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Phase transitions of McKean-Vlasov processes in double-wells landscape*

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Abstract

We aim to establish results about a particular class of inhomogeneous processes, the so-called McKean-Vlasov diffusions. Such a diffusion corresponds to the hydrodynamical limit of an interacting particle system, the mean-field one. Existence and uniqueness of the invariant probability are classical results provided that the external force corresponds to the gradient of a convex potential. However, previous results, see [Herrmann, Tugaut|2010], state that the non-convexity of this potential implies the nonuniqueness of the invariant probabilities under easily checked assumptions. Here, we prove that there exists phase transitions, that is under a critical value, there are exactly three invariant probabilities and over another critical value, there is exactly one. Under simple assumptions, these two critical values coincide and it is characterized by a simple implicit equation. We exhibit other cases in which phase transitions occur. Finally, we proceed numerical estimations of the critical values.

Key words and phrases: phase transitions ; McKean-Vlasov diffusions ; invariant probabilities ; interacting particles systems ; free-energy.

2000 AMS subject classifications: primary 60H10; secondary: 60G10, 60J60, 65C05, 82C22

Introduction

The aim of this work is to study a particular class of inhomogeneous process, the so-called McKean-Vlasov diffusion. This kind of processes are obtained by

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taking the hydrodynamic limit of an interacting particle system.

More precisely, we focus on a system of N diffusions in \mathbb{R}^d with independent d -dimensional Wiener processes. We add a friction term ∇V and an interaction between each pair of particles. We assume that the interaction depends only on the distance between two particles and the further are the particles, the stronger is the attraction. In other words, we study the hydrodynamic limit of the mean-field system

$$\begin{cases} X_t^1 = X_0^1 + \sqrt{\epsilon} B_t^1 - \int_0^t \nabla V(X_s^1) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(X_s^1 - X_s^j) ds, \\ \vdots \\ X_t^i = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t \nabla V(X_s^i) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(X_s^i - X_s^j) ds, \\ \vdots \\ X_t^N = X_0^N + \sqrt{\epsilon} B_t^N - \int_0^t \nabla V(X_s^N) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t \nabla F(X_s^N - X_s^j) ds. \end{cases}$$

The function V here is called the confining potential and F is the interacting one. Let us remark that the diffusion (X^1, \dots, X^N) in $(\mathbb{R}^d)^N$ simply is a diffusion evolving in the potential $N\Upsilon_0^N$ with

$$\Upsilon_0^N(X_1, \dots, X_N) := \frac{1}{N} \sum_{i=1}^N V(X_i) + \frac{1}{2N^2} \sum_{i=1}^N \sum_{j=1}^N F(X_i - X_j).$$

By $\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}$, we denote the empirical measure of the particle system. By using Itô formula, we are able to obtain

$$\frac{d}{dt} \mathbb{E} \left\{ \int_{\mathbb{R}^d} f \mu_t^N \right\} = \mathbb{E} \left\{ \frac{\epsilon}{2} \int_{\mathbb{R}^d} \Delta f \mu_t^N - \int_{\mathbb{R}^d} \langle \nabla f; \nabla V + \nabla F * \mu_t^N \rangle \mu_t^N \right\}$$

where f is a smooth function with compact support from \mathbb{R}^d to \mathbb{R} . If μ_t^N is a deterministic measure, the previous equality leads to

$$\frac{\partial}{\partial t} \mu_t^N = \operatorname{div} \left\{ \frac{\epsilon}{2} \nabla \mu_t^N + (\nabla V + \nabla F * \mu_t^N) \mu_t^N \right\},$$

in a distributional sense. This motivates to look at the limit as N goes to infinity. Indeed, we can then prove that the empirical measure converges to a deterministic measure satisfying the previous partial differential equation.

The idea of the propagation of chaos is the following. Let us assume that $(X_0^i)_{i \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables with common law μ_0 . The law of large numbers implies that the empirical measure at time 0, that is $\mu_0^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_0^i}$, converges to μ_0 as N goes to infinity. We say that propagation of chaos holds on interval $[0; T]$ if $(\mu_t^N)_{t \in [0; T]}$ converges to $(\mu_t)_{t \in [0; T]}$ where μ_t is a deterministic probability measure on \mathbb{R}^d , for all $t \in [0; T]$. Let us stress that such a measure, if it exists, is necessary a solution of the partial differential equation

$$\frac{\partial}{\partial t} \mu_t = \operatorname{div} \left\{ \frac{\epsilon}{2} \nabla \mu_t + (\nabla V + \nabla F * \mu_t) \mu_t \right\},$$

the so-called granular media equation. About the propagation of chaos, we refer the reader to [Szn91, Mel96, BGV07]. According to the equations satisfied by μ_t^N and μ_t , we remark that μ_t plays the role of the expectation of μ_t^N for large N . We can go further than the propagation of chaos and prove some large deviations results, see [DG87].

