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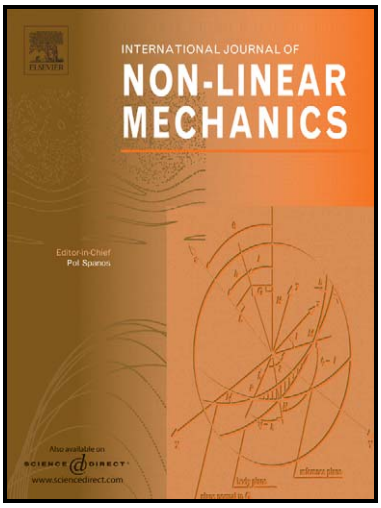

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Micromorphic continuum. Part I: Strain and stress tensors and their associated rates

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Abstract

Micropolar and micromorphic solids are continuum mechanics models, which take into account, in some sense, the microstructure of the considered real material. The characteristic property of such continua is that the state functions depend, besides the classical deformation of the macroscopic material body, also upon the deformation of the microcontinuum modelling the microstructure, and its gradient with respect to the space occupied by the material body. While micropolar plasticity theories, including nonlinear isotropic and nonlinear kinematic hardening, have been formulated, even for nonlinear geometry, few works are known yet about the formulation of (finite deformation) micromorphic plasticity. It is the aim of the three papers (Part I, Part II and Part III) to demonstrate how micromorphic plasticity theories may be formulated in a thermodynamically consistent way.

In the present article we start by outlining the framework of the theory. Especially, we confine attention to the theory of Mindlin on continua with microstructure, which is formulated for small deformations. After precising some conceptual aspects

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concerning the notion of microcontinuum, we work out a finite deformation version of theory, suitable for our aims. It is examined that resulting basic field equations are the same as in the nonlinear theory of Eringen, which deals with a different definition of the microcontinuum. Furthermore, geometrical interpretations of strain and curvature tensors are elaborated. This allows to find out associated rates in a natural manner. Dual stress and double stress tensors, as well as associated rates, are then defined on the basis of the stress powers. This way, it is possible to relate strain tensors (respectively micromorphic curvature tensors) and stress tensors (respectively double stress tensors), as well as associated rates, independently of the particular constitutive properties.

Key words: micromorphic plasticity; finite deformation; isotropic and kinematic hardening rules; isotropic continuum damage mechanics; finite element implementation;

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1 Introduction

It is well recognized that nonlocality effects have to be involved in a plasticity theory when discussing localization phenomena or size effects in the material response. One possibility to augment classical theories to capture nonlocality aspects is to incorporate higher order gradients of the kinematical and dynamical variables (see e.g. Aifantis, 1992, 2003; Fleck and Hutchinson, 1993, 2001; Toupin, 1962; Mindlin and Tiersten, 1962; Mindlin, 1964; Eringen and Suhubi, 1964; Eringen, 1968, 1999; Steinmann, 1994; Steinmann and Willam,

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1991; Forest, 1998; Forest and Cailletaud, 2000; de Borst, 1991; de Borst and Mühlhaus, 1992; Dietsche and Willam, 1997; Mühlhaus and Vardoulakis, 1987; Grammenoudis and Tsakmakis, 2001, 2005a,b).

Among the continuum theories involving higher order of gradients there are some continuum models which take into account, in some sense, the microstructure of the real material (continua with microstructure), as e.g. the micropolar and the micromorphic ones. The formulation of micropolar plasticity is progressed as can be recognized, e.g., from the works Forest and Cailletaud (2000); de Borst (1991, 1993); Steinmann (1994); Ristinmaa and Vecchi (1996); Grammenoudis and Tsakmakis (2001, 2005a), and the references cited there. On the other hand, some interesting ideas concerning finite deformation micromorphic plasticity have been elaborated in Forest and Sievert (2003, 2006); Sansour (1998); Hirschberger and Steinmann (2007). In particular the comprehensive work of Forest and Sievert (2003), provides a unified thermomechanical framework for the development of micromorphic plasticity. Nevertheless, several aspects of (finite deformation) micromorphic plasticity are not broadly investigated, concerning among others geometrical issues of deformation decompositions into elastic and plastic parts, or formulation of hardening laws like kinematic hardening rules. Thus, the aim of the first two works (Part I and Part II) is to sketch how (finite deformation) micromorphic plasticity models may be formulated in a thermodynamically consistent way, while Part III is concerned with the discussion of examples which are calculated numerically. In the first paper, we shall present the framework of our micromorphic theory, and the variables which are chosen as appropriate for formulating the constitutive laws.

Eringen and coworkers (see e.g. Eringen and Suhubi, 1964) introduced and

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discussed micromorphic theories, which capture the microstructure of the real material by assuming a microvolume to be included in each material particle of the macroscopic body ("Macro-elements are constructed by micro-elements"). On the other hand, Mindlin proposed an elasticity theory, in which also the microstructure of the real material is modeled by embedding a microvolume in each particle of the macro-material. Mindlin's theory differs from that one according to Eringen in that both the macro- and the micro-material contribute to the kinetic energy-density of the overall material. The seminal works of Eringen and Mindlin are today the basis for every micromorphic continuum theory, and provide relevant field equations. Now, for a clear formulation of the basic concepts, it is perhaps helpful to address the question how small or how large can be the micro-volume. If the micro-volume is finite, what happens with material points in the neighborhood of the boundary of the continuum. Is there allowed for a part of the micro-volume to be not included in the macro-volume element or even in the range of the continuum? In attempting to clarify such questions the authors were leaded to postulate the concept of microstructure something others than in the aforementioned works. But otherwise, as we shall see, not new basic equations may be gained by using this method.

In the present article, we adopt the proposal of Mindlin for establishing the balance laws of momentum and moment of momentum, in a fashion which allows the microcontinuum to exhibit arbitrary finite dimensions. In our opinion this can be a convenient way, when modeling microphysical properties in the framework of phenomenological continuum mechanics. It is examined that basic field equations established for nonlinear geometry by Eringen's theory, may be recovered by the version of Mindlin's theory as accommodated here.

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Moreover, we elaborate geometrical interpretations for the strain and curvature tensors, which enter in the constitutive theory to be presented in Part II. This allows to find out associated rates in a natural manner. The method we pursue here is similar to the one developed by Haupt and Tsakmakis (1996) in the context of classical continuum mechanics. Dual stress and double stress tensors, as well as associated rates, are then introduced on the basis of the stress powers. Thus, we relate strain (respectively micromorphic curvature) and stress (respectively double stress) tensors with each other, as well as associated rates, independently of the particular constitutive properties. It is shown, that our approach is a generalization of the known method of conjugate variables in classical continuum mechanics. In fact, we extend the method of conjugate variables, on the one hand by covering also spacial variables, and on the other hand by dealing with double stress tensors.

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2 Preliminaries

We consider isothermal deformations and write $\mathbb{R}$ for the axis of real numbers, and $\dot{\varphi}(t)$ for the material time derivative of a function $\varphi(t)$, where t is the time. An explicit reference to space will be dropped in most part of the paper. Commonly, the same symbol is used to designate a function and the value of that function at a point. However, if we deal with different representations of the same function, then use will often be made of different symbols. For real x, $\langle x \rangle$ denotes the function

$$\langle x \rangle := \begin{cases} x & \text{if } x \geq 0 \quad , \\ 0 & \text{if } x < 0 \quad . \end{cases} \tag{1}$$

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Second-order tensors, like vectors, are denoted by bold-face Latin or Greek letters. In particular, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}$ denote the inner product and the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. For second-order tensors $\mathbf{A}$ and $\mathbf{B}$, we write $\mathrm{tr}\mathbf{A}$, $\det\mathbf{A}$ and $\mathbf{A}^T$ for the trace, the determinant and the transpose of $\mathbf{A}$, respectively, while $\mathbf{A}\cdot\mathbf{B}=\mathrm{tr}(\mathbf{A}\mathbf{B}^T)$ is the inner product between $\mathbf{A}$ and $\mathbf{B}$, and $\|\mathbf{A}\|=\sqrt{\mathbf{A}\cdot\mathbf{A}}$ is the Euclidean norm of $\mathbf{A}$. Furthermore,

$$\mathbf{1}=\delta_{ij}\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_j \quad , \quad i,j=1,2,3 \quad , \tag{2}$$

represents the identity tensor of second-order, where $\delta_{ij}=\delta^i_{\ j}=\delta_i^{\ j}$ is the Kronecker delta and $\{\bar{\mathbf{e}}_i\}$ is an orthonormal basis in the three-dimensional Euclidean vector space we deal with. Also, we use the notations $\mathbf{A}^D=\mathbf{A}-\frac{1}{3}(\mathrm{tr}\mathbf{A})\mathbf{1}$ for the deviator of $\mathbf{A}$ and $\mathbf{A}^{T-1}=(\mathbf{A}^{-1})^T$, provided $\mathbf{A}^{-1}$ exists.

Third- and fourth-order tensors are denoted by bold face calligraphic and double-stroke letters, respectively. Let $\mathbb{A}$, $\mathbb{B}$ be fourth-order tensors, $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ third-order tensors, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ second-order tensors and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ vectors. With respect to the orthonormal basis $\{\bar{\mathbf{e}}_i\}$, the components of $\mathbb{A}$, $\mathbb{B}$, $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are $\mathbb{A}_{ijkl}$, $\mathbb{B}_{ijkl}$, $\mathcal{A}_{ijk}$, $\mathcal{B}_{ijk}$, A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, a_i, b_i, c_i, d_i (often use will be made of notations of the form $A_{ij}=(\mathbf{A})_{ij}$). Then, we have

$$\mathbb{A}\mathbb{B}=\mathbb{A}_{ijmn}\mathbb{B}_{mnkl}\,\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_j\otimes\bar{\mathbf{e}}_k\otimes\bar{\mathbf{e}}_l \quad , \tag{3}$$

$$\mathbb{A}^T=\mathbb{A}_{klij}\,\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_j\otimes\bar{\mathbf{e}}_k\otimes\bar{\mathbf{e}}_l \quad , \tag{4}$$

$$\mathbb{A}[\mathbf{B}]=\mathbb{A}_{ijmn}B_{mn}\,\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_j \quad , \tag{5}$$

$$\mathbf{B}\cdot\mathbb{A}[\mathbf{C}]=\mathbb{A}^T[\mathbf{B}]\cdot\mathbf{C} \quad , \tag{6}$$

$$\mathbf{A}^2=\mathbf{A}\mathbf{A}=A_{ij}A_{jk}\,\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_k \quad , \quad (\mathbf{A}^{-2}=\mathbf{A}^{-1}\mathbf{A}^{-1}) \quad , \tag{7}$$

$$\mathbf{A}\mathbf{a}\equiv\mathbf{A}[\mathbf{a}]=A_{ij}a_j\,\bar{\mathbf{e}}_i \quad , \tag{8}$$

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$$\mathbf{AB} = B_{ij}(\mathbf{A}\bar{\mathbf{e}}_i) \otimes \bar{\mathbf{e}}_j = A_{ij}\bar{\mathbf{e}}_i \otimes (\mathbf{B}^T\bar{\mathbf{e}}_j) \quad , \tag{9}$$

$$\mathbf{A}[\mathbf{a},\mathbf{b}] = A_{ij}(\mathbf{a}\cdot\bar{\mathbf{e}}_i)(\mathbf{b}\cdot\bar{\mathbf{e}}_j) = \mathbf{a}\cdot\mathbf{Ab} = A_{ij}a_ib_j \quad , \tag{10}$$

$$\boldsymbol{\mathcal{A}}[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathcal{A}_{ijk}(\mathbf{a}\cdot\bar{\mathbf{e}}_i)(\mathbf{b}\cdot\bar{\mathbf{e}}_j)(\mathbf{c}\cdot\bar{\mathbf{e}}_k) = \mathcal{A}_{ijk}a_ib_jc_k \quad , \tag{11}$$

$$\boldsymbol{\mathcal{A}}[\mathbf{a}] = \mathcal{A}_{ijk}\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j(\mathbf{a}\cdot\bar{\mathbf{e}}_k) = \mathcal{A}_{ijk}a_k\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j \quad , \tag{12}$$

$$(\mathbf{A}\otimes\mathbf{d})[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\cdot\mathbf{Ab})(\mathbf{d}\cdot\mathbf{c}) \quad , \tag{13}$$

$$\boldsymbol{\mathcal{A}}\cdot\boldsymbol{\mathcal{B}} = \mathcal{A}_{ijk}\mathcal{B}_{lmn}(\bar{\mathbf{e}}_i\cdot\bar{\mathbf{e}}_l)(\bar{\mathbf{e}}_j\cdot\bar{\mathbf{e}}_m)(\bar{\mathbf{e}}_k\cdot\bar{\mathbf{e}}_n) = \mathcal{A}_{ijk}\mathcal{B}_{ijk} \quad , \tag{14}$$

$$\|\boldsymbol{\mathcal{A}}\| = \sqrt{\boldsymbol{\mathcal{A}}\cdot\boldsymbol{\mathcal{A}}} \quad . \tag{15}$$

The products $\mathbf{A}\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{A}}\mathbf{A}$, $\boldsymbol{\mathcal{A}}\diamond\mathbf{A} \equiv \mathbf{A}^T\diamond\boldsymbol{\mathcal{A}}$ are defined to represent third-order tensors given by

$$\mathbf{A}\boldsymbol{\mathcal{A}} := \mathcal{A}_{ijk}(\mathbf{A}\bar{\mathbf{e}}_i)\otimes\bar{\mathbf{e}}_j\otimes\bar{\mathbf{e}}_k = A_{pi}\mathcal{A}_{ijk}\bar{\mathbf{e}}_p\otimes\bar{\mathbf{e}}_j\otimes\bar{\mathbf{e}}_k \quad , \tag{16}$$

$$\boldsymbol{\mathcal{A}}\mathbf{A} := \mathcal{A}_{ijk}\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_j\otimes(\mathbf{A}^T\bar{\mathbf{e}}_k) = \mathcal{A}_{ijk}A_{kp}\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_j\otimes\bar{\mathbf{e}}_p \quad , \tag{17}$$

$$\boldsymbol{\mathcal{A}}\diamond\mathbf{A} \equiv \mathbf{A}^T\diamond\boldsymbol{\mathcal{A}} := \mathcal{A}_{ijk}\bar{\mathbf{e}}_i\otimes(\mathbf{A}^T\bar{\mathbf{e}}_j)\otimes\bar{\mathbf{e}}_k = \mathcal{A}_{ijk}A_{jp}\bar{\mathbf{e}}_i\otimes\bar{\mathbf{e}}_p\otimes\bar{\mathbf{e}}_k \quad , \tag{18}$$

while $\boldsymbol{\mathcal{A}}[\mathbf{A}]$ represents the vector

$$\boldsymbol{\mathcal{A}}[\mathbf{A}] = \mathcal{A}_{ijk}\{(\bar{\mathbf{e}}_j\otimes\bar{\mathbf{e}}_k)\cdot\mathbf{A}\}\bar{\mathbf{e}}_i = \mathcal{A}_{ijk}A_{jk}\bar{\mathbf{e}}_i \quad . \tag{19}$$

We introduce a linear operator $(\mathbf{A},\mathbf{B},\mathbf{C}) \mapsto \mathcal{L}(\mathbf{A},\mathbf{B},\mathbf{C})$, acting on the space of all third-order tensors, by

$$\mathcal{L}(\mathbf{A},\mathbf{B},\mathbf{C}) : \boldsymbol{\mathcal{A}} \mapsto \mathcal{L}(\mathbf{A},\mathbf{B},\mathbf{C})[\boldsymbol{\mathcal{A}}] = \mathcal{A}_{ijk}(\mathbf{A}\bar{\mathbf{e}}_i)\otimes(\mathbf{B}\bar{\mathbf{e}}_j)\otimes(\mathbf{C}\bar{\mathbf{e}}_k) \quad , \tag{20}$$

or, with respect to the orthonormal basis $\{\bar{\mathbf{e}}_i\}$,

$$(\mathcal{L}(\mathbf{A},\mathbf{B},\mathbf{C})[\boldsymbol{\mathcal{A}}])_{mnp} = A_{mi}B_{nj}C_{pk}\mathcal{A}_{ijk} \quad . \tag{21}$$

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It can be seen that

$$\mathcal{L}(\mathbf{AB},\mathbf{CD},\mathbf{EF}) = \mathcal{L}(\mathbf{A},\mathbf{C},\mathbf{E})\mathcal{L}(\mathbf{B},\mathbf{D},\mathbf{F}) \quad , \tag{22}$$

$$\boldsymbol{\mathcal{A}} \cdot \mathcal{L}(\mathbf{A},\mathbf{B},\mathbf{C})[\boldsymbol{\mathcal{B}}] = \mathcal{L}^T(\mathbf{A},\mathbf{B},\mathbf{C})[\boldsymbol{\mathcal{A}}] \cdot \boldsymbol{\mathcal{B}} = \mathcal{L}(\mathbf{A}^T,\mathbf{B}^T,\mathbf{C}^T)[\boldsymbol{\mathcal{A}}] \cdot \boldsymbol{\mathcal{B}} \tag{23}$$

and

$$\mathcal{L}^{-1}(\mathbf{A},\mathbf{B},\mathbf{C}) = \mathcal{L}(\mathbf{A}^{-1},\mathbf{B}^{-1},\mathbf{C}^{-1}) \quad , \tag{24}$$

provided $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $\mathbf{C}^{-1}$ exist. For the particular case where $\boldsymbol{\mathcal{A}} = \mathbf{D} \otimes \mathbf{a}$, we have

$$\mathcal{L}(\mathbf{A},\mathbf{B},\mathbf{C})[\mathbf{D} \otimes \mathbf{a}] = \mathbf{ADB}^T \otimes (\mathbf{Ca}) \quad . \tag{25}$$

In this case, we have also

$$\boldsymbol{\mathcal{A}}[\mathbf{b},\mathbf{c},\mathbf{d}] = (\mathbf{D} \otimes \mathbf{a})[\mathbf{b},\mathbf{c},\mathbf{d}] = \mathbf{D}[\mathbf{b},\mathbf{c}](\mathbf{a} \cdot \mathbf{d}) = (\mathbf{b} \cdot \mathbf{Dc})(\mathbf{a} \cdot \mathbf{d}) \quad . \tag{26}$$

We write $\mathbb{I}$ for the fourth-order identity tensor,

$$\mathbb{I} = \delta_{im}\delta_{jn}\, \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_m \otimes \bar{\mathbf{e}}_n \quad , \tag{27}$$

which satisfies the property

$$\mathbb{I} = \mathbb{E} + \mathbb{J} \quad , \tag{28}$$

$$\mathbb{E} = \mathbb{E}_{imjn}\, \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_m \otimes \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_n = \frac{1}{2}(\delta_{ij}\delta_{mn} + \delta_{in}\delta_{mj})\, \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_m \otimes \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_n \quad , \tag{29}$$

$$\mathbb{J} = \mathbb{J}_{imjn}\, \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_m \otimes \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_n = \frac{1}{2}(\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj})\, \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_m \otimes \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_n \quad . \tag{30}$$

Hence, for the symmetric and the skew-symmetric (anti-symmetric) part of the second-order tensor $\mathbf{A}$, denoted respectively by $\mathbf{A}_S$ and $\mathbf{A}_A$, we have

$$\mathbf{A}_S = \mathbb{E}[\mathbf{A}] \quad , \quad \mathbf{A}_A = \mathbb{J}[\mathbf{A}] \quad , \tag{31}$$

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while

$$\mathbb{I}[\mathbf{A}] = \mathbf{A} \quad . \tag{32}$$

We write $\mathbb{S}$ for the fourth-order tensor with the property

$$\mathbb{S}[\mathbf{A}] = \mathbf{A}^T \quad . \tag{33}$$

Thus, every isotropic fourth-order tensor $\mathbb{A}$ possesses the representation

$$\mathbb{A} = \alpha_1 \mathbf{1} \otimes \mathbf{1} + \alpha_2 \mathbb{I} + \alpha_3 \mathbb{S} \quad , \tag{34}$$

where α_1, α_2 and α_3 are scalars.

