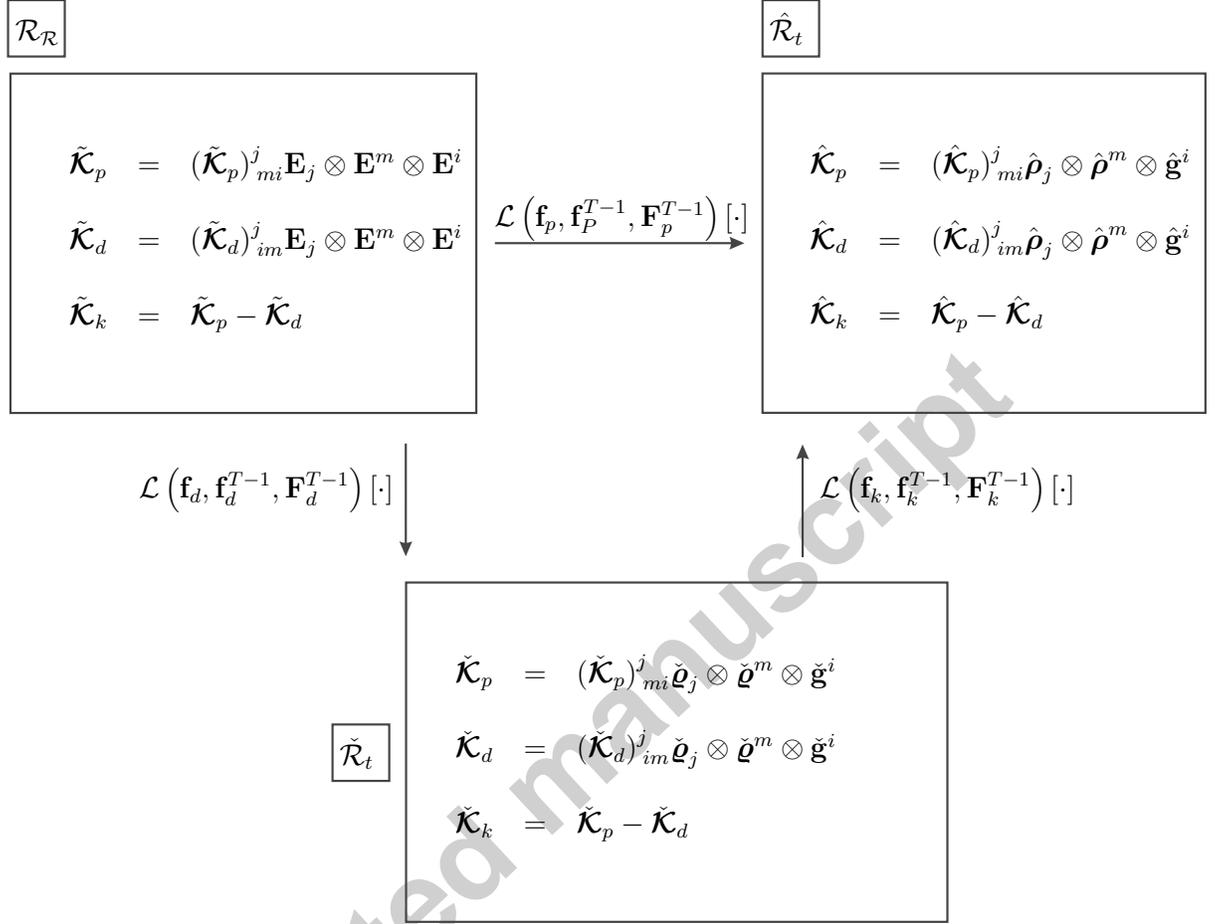
$(\cdot)^\cdot$ : relative to  $\mathcal{R}_{\mathcal{R}}$

$(\cdot)^\Delta = (\cdot)^\cdot - \check{\mathbf{l}}_d(\cdot) + (\cdot) \check{\mathbf{L}}_d$ : relative to  $\hat{\mathcal{R}}'_t$

$(\cdot)^\Delta = (\cdot)^\cdot - \hat{\mathbf{l}}_p(\cdot) + (\cdot) \hat{\mathbf{L}}_p$ : relative to  $\hat{\mathcal{R}}_t$



D.5 Kinematic hardening – decomposition of plastic micromorphic curvature tensors