The McKean-Vlasov diffusion is defined as the hydrodynamic limit of the mean-field system. As N goes to infinity, the empirical law μ_t^N converges to μ_t and each particle X_t^i follows a law close to μ_t . Consequently, the McKean-Vlasov diffusion in \mathbb{R} consists in the solution of the inhomogeneous stochastic differential equation

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon}B_t - \int_0^t V'(X_s) ds - \int_0^t F' * \mu_s(X_s) ds, \\ \mu_s = \mathcal{L}(X_s). \end{cases} \quad (\text{I})$$

This kind of processes have been introduced by McKean, see [McK67]. Here, by $*$, we denote the convolution. The particularity of this process is the influence of the law on the drift term. Let us notice that X_t and μ_t do depend on ϵ . We do not write it in order to simplify.

The existence problem can be solved by two different methods. The first one consists in the application of the propagation of chaos, see for example [Mel96]. The other method is the following. We consider a class of functions Λ_T from $[0; T] \times \mathbb{R}$ to \mathbb{R} . For each $b \in \Lambda_T$, we construct the diffusion $X^{(b)}$ which is the unique solution of the stochastic differential equation

$$X_t^{(b)} = X_0 + \sqrt{\epsilon}B_t - \int_{\mathbb{R}} V'(X_s^{(b)}) ds - \int_{\mathbb{R}} b(s, X_s^{(b)}) ds.$$

Then we introduce the function $\Gamma(b)(s, x) := \mathbb{E} \left[F'(x - X_s^{(b)}) \right]$. With sufficiently good properties on the set Λ_T and on the functional Γ , the application of a fixed-point theorem provides a solution of Eq. (I) on a finite interval. We extend to \mathbb{R}_+ by classical arguments. We refer to [BRTV98, HIP08].

Let us remark that for all $T < \infty$, we have

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t - X_t^1|^2 \right] = 0$$

if $B^1 = B$. See [Szn91] under Lipschitz properties, [BRTV98] if V is a constant, [Mal01, Mal03] when both potentials are convex. Under some convexity properties, this coupling (which is equivalent to the propagation of chaos) holds on \mathbb{R}_+ , see [CGM08].

Under appropriate regularity conditions, it is well-known, see [McK67] that the law $\mu_t := \mathcal{L}(X_t)$ is absolutely continuous with respect to the Lebesgue measure for all $t > 0$. By u_t , we denote its density. Moreover, it satisfies the granular media equation,

$$\frac{\partial}{\partial t} u_t = \frac{\partial}{\partial x} \left\{ \frac{\epsilon}{2} \frac{\partial}{\partial x} u_t + u_t (V' + F' * u_t) \right\}. \quad (\text{II})$$

Thus, we can study Equation (II) by probabilistic methods involving Eq. (I) or the associated mean-field system of particles, see [CGM08, Fun84, Mal03]. Reciprocally, Equation (II) is a useful tool to describe the invariant probabilities and the long-time behavior, see [BRTV98, BRV98, Tam84, Tam87, Ver06]. The partial differential equation is nonlinear. We especially remark the following fact. If μ is not a Dirac measure, nothing ensures us that

$$\mathbb{E}[f(X_t^\mu)] = \int_{\mathbb{R}} \mathbb{E}\left[f\left(X_t^{\delta_x}\right)\right] d\mu(x),$$

where X^ν is a solution of Eq. (I) with $\mathcal{L}(X_0^\nu) = \nu$.

When the confining potential V and the interacting potential F are both convex, there is a unique invariant probability. Furthermore, it is possible to prove by several methods (see [BRV98, CGM08, CMV03]) the convergence to this stationary measure.