If $\mathcal{M}$ is a manifold, then $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$,

$$T\mathcal{M} := \bigcup_{p \in \mathcal{M}} T_p\mathcal{M} \quad , \tag{35}$$

where $T_p\mathcal{M}$ is the tangent space to $\mathcal{M}$ at p.

3 Kinematic

3.1 Micromorphic continuum

Consider a material body $\mathcal{B}$ (macroscopic continuum, or macrocontinuum, or macroscopic material, or overall material body), with elements $\mathcal{X}$, $\mathcal{Y}$, ..., which may be mapped into a region of the three dimensional Euclidean space $\mathcal{E}$. With an origin O fixed in $\mathcal{E}$, every point $P \in \mathcal{E}$ may be identified by a position vector $\mathbf{p}$, which belongs to the tangent space to $\mathcal{E}$ at O. As usually in classical continuum mechanics, we shall often set $\mathbf{p}$ equal to point P, and we shall speak of the point $\mathbf{p} \in \mathcal{E}$.

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Microphysically, real materials like metals indicate some kind of patterning with discrete distributed mass. This may be addressed, when formulating constitutive properties of a material point, by taking into account not only the material point itself, but rather an entire neighborhood of the point. We may realize this by attaching to each material point $\mathcal{X} \in \mathcal{B}$, a material body $\mathcal{B}'(\mathcal{X})$ (microcontinuum, or microstructure), which serves to model the microphysical (microstructural) properties of the overall material body. It is assumed, that the same body $\mathcal{B}'$, with elements $\mathcal{X}'$, $\mathcal{Y}'$, ..., is attached at every $\mathcal{X}$. A configuration of the body $\mathcal{B}$ and its microstructure $\mathcal{B}'$ is a map

$$(k, k') \quad : \quad (\mathcal{B}, \mathcal{B}') \quad \rightarrow \quad \mathcal{E} \times \mathcal{E} \quad , \tag{36}$$

$$(\chi, \chi') \quad \mapsto \quad \begin{pmatrix} k = k(\mathcal{X}) \\ \\ k' = k'(\mathcal{X}, \mathcal{X}') \end{pmatrix} \quad , \tag{37}$$

with $k(\mathcal{B})$, $k'(\mathcal{X}, \mathcal{B}')$ open and simply connected subsets of $\mathcal{E}$. $k(\mathcal{B})$ and $k'(\mathcal{X}, \mathcal{B}')$ are denoted as the ranges in $\mathcal{E}$ occupied respectively under the configuration (k, k'). We shall also write $(k, k') \equiv (\mathbf{k}, \mathbf{k}')$, and we shall set $k(\mathcal{B}) \equiv \mathbf{k}(\mathcal{B})$, $k'(\mathcal{X}, \mathcal{B}') \equiv \mathbf{k}'(\mathcal{X}, \mathcal{B}')$, where $\mathbf{k}(\mathcal{X})$ is a position vector with respect to origin O and $\mathbf{k}'(\mathcal{X}, \mathcal{X}')$ is a position vector emanated from point $\mathbf{k}(\mathcal{X}) \in \mathcal{E}$ and leading to the point in space occupied by $\mathcal{X}'$.

A fixed chosen configuration $(\boldsymbol{\chi}_R, \boldsymbol{\chi}'_R)$ is called reference configuration of $(\mathcal{B}, \mathcal{B}')$,

$$(\mathcal{X}, \mathcal{X}') \quad \mapsto \quad \begin{pmatrix} \mathbf{X} = \boldsymbol{\chi}_R(\mathcal{X}) \\ \\ \mathbf{X}' = \boldsymbol{\chi}'_R(\mathcal{X}, \mathcal{X}') \end{pmatrix} \quad , \tag{38}$$

while a motion of $(\mathcal{B}, \mathcal{B}')$ in $\mathcal{E} \times \mathcal{E}$ is an one parameter family of configurations $(\boldsymbol{\chi}, \boldsymbol{\chi}')$, parameterized with time $t \in I$ ($I \subset \mathbb{R}$, I: interval),

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$$(\boldsymbol{\chi}, \boldsymbol{\chi}') \quad : \quad \mathcal{B} \times \mathcal{B}' \times I \quad \rightarrow \quad \mathcal{E} \times \mathcal{E} \quad , \tag{39}$$

$$(\mathcal{X}, \mathcal{X}', t) \quad \mapsto \quad \begin{pmatrix} \mathbf{x} = \boldsymbol{\chi}(\mathcal{X}, t) \\ \\ \mathbf{x}' = \boldsymbol{\chi}'(\mathcal{X}, \mathcal{X}', t) \end{pmatrix} \quad . \tag{40}$$

It is supposed, that for fixed time t, the map $(\boldsymbol{\chi}, \boldsymbol{\chi}')$ possesses an inverse, so that $\mathcal{X}$, $\mathcal{X}'$ may be expressed in terms of $\mathbf{x}$, $\mathbf{x}'$. If the motion of $(\mathcal{B}, \mathcal{B}')$ starts at time t_0, then the configuration $(\boldsymbol{\chi}(\cdot, t_0), \boldsymbol{\chi}'(\cdot, \cdot, t_0))$ is called the initial configuration. Accordingly, the configuration $(\boldsymbol{\chi}(\cdot, t), \boldsymbol{\chi}'(\cdot, \cdot, t))$ is denoted as actual or current or Eulerian configuration. In this article, we assume the initial configuration at time t_0 to coincide with the reference configuration,

$$\mathbf{x} = \boldsymbol{\chi}(\mathcal{X}, t_0) \equiv \boldsymbol{\chi}_R(\mathcal{X}) \quad , \tag{41}$$

$$\mathbf{x}' = \boldsymbol{\chi}'(\mathcal{X}, \mathcal{X}', t_0) \equiv \boldsymbol{\chi}'_R(\mathcal{X}, \mathcal{X}') \quad . \tag{42}$$

It is common to call configurations different than the reference configuration as spatial ones. Also, it is assumed that, for fixed t, all inverse functions exist, so that the motion can be expressed in terms of $\mathbf{X}$, $\mathbf{X}'$. If no confusion may arise, following common praxis, we shall use the same symbols for these functions as in Equation (40),

$$(\mathbf{X}, \mathbf{X}', t) \quad \mapsto \quad \begin{pmatrix} \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \equiv \boldsymbol{\chi}_t(\mathbf{X}) \\ \\ \mathbf{x}' = \boldsymbol{\chi}'(\mathbf{X}, \mathbf{X}', t) \equiv \boldsymbol{\chi}'_t(\mathbf{X}, \mathbf{X}') \end{pmatrix} \quad . \tag{43}$$

Also, we shall write

$$\mathbf{u} = \mathbf{u}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X} \tag{44}$$

for the macroscopic displacement vector. Functions $(\boldsymbol{\chi}, \boldsymbol{\chi}')$ in this equation, for fixed time t, are referred to as deformation from the reference to the actual configuration. We refer to the ranges in $\mathcal{E}$ $R_R := \boldsymbol{\chi}_R(\mathcal{B})$, $R_t := \boldsymbol{\chi}_t(\mathcal{B})$,

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$R'_R(\mathbf{X}) := \boldsymbol{\chi}'_R(\mathcal{X}, \mathcal{B}')$, $R'_t(\mathbf{x}) := \boldsymbol{\chi}'_t(\mathcal{X}, \mathcal{B}')$.

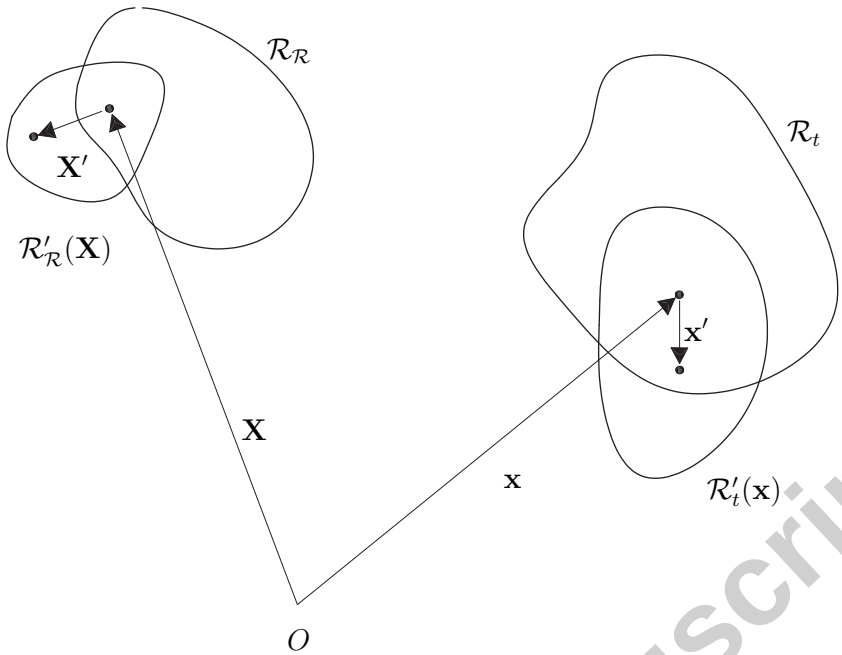

Fig. 1. The region $\mathcal{R}'_{\mathcal{R}}(\mathbf{X})$ (respectively $\mathcal{R}'_t(\mathbf{x})$) must not necessarily be subset of the region $\mathcal{R}_{\mathcal{R}}$ (respectively $\mathcal{R}_t$).

The introduction of microcontinua into the theory goes back essentially to Mindlin (1964) and Eringen (1968). Here, the microcontinuum is supposed to be, in some sense, mechanically (we are dealing with isothermal processes and uniform distributed temperature only) equivalent to some patterned material neighborhood around the considered point. The mass in the microcontinuum is assumed to be continuously distributed. Generally, the microcontinuum as adopted in the present article, is a fictitious (conceptual) one, which may have arbitrary finite dimensions (see Fig. 1), i.e. the region in $\mathcal{E}$ occupied by the microcontinuum at a material point of the macroscopic material must not necessarily be subset of the region occupied by the macroscopic material itself (see also Grammenoudis and Tsakmakis (2007)), where this kind of microcontinuum has been invoked in a micropolar plasticity theory). Following Eringen

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(see e.g. Eringen (1999)), we define a micromorphic material to be a material body with a microcontinuum at each point, which suffers only homogeneous deformations.

3.2 Deformation

As for a classical continuum, the deformation of the macrocontinuum will be described by the (macro-) deformation gradient tensor

$$\mathbf{F} = \mathbf{F}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}} = \mathrm{GRAD}\boldsymbol{\chi}(\mathbf{X}, t) \quad , \tag{45}$$

where $\det \mathbf{F} > 0$ is assumed. We distinguish between the operators GRAD and grad, representing the gradients with respect to $\mathbf{X}$ and $\mathbf{x}$, respectively. Under arbitrary rigid body rotations $\mathbf{Q}$ superposed on the actual configuration, $\mathbf{F}$ transforms according to

$$\mathbf{F} \;\rightarrow\; \mathbf{F}^* = \mathbf{Q}\mathbf{F} \quad . \tag{46}$$

The right Cauchy-Green deformation tensor $\mathbf{C}$ and the left Cauchy-Green deformation tensor $\mathbf{B}$ are given by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \quad , \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \quad , \tag{47}$$

in which $\mathbf{U}$ and $\mathbf{V}$, called respectively the right and the left stretch tensors, are symmetric and positive definite. They appear in the polar decomposition of $\mathbf{F}$,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \quad , \tag{48}$$

where $\mathbf{R}$ is a proper orthogonal second-order tensor. The velocity gradient tensor is denoted by $\mathbf{L}$,

$$\mathbf{L} := \mathrm{grad}\dot{\mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1} \quad , \tag{49}$$

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with

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \quad , \quad \mathbf{D} := \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \quad , \quad \mathbf{W} := \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \quad . \tag{50}$$

Similar to Eq. (45), the deformation of the microscopic body will be described by the microdeformation gradient tensor

$$\mathbf{f} = \mathbf{f}(\mathbf{X}, \mathbf{X}', t) := \frac{\partial \boldsymbol{\chi}'(\mathbf{X}, \mathbf{X}', t)}{\partial \mathbf{X}'} \quad , \tag{51}$$

with $\det \mathbf{f} > 0$ being assumed. As in the micromorphic continuum the microstructure suffers homogeneous deformations only, we have $\mathbf{f} = \mathbf{f}(\mathbf{X}, t)$. Geometrically, the macrodeformation gradient $\mathbf{F}(\mathbf{X}, t)$ is a two point tensor, i.e. it acts on a vector $\mathbf{A} \in T_{\mathbf{X}}\mathcal{R}_{\mathcal{R}}$ and furnishes a vector $\mathbf{a} = \mathbf{F}\mathbf{A} \in T_{\mathbf{x}}\mathcal{R}_t$. On the other hand, since $\mathbf{f}$ is homogeneous deformation, $\mathbf{f}$ may also be thought to be a two point tensor, which acts on vectors $\mathbf{A} \in T_{\mathbf{X}}\mathcal{R}_{\mathcal{R}}$ and furnishes vectors $\boldsymbol{\alpha} \in T_{\mathbf{x}}\mathcal{R}_t$. Assume $\mathbf{A}$, $\boldsymbol{\alpha}$ to be objective Lagrangean and objective Eulerian vectors respectively, i.e., under arbitrary rigid body rotations $\mathbf{Q}$ superposed on the actual configuration (of the macroscopic body), $\mathbf{A}$, $\boldsymbol{\alpha}$ transforms according to $\mathbf{A}^* = \mathbf{A}$ and $\boldsymbol{\alpha}^* = \mathbf{Q}\boldsymbol{\alpha} = \mathbf{Q}\mathbf{f}\mathbf{A}$. Then, $\mathbf{f}$ has to transform according to

$$\mathbf{f} \;\rightarrow\; \mathbf{f}^* = \mathbf{Q}\mathbf{f} \quad . \tag{52}$$

Keeping in mind $\det \mathbf{f} > 0$, the polar decomposition

$$\mathbf{f} = \mathbf{r}\mathbf{u} = \mathbf{v}\mathbf{r} \tag{53}$$

holds, with $\mathbf{r}$,$\mathbf{u}$,$\mathbf{v}$ being second-order tensors corresponding respectively to the tensors $\mathbf{R}$,$\mathbf{U}$,$\mathbf{V}$ in the polar decomposition (48). We use the symbol $\mathbf{l}$ for the

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"microvelocity" gradient tensor,

$$\mathbf{l} := \dot{\mathbf{f}}\mathbf{f}^{-1} = \mathbf{d} + \mathbf{w} \quad , \tag{54}$$

$$\mathbf{d} := \frac{1}{2}(\mathbf{l} + \mathbf{l}^T) \quad , \quad \mathbf{w} := \frac{1}{2}(\mathbf{l} - \mathbf{l}^T) \quad . \tag{55}$$

(According to Eringen (1999, p.24), $\mathbf{l}$ is called the microgyration tensor.)