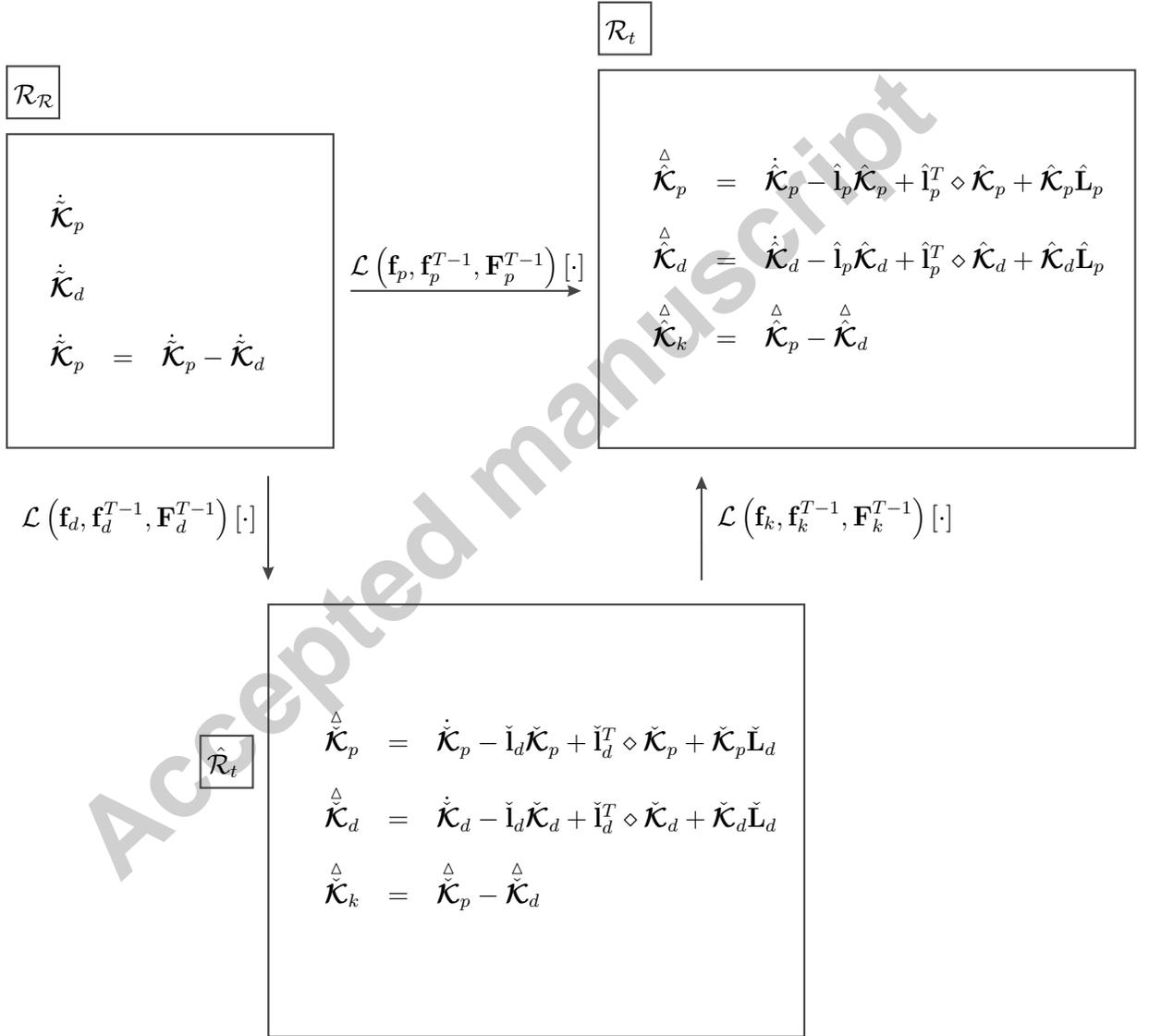


D.6 Kinematic hardening – decomposition of plastic micromorphic curvature rates

$(\cdot)^\cdot$ : relative to  $\mathcal{R}_{\mathcal{R}}$

$(\cdot)^\Delta = (\cdot)^\cdot - \check{\mathbf{l}}_d(\cdot) + \check{\mathbf{l}}_d^T \diamond (\cdot) + (\cdot) \check{\mathbf{L}}_d$ : relative to  $\check{\mathcal{R}}_t$

$(\cdot)^\Delta = (\cdot)^\cdot - \hat{\mathbf{l}}_p(\cdot) + \hat{\mathbf{l}}_p^T \diamond (\cdot) + (\cdot) \hat{\mathbf{L}}_p$ : relative to  $\hat{\mathcal{R}}_t$



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