However, in [HT10a], it has been proved under easily checked assumptions that Diffusion (I) admits several invariant probabilities. Moreover, there are exactly three ones if the diffusion coefficient is sufficiently small. The thirdness result in the small-noise case has been established with different sets of assumptions in [HT10b, HT09, Tug10, Tug11]. The long-time behavior of the law μ_t in the non-convex case has been the subject of [Tug10]. Under simple assumptions, μ_t converges weakly to an invariant probability. Furthermore, the basin of attraction of each invariant probability is not a singleton.

The purpose of the work is to provide a more complete picture about the enumeration of the invariant probabilities of Diffusion (I) in function of the noise ϵ . First of all, we present the material used in this work. Then, we look at the small-noise case when V is not even and F is not quadratic. Particularly, we prove the existence of invariant probabilities which are not close to any Dirac measure of a wells of the confining potential V . Then, we look at the number of invariant probabilities when the diffusion coefficient is not small. We particularly establish, in a simple case, the existence of a critical value ϵ_c under which there are three invariant probabilities and over which there is exactly one. Finally, we provide numerical estimations of the phases transitions by different methods.

Assumption (M): *We say that the confining potential V and the interacting potential F satisfy the set of assumptions (M) if*

(M-1) *V is a polynomial function with $\deg(V) \geq 4$.*

(M-2) *V has three critical points, $a_1 < a_2 < a_3$. Furthermore, $V''(a_1) > 0$, $V''(a_3) > 0$ and $V''(a_2) < 0$.*

(M-3) *The second derivative of V goes to infinity at infinity. Moreover, V'' is positive outside the compact interval $[a_1; a_3]$.*

(M-4) *F is an even polynomial function with $\deg(F) =: 2n \geq 2$.*

(M-5) *Both functions F and F'' are convex.*

(M-6) *$V(0) = F(0) = 0$.*

In the current work, some results require more hypotheses. Let us present these

additional assumptions.

Assumption (E): V is even and for all $k \geq 2$, $V^{(2k)}(0) \geq 0$. By a , we denote the unique positive critical point of V . Thus, $a = a_3$, $a_1 = -a$ and $a_2 = 0$.

Assumption (O): There exists $\lambda, \mu, \gamma \in \mathbb{R}$ and another function \widehat{V} satisfying (E) such that $V(x) = \widehat{V}(x - \lambda) + \mu x + \gamma$ for all $x \in \mathbb{R}$.

Assumption (Q): The interacting potential F is quadratic. There exists $\alpha > 0$ such that $F(x) := \frac{\alpha}{2}x^2$ for all $x \in \mathbb{R}$.

In the following, by \mathcal{S}_ϵ , we denote the set of the invariant probabilities of Diffusion (I). In order to study the number of elements of \mathcal{S}_ϵ , we introduce the following functional, the so-called free-energy, already used in previous works,

$$\Upsilon_\epsilon(\mu) := \int_{\mathbb{R}} \left\{ \frac{\epsilon}{2} \log(u(x)) + V(x) + \frac{1}{2} F * u(x) \right\} u(x) dx$$

for all the probability measures μ absolutely continuous with respect to the Lebesgue measure, with density equal to u .

1 Invariant probabilities in the small-noise case

The aim of this section is to study the invariant probabilities with arbitrarily small diffusion coefficient. In [HT10a], it has been proved that there is a symmetric invariant probability when the confining potential V is even.

Moreover, under some easily checked assumptions on F , there are also two asymmetric invariant probabilities if ϵ is small enough.

The method used in this work consists in a parametrisation. Indeed, Lemma 2.2 in [HT10a] provides an implicit equation on each invariant probability,

$$\mu^\epsilon(dx) = \frac{\exp \left[-\frac{2}{\epsilon} (V(x) + F * \mu^\epsilon(x)) \right]}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} (V(y) + F * \mu^\epsilon(y)) \right] dy} dx.$$

To find an invariant probability is thus equivalent to solve a fixed-point problem in a space which dimension is infinity. However, F is a polynomial function. Consequently, it is reduced to a fixed-point problem in a space which dimension is $2n - 1 =: \deg(F) - 1$ as noticed in [HT10a]. The parity of the potential V has not been used in the proof of Lemma 2.2 in [HT10a]. Hence, the above remark still holds.