If we set $\mathbf{x}' = \mathbf{f}(\mathbf{X}, t)\mathbf{X}'$ in Eq. (43), then the motion can be described by the map

$$(\mathbf{X}, \mathbf{X}', t) \quad \mapsto \quad \begin{pmatrix} \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \\ \\ \mathbf{x}' = \mathbf{f}(\mathbf{X}, t)\mathbf{X}' \end{pmatrix} \quad . \tag{56}$$

All what says this map can be reflected by considering instead of arbitrary vectors $\mathbf{X}'$, position vectors $\boldsymbol{\Phi}$ for the microcontinuum, which are arbitrary fields of $\mathbf{X}$, $\boldsymbol{\Phi} = \boldsymbol{\Phi}(\mathbf{X})$, and which are mapped to vectors $\boldsymbol{\varphi}(\mathbf{x}, t) = \mathbf{f}(\mathbf{X}, t)\boldsymbol{\Phi}(\mathbf{X})$. Then, for arbitrary but fixed $\boldsymbol{\Phi}(\cdot)$, Eq. (56) yields

$$(\mathbf{X}, t) \quad \mapsto \quad \begin{pmatrix} \mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \\ \\ \boldsymbol{\varphi}(\mathbf{x}, t) = \mathbf{f}(\mathbf{X}, t)\boldsymbol{\Phi}(\mathbf{X}) \end{pmatrix} \quad . \tag{57}$$

3.3 Coordinate systems

As usually, the tangent space at any point of a manifold is defined to be an Euclidean vector space. The inner product in this space is denoted by a dot. Clearly, in the tangent space of every point there exists always an orthonormal basis, so that, with respect to this (perhaps local) basis, the components of the metric tensor will be given by the Kronecker delta symbol. If these bases form a field of coordinate basis vectors, tangent to a global coordinate system, then

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the metric coefficients of the metric on the manifold, will be given everywhere by the Kronecker delta symbol. In this case, the metric tensor on the manifold is everywhere the identity tensor of the second-order, and the manifold will be Euclidean. However, if it is not possible to select such a coordinate system, then the tangent vector spaces will still be Euclidean, but the metric of the manifold, and hence the manifold itself, will be not Euclidean. In this case, if we are given the metric coefficients at every point on the manifold, then there do not exist some coordinate transformations rendering the metric coefficients equal to the Kronecker delta everywhere. Nevertheless, the components of tensorial quantities will be expressed in terms of the Euclidean product, which holds always in the tangent space at every point.

It is assumed that $\mathcal{R}_{\mathcal{R}}$ and $\mathcal{R}_t$ are Euclidean manifolds, and that they can be covered by coordinate lines of single coordinate systems, respectively. Let $\{X^i\}$, $\{x^i\}$ be Cartesian coordinate systems for $\mathcal{R}_{\mathcal{R}}$ and $\mathcal{R}_t$, inducing the coordinate bases $\mathbf{E}_i \equiv \mathbf{E}^i$, $\mathbf{e}_i \equiv \mathbf{e}^i$, respectively,

$$\mathbf{E}^i \cdot \mathbf{E}_j = \delta^i_j \quad , \quad \mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j \quad . \tag{58}$$

It is convenient to use the coordinate system $\{X^i\}$ as a convective one. Then the coordinate lines in $\mathcal{R}_{\mathcal{R}}$ of the coordinate system $\{X^i\}$ will represent material lines, which will be deformed in $\mathcal{R}_t$, to form the coordinate lines of the convective coordinate system. To a material point, it will be assigned in $\mathcal{R}_{\mathcal{R}}$ and $\mathcal{R}_t$ the same values of convective coordinates $\{X^i\}$, but the corresponding local coordinate basis will be different. If $\mathbf{E}_i$ and $\mathbf{g}_i$ are the coordinate basis vectors for the same material point in $\mathcal{R}_{\mathcal{R}}$ and $\mathcal{R}_t$, respectively, then

$$\mathbf{g}_i = \mathbf{F}\mathbf{E}_i \quad , \quad \mathbf{g}^i = \mathbf{F}^{T-1}\mathbf{E}^i \quad , \quad \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j \quad , \tag{59}$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{E}_i \cdot \mathbf{C}\mathbf{E}_j \quad , \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = \mathbf{E}^i \cdot \mathbf{C}^{-1}\mathbf{E}^j \quad . \tag{60}$$

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Between the two basis fields $\{\mathbf{e}_i\}$ and $\{\mathbf{g}_i\}$, assigned to the manifold $\mathcal{R}_t$, there are the relations

$$\mathbf{g}_j = \frac{\partial x^i}{\partial X^j}\mathbf{e}_i \quad , \quad \mathbf{g}^i = \frac{\partial X^i}{\partial x^j}\mathbf{e}^i \quad . \tag{61}$$

This, together with the formula $\mathbf{e}^k \dfrac{\partial}{\partial x^k} = \mathbf{e}^k \dfrac{\partial X^m}{\partial x^k}\dfrac{\partial}{\partial X^m}$, imply

$$\mathbf{e}^k \frac{\partial}{\partial x^k} = \mathbf{g}^m \frac{\partial}{\partial X^m} \quad . \tag{62}$$

Then,

$$\mathbf{F} = F^i_{\ j}\mathbf{e}_i \otimes \mathbf{E}^j = \delta^i_{\ j}\mathbf{g}_i \otimes \mathbf{E}^j \quad , \quad F^i_{\ j} \equiv F_{ij} = \frac{\partial x^i}{\partial X^j} \quad , \tag{63}$$

$$\mathbf{F}^{-1} = (\mathbf{F}^{-1})^i_{\ j}\mathbf{E}_i \otimes \mathbf{e}^j = \delta^i_{\ j}\mathbf{g}_i \otimes \mathbf{E}^j \quad , \quad (\mathbf{F}^{-1})^i_{\ j} \equiv (\mathbf{F}^{-1})_{ij} = \frac{\partial X^i}{\partial x^j} \quad . \tag{64}$$

In analogy to (59), (60), an additional basis field $\boldsymbol{\varrho}_i = \boldsymbol{\varrho}_i(\mathbf{x}, t)$ may be introduced on $\mathcal{R}_t$, by

$$\boldsymbol{\varrho}_i := \mathbf{f}\mathbf{E}_i \quad , \quad \boldsymbol{\varrho}^i := \mathbf{f}^{T-1}\mathbf{E}^i \quad , \quad \boldsymbol{\varrho}^i \cdot \boldsymbol{\varrho}_j = \delta^i_{\ j} \quad . \tag{65}$$

Note that, in opposite to $\{\mathbf{g}_i\}$, the basis $\{\boldsymbol{\varrho}_i\}$ is anholonomic, since $\mathbf{f}$ does not satisfy some compatibility conditions with respect to the coordinates $\{X^i\}$. In other words, $\mathbf{f}$ may be thought to be only a local deformation for the macroscopic body.

3.4 Strain energy function in micromorphic hyperelasticity

The definition of kinematical measures typical for micromorphic continua can be motivated by considering first pure elasticity. Suppose the specific (per unit mass of the macroscopic continuum) strain energy function Ψ for an elastic micromorphic material to depend on $\mathbf{F}$, $\mathbf{f}$, and $\mathrm{GRAD}\mathbf{f}$,

$$\Psi = \bar{\Psi}(\mathbf{F}, \mathbf{f}, \mathrm{GRAD}\mathbf{f}) \quad . \tag{66}$$

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It is worth regarding, that GRAD$\mathbf{f}$ introduces an internal length into the theory. According to Eq. (66), an elastic micromorphic continuum models microphysical properties of the real material by taking into account, beyond the classical deformation gradient tensor $\mathbf{F}$, the microdeformation gradient tensor $\mathbf{f}$ and its gradient GRAD$\mathbf{f}$ (see Eringen (1968)).

With respect to the convective coordinate system $\{X^i\}$, we have

$$\mathrm{GRAD}\mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{X}} = \frac{\partial \mathbf{f}}{\partial X^i} \otimes \mathbf{E}^i \quad . \tag{67}$$

In view of (52), the third-order tensor GRAD$\mathbf{f}$ obeys the transformation law

$$(\mathrm{GRAD}\mathbf{f})^* \equiv \mathrm{GRAD}(\mathbf{Q}\mathbf{f}) = \mathbf{Q}\frac{\partial \mathbf{f}}{\partial X^k} \otimes \mathbf{E}^k = \mathbf{Q}\mathrm{GRAD}\mathbf{f} \quad , \tag{68}$$

where, as usual, $\mathbf{X}^* \equiv \mathbf{X}$ has been assumed. On requiring $\Psi^* = \bar{\Psi}(\mathbf{F}^*, \mathbf{f}^*, \mathrm{GRAD}\mathbf{f}^*) = \Psi$, we get

$$\Psi = \bar{\Psi}(\mathbf{Q}\mathbf{F}, \mathbf{Q}\mathbf{f}, \mathbf{Q}\mathrm{GRAD}\mathbf{f}) \quad , \tag{69}$$

which must hold for every proper orthogonal $\mathbf{Q}$. By setting $\mathbf{Q} = \mathbf{r}^{-1}$,

$$\begin{aligned} \Psi &= \bar{\Psi}(\mathbf{r}^{-1}\mathbf{F}, \mathbf{u}, \mathbf{r}^{-1}\mathrm{GRAD}\mathbf{f}) \\ &= \bar{\Psi}(\mathbf{u}\mathbf{f}^{-1}\mathbf{F}, \mathbf{u}, \mathbf{u}\mathbf{f}^{-1}\mathrm{GRAD}\mathbf{f}) \\ &= \bar{\bar{\Psi}}(\mathbf{f}^{-1}\mathbf{F}, \mathbf{u}^2, \mathbf{f}^{-1}\mathrm{GRAD}\mathbf{f}) \quad , \end{aligned} \tag{70}$$

or

$$\Psi = \tilde{\Psi}(\tilde{\boldsymbol{\epsilon}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\mathcal{K}}}) \quad , \tag{71}$$

where

$$\tilde{\boldsymbol{\epsilon}} := \mathbf{f}^{-1}\mathbf{F} - \mathbf{1} \quad , \quad \tilde{\boldsymbol{\beta}} := \frac{1}{2}(\mathbf{u}^2 - \mathbf{1}) \quad , \quad \tilde{\boldsymbol{\mathcal{K}}} := \mathbf{f}^{-1}\mathrm{GRAD}\mathbf{f} = \mathbf{f}^{-1}\frac{\partial \mathbf{f}}{\partial X^k} \otimes \mathbf{E}^k \quad . \tag{72}$$

Variables $\tilde{\boldsymbol{\epsilon}}$, $\tilde{\boldsymbol{\beta}}$, and $\tilde{\boldsymbol{\mathcal{K}}}$ are Lagrangean measures, which represent a second-order micromorphic strain tensor, a second-order classical strain tensor for the

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microstructure and a third-order micromorphic curvature tensor, respectively. This set of variables has been introduced by Eringen (see e.g. Eringen, 1999, p.15) in order to formulate micromorphic elasticity. Especially, $\tilde{\boldsymbol{\mathcal{K}}}$ is called by Eringen the wryness tensor. Note that (72) is not the only set of variables appropriate for formulating the constitutive theory. Alternatives arise by setting in (69) $\mathbf{Q}$ equal to $\mathbf{r}^T$ or $\mathbf{R}^{-1}$ or $\mathbf{R}^T$. Then, by following similar steps as above, one can readily prove that the sets (cf. Eringen, 1999, p.14 and 15)

$$\left(\mathbf{f}^{-1}\mathbf{F}-\mathbf{1},\frac{1}{2}(\mathbf{u}^2-\mathbf{1}),\mathbf{F}^{-1}\mathrm{GRAD}\mathbf{f}\right) \quad , \tag{73}$$

$$\left(\mathbf{f}^{T}\mathbf{F}-\mathbf{1},\frac{1}{2}(\mathbf{u}^2-\mathbf{1}),\mathbf{f}^{T}\mathrm{GRAD}\mathbf{f}\right) \quad , \tag{74}$$

$$\left(\mathbf{f}^{T}\mathbf{F}-\mathbf{1},\frac{1}{2}(\mathbf{u}^2-\mathbf{1}),\mathbf{F}^{-1}\mathrm{GRAD}\mathbf{f}\right) \quad , \tag{75}$$

$$\left(\mathbf{f}^{T}\mathbf{F}-\mathbf{1},\frac{1}{2}(\mathbf{U}^2-\mathbf{1}),\mathbf{F}^{T}\mathrm{GRAD}\mathbf{f}\right) \quad , \tag{76}$$

$$\left(\mathbf{f}^{T}\mathbf{F}-\mathbf{1},\frac{1}{2}(\mathbf{U}^2-\mathbf{1}),\mathbf{f}^{-1}\mathrm{GRAD}\mathbf{f}\right) \quad , \tag{77}$$

$$\left(\mathbf{F}^{-1}\mathbf{f}-\mathbf{1},\frac{1}{2}(\mathbf{U}^2-\mathbf{1}),\mathbf{F}^{-1}\mathrm{GRAD}\mathbf{f}\right) \quad , \tag{78}$$

$$\vdots$$

may also be used as Lagrangean kinematical variables for formulating constitutive theories of micromorphic elasticity.

4 Balance laws

4.1 Conservation of mass for the micro- and the macroscopic continuum

Let $V'(\mathbf{X})$, $v'(\mathbf{x},t)$ be the volumes of the space ranges $\mathcal{R}'_{\mathcal{R}}(\mathbf{X})$ and $\mathcal{R}'_t(\mathbf{x})$, respectively. In the ensuing analysis we shall often suppress the argument $\mathbf{X}$ in functions $V'(\mathbf{X})$, $\mathcal{R}'_{\mathcal{R}}(\mathbf{X})$, the argument $\mathbf{x}$ in function $\mathcal{R}'_t(\mathbf{x})$, the arguments

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$\mathbf{x}, t$ in function $v'(\mathbf{x}, t)$, the arguments $\mathbf{X}, t$ in function $\mathbf{f}(\mathbf{X}, t)$, and so forth. Thus, for corresponding volume elements $\mathrm{d}V'$ and $\mathrm{d}v'$, we have

$$\mathrm{d}v' = (\det \mathbf{f})\, \mathrm{d}V' \quad , \tag{79}$$

or equivalently

$$v' = (\det \mathbf{f}) V' \quad . \tag{80}$$

The mass in the microcontinuum at $\mathbf{X}$, which models a neighborhood around $\mathbf{X}$ of the real material, is assumed to be continuously distributed, so that a mass density $\varrho'(\mathbf{x}, \mathbf{x}', t)$ is assigned to each point in $\mathcal{R}'_t(\mathbf{x})$, the corresponding mass density in $\mathcal{R}'_{\mathcal{R}}(\mathbf{X})$ being $\varrho'_{\mathcal{R}}(\mathbf{X}, \mathbf{X}') \equiv \varrho'(\mathbf{X}, \mathbf{X}', 0)$. Conservation of mass for the microcontinuum is assumed to apply, so that

$$\varrho'(\mathbf{x}, \mathbf{x}', t) = \frac{\varrho'_{\mathcal{R}}(\mathbf{X}, \mathbf{X}')}{\det \mathbf{f}(\mathbf{X}, t)} \quad . \tag{81}$$

Let $\mathrm{d}V$, $\varrho_{\mathcal{R}}(\mathbf{X})$ be the volume element and the mass density of the macroscopic continuum in the reference configuration at point $\mathbf{X}$. Denote by $\mathrm{d}v$ the corresponding volume element in the actual configuration at point $\mathbf{x}$. Then

$$\mathrm{d}v = (\det \mathbf{F})\, \mathrm{d}V \quad . \tag{82}$$

The volumes of $\mathcal{R}_{\mathcal{R}}$ and $\mathcal{R}_t$ are denoted by V and v, respectively.

We suppose $\varrho_{\mathcal{R}}$ to be given by the volume average

$$\varrho_{\mathcal{R}}(\mathbf{X}) = \langle \varrho'_{\mathcal{R}}(\mathbf{X}, \mathbf{X}') \rangle_{\mathcal{R}'_{\mathcal{R}}} := \frac{1}{V'} \int_{\mathcal{R}'_{\mathcal{R}}} \varrho'_{\mathcal{R}}(\mathbf{X}, \mathbf{X}')\, \mathrm{d}V' \quad . \tag{83}$$

The mass density of the macroscopic continuum in the actual configuration is referred to as $\varrho(\mathbf{x}, t)$, and conservation of mass for the macroscopic continuum

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is required,

$$\varrho(\mathbf{x},t) = \frac{\varrho_{\mathcal{R}}(\mathbf{X})}{\det \mathbf{F}(\mathbf{X},t)} \quad . \tag{84}$$

The latter, together with (79)-(83) yields

$$\varrho(\mathbf{x},t) = \langle \varrho'(\mathbf{x},\mathbf{x}',t)\rangle_{\mathcal{R}'_t} := \frac{1}{v'}\int_{\mathcal{R}'_t} \chi(\mathbf{x},t)\varrho'(\mathbf{x},\mathbf{x}',t)\,\mathrm{d}v' \quad , \tag{85}$$

with the weight function χ being defined by

$$\chi(\mathbf{x},t) := \frac{\det \mathbf{f}(\mathbf{X},t)}{\det \mathbf{F}(\mathbf{X},t)} \quad . \tag{86}$$

In other words, the mass density of the macroscopic continuum is given by the weighted volume average of the mass density of the microcontinuum. The weight function χ captures both the deformation of the macroscopic continuum and the deformation of the microcontinuum. On the other hand, one may thoughtthink the mass density of the macroscopic continuum to be defined by (85). Then, as $\varrho'(\mathbf{X},\mathbf{X}',0) = \varrho'_{\mathcal{R}}(\mathbf{X},\mathbf{X}')$, $v'(\mathbf{X},0) = V'(\mathbf{X})$, $\mathbf{f}(\mathbf{X},0) = \mathbf{F}(\mathbf{X},0) = \mathbf{1}$, and hence $\chi(\mathbf{X},0) = 1$, we see that $\varrho(\mathbf{X},0) = \varrho_{\mathcal{R}}(\mathbf{X})$, with $\varrho_{\mathcal{R}}(\mathbf{X})$ given by (83), and Eq. (84) will be recovered.

4.2 Balance laws of momentum and moment of momentum

By taking into account the motion of the microcontinuum, Mindlin (1964) elaborated rigorous derivations for the balance laws for momentum (linear moment) and moment of momentum (angular moment) for the case of small deformations. Following steps similar to those in Mindlin's approach, but adjusted to the finite deformation version of the theory adopted here, one may derive in $\mathcal{R}_t$, relative to the Cartesian coordinate system $\{x_i\}$, the balance of

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momentum

$$\frac{\partial T_{ij}}{\partial x_j} + b_i = \varrho \ddot{x}_i \quad \text{in } \mathcal{R}_t \quad , \tag{87}$$

the balance of moment of momentum

$$\frac{\partial \mathcal{T}_{ijk}}{\partial x_k} + T_{ij} - \Sigma_{ij} + b_{ij}^{(d)} = \varrho \lambda_{ij} \quad \text{in } \mathcal{R}_t \quad , \tag{88}$$

and appropriate boundary conditions. (For reasons of completeness, the proof of this assertion is given in Appendix A). In (87), (88), $\mathrm{div}\mathbf{S} = \dfrac{\partial S_{ij}}{\partial x_j}\mathbf{e}_i$ for an Eulerian second-order tensor field $\mathbf{S} = \mathbf{S}(\mathbf{x})$, $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is the Cauchy stress tensor (non-symmetric), $\boldsymbol{\Sigma} = \Sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is a symmetric stress tensor responsible for the microcontinuum, $\boldsymbol{\mathcal{T}} = \mathcal{T}_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ is a so-called double stress tensor, $\mathbf{b} = b_i\mathbf{e}_i$ and $\mathbf{b}^{(d)} = b_{ij}^{(d)}\mathbf{e}_i \otimes \mathbf{e}_j$ are respectively the body force and the double body force per unit volume of the actual configuration of the macroscopic continuum, and $\mathbf{t} = t_i\mathbf{e}_i$, $\mathbf{t}^{(d)} = t_{ij}^{(d)}\mathbf{e}_i \otimes \mathbf{e}_j$ are respectively the surface force (traction) and the double surface force (double traction) per unit area of the actual configuration of the macroscopic continuum. The second-order tensor $\boldsymbol{\lambda}$ is defined by

$$\boldsymbol{\lambda}(\mathbf{x}, t) := \frac{1}{v'} \int_{\mathcal{R}'_t} \ddot{\mathbf{x}}' \otimes \mathbf{x}' \, \mathrm{d}v' \quad , \tag{89}$$

if $\mathbf{X}' = \mathbf{0}$ is volume centroid of the microcontinuum in the reference configuration, or by

$$\boldsymbol{\lambda}(\mathbf{x}, t) := \frac{1}{\varrho(\mathbf{x}, t)v'} \int_{\mathcal{R}'_t} \chi(\mathbf{x}, t)(\ddot{\mathbf{x}}' \otimes \mathbf{x}')\varrho'(\mathbf{x}, \mathbf{x}', t) \, \mathrm{d}v' \quad , \tag{90}$$

if $\mathbf{X}' = \mathbf{0}$ is center of mass of the microcontinuum in the reference configuration. Tensor $\boldsymbol{\lambda}$ in (89) or (90) is called specific (per unit mass of the microscopic continuum) spin inertia tensor. It is analogously but not equal to a corresponding tensor introduced by Eringen and Suhubi (1964). Note that $\boldsymbol{\lambda}$ obeys the

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representation (see Appendix A)

$$\boldsymbol{\lambda} = (\dot{\mathbf{l}} + \mathbf{l}\mathbf{l})\boldsymbol{\theta} \quad , \tag{91}$$

with

$$\boldsymbol{\theta} = \mathbf{f}\boldsymbol{\Theta}\mathbf{f}^T \tag{92}$$

and

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{X}) := \frac{1}{V'}\int_{\mathcal{R}'_{\mathcal{R}}} (\mathbf{X}' \otimes \mathbf{X}')\,\mathrm{d}V' = \boldsymbol{\Theta}^T \quad , \tag{93}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{x}, t) := \frac{1}{v'}\int_{\mathcal{R}'_t} (\mathbf{x}' \otimes \mathbf{x}')\,\mathrm{d}v' = \boldsymbol{\theta}^T \tag{94}$$

if $\mathbf{X}' = \mathbf{0}$ is volume centroid of the microcontinuum in the reference configuration, or

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{X}) := \frac{1}{\varrho_{\mathcal{R}}(\mathbf{X})V'}\int_{\mathcal{R}'_{\mathcal{R}}} (\mathbf{X}' \otimes \mathbf{X}')\varrho'_{\mathcal{R}}(\mathbf{X}, \mathbf{X}')\,\mathrm{d}V' \quad , \tag{95}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{x}, t) := \frac{1}{\varrho(\mathbf{x}, t)v'}\int_{\mathcal{R}'_t} \chi(\mathbf{x}, t)(\mathbf{x}' \otimes \mathbf{x}')\varrho'(\mathbf{x}, \mathbf{x}', t)\,\mathrm{d}v' \tag{96}$$

if $\mathbf{X}' = \mathbf{0}$ is center of mass of the microcontinuum in the reference configuration. Again, the tensors $\boldsymbol{\Theta}$, $\boldsymbol{\theta}$ defined by (93), (94) or (95), (96) are in essence the same as the so-called microinertia tensors introduced by Eringen (see e.g. Eringen, 1999, p.32). Furthermore, starting from the motion of a microcontinuum included in each material particle of the macroscopic continuum, Eringen and coworkers (see e.g. Eringen and Suhubi, 1964) proposed, by using different approaches than Mindlin, balance laws of momentum and moment of momentum for the macroscopic continuum, and related boundary conditions. If one chooses a fixed form for $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{X})$ in (92), then, regardless of the way $\boldsymbol{\Theta}$ is defined, these laws are exactly the same as those in (87), (88). Also, for small deformations, apart from definition of $\boldsymbol{\Theta}$, relations (87), (88) are exactly the same relations obtained by Mindlin (1964) for a material

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composed wholly of unit cells.

4.3 Balance of mechanical energy

It is well known in continuum mechanics that the balance law of mechanical energy is derivable from the balance laws of momentum and moment of momentum (cf. corresponding relations in Eringen, 1999; Eringen and Suhubi, 1964). Since the resulting equation is important for our aims, and in order to make the paper self-contained, we discuss briefly the derivation of this balance law.

We take the product of Eq. (87) (respectively Eq. (88)) with $\dot{x}_i$ (respectively l_{ij}):

$$\frac{\partial T_{ij}}{\partial x_j}\dot{x}_i + b_i\dot{x}_i = \varrho\ddot{x}_i\dot{x}_i = \frac{1}{2}\varrho\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}_i\dot{x}_i) \quad , \tag{97}$$

$$\frac{\partial \mathcal{T}_{ijk}}{\partial x_k}l_{ij} + T_{ij}l_{ij} - \Sigma_{ij}l_{ij} + b^{(d)}_{ij}l_{ij} = \varrho\lambda_{ij}l_{ij} \quad . \tag{98}$$

Employing the identities

$$\frac{\partial T_{ij}}{\partial x_j}\dot{x}_i = \frac{\partial(T_{ij}\dot{x}_i)}{\partial x_j} - T_{ij}L_{ij} \quad , \tag{99}$$

$$\frac{\partial \mathcal{T}_{ijk}}{\partial x_k}l_{ij} = \frac{\partial(\mathcal{T}_{ijk}l_{ij})}{\partial x_k} - \mathcal{T}_{ijk}(\mathrm{grad}\mathbf{l})_{ijk} \quad , \tag{100}$$

and integrating (97), (98) over $\mathcal{R}_t$ (with boundary $\partial\mathcal{R}_t$), followed by the use of the divergence theorem,

$$\int_{\partial\mathcal{R}_t}\mathbf{t}\cdot\dot{\mathbf{x}}\,\mathrm{d}a + \int_{\mathcal{R}_t}\mathbf{b}\cdot\dot{\mathbf{x}}\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}_t}\frac{1}{2}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\varrho\,\mathrm{d}v + \int_{\mathcal{R}_t}\mathbf{T}\cdot\mathbf{L}\,\mathrm{d}v \quad , \tag{101}$$

$$\begin{aligned}\int_{\partial\mathcal{R}_t}\mathbf{t}^{(d)}\cdot\mathbf{l}\,\mathrm{d}a + \int_{\mathcal{R}_t}\mathbf{b}^{(d)}\cdot\mathbf{l}\,\mathrm{d}v \\ = \int_{\mathcal{R}_t}\boldsymbol{\lambda}\cdot\mathbf{l}\varrho\,\mathrm{d}v + \int_{\mathcal{R}_t}(\boldsymbol{\Sigma}-\mathbf{T})\cdot\mathbf{l}\,\mathrm{d}v + \int_{\mathcal{R}_t}\boldsymbol{\mathcal{T}}\cdot\mathrm{grad}\mathbf{l}\,\mathrm{d}v \quad .\end{aligned} \tag{102}$$

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We write $\mathrm{d}\mathbf{A} = \mathrm{d}A\mathbf{n}_R$ for a material surface element of the macroscopic continuum at $\mathbf{X}$ in $\mathcal{R}_\mathcal{R}$, the corresponding material surface element at $\mathbf{x}$ in $\mathcal{R}_t$ being $\mathrm{d}\mathbf{a} = \mathrm{d}a\mathbf{n}$, where $\mathbf{n}_R$ and $\mathbf{n}$ are the (positive) unit normals to the boundaries of $\mathcal{R}_\mathcal{R}$ and $\mathcal{R}_t$, respectively. After adding Eq. (101) to Eq. (102),

$$\begin{aligned}&\int_{\partial\mathcal{R}_t}(\mathbf{t}\cdot\dot{\mathbf{x}}+\mathbf{t}^{(d)}\cdot\mathbf{l})\,\mathrm{d}a+\int_{\mathcal{R}_t}(\mathbf{b}\cdot\dot{\mathbf{x}}+\mathbf{b}^{(d)}\cdot\mathbf{l})\,\mathrm{d}v\\&=\int_{\mathcal{R}_t}\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})+\boldsymbol{\lambda}\cdot\mathbf{l}\right]\varrho\,\mathrm{d}v+\int_{\mathcal{R}_t}[\mathbf{T}\cdot(\mathbf{L}-\mathbf{l})+\boldsymbol{\Sigma}\cdot\mathbf{l}+\boldsymbol{\mathcal{T}}\cdot\mathrm{grad}\mathbf{l}]\,\mathrm{d}v\quad,\end{aligned}\tag{103}$$

which is the resulting balance of mechanical energy. The terms on the left-hand side represent the rate of working of the external (applied) forces. The first integral on the right-hand side is the rate of change of the kinetic energy of the body and

$$\int_{\mathcal{R}_t}[\mathbf{T}\cdot(\mathbf{L}-\mathbf{l})+\boldsymbol{\Sigma}\cdot\mathbf{l}+\boldsymbol{\mathcal{T}}\cdot\mathrm{grad}\mathbf{l}]\,\mathrm{d}v=\int_{\mathcal{R}_\mathcal{R}}[\mathbf{S}\cdot(\mathbf{L}-\mathbf{l})+\boldsymbol{\sigma}\cdot\mathbf{l}+\boldsymbol{\mathcal{S}}\cdot\mathrm{grad}\mathbf{l}]\,\mathrm{d}V\quad,\tag{104}$$

is the rate of working of the internal forces, where

$$\mathbf{S}:=(\det\mathbf{F})\mathbf{T}\quad,\quad\boldsymbol{\sigma}:=(\det\mathbf{F})\boldsymbol{\Sigma}\quad,\quad\boldsymbol{\mathcal{S}}:=(\det\mathbf{F})\boldsymbol{\mathcal{T}}\tag{105}$$

are the weighted Cauchy stress tensor, the weighted stress tensor for the micro-continuum and the weighted double stress tensor, respectively. From Eq. (104), we recognize that

$$w:=\mathbf{S}\cdot(\mathbf{L}-\mathbf{l})\quad,\quad w':=\boldsymbol{\sigma}\cdot\mathbf{l}\equiv\boldsymbol{\sigma}\cdot\mathbf{d}\quad,\quad w_c:=\boldsymbol{\mathcal{S}}\cdot\mathrm{grad}\mathbf{l}\tag{106}$$

represent stress powers per unit volume of the reference configuration.

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5 Dual Variables

5.1 Equivalent classes of strain and micromorphic curvature tensors

In this section we shall interpret geometrically the set of kinematical variables $\tilde{\boldsymbol{\epsilon}}$, $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{\mathcal{K}}}$ appearing in Eq. (72). To this end we shall made use of scalar valued differences of geometrical measures. The interpretation of other sets of variables like these in Eqs. (73)–(78) will be established in a similar fashion.

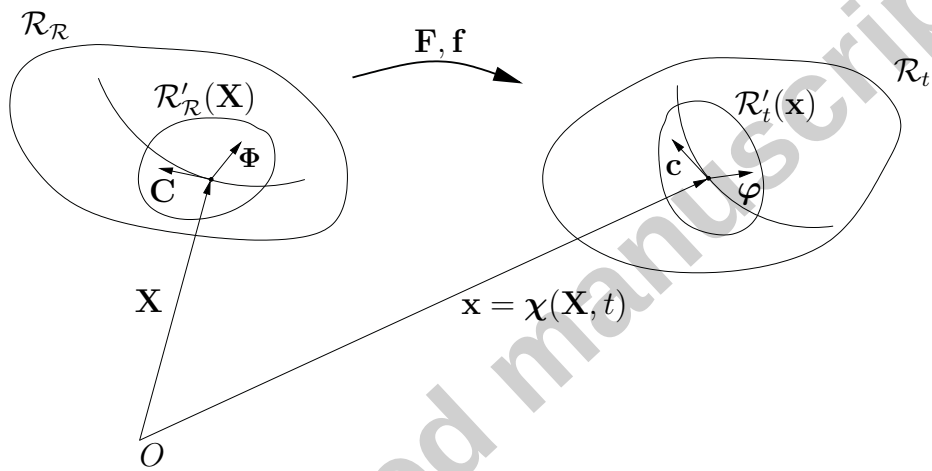

Fig. 2. $\mathbf{F}(\mathbf{X}, t)$, $\mathbf{f}(\mathbf{X}, t)$ are two-point deformation tensors, mapping vectors at $\mathbf{X}$ in the reference configuration to vectors at $\mathbf{x}$ in the actual configuration.

Consider a material line on $\mathcal{R}_{\mathcal{R}}$ passing through an arbitrary point $\mathbf{X}$ and having there tangent vector $\mathbf{C} = \mathbf{C}(\mathbf{X})$ (see Fig. 2). The corresponding tangent vector on the same material line on $\mathcal{R}_t$ at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ is $\mathbf{c} = \mathbf{c}(\mathbf{x}, t)$. Further, assume $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{X})$ to be a vector at $\mathbf{X}$, which is position vector to some material point $\mathcal{X}' \in \mathcal{R}'_{\mathcal{R}}(\mathbf{X})$, the corresponding vector at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ being $\boldsymbol{\varphi}(\mathbf{x}, t)$. Then,

$$\mathbf{c} = \mathbf{F}\mathbf{C} \quad , \quad \boldsymbol{\varphi} = \mathbf{f}\mathbf{\Phi} \quad . \tag{107}$$

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On the other hand, one may consider vectors $\boldsymbol{\Xi} = \boldsymbol{\Xi}(\mathbf{X})$ and $\mathbf{Z} = \mathbf{Z}(\mathbf{X})$, which are normal at $\mathbf{X}$, to material surfaces in the macroscopic and the microscopic continuum, respectively. The corresponding vectors normal to the same material surfaces in the actual configuration are respectively $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, t)$ and $\boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{x}, t)$ and we have

$$\boldsymbol{\xi} = \mathbf{F}^{T-1}\boldsymbol{\Xi} \quad , \quad \boldsymbol{\zeta} = \mathbf{f}^{T-1}\mathbf{Z} \quad . \tag{108}$$

More generally, one can consider regular linear transformations $\mathbf{F}_a = \mathbf{F}_a(\mathbf{X}, t)$ for the macroscopic continuum, and regular linear transformations $\mathbf{f}_a = \mathbf{f}_a(\mathbf{X}, t)$ for the microscopic continuum, both going from the reference to the same, but otherwise arbitrary configuration. On designating the counterparts of $\mathbf{C}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Xi}$, $\mathbf{Z}$ with respect to these configurations respectively by $\mathbf{c}_a$, $\boldsymbol{\varphi}_a$, $\boldsymbol{\xi}_a$, $\boldsymbol{\zeta}_a$,

$$\mathbf{c}_a = \mathbf{F}_a\mathbf{C} \quad , \quad \boldsymbol{\varphi}_a = \mathbf{f}_a\boldsymbol{\Phi} \quad , \tag{109}$$

$$\boldsymbol{\xi}_a = \mathbf{F}_a^{T-1}\boldsymbol{\Xi} \quad , \quad \boldsymbol{\zeta}_a = \mathbf{f}_a^{T-1}\mathbf{Z} \quad . \tag{110}$$

Particular examples of such transformations are discussed in Part II in the framework of multiplicative decompositions of $\mathbf{F}$ and $\mathbf{f}$ into elastic and plastic parts.

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5.1.1 Strain tensors

As in classical continuum mechanics, the state of strain in the microstructure at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, for fixed time t, may be expressed in terms of the scalar valued difference

$$\Delta'_s = \Delta'_s(\mathbf{X}, t) := \frac{1}{2}(\boldsymbol{\varphi} \cdot \boldsymbol{\varphi} - \boldsymbol{\Phi} \cdot \boldsymbol{\Phi}) \quad . \tag{111}$$

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With respect to the reference configuration, on using Eq. $(107)_2$, we have

$$\Delta_s' = \mathbf{\Phi} \cdot \tilde{\boldsymbol{\beta}} \mathbf{\Phi} \quad , \tag{112}$$

where

$$\tilde{\boldsymbol{\beta}} := \frac{1}{2}(\mathbf{f}^T \mathbf{f} - \mathbf{1}) \tag{113}$$

represents a Green strain tensor for the microstructure. Various counterparts of $\tilde{\boldsymbol{\beta}}$ may be introduced by requiring from Δ_s' to remain form-invariant with respect to the chosen configuration. (This method for defining strain tensors has been discussed intensively in Haupt and Tsakmakis (1996)). For example, relative to the actual configuration,

$$\Delta_s' = \boldsymbol{\varphi} \cdot \boldsymbol{\beta} \boldsymbol{\varphi} \quad , \tag{114}$$

where

$$\boldsymbol{\beta} := \mathbf{f}^{T-1} \tilde{\boldsymbol{\beta}} \mathbf{f}^{-1} \equiv \frac{1}{2}(\mathbf{1} - \mathbf{v}^{-2}) \tag{115}$$

is Eulerian counterpart of $\tilde{\boldsymbol{\beta}}$, and is called Almansi strain tensor (for the microstructure). An equivalence class of strain tensors $\boldsymbol{\beta}_a$ may be generated by representing Δ_s' with respect to configurations induced by $\mathbf{F}_a$, $\mathbf{f}_a$,

$$\Delta_s' = \boldsymbol{\varphi}_a \cdot \boldsymbol{\beta}_a \boldsymbol{\varphi}_a \quad , \quad \boldsymbol{\beta}_a = \mathbf{f}_a^{T-1} \tilde{\boldsymbol{\beta}} \mathbf{f}_a^{-1} \quad . \tag{116}$$

In other words, Δ_s' is represented form-invariantly, with respect to configurations induced by $\mathbf{f}_a$, by means of the strain tensors $\boldsymbol{\beta}_a$, which are obtained by push-forward transformations of $\tilde{\boldsymbol{\beta}}$.

In order to interpret the micromorphic strain tensor $\tilde{\boldsymbol{\epsilon}}$, we enter into relation the deformations of the micro- and the macrocontinuum by introducing the scalar valued difference

$$\Delta_s = \Delta_s(\mathbf{X}, t) := \boldsymbol{\zeta} \cdot \mathbf{c} - \mathbf{Z} \cdot \mathbf{C} \quad . \tag{117}$$

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Then, by virtue of $(107)_1$ and $(108)_2$, we get relative to the reference configuration

$$\Delta_s = \mathbf{Z} \cdot \tilde{\boldsymbol{\epsilon}} \mathbf{C} \quad . \tag{118}$$

With respect to arbitrary configurations induced by $\mathbf{F}_a$, $\mathbf{f}_a$, we have

$$\Delta_s = \boldsymbol{\zeta}_a \cdot \boldsymbol{\epsilon}_a \mathbf{c}_a \quad . \tag{119}$$

The strain tensors $\boldsymbol{\epsilon}_a$ are defined by the push-forward transformations

$$\boldsymbol{\epsilon}_a = \mathbf{f}_a \tilde{\boldsymbol{\epsilon}} \mathbf{F}_a^{-1} \quad , \tag{120}$$

and form an equivalence class of micromorphic strain tensors. For the particular choice $\mathbf{f}_a = \mathbf{f}$, $\mathbf{F}_a = \mathbf{F}$, we obtain the Eulerian micromorphic strain tensor

$$\boldsymbol{\epsilon} := \mathbf{f} \tilde{\boldsymbol{\epsilon}} \mathbf{F}^{-1} = \mathbf{f}(\mathbf{f}^{-1}\mathbf{F} - \mathbf{1})\mathbf{F}^{-1} = \mathbf{1} - \mathbf{f}\mathbf{F}^{-1} \quad , \tag{121}$$

for which

$$\Delta_s = \boldsymbol{\zeta} \cdot \boldsymbol{\epsilon} \mathbf{c} \quad . \tag{122}$$

It is perhaps of interest to remark, that as $\boldsymbol{\Phi}(\mathbf{X})$, $\mathbf{Z}(\mathbf{X}) \in T_{\mathbf{X}}\mathcal{R}_{\mathcal{R}}$, $\tilde{\boldsymbol{\beta}}$, $\tilde{\boldsymbol{\epsilon}}$ can be imagined as second-order tensor fields on $\mathcal{R}_{\mathcal{R}}$, i.e. $\tilde{\boldsymbol{\beta}}(\mathbf{X}, t)$, $\tilde{\boldsymbol{\epsilon}}(\mathbf{X}, t) \ : \ T_{\mathbf{X}}\mathcal{R}_{\mathcal{R}} \times T_{\mathbf{X}}\mathcal{R}_{\mathcal{R}} \to \mathbb{R}$. Similarly, $\boldsymbol{\beta}$, $\boldsymbol{\epsilon}$ are second-order tensor fields on $\mathcal{R}_t$, i.e. $\boldsymbol{\beta}(\mathbf{x}, t)$, $\boldsymbol{\epsilon}(\mathbf{x}, t) \ : \ T_{\mathbf{x}}\mathcal{R}_t \times T_{\mathbf{x}}\mathcal{R}_t \to \mathbb{R}$ and so on.

As mentioned at the beginning of Sect. 5.1, all strain tensors may be introduced geometrically by considering appropriate scalar valued differences like Δ_s' and Δ_s. For example, the micromorphic strain tensors in the sets (73)–(78) can be obtained by considering, relative to the reference configuration, differences of the form

$$\boldsymbol{\varphi} \cdot \mathbf{c} - \boldsymbol{\Phi} \cdot \mathbf{C} \quad , \quad \boldsymbol{\xi} \cdot \boldsymbol{\varphi} - \boldsymbol{\Xi} \cdot \boldsymbol{\Phi} \quad , \quad \ldots \quad . \tag{123}$$

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5.1.2 Micromorphic curvature tensors

For interpreting geometrically the micromorphic curvature tensor $\tilde{\boldsymbol{\mathcal{K}}}$ (see Eq. $(72)_3$), suppose $\mathcal{X}_i'$, $i = 1, 2, 3$, to be mutually different material points of the microstructure attached to $\mathcal{X}$, which are not all in a plane, and are different that $\mathcal{X}$. Let $\boldsymbol{\Phi}_i = \boldsymbol{\Phi}_i(\mathbf{X}) \equiv \mathbf{X}_i'$ be the (position) vectors assigned to the pairs $(\mathcal{X}, \mathcal{X}_i')$. Clearly, $\boldsymbol{\Phi}_i$ are three time- and linear independent vectors (directors) at $\mathbf{X}$ ($\boldsymbol{\Phi}_i \in T_{\mathbf{X}}\mathcal{R}_{\mathcal{R}}$), which form a basis at $\mathbf{X}$, the reciprocal basis being $\boldsymbol{\Phi}^i = \boldsymbol{\Phi}^i(\mathbf{X})$, $\boldsymbol{\Phi}^i \cdot \boldsymbol{\Phi}_j = \delta^i_{\ j}$. On the other hand, $\boldsymbol{\Phi}_i$ may be thought to be tangent vectors to material lines of the microstructure at $\mathbf{X}$. Then, the reduced convective basis for the microcontinuum at $\mathbf{x}$ will be given by

$$\boldsymbol{\varphi}_i = \boldsymbol{\varphi}_i(\mathbf{x}, t) = \mathbf{f}\boldsymbol{\Phi}_i \in T_{\mathbf{x}}\mathcal{R}_t \quad , \tag{124}$$

with reciprocal basis

$$\boldsymbol{\varphi}^i = \boldsymbol{\varphi}^i(\mathbf{x}, t) = \mathbf{f}^{T^{-1}}\boldsymbol{\Phi}^i \in T_{\mathbf{x}}\mathcal{R}_t \quad . \tag{125}$$

In the particular case where $\boldsymbol{\Phi}_i = \mathbf{E}_i$, the basis $\{\boldsymbol{\varphi}_i\}$ will coincide with the basis $\{\boldsymbol{\varrho}_i\}$ (cf. Eg. (65)). However, in the following it is convenient to left $\{\boldsymbol{\Phi}_i\}$ arbitrary. Evidently, the basis fields $\boldsymbol{\varphi}_i(\mathbf{x}, t)$ and $\boldsymbol{\Phi}_i(\mathbf{X})$, induced by the convective coordinate systems in the microstructure, can be invoked to characterize the deformation of the microcontinuum. This is analogous to the macrocontinuum, the deformation of which can be reflected by the basis vector fields $\mathbf{E}_i(\mathbf{X})$ and $\mathbf{g}_i(\mathbf{x}, t)$, induced by the convective coordinate system $\{X^i\}$.

Next, we define a scalar-valued difference Δ_c by

$$\Delta_c = \Delta_c(\mathbf{X}, t) := \boldsymbol{\varphi}^1 \cdot (\nabla_{\mathcal{R}_t}\boldsymbol{\varphi}_2)[\mathbf{g}_3] - \boldsymbol{\Phi}^1 \cdot (\nabla_{\mathcal{R}_{\mathcal{R}}}\boldsymbol{\Phi}_2)[\mathbf{E}_3] \quad , \tag{126}$$

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where

$$\nabla_{\mathcal{R}_t}\boldsymbol{\varphi}_2 := \mathrm{grad}\boldsymbol{\varphi}_2 = \frac{\partial \boldsymbol{\varphi}_2}{\partial X^k} \otimes \mathbf{g}^k \quad , \tag{127}$$

$$\nabla_{\mathcal{R}_\mathcal{R}}\boldsymbol{\Phi}_2 := \mathrm{GRAD}\boldsymbol{\Phi}_2 = \frac{\partial \boldsymbol{\Phi}_2}{\partial X^k} \otimes \mathbf{E}^k \quad . \tag{128}$$

Δ_c is a measure for the deformation of the microstructure at a material point, which takes into account the deformation of the microstructure assigned to points in the neighborhood.

Our aim is to represent Δ_c first by means of the curvature tensor $\tilde{\boldsymbol{\mathcal{K}}}$. To this end we express $\boldsymbol{\varphi}^1$, $\boldsymbol{\varphi}_2$ and $\mathbf{g}_3$ in Eq. (126) in terms of $\boldsymbol{\Phi}^1$, $\boldsymbol{\Phi}_2$ and $\mathbf{E}_3$,

$$\begin{aligned}\Delta_c &= \mathbf{f}^{T-1}\boldsymbol{\Phi}^1 \cdot \left(\frac{\partial(\mathbf{f}\boldsymbol{\Phi}_2)}{\partial X^k} \otimes \mathbf{g}^k\right)[\mathbf{g}_3] - \boldsymbol{\Phi}^1 \cdot \left(\frac{\partial \boldsymbol{\Phi}_2}{\partial X^k} \otimes \mathbf{E}^k\right)[\mathbf{E}_3] \\ &= \boldsymbol{\Phi}^1 \cdot \left\{\mathbf{f}^{-1}\frac{\partial \mathbf{f}}{\partial X^k}\boldsymbol{\Phi}_2\right\}(\mathbf{g}^k \cdot \mathbf{g}_3) + \boldsymbol{\Phi}^1 \cdot \frac{\partial \boldsymbol{\Phi}_2}{\partial X^k}(\mathbf{g}^k \cdot \mathbf{g}_3) - \boldsymbol{\Phi}^1 \cdot \frac{\partial \boldsymbol{\Phi}_2}{\partial X^k}(\mathbf{E}^k \cdot \mathbf{E}_3) \quad ,\end{aligned} \tag{129}$$

or, in view of $\mathbf{g}^k \cdot \mathbf{g}_3 = \mathbf{E}^k \cdot \mathbf{E}_3$,

$$\Delta_c = \boldsymbol{\Phi}^1 \cdot \left(\mathbf{f}^{-1}\frac{\partial \mathbf{f}}{\partial X^k}\boldsymbol{\Phi}_2\right)(\mathbf{E}^k \cdot \mathbf{E}_3) = \left(\mathbf{f}^{-1}\frac{\partial \mathbf{f}}{\partial X^k} \otimes \mathbf{E}^k\right)[\boldsymbol{\Phi}^1, \boldsymbol{\Phi}_2, \mathbf{E}_3] \quad . \tag{130}$$

Hence, (cf. definition (72))$_3$

$$\Delta_c = \tilde{\boldsymbol{\mathcal{K}}}[\boldsymbol{\Phi}^1, \boldsymbol{\Phi}_2, \mathbf{E}_3] \quad . \tag{131}$$

It straightforward to verify that, with respect to the actual configuration,

$$\Delta_c = \boldsymbol{\mathcal{K}}[\boldsymbol{\varphi}^1, \boldsymbol{\varphi}_2, \mathbf{g}_3] \quad , \tag{132}$$

where

$$\boldsymbol{\mathcal{K}} := \frac{\partial \mathbf{f}}{\partial X^k}\mathbf{f}^{-1} \otimes \mathbf{g}^k \equiv (\mathrm{grad}\mathbf{f}) \diamond \mathbf{f}^{-1} \tag{133}$$

and

$$\boldsymbol{\mathcal{K}} = \mathcal{L}(\mathbf{f}, \mathbf{f}^{T-1}, \mathbf{F}^{T-1})[\tilde{\boldsymbol{\mathcal{K}}}] \quad . \tag{134}$$

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That means, $\boldsymbol{\mathcal{K}}$ can be derived from $\tilde{\boldsymbol{\mathcal{K}}}$ by push-forward transformation generated by $\mathcal{L}(\mathbf{f}, \mathbf{f}^{T-1}, \mathbf{F}^{T-1})$.

Result (134) can be generalized to arbitrary configurations induced by the deformations $\mathbf{F}_a$, $\mathbf{f}_a$. In fact, if we define

$$(\mathbf{g}_a)_k := \mathbf{F}_a \mathbf{E}_k \quad , \quad (\mathbf{g}_a)^k = \mathbf{F}_a^{T-1} \mathbf{E}^k \quad , \tag{135}$$

$$(\boldsymbol{\varphi}_a)_k := \mathbf{f}_a \boldsymbol{\Phi}_k \quad , \quad (\boldsymbol{\varphi}_a)^k = \mathbf{f}_a^{T-1} \boldsymbol{\Phi}^k \quad , \tag{136}$$

then an equivalence class of tensors $\boldsymbol{\mathcal{K}}_a$ can be constructed, such that

$$\Delta_c = \boldsymbol{\mathcal{K}}_a[(\boldsymbol{\varphi}_a)^1, (\boldsymbol{\varphi}_a)_2, (\mathbf{g}_a)_3] \tag{137}$$

and

$$\boldsymbol{\mathcal{K}}_a = \mathcal{L}(\mathbf{f}_a, \mathbf{f}_a^{T-1}, \mathbf{F}_a^{T-1})[\tilde{\boldsymbol{\mathcal{K}}}] \tag{138}$$

or

$$\tilde{\boldsymbol{\mathcal{K}}} = \mathcal{L}(\mathbf{f}_a^{-1}, \mathbf{f}_a^{T}, \mathbf{F}_a^{T})[\boldsymbol{\mathcal{K}}_a] \quad . \tag{139}$$

Especially, for $\mathbf{F}_a = \mathbf{F}$ and $\mathbf{f}_a = \mathbf{f}$,

$$\mathcal{K}_{mnr} = (\mathbf{f})_{mi} (\mathbf{f}^{T-1})_{nj} (\mathbf{F}^{T-1})_{rk} \tilde{\mathcal{K}}_{ijk} \quad , \tag{140}$$

$$\tilde{\mathcal{K}}_{mnr} = (\mathbf{f}^{-1})_{mi} (\mathbf{f}^{T})_{nj} (\mathbf{F}^{T})_{rk} \mathcal{K}_{ijk} \quad , \tag{141}$$

with respect to the Cartesian coordinate systems $\{X_i\}$ for $\mathcal{R}_{\mathcal{R}}$ and $\{x_i\}$ for $\mathcal{R}_t$, inducing the basis $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$, respectively. Evidently, $\tilde{\boldsymbol{\mathcal{K}}}(\mathbf{X}, t)$ is a third-order tensor on $\mathcal{R}_{\mathcal{R}}$, $\boldsymbol{\mathcal{K}}(\mathbf{x}, t)$ is a third-order tensor on $\mathcal{R}_t$, and so on.

Summarizing, by representing the scalar differences Δ_s', Δ_s and Δ_c in a form-invariant manner with respect to the chosen configuration, equivalent classes of strain tensors for the microcontinuum, micromorphic strain tensors and micromorphic curvature tensors can be obtained. This also provides the geometrical interpretation of the considered set of strain and curvature tensors.

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The geometrical interpretation of other sets of variables can be established in a similar fashion. However, in the remaining of this paper, and in Part II, we shall concentrate ourself on the equivalent classes of strain and curvature tensors produced by the set of variables in (72). This set of variables is also favorized by Eringen (1999, p. 15 and 16). For the tensors in these equivalent classes, we shall show how to find out associated time derivatives and dual stresses. For other sets of variables the approach will be quite similar.

5.2 Associated rates for strain and micromorphic curvature tensors

For every strain or micromorphic curvature tensor, a specific rate (associated rate) may be uniquely determined by requiring from $\dot{\Delta}'_s$, $\dot{\Delta}_s$, and $\dot{\Delta}_c$ to remain also form-invariant with respect to the chosen configuration (cf. Haupt and Tsakmakis (1996) for similar approaches in classical continuum mechanics). It is worth mentioning that this method for assigning to each strain or micromorphic curvature tensor an associated rate is independent of particular material properties.

To illustrate the method, we restrict attention to Δ'_s, take the material time derivative of (111), (114) or (116), and summarize the results as follows

$$\dot{\Delta}'_s = \boldsymbol{\Phi} \cdot \overset{\triangle}{\tilde{\boldsymbol{\beta}}} \boldsymbol{\Phi} \quad , \quad \overset{\triangle}{\tilde{\boldsymbol{\beta}}} := \dot{\tilde{\boldsymbol{\beta}}} \quad , \tag{142}$$

$$\dot{\Delta}'_s = \boldsymbol{\varphi} \cdot \overset{\triangle}{\boldsymbol{\beta}} \boldsymbol{\varphi} \quad , \quad \overset{\triangle}{\boldsymbol{\beta}} := \dot{\boldsymbol{\beta}} + \mathbf{l}^T \boldsymbol{\beta} + \boldsymbol{\beta} \mathbf{l} = \mathbf{d} \quad , \tag{143}$$

or generally

$$\dot{\Delta}'_s = \boldsymbol{\varphi}_a \cdot \overset{\triangle}{\boldsymbol{\beta}}_a \boldsymbol{\varphi}_a \quad , \quad \overset{\triangle}{\boldsymbol{\beta}}_a := \dot{\boldsymbol{\beta}}_a + (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1})^T \boldsymbol{\beta}_a + \boldsymbol{\beta}_a (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1}) \quad . \tag{144}$$

We refer to $\overset{\triangle}{\boldsymbol{\beta}}_a$ as the rate associated to $\boldsymbol{\beta}_a$. Obviously, $\dot{\Delta}'_s$, like Δ'_s, is rep-

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resented form-invariantly with respect to the chosen configuration. To each strain $\boldsymbol{\beta}_a$, operating in configurations generated by $\mathbf{F}_a$, $\mathbf{f}_a$, there is assigned a specific rate $\overset{\triangle}{\boldsymbol{\beta}}_a$, which represents a generalized Oldroyd time derivative. With respect to the reference configuration, the associated rate is the material time derivative, while relative to the actual configuration, the associated rate corresponds to a classical Oldroyd derivative. It is of interest, and also of practical importance, to remark that the associated rates $\overset{\triangle}{\boldsymbol{\beta}}_a$ arise from the rate $\dot{\tilde{\boldsymbol{\beta}}}$ by the same push-forward transformations as between $\boldsymbol{\beta}_a$ and $\tilde{\boldsymbol{\beta}}$ (cf. Eq. $(116)_2$,

$$\overset{\triangle}{\boldsymbol{\beta}}_a = \mathbf{f}_a^{T-1}\dot{\tilde{\boldsymbol{\beta}}}\mathbf{f}_a^{-1} \quad . \tag{145}$$

Without proof, we mention that under rigid body rotations superposed on the configuration induced by $\mathbf{F}_a$, $\mathbf{f}_a$, $\overset{\triangle}{\boldsymbol{\beta}}_a$ transforms like $\boldsymbol{\beta}_a$. Moreover, higher associated rates may be introduced by postulating the time rates $\ddot{\Delta}'_s$, $\dddot{\Delta}'_s$, ... to be form-invariant with respect to the chosen configuration as well. This means, our approach for introducing associated rates is the same as that one used to construct the well known Rivlin-Ericksen tensors (see Malvern, 1969, p. 403).

In a similar way, Eqs. (117)–(122) yield

$$\dot{\Delta}_s = \mathbf{Z}\cdot\overset{\triangle}{\tilde{\boldsymbol{\epsilon}}}\mathbf{C} \quad , \quad \overset{\triangle}{\tilde{\boldsymbol{\epsilon}}} := \dot{\tilde{\boldsymbol{\epsilon}}} \quad , \tag{146}$$

$$\dot{\Delta}_s = \boldsymbol{\zeta}\cdot\overset{\triangle}{\boldsymbol{\epsilon}}\mathbf{c} \quad , \quad \overset{\triangle}{\boldsymbol{\epsilon}} := \dot{\boldsymbol{\epsilon}} - \mathbf{l}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{L} = \mathbf{L} - \mathbf{l} = \mathbf{f}\dot{\tilde{\boldsymbol{\epsilon}}}\mathbf{F}^{-1} \quad , \tag{147}$$

or generally

$$\dot{\Delta}_s = \boldsymbol{\zeta}_a\cdot\overset{\triangle}{\boldsymbol{\epsilon}}_a\mathbf{c}_a \quad , \quad \overset{\triangle}{\boldsymbol{\epsilon}}_a := \dot{\boldsymbol{\epsilon}}_a - (\dot{\mathbf{f}}_a\mathbf{f}_a^{-1})\boldsymbol{\epsilon}_a + \boldsymbol{\epsilon}_a(\dot{\mathbf{F}}_a\mathbf{F}_a^{-1}) = \mathbf{f}_a\dot{\tilde{\boldsymbol{\epsilon}}}\mathbf{F}_a^{-1} \quad . \tag{148}$$

For the associated rates of micromorphic curvature tensors, we deduce from

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Eqs. (131)–(141)

$$\dot{\Delta}_c = \overset{\triangle}{\dot{\tilde{\boldsymbol{\mathcal{K}}}}}[\boldsymbol{\Phi}^1, \boldsymbol{\Phi}_2, \mathbf{E}_3] \quad , \quad \overset{\triangle}{\dot{\tilde{\boldsymbol{\mathcal{K}}}}} := \dot{\tilde{\boldsymbol{\mathcal{K}}}} \quad , \tag{149}$$

$$\dot{\Delta}_c = \overset{\triangle}{\boldsymbol{\mathcal{K}}}[\boldsymbol{\varphi}^1, \boldsymbol{\varphi}_2, \mathbf{g}_3] \quad , \quad \overset{\triangle}{\boldsymbol{\mathcal{K}}} := \dot{\boldsymbol{\mathcal{K}}} - \mathbf{l}\boldsymbol{\mathcal{K}} + \mathbf{l}^T \diamond \boldsymbol{\mathcal{K}} + \boldsymbol{\mathcal{K}}\mathbf{l} = \mathrm{grad}\mathbf{l}$$

$$= \mathcal{L}(\mathbf{f}, \mathbf{f}^{T-1}, \mathbf{F}^{T-1})[\dot{\tilde{\boldsymbol{\mathcal{K}}}}] \quad . \tag{150}$$

With respect to the orthonormal bases $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$, induced by the Cartesian coordinate systems $\{X_i\}$ for $\mathcal{R}_{\mathcal{R}}$ and $\{x_i\}$ for $\mathcal{R}_t$,

$$\dot{\tilde{\mathcal{K}}}_{ijm} = \left((\mathbf{f}^{-1})_{ir} \frac{\partial f_{rj}}{\partial X_m} \right)^{\cdot} = (\mathbf{f}^{-1})_{ir} (\mathbf{f}^T)_{jp} (\mathbf{F}^T)_{ms} \frac{\partial l_{rp}}{\partial x_s} \quad , \tag{151}$$

$$\overset{\triangle}{\mathcal{K}}_{kln} = (\mathbf{f})_{ki} (\mathbf{f}^{T-1})_{lj} (\mathbf{F}^{T-1})_{nm} \dot{\tilde{\mathcal{K}}}_{ijm} = \frac{\partial l_{kl}}{\partial x_n} \quad . \tag{152}$$

More generally,

$$\dot{\Delta}_c = \overset{\triangle}{\boldsymbol{\mathcal{K}}}_a[(\boldsymbol{\varphi}_a)^1, (\boldsymbol{\varphi}_a)_2, (\mathbf{g}_a)_3] \quad , \tag{153}$$

$$\overset{\triangle}{\boldsymbol{\mathcal{K}}}_a = \dot{\boldsymbol{\mathcal{K}}}_a - (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1}) \boldsymbol{\mathcal{K}}_a + (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1})^T \diamond \boldsymbol{\mathcal{K}}_a + \boldsymbol{\mathcal{K}}_a (\dot{\mathbf{F}}_a \mathbf{F}_a^{-1})$$

$$= \mathcal{L}(\mathbf{f}_a, \mathbf{f}_a^{T-1}, \mathbf{F}_a^{T-1})[\dot{\tilde{\boldsymbol{\mathcal{K}}}}] \quad . \tag{154}$$

Of course, higher rates for the micromorphic strain and the micromorphic curvature tensors may be introduced in a natural manner, by requiring from the rates $\ddot{\Delta}_s$, $\dddot{\Delta}_s$, ..., $\ddot{\Delta}_c$, $\dddot{\Delta}_c$, ... to be form-invariant with respect to the chosen configuration. Concluding, we remark that also for the micromorphic strain and the micromorphic curvature tensors, the associated rates transform, under rigid body rotations superposed on the configuration induced by $\mathbf{F}_a$, $\mathbf{f}_a$, as the tensors themselves.

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5.3 Dual stress tensors and their associated rates

Generally, strain and stress tensors are not a priori related to each other, raising the question of whether there exists a method to connect with each strain tensor a stress tensor independently of specific material properties. The stress power is commonly the convenient framework for answering this question. In the context of classical continuum mechanics, Hill (1968) developed the concept of conjugate variables on the basis of the stress power $\bar{w}$ (e.g. per unit volume of reference configuration). According to this, a stress tensor is postulated to be conjugate to a given strain tensor $\mathbf{e}$, if the scalar product of $\mathbf{t}$ with the material time derivative of $\mathbf{e}$ yields the stress power $\bar{w}$,

$$\bar{w} = \mathbf{t} \cdot \dot{\mathbf{e}} \quad . \tag{155}$$

Hill's conjugacy concept is meaningful only for Lagrangean variables. In fact, for the Eulerian Cauchy stress tensor a conjugate strain tensor does not exist (see e.g. Ogden, 1983, p.159). To overcome this difficulty in classical continuum mechanics, Haupt and Tsakmakis (1996) proposed the concept of dual variables. We shall adopt this concept and we shall extend it to cover micromorphic continua as well. For simplicity, we shall define the notion dual variables only with reference to the three classes of strain and micromorphic curvature tensors introduced in Sect. (5.1). But it is emphasized that for other classes one has to go on analogously.

We first concentrate ourself to the class of strain tensors $\boldsymbol{\beta}_a$ and discuss in full-length the main issues of the concept. Recall that for defining these tensors and their associated rates, use is made of the scalar quantities Δ'_s, $\dot{\Delta}'_s$, ... These scalars were required to be form-invariant with respect to the chosen

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configuration. Now, we consider the stress power w' defined by $(106)_2$, and its rates $\dot{w}'$, $\ddot{w}'$, ..., and require from these scalar quantities to be also form-invariant with respect to the chosen configuration. Keeping in mind (143), it follows that

$$w' = \boldsymbol{\sigma} \cdot \overset{\triangle}{\boldsymbol{\beta}} \tag{156}$$

relative to the actual configuration, or

$$w' = \boldsymbol{\sigma} \cdot \mathbf{f}^{T-1} \dot{\tilde{\boldsymbol{\beta}}} \mathbf{f}^{-1} = \mathbf{f}^{-1} \boldsymbol{\sigma} \mathbf{f}^{T-1} \cdot \dot{\tilde{\boldsymbol{\beta}}} \quad , \tag{157}$$

and therefore

$$w' = \tilde{\boldsymbol{\sigma}} \cdot \dot{\tilde{\boldsymbol{\beta}}} \tag{158}$$

relative to the reference configuration, where (cf. Eq. (A.46))

$$\tilde{\boldsymbol{\sigma}} = \mathbf{f}^{-1} \boldsymbol{\sigma} \mathbf{f}^{T-1} \tag{159}$$

represents a second Piola-Kirchhoff stress tensor for the microcontinuum. Substituting $\dot{\tilde{\boldsymbol{\beta}}}$ from (145) into (158),

$$w' = \mathbf{f}_a \tilde{\boldsymbol{\sigma}} \mathbf{f}_a^T \cdot \overset{\triangle}{\boldsymbol{\beta}}_a \quad . \tag{160}$$

On defining the stress tensor $\boldsymbol{\sigma}_a$ through

$$\boldsymbol{\sigma}_a := \mathbf{f}_a \tilde{\boldsymbol{\sigma}} \mathbf{f}_a^T \quad , \tag{161}$$

we obtain, with respect to configurations generated by $\mathbf{F}_a$, $\mathbf{f}_a$,

$$w' = \boldsymbol{\sigma}_a \cdot \overset{\triangle}{\boldsymbol{\beta}}_a \quad . \tag{162}$$

The latter reveals that w' exhibits a form-invariant representation with respect to every configuration induced by $\mathbf{F}_a$, $\mathbf{f}_a$. If $\mathbf{F}_a = \mathbf{f}_a = \mathbf{1}$ (reference configuration), then w' is given by (158), while for $\mathbf{F}_a = \mathbf{F}$ and $\mathbf{f}_a = \mathbf{f}$ (actual configuration) w' is given by (156). Pairs ($\boldsymbol{\beta}_a$,$\boldsymbol{\sigma}_a$) of strain and stress tensors

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satisfying (162) are said to be dual strain and stress tensors (with respect to w').

To determine the time derivative which is associated with the stress tensor $\boldsymbol{\sigma}_a$, we take the material time derivative of w' in (158),

$$\dot{w}' = \dot{\tilde{\boldsymbol{\sigma}}} \cdot \dot{\tilde{\boldsymbol{\beta}}} + \tilde{\boldsymbol{\sigma}} \cdot \ddot{\tilde{\boldsymbol{\beta}}} \quad . \tag{163}$$

Using the stress tensors $\boldsymbol{\sigma}_a$ and the strain tensors $\boldsymbol{\beta}_a$, the term $\tilde{\boldsymbol{\sigma}} \cdot \ddot{\tilde{\boldsymbol{\beta}}}$ can be written in the form

$$\tilde{\boldsymbol{\sigma}} \cdot \ddot{\tilde{\boldsymbol{\beta}}} = \boldsymbol{\sigma}_a \cdot \overset{\triangle\triangle}{\boldsymbol{\beta}}_a \quad , \tag{164}$$

where

$$\overset{\triangle\triangle}{\boldsymbol{\beta}}_a = \mathbf{f}_a^{T-1} \ddot{\tilde{\boldsymbol{\beta}}} \mathbf{f}_a^{-1} = (\overset{\triangle}{\boldsymbol{\beta}}_a)^{\cdot} + (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1})^T \overset{\triangle}{\boldsymbol{\beta}}_a + \overset{\triangle}{\boldsymbol{\beta}}_a (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1}) \quad . \tag{165}$$

Clearly, Equation (164) represents a scalar, which is expressible form-invariantly with respect to the chosen configuration. Consequently, the scalar

$$w'_{incr.} := \dot{\tilde{\boldsymbol{\sigma}}} \cdot \dot{\tilde{\boldsymbol{\beta}}} \quad , \tag{166}$$

which is called the incremental stress power (per unit volume of the reference configuration) for the microcontinuum, must also be form-invariant with respect to the chosen configuration. Because,

$$w'_{incr.} = \dot{\tilde{\boldsymbol{\sigma}}} \cdot \mathbf{f}_a^T \overset{\triangle}{\boldsymbol{\beta}}_a \mathbf{f}_a = \mathbf{f}_a \dot{\tilde{\boldsymbol{\sigma}}} \mathbf{f}_a^T \cdot \overset{\triangle}{\boldsymbol{\beta}}_a \quad , \tag{167}$$

$\dot{w}'$ will be form-invariant,

$$w'_{incr.} = \overset{\triangledown}{\boldsymbol{\sigma}}_a \cdot \overset{\triangle}{\boldsymbol{\beta}}_a \quad , \tag{168}$$

whenever

$$\overset{\triangledown}{\boldsymbol{\sigma}}_a := \dot{\boldsymbol{\sigma}}_a - (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1}) \boldsymbol{\sigma}_a - \boldsymbol{\sigma}_a (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1})^T = \mathbf{f}_a \dot{\tilde{\boldsymbol{\sigma}}} \mathbf{f}_a^T \quad . \tag{169}$$

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If $\mathbf{F}_a = \mathbf{F}$, $\mathbf{f}_a = \mathbf{f}$ (actual configuration), then

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^T \quad . \tag{170}$$

This way, we can associate with each stress tensor $\boldsymbol{\sigma}_a$ a specific time derivative $\overset{\triangledown}{\boldsymbol{\sigma}}_a$, which is referred to as associated rate. Concerning the properties of the rates $\overset{\triangledown}{\boldsymbol{\sigma}}_a$, it is easy to prove that under rigid body rotations superposed on the configurations induced by $\mathbf{F}_a$, $\mathbf{f}_a$, $\overset{\triangledown}{\boldsymbol{\sigma}}_a$ transform like $\boldsymbol{\sigma}_a$. Also, Eqs. (169) and (161) suggest that $\overset{\triangledown}{\boldsymbol{\sigma}}_a$ arises from $\dot{\tilde{\boldsymbol{\sigma}}}$ by the same push-forward transformation as between $\boldsymbol{\sigma}_a$ and $\tilde{\boldsymbol{\sigma}}$.

The method for determining stress tensors dual to the micromorphic strains $\boldsymbol{\epsilon}_a$, as well as associated rates, is quite similar. The main results read as follows,

$$w = \tilde{\mathbf{S}} \cdot \dot{\tilde{\boldsymbol{\epsilon}}} = \mathbf{S} \cdot \overset{\triangle}{\boldsymbol{\epsilon}} = \mathbf{S}_a \cdot \overset{\triangle}{\boldsymbol{\epsilon}}_a \quad , \tag{171}$$

with (cf. (A.46))

$$\mathbf{S} = \mathbf{f}^{T-1}\tilde{\mathbf{S}}\mathbf{F}^T \quad , \tag{172}$$

or generally

$$\mathbf{S}_a := \mathbf{f}_a^{T-1}\tilde{\mathbf{S}}\mathbf{F}_a^T \quad . \tag{173}$$

$\mathbf{S}_a$ is called dual to $\boldsymbol{\epsilon}_a$. The associated rates $\overset{\triangledown}{\mathbf{S}}_a$ are given by

$$\overset{\triangledown}{\mathbf{S}}_a = \mathbf{f}_a^{T-1}\dot{\tilde{\mathbf{S}}}\mathbf{F}_a^T = \dot{\mathbf{S}}_a + (\dot{\mathbf{f}}_a\mathbf{f}_a^{-1})^T\mathbf{S}_a - \mathbf{S}_a(\dot{\mathbf{F}}_a\mathbf{F}_a^{-1})^T \quad . \tag{174}$$

For $\mathbf{F}_a = \mathbf{F}$, $\mathbf{f}_a = \mathbf{f}$ (actual configuration)

$$\overset{\triangledown}{\mathbf{S}} = \mathbf{f}^{T-1}\dot{\tilde{\mathbf{S}}}\mathbf{F}^T = \dot{\mathbf{S}} + \mathbf{l}^T\mathbf{S} - \mathbf{S}\mathbf{L}^T \quad . \tag{175}$$

Finally, for the double stress tensors dual to the micromorphic curvature tensors, we have

$$w_c = \tilde{\boldsymbol{\mathcal{S}}} \cdot \dot{\tilde{\boldsymbol{\mathcal{K}}}} = \boldsymbol{\mathcal{S}} \cdot \overset{\triangle}{\boldsymbol{\mathcal{K}}} = \boldsymbol{\mathcal{S}}_a \cdot \overset{\triangle}{\boldsymbol{\mathcal{K}}}_a \quad , \tag{176}$$

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with (cf. (A.46))

$$\boldsymbol{\mathcal{S}} = \mathcal{L}(\mathbf{f}^{T-1}, \mathbf{f}, \mathbf{F})[\tilde{\boldsymbol{\mathcal{S}}}] \quad , \tag{177}$$

or generally

$$\boldsymbol{\mathcal{S}}_a := \mathcal{L}(\mathbf{f}_a^{T-1}, \mathbf{f}_a, \mathbf{F}_a)[\tilde{\boldsymbol{\mathcal{S}}}] \quad . \tag{178}$$

Micromorphic curvature tensors $\boldsymbol{\mathcal{K}}_a$ and double stress tensors $\boldsymbol{\mathcal{S}}_a$ are said to be dual to each other. The associated rates $\overset{\triangledown}{\boldsymbol{\mathcal{S}}}_a$ are given by

$$\begin{aligned} \overset{\triangledown}{\boldsymbol{\mathcal{S}}}_a &:= \mathcal{L}(\mathbf{f}_a^{T-1}, \mathbf{f}_a, \mathbf{F}_a)[\dot{\tilde{\boldsymbol{\mathcal{S}}}}] \\ &= \dot{\boldsymbol{\mathcal{S}}}_a + (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1})\boldsymbol{\mathcal{S}}_a - \boldsymbol{\mathcal{S}}_a \diamond (\dot{\mathbf{f}}_a \mathbf{f}_a^{-1})^T - \boldsymbol{\mathcal{S}}_a(\dot{\mathbf{F}}_a \mathbf{F}_a^{-1}) \quad . \end{aligned} \tag{179}$$

In particular, for $\mathbf{F}_a = \mathbf{F}$ and $\mathbf{f}_a = \mathbf{f}$,

$$\overset{\triangledown}{\boldsymbol{\mathcal{S}}} = \mathcal{L}(\mathbf{f}^{T-1}, \mathbf{f}, \mathbf{F})[\dot{\tilde{\boldsymbol{\mathcal{S}}}}] = \dot{\boldsymbol{\mathcal{S}}} + \mathbf{l}\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}} \diamond \mathbf{l}^T - \boldsymbol{\mathcal{S}}\mathbf{L} \quad . \tag{180}$$

We shall employ in Part II the strain and micromorphic curvature tensors, as well as their corresponding dual stress tensors to formulate micromorphic plasticity.

A Derivation of balance laws for momentum and moment of momentum according to Mindlin's approach

We shall extend, from small to finite deformations, the approach of Mindlin for establishing the balance laws of momentum and moment of momentum. We start from the definition of the microcontinuum, the kinematical relations, the equations describing conservation of mass (see Sect. 4.1) and Hamilton's principle. Latter is considered only for conservative mechanical systems and is equivalent to the local equations of motion, provided all functions involved are sufficiently smooth. However, although the local equations of motion will be

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derived in the framework of conservative systems, they still apply to every mechanical system governed by similar higher order stresses. Consequently, they can be utilized for the elastic-plastic materials addressed in Part II. Another important reason for employing this approach is to derive rigorously the local equations of motion for the macrocontinuum by using appropriate averages of the microcontinuum, as it shown in the remainder.

Before going any further we would like to mention two articles, which come into our knowledge by one of the reviewers. The first one is of Germain (1973) and deals with the virtual power method to derive relevant field equations among others also for micromorphic materials. According to this method, which has been recently applied by Forest and Sievert (2003), it is not necessary for the material behavior to be hyperelastic, which make it appear to be an advantage, or to be more general, than other energetical principles dealing with hyperelastic material behavior. However, the virtual power method requires some a priori knowledge, which may be available only through experience. The second article goes back on Chen (2007) and proposes to gain the field equations from Hamilton's principle under the framework of Poison bracket formalisms. There are some similarities between this article and our work, in what concerns the weight functions in the averaging procedures. But otherwise the article of Chen (2007) relies upon Eringen's definition on microcontinuum, so that differences exist, e.g., in the definition of the spin inertia tensor.

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A.1 Hamilton's principle for pure elastic materials

Hamilton's principle for independent variations $\delta\mathbf{u}$ and $\delta\mathbf{f}$ of displacement $\mathbf{u} := \mathbf{x} - \mathbf{X}$ and microdeformation $\mathbf{f}$, and fixed times t_0, t_1, reads

$$\delta\left(\int_{t_0}^{t_1} K dt + \int_{t_0}^{t_1} W_e dt\right) = \delta \int_{t_0}^{t_1} W dt \quad , \tag{A.1}$$

or

$$\int_{t_0}^{t_1} \delta K dt + \int_{t_0}^{t_1} \delta W_e dt = \int_{t_0}^{t_1} \delta W dt \quad . \tag{A.2}$$

Here, K and W_e are the total kinetic energy and the work done by external forces for the macrocontinuum, respectively, while W designates the work of the internal forces. Variations $\delta\mathbf{u}$ and $\delta\mathbf{f}$, as well as quantities K, W_e, W are defined in the following sections. In doing this, it suffices to concentrate ourself on the Cartesian coordinates $\{x_i\}$ for $\mathcal{R}_t$ and $\{X_i\}$ for $\mathcal{R}_\mathcal{R}$, inducing the orthonormal bases $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_i\}$, respectively. All tensorial components are referred to these coordinate systems.

A.2 Variation of $\mathbf{u}$ *and* $\mathbf{f}$

Let $\partial\mathcal{R}_t$ be the boundary of $\mathcal{R}_t$, and denote by $\partial\mathcal{R}_t^{u_i}$ the part of $\partial\mathcal{R}_t$ where the displacement components u_i are prescribed, $u_i = \bar{u}_i$ on $\partial\mathcal{R}_t^{u_i}$. Variations $\delta\mathbf{u} = \delta\mathbf{u}(\mathbf{x}, t)$ are defined to be, as sufficiently as needed, smooth functions vanishing on $\partial\mathcal{R}_t^{u_i}$, i.e. $\delta u_i = 0$ on $\partial\mathcal{R}_t^{u_i}$. Moreover, $\delta\mathbf{u}$ have to vanish everywhere at times t_0 and t_1, $\delta\mathbf{u} \equiv \mathbf{0}$ in $\mathcal{R}_{t_0}$ or $\mathcal{R}_{t_1}$.

Let $\partial\mathcal{R}_t^{f_{ij}}$ be the part of $\partial\mathcal{R}_t$ where components of the microdeformation f_{ij} are prescribed, $f_{ij} = \bar{f}_{ij}$ on $\partial\mathcal{R}_t^{f_{ij}}$. Variations $\delta\mathbf{f} = \delta\mathbf{f}(\mathbf{x}, t)$ are defined to be, as sufficiently as needed, smooth functions vanishing on $\partial\mathcal{R}_t^{f_{ij}}$, i.e. $\delta f_{ij} = 0$ on

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$\partial\mathcal{R}_t^{f_{ij}}$, and to satisfy $\delta\mathbf{f} = \mathbf{0}$ in in $\mathcal{R}_{t_0}$ or $\mathcal{R}_{t_1}$.

A.3 Kinetic energy of the macroscopic continuum

The total kinetic energy of the macrocontinuum is given by (cf. Mindlin (1964))

$$K := \int_{\mathcal{R}_\mathcal{R}} T dV \equiv \int_{\mathcal{R}_t} \tau dv \quad , \tag{A.3}$$

where T is the density of kinetic energy of the macroscopic continuum at $\mathbf{X}$ per unit volume of the reference configuration of the macroscopic continuum, and τ is the density of kinetic energy of the macroscopic continuum at $\mathbf{x}$ per unit volume of the actual configuration of the macroscopic continuum. We define

$$\begin{aligned} T = T(\mathbf{X}, t) &:= \frac{1}{2}\langle \varrho'_\mathcal{R}(\mathbf{X}, \mathbf{X}')\rangle_{\mathcal{R}_\mathcal{R}} \langle (\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_\mathcal{R}} \\ &= \frac{\varrho_\mathcal{R}}{2}\langle (\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_\mathcal{R}} \quad , \end{aligned} \tag{A.4}$$

$$\begin{aligned} \tau = \tau(\mathbf{x}, t) &:= \frac{1}{2}\langle \varrho'(\mathbf{x}, \mathbf{x}', t)\rangle_{\mathcal{R}_t} \langle (\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_t} \\ &= \frac{\varrho}{2}\langle (\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_t} \quad , \end{aligned} \tag{A.5}$$

where $\langle(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_\mathcal{R}}$ and $\langle(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_t}$ are respectively averages of squares of velocities to be defined appropriately, and use has been made of (83) and (85). In order for definitions (A.4) and (A.5) to be compatible with (A.3), we have to prove that

$$T\,\mathrm{d}V = \tau\,\mathrm{d}v \quad . \tag{A.6}$$

In addition, we shall show that T and τ obey the representations

$$T = \frac{\varrho_\mathcal{R}}{2}(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}) + \frac{\varrho_\mathcal{R}}{2}\dot{\mathbf{f}}^T\dot{\mathbf{f}} \cdot \mathbf{\Theta} \quad , \tag{A.7}$$

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$$\tau = \frac{\varrho}{2}(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}) + \frac{\varrho}{2}\mathbf{l}^T\mathbf{l} \cdot \boldsymbol{\theta} \quad , \tag{A.8}$$

where $\boldsymbol{\Theta}$ and $\boldsymbol{\theta}$ are Lagrangean and Eulerian second-order tensors, respectively, to be given below. They fulfill the transformation law

$$\boldsymbol{\theta} = \mathbf{f}\boldsymbol{\Theta}\mathbf{f}^T \quad . \tag{A.9}$$

Proceeding to prove (A.6)–(A.9), we consider two possibilities for the material point $\mathbf{X}' = \mathbf{0}$ of the microcontinuum at $\mathbf{X}$.

In the first possibility, we assume this point to be the volume centroid of the microcontinuum, i.e.

$$\int_{\mathcal{R}'_{\mathcal{R}}} \mathbf{X}' \, \mathrm{d}V' = \mathbf{0} \quad . \tag{A.10}$$

As the deformation of the microcontinuum is homogeneous, we have $\mathbf{X}' = \mathbf{f}^{-1}\mathbf{x}'$, so that (A.10) is equivalent to

$$\frac{1}{\det \mathbf{f}}\mathbf{f}^{-1} \int_{\mathcal{R}'_t} \mathbf{x}' \, \mathrm{d}v' = \mathbf{0} \quad , \tag{A.11}$$

where (79) has been taken into account. Since $\mathbf{f}$ is a regular mapping, the linear equation (A.11) possesses only the trivial solution

$$\int_{\mathcal{R}'_t} \mathbf{x}' \, \mathrm{d}v' = \mathbf{0} \quad . \tag{A.12}$$

In other words, the material point of the microcontinuum which is volume centroid in the reference configuration remains volume centroid in the actual configuration as well. From (A.12);

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{R}'_t} \mathbf{x}' \, \mathrm{d}v' = \int_{\mathcal{R}'_t} (\dot{\mathbf{x}}' + \mathbf{x}'(\mathrm{tr}\mathbf{l}) \, \mathrm{d}v' = \mathbf{0} \quad , \tag{A.13}$$

and hence

$$\int_{\mathcal{R}'_t} \dot{\mathbf{x}}' \, \mathrm{d}v' = \mathbf{0} \quad \text{or} \quad \int_{\mathcal{R}'_{\mathcal{R}}} \dot{\mathbf{x}}' \, \mathrm{d}V' = \mathbf{0} \quad . \tag{A.14}$$

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Now, we define

$$\langle(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_{\mathcal{R}}} := \frac{1}{V'}\int_{\mathcal{R}'_{\mathcal{R}}}(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}dV' \quad , \tag{A.15}$$

$$\langle(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\rangle_{\mathcal{R}_t} := \frac{1}{v'}\int_{\mathcal{R}'_t}(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}dv' \quad . \tag{A.16}$$

It follows from (A.4), (A.5), by virtue of (A.14), that

$$\begin{aligned} T = T(\mathbf{X},t) &= \frac{\varrho_{\mathcal{R}}(\mathbf{X})}{2}\left\{\frac{1}{V'}\int_{\mathcal{R}'_{\mathcal{R}}}(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\,\mathrm{d}V'\right\} \\ &= \frac{\varrho_{\mathcal{R}}}{2}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) + \frac{\varrho_{\mathcal{R}}}{2}\left\{\frac{1}{V'}\int_{\mathcal{R}'_{\mathcal{R}}}(\dot{\mathbf{x}}'\cdot\dot{\mathbf{x}}')\,\mathrm{d}V'\right\} \quad , \end{aligned} \tag{A.17}$$

and

$$\begin{aligned} \tau = \tau(\mathbf{x},t) &= \frac{\varrho(\mathbf{x},t)}{2}\left\{\frac{1}{v'}\int_{\mathcal{R}'_t}(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\,\mathrm{d}v'\right\} \\ &= \frac{\varrho}{2}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) + \frac{\varrho}{2}\left\{\frac{1}{v'}\int_{\mathcal{R}'_t}(\dot{\mathbf{x}}'\cdot\dot{\mathbf{x}}')\,\mathrm{d}v'\right\} \quad . \end{aligned} \tag{A.18}$$

Note in passing, that (A.17) corresponds to the kinetic energy density proposed by Mindlin (1964), when "the material is composed wholly of unit cells". Also, it is not difficult (by using (79)–(86)) to verify that (A.17) and (A.18) satisfy the equivalence relation (A.6).

In order to recast T and τ, we introduce the second-order tensors $\mathbf{\Theta}$ and $\boldsymbol{\theta}$ by the volume averages

$$\mathbf{\Theta} = \mathbf{\Theta}(\mathbf{X}) := \frac{1}{V'}\int_{\mathcal{R}'_{\mathcal{R}}}(\mathbf{X}'\otimes\mathbf{X}')\,\mathrm{d}V' = \mathbf{\Theta}^T \quad , \tag{A.19}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{x},t) := \frac{1}{v'}\int_{\mathcal{R}'_t}(\mathbf{x}'\otimes\mathbf{x}')\,\mathrm{d}v' = \boldsymbol{\theta}^T \quad . \tag{A.20}$$

Clearly, these definitions satisfy the transformation rule (A.9), and recalling that $\dot{\mathbf{x}}' = \mathbf{l}\mathbf{x}'$, we find

$$\frac{1}{V'}\int_{\mathcal{R}'_{\mathcal{R}}}(\dot{\mathbf{x}}'\cdot\dot{\mathbf{x}}')\,\mathrm{d}V' \equiv \frac{1}{v'}\int_{\mathcal{R}'_t}(\dot{\mathbf{x}}'\cdot\dot{\mathbf{x}}')\,\mathrm{d}v' = \dot{\mathbf{f}}^T\dot{\mathbf{f}}\cdot\mathbf{\Theta} = \mathbf{l}^T\mathbf{l}\cdot\boldsymbol{\theta} \quad . \tag{A.21}$$

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After inserting in (A.17) and (A.18), we get Eqs. (A.7) and (A.8), which proves the assertion in the context of the first possibility.

According to the second possibility, we assume the point $\mathbf{X}' = \mathbf{0}$ of the microcontinuum at $\mathbf{X}$, to be the center of mass, i.e.

$$\int_{\mathcal{R}'_{\mathcal{R}}} \mathbf{X}' \varrho'_{\mathcal{R}} \, \mathrm{d}V' = \mathbf{0} \quad , \tag{A.22}$$

which is equivalent to

$$\int_{\mathcal{R}'_t} \mathbf{x}' \varrho' \, \mathrm{d}v' = \mathbf{0} \quad . \tag{A.23}$$

That means, the material point of the microcontinuum which is center of mass in the reference configuration remains center of mass in every configuration during the motion of the material body. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{R}'_t} \mathbf{x}' \varrho' \, \mathrm{d}v' = \int_{\mathcal{R}'_t} \dot{\mathbf{x}}' \varrho' \, \mathrm{d}v' = \mathbf{0} \quad . \tag{A.24}$$

(These results go back to Eringen (see e.g. Eringen, 1999, p. 31).)

Now, we define

$$\begin{aligned} &\langle (\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot} \rangle_{\mathcal{R}_{\mathcal{R}}} \\ &\quad := \frac{1}{\int_{\mathcal{R}'_{\mathcal{R}}} \varrho'_{\mathcal{R}}(\mathbf{X},\mathbf{X}') dV'} \int_{\mathcal{R}'_{\mathcal{R}}} (\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot} \varrho'_{\mathcal{R}}(\mathbf{X},\mathbf{X}') dV' \quad , \end{aligned} \tag{A.25}$$

$$\begin{aligned} &\langle (\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot} \rangle_{\mathcal{R}_t} \\ &\quad := \frac{1}{\int_{\mathcal{R}'_t} \chi(\mathbf{x},t) \varrho'(\mathbf{x},\mathbf{x}',t) dv'} \int_{\mathcal{R}'_t} \chi(\mathbf{x},t) [(\mathbf{x}+\mathbf{x}')^{\cdot} \cdot (\mathbf{x}+\mathbf{x}')^{\cdot}] \varrho'(\mathbf{x},\mathbf{x}',t) dv' \quad . \end{aligned} \tag{A.26}$$

After inserting in (A.4), (A.5), and applying (A.22)–(A.24) and the relations (cf. Eqs. (83), (85))

$$\varrho_{\mathcal{R}} V' = \int_{\mathcal{R}'_{\mathcal{R}}} \varrho'_{\mathcal{R}} dV' \quad , \quad \varrho v' = \int_{\mathcal{R}'_t} \chi \varrho' dv' \quad , \tag{A.27}$$

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we conclude that

$$T = \frac{\varrho_{\mathcal{R}}}{2}\left\{\frac{1}{\varrho_{\mathcal{R}}V'}\int_{\mathcal{R}'_{\mathcal{R}}}(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\varrho'_{\mathcal{R}}dV'\right\}$$
$$= \frac{\varrho_{\mathcal{R}}}{2}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) + \frac{\varrho_{\mathcal{R}}}{2}\left\{\frac{1}{\varrho_{\mathcal{R}}V'}\int_{\mathcal{R}'_{\mathcal{R}}}(\dot{\mathbf{x}}'\cdot\dot{\mathbf{x}}')\varrho'_{\mathcal{R}}dV'\right\} \tag{A.28}$$

and

$$\tau = \frac{\varrho}{2}\left\{\frac{1}{\varrho v'}\int_{\mathcal{R}'_t}\chi(\mathbf{x}+\mathbf{x}')^{\cdot}\cdot(\mathbf{x}+\mathbf{x}')^{\cdot}\varrho' dv'\right\}$$
$$= \frac{\varrho}{2}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}) + \frac{\varrho}{2}\left\{\frac{1}{\varrho v'}\int_{\mathcal{R}'_t}\chi(\dot{\mathbf{x}}'\cdot\dot{\mathbf{x}}')\varrho' dv'\right\} \quad . \tag{A.29}$$

It is easy to confirm (by using relations (79)–(86)), on the one hand, that (A.28) and (A.29) satisfy Eq. (A.6), and on the other hand, that T and τ may be represented by (A.7), (A.8), provided $\boldsymbol{\Theta}$ and $\boldsymbol{\theta}$ are now defined by the mass averages

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}(\mathbf{X}) := \frac{1}{\varrho_{\mathcal{R}}(\mathbf{X})V'(\mathbf{X})}\int_{\mathcal{R}'_{\mathcal{R}}}(\mathbf{X}'\otimes\mathbf{X}')\varrho'_{\mathcal{R}}(\mathbf{X},\mathbf{X}')\,\mathrm{d}V' = \boldsymbol{\Theta}^T \quad , \tag{A.30}$$
$$\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{x},t) := \frac{1}{\varrho(\mathbf{x},t)v'(\mathbf{x},t)}\int_{\mathcal{R}'_t}\chi(\mathbf{x},t)(\mathbf{x}'\otimes\mathbf{x}')\varrho'(\mathbf{x},\mathbf{x}',t)\,\mathrm{d}v' = \boldsymbol{\theta}^T \quad , \tag{A.31}$$

which proves the assertion in the context of the second possibility.

Note that the second-order tensors $\boldsymbol{\Theta}$ and $\boldsymbol{\theta}$ in Eqs. (A.30), (A.31) correspond, but are not equal, to the microinertia tensors introduced by Eringen (see e.g. Eringen, 1999, p. 32).

From (A.3) and (A.7)

$$\int_{t_0}^{t_1}\delta K\,\mathrm{d}t = -\int_{t_0}^{t_1}\left\{\int_{\mathcal{R}_{\mathcal{R}}}(\varrho_{\mathcal{R}}\ddot{\mathbf{x}}\cdot\delta\mathbf{u} + \varrho_{\mathcal{R}}\boldsymbol{\Theta}\cdot\ddot{\mathbf{f}}^T\delta\mathbf{f})\,\mathrm{d}V\right\}\mathrm{d}t \quad , \tag{A.32}$$

where use is made of partial integration and of the fact that $\delta\mathbf{u}$ and $\delta\mathbf{f}$ vanish at times t_0 and t_1.

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Similar to Eringen (1999, p. 33), we define a spin inertia second-order tensor $\boldsymbol{\lambda}$ by

$$\boldsymbol{\lambda}(\mathbf{x},t) := \frac{1}{v'}\int_{\mathcal{R}'_t} (\ddot{\mathbf{x}}' \otimes \mathbf{x}')\,\mathrm{d}v' \quad , \tag{A.33}$$

if $\mathbf{X}' = \mathbf{0}$ is volume centroid, or by

$$\boldsymbol{\lambda}(\mathbf{x},t) := \frac{1}{\varrho(\mathbf{x},t)v'}\int_{\mathcal{R}'_t} \chi(\mathbf{x},t)(\ddot{\mathbf{x}}' \otimes \mathbf{x}')\varrho'(\mathbf{x},\mathbf{x}',t)\,\mathrm{d}v' \quad , \tag{A.34}$$

if $\mathbf{X}' = \mathbf{0}$ is center of mass. Since the microcontinuum undergoes homogeneous deformations, the relation $\ddot{\mathbf{x}}' = (\dot{\mathbf{l}} + \mathbf{ll})\mathbf{x}'$ applies, so that in every case

$$\boldsymbol{\lambda} = (\dot{\mathbf{l}} + \mathbf{ll})\boldsymbol{\theta} \quad , \tag{A.35}$$

in view of (A.20) and (A.31). Further, it is readily shown that

$$\boldsymbol{\Theta}\cdot\ddot{\mathbf{f}}^T\delta\mathbf{f} = (\dot{\mathbf{l}} + \mathbf{ll})\boldsymbol{\theta}\cdot(\delta\mathbf{f})\mathbf{f}^{-1} \quad , \tag{A.36}$$

so that, after substitution in (A.32),

$$\int_{t_0}^{t_1} \delta K\,\mathrm{d}t = -\int_{t_0}^{t_1}\left\{\int_{\mathcal{R}_t}[\varrho\ddot{\mathbf{x}}\cdot\delta\mathbf{u} + \varrho\boldsymbol{\lambda}\cdot(\delta\mathbf{f})\mathbf{f}^{-1}]\,\mathrm{d}v\right\}\mathrm{d}t \quad , \tag{A.37}$$

or,

$$\int_{t_0}^{t_1} \delta K\,\mathrm{d}t = -\int_{t_0}^{t_1}\left\{\int_{\mathcal{R}_t}[\varrho\ddot{x}_i\delta u_i + \varrho\lambda_{ij}(\delta f_{im})(\mathbf{f}^{-1})_{mj}]\,\mathrm{d}v\right\}\mathrm{d}t \quad . \tag{A.38}$$

A.4 Work of the internal forces

The work of the internal forces will be stored in the material as potential energy W,

$$W := \int_{\mathcal{R}_\mathcal{R}} \varrho_\mathcal{R}\Psi dV \equiv \int_{\mathcal{R}_t} \varrho\Psi dv \quad , \tag{A.39}$$

with Ψ being given by (71). We define the second-order stress tensors

$$\tilde{\mathbf{S}} := \varrho_\mathcal{R}\frac{\partial\tilde{\Psi}}{\partial\tilde{\boldsymbol{\epsilon}}} \quad , \quad \tilde{\boldsymbol{\sigma}} := \varrho_\mathcal{R}\frac{\partial\tilde{\Psi}}{\partial\tilde{\boldsymbol{\beta}}} = \tilde{\boldsymbol{\sigma}}^T \tag{A.40}$$

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and the third-order double stress tensor

$$\tilde{\boldsymbol{\mathcal{S}}} := \varrho_{\mathcal{R}} \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{\mathcal{K}}}} \quad . \tag{A.41}$$

It follows from (A.39), that

$$\delta W = \int_{\mathcal{R}_{\mathcal{R}}} (\tilde{\mathbf{S}} \cdot \delta \tilde{\boldsymbol{\epsilon}} + \tilde{\boldsymbol{\sigma}} \cdot \delta \tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\mathcal{S}}} \cdot \delta \tilde{\boldsymbol{\mathcal{K}}}) \, \mathrm{d}V \quad . \tag{A.42}$$

After some lengthy, but otherwise straightforward calculations,

$$\tilde{\mathbf{S}} \cdot \delta \tilde{\boldsymbol{\epsilon}} = \mathbf{S} \cdot \{(\delta \mathbf{F})\mathbf{F}^{-1} - (\delta \mathbf{f})\mathbf{f}^{-1}\} \quad , \tag{A.43}$$

$$\tilde{\boldsymbol{\sigma}} \cdot \delta \tilde{\boldsymbol{\beta}} = \boldsymbol{\sigma} \cdot \{(\delta \mathbf{f})\mathbf{f}^{-1}\} \quad , \tag{A.44}$$

$$\tilde{\boldsymbol{\mathcal{S}}} \cdot \delta \tilde{\boldsymbol{\mathcal{K}}} = \boldsymbol{\mathcal{S}} \cdot \frac{\partial \{(\delta \mathbf{f})\mathbf{f}^{-1}\}}{\partial \mathbf{x}} \quad , \tag{A.45}$$

where

$$\mathbf{S} := \mathbf{f}^{T-1} \tilde{\mathbf{S}} \mathbf{F}^{T} \quad , \quad \boldsymbol{\sigma} := \mathbf{f} \tilde{\boldsymbol{\sigma}} \mathbf{f}^{T} \quad , \quad \boldsymbol{\mathcal{S}} := \boldsymbol{\mathcal{L}}(\mathbf{f}^{T-1}, \mathbf{f}, \mathbf{F})[\tilde{\boldsymbol{\mathcal{S}}}] \quad . \tag{A.46}$$

Substitution in (A.42), and integrating with respect to the actual configuration,

$$\delta W = \int_{\mathcal{R}_t} \left\{ \mathbf{T} \cdot \{(\delta \mathbf{F})\mathbf{F}^{-1}\} - \mathbf{T} \cdot \{(\delta \mathbf{f})\mathbf{f}^{-1}\} + \boldsymbol{\Sigma} \cdot \{(\delta \mathbf{f})\mathbf{f}^{-1}\} + \boldsymbol{\mathcal{T}} \cdot \frac{\partial \{(\delta \mathbf{f})\mathbf{f}^{-1}\}}{\partial \mathbf{x}} \right\} \mathrm{d}v \quad , \tag{A.47}$$

or equivalently

$$\begin{aligned} \delta W = \int_{\mathcal{R}_t} \Big\{ & T_{ij} (\delta F_{im}) (\mathbf{F}^{-1})_{mj} - T_{ij} (\delta f_{im}) (\mathbf{f}^{-1})_{mj} + \Sigma_{ij} (\delta f_{im}) (\mathbf{f}^{-1})_{mj} \\ & + \mathcal{T}_{ijk} \frac{\partial \{(\delta f_{im})(\mathbf{f}^{-1})_{mj}\}}{\partial x_k} \Big\} \, \mathrm{d}v \quad , \end{aligned} \tag{A.48}$$

where

$$\mathbf{T} := \frac{1}{\det \mathbf{F}} \mathbf{S} \quad , \quad \boldsymbol{\Sigma} := \frac{1}{\det \mathbf{F}} \boldsymbol{\sigma} \quad , \quad \boldsymbol{\mathcal{T}} := \frac{1}{\det \mathbf{F}} \boldsymbol{\mathcal{S}} \quad . \tag{A.49}$$

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We notice the relations

$$T_{ij}(\delta F_{im})(\mathbf{F}^{-1})_{mj} = \frac{\partial(T_{ij}\delta u_i)}{\partial x_j} - \frac{\partial T_{ij}}{\partial x_j}\delta u_i \quad , \tag{A.50}$$

$$\mathcal{T}_{ijk}\frac{\partial\{(\delta f_{im})(\mathbf{f}^{-1})_{mj}\}}{\partial x_k} = \frac{\partial\{\mathcal{T}_{ijk}(\delta f_{im})(\mathbf{f}^{-1})_{mj}\}}{\partial x_k} - \frac{\partial \mathcal{T}_{ijk}}{\partial x_k}(\delta f_{im})(\mathbf{f}^{-1})_{mj} \quad . \tag{A.51}$$

On inserting in (A.48), and employing the divergence theorem,

$$\delta W = \int_{\mathcal{R}_t}\left(\Sigma_{ij} - T_{ij} - \frac{\partial \mathcal{T}_{ijk}}{\partial x_k}\right)(\delta f_{im})(\mathbf{f}^{-1})_{mj}\,\mathrm{d}v - \int_{\mathcal{R}_t}\frac{\partial T_{ij}}{\partial x_j}(\delta u_i)\,\mathrm{d}v + \int_{\partial\mathcal{R}_t} T_{ij}n_j(\delta u_i)\,\mathrm{d}a + \int_{\partial\mathcal{R}_t}\mathcal{T}_{ijk}n_k(\delta f_{im})(\mathbf{f}^{-1})_{mj}\,\mathrm{d}a \quad . \tag{A.52}$$

with $\partial\mathcal{R}_t$ being the boundary of $\mathcal{R}_t$.

A.5 Work of the external forces

As suggested by Mindlin (1964), the form of (A.52) motivates to adopt the following form for δW_e,

$$\delta W_e = \int_{\mathcal{R}_t} b_i\delta u_i\,\mathrm{d}v + \int_{\mathcal{R}_t} b_{ij}^{(d)}(\delta f_{im})(\mathbf{f}^{-1})_{mj}\,\mathrm{d}v + \int_{\partial\mathcal{R}_t} t_i\delta u_i\,\mathrm{d}a + \int_{\partial\mathcal{R}_t} t_{ij}^{(d)}(\delta f_{im})(\mathbf{f}^{-1})_{mj}\,\mathrm{d}a \quad , \tag{A.53}$$

where $\mathbf{b} = b_i\mathbf{e}_i$, $\mathbf{b}^{(d)} = b_{ij}^{(d)}\mathbf{e}_i \otimes \mathbf{e}_j$ are the body force and double body force per unit volume of the actual configuration of the macroscopic continuum and $\mathbf{t} = t_i\mathbf{e}_i$, $\mathbf{t}^{(d)} = t_{ij}^{(d)}\mathbf{e}_i \otimes \mathbf{e}_j$ are the surface force (traction) and the double surface force (double traction) per unit area of the actual configuration of the macroscopic continuum, respectively.

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A.6 Local equations of motion

We now insert Eqs. (A.53), (A.52) and (A.38) into Eq. (A.2) and drop the integration with respect to time,

$$\begin{aligned}
&\int_{\mathcal{R}_t} \left(\frac{\partial T_{ij}}{\partial x_j} + b_i - \varrho \ddot{x}_i \right) \delta u_i \, \mathrm{d}v \\
&+ \int_{\mathcal{R}_t} \left(\frac{\partial \mathcal{T}_{ijk}}{\partial x_k} + T_{ij} - \Sigma_{ij} + b_{ij}^{(d)} - \varrho \lambda_{ij} \right) (\delta f_{im}) (\mathbf{f}^{-1})_{mj} \, \mathrm{d}v \\
&+ \int_{\partial \mathcal{R}_t} (t_i - T_{ij} n_j) \delta u_i \, \mathrm{d}a + \int_{\partial \mathcal{R}_t} (t_{ij}^{(d)} - \mathcal{T}_{ijk} n_k)(\delta f_{im})(\mathbf{f}^{-1})_{mj} \, \mathrm{d}a = 0 \quad . \qquad \text{(A.54)}
\end{aligned}$$

The necessary and sufficient conditions in order for (A.54) to be satisfied for arbitrary variations $\delta \mathbf{u}$, $\delta \mathbf{f}$, are the local equations of motion

$$\frac{\partial T_{ij}}{\partial x_j} + b_i = \varrho \ddot{x}_i \quad \text{in } \mathcal{R}_t \quad , \qquad \text{(A.55)}$$

$$\frac{\partial \mathcal{T}_{ijk}}{\partial x_k} + T_{ij} - \Sigma_{ij} + b_{ij}^{(d)} = \varrho \lambda_{ij} \quad \text{in } \mathcal{R}_t \quad , \qquad \text{(A.56)}$$

together with the boundary conditions

$$T_{ij} n_j = t_i = \bar{t}_i \quad \text{on } \partial \mathcal{R}_t^{t_i} = \partial \mathcal{R}_t \setminus \partial \mathcal{R}_t^{u_i} \quad , \qquad \text{(A.57)}$$

$$\mathcal{T}_{ijk} n_k = t_{ij}^{(d)} = \bar{t}_{ij}^{(d)} \quad \text{on } \partial \mathcal{R}_t^{t_{ij}^{(d)}} = \partial \mathcal{R}_t \setminus \partial \mathcal{R}_t^{f_{ij}} \quad , \qquad \text{(A.58)}$$

$$\delta u_i = 0 \quad \& \quad u_i = \bar{u}_i \quad \text{on} \quad \partial \mathcal{R}_t^{u_i} \quad , \qquad \text{(A.59)}$$

$$\delta f_{ij} = 0 \quad \& \quad f_{ij} = \bar{f}_{ij} \quad \text{on} \quad \partial \mathcal{R}_t^{f_{ij}} \quad . \qquad \text{(A.60)}$$

Thereby,

$$\partial \mathcal{R}_t^{u_i} \cup \partial \mathcal{R}_t^{t_i} = \partial \mathcal{R}_t \quad , \quad \partial \mathcal{R}_t^{u_i} \cap \partial \mathcal{R}_t^{t_i} = \emptyset \quad , \qquad \text{(A.61)}$$

$$\partial \mathcal{R}_t^{f_{ij}} \cup \partial \mathcal{R}_t^{t_{ij}^{(d)}} = \partial \mathcal{R}_t \quad , \quad \partial \mathcal{R}_t^{f_{ij}} \cap \partial \mathcal{R}_t^{t_{ij}^{(d)}} = \emptyset \quad . \qquad \text{(A.62)}$$

Concluding, we emphasize once more that relations (A.55)–(A.62) have been established here by confining to pure elasticity, but otherwise they are valid for

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all micromorphic materials, irrespective of particular constitutive properties.

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