Let us describe the method. For all $m := (m_1, \dots, m_{2n-1}) \in \mathbb{R}^{2n-1}$, we introduce the potential

$$W_m := V + \sum_{p=0}^{2n-1} \frac{(-1)^p}{p!} m_p F^{(p)} \text{ with } m_0 := 1.$$

The potential W_m is the sum of the confining potential V and the convolution of the interacting potential F with a measure μ satisfying $\int x^k \mu(dx) = m_k$ for

$1 \leq k \leq 2n - 1$. Let us notice that such a measure always exists but it is not necessary a probability measure. Then, we consider the measure

$$\mu_m^\epsilon(dx) := \frac{\exp\left[-\frac{2}{\epsilon}W_m(x)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}W_m(y)\right] dy} dx. \quad (1.1)$$

The implicit equation on the invariant probabilities yields the existence for all $v^\epsilon \in \mathcal{S}_\epsilon$ of a vector $m \in \mathbb{R}^{2n-1}$ such that $v^\epsilon = \mu_m^\epsilon$. It remains then to solve the system of equations

$$m_p = \int_{\mathbb{R}} x^p \mu_m^\epsilon(dx) = \frac{\int_{\mathbb{R}} x^p \exp\left[-\frac{2}{\epsilon}\left(V(x) + \sum_{k=0}^{2n-1} \frac{(-1)^k}{k!} m_k F^{(k)}(x)\right)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(x) + \sum_{k=0}^{2n-1} \frac{(-1)^k}{k!} m_k F^{(k)}(x)\right)\right] dx},$$

for all $1 \leq p \leq 2n - 1$.

Remark 1.1. *We can use another tool for this study, the free-energy, see [Tug11]. The main advantage is that it requires less hypotheses in the non-quadratic interaction case. But, it is only available for invariant probabilities which are around local minima of the confining potential V . Moreover, in the quadratic interaction case, the method presented here requires less hypotheses.*

Let us notice that the results of current section hold even if V has more than three critical points.

1.1 Quadratic interaction

Here, we assume (Q), that is $F(x) := \frac{\alpha}{2}x^2$ with $\alpha > 0$. The research of an invariant probability is reduced *in fine* to a parametrisation problem on \mathbb{R} . Indeed, according to the previous remarks, v^ϵ is an invariant probability if and only if there exists $m \in \mathbb{R}$ such that $v^\epsilon = \mu_m^\epsilon$ and $m = \int_{\mathbb{R}} x \mu_m^\epsilon(dx)$. In other words, m is a zero of the function (already used in [HT10a]),

$$\chi_\epsilon(m) := \frac{\int_{\mathbb{R}} x \exp\left[-\frac{2}{\epsilon}\left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx\right)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx\right)\right] dx} - m.$$

We now are able to prove the existence of an invariant probability around each critical point, under some assumptions.

Proposition 1.2. *We assume that the confining potential V and the interacting potential F satisfy Assumptions (M) and (Q). Let a_0 be a critical point of V such that $\alpha + V''(a_0) > 0$ and*

$$\alpha > 2 \sup_{x \neq a_0} \frac{V(a_0) - V(x)}{(a_0 - x)^2}. \quad (1.2)$$

Then, for all $\delta \in]0; 1[$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$, Diffusion (I) admits an invariant probability μ^ϵ verifying

$$\left| \int_{\mathbb{R}} x \mu^\epsilon(dx) - a_0 + \frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))} \epsilon \right| \leq \delta \epsilon.$$

Proof. The idea consists in proving the existence of a zero m_ϵ of χ_ϵ such that

$$\left| m_\epsilon - a_0 + \frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))} \epsilon \right| \leq \delta \epsilon.$$

By τ_0 , we denote the expression $\frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))}$. We approximate the quantity $\chi_\epsilon(a_0 - \tau_0(1 \pm \delta)\epsilon)$. Applying Lemma A.3 in [HT10a] to $f(x) := -2\alpha\tau_0(1 \pm \delta)x$, $n := 1$, $U(x) := V(x) + \frac{\alpha}{2}x^2 - \alpha a_0 x$ and $\mu := 0$ yields

$$\chi_\epsilon(a_0 - \tau^0(1 \pm \delta)\epsilon) = \pm \delta \frac{V''(a_0)}{\alpha + V''(a_0)} \tau^0 \epsilon + o(\epsilon).$$

We deduce that the product $\chi_\epsilon(a_0 - \tau_0(1 - \delta)\epsilon) \chi_\epsilon(a_0 - \tau_0(1 + \delta)\epsilon)$ is negative for sufficiently small ϵ . Since χ_ϵ is a continuous function, the intermediate value theorem yields that the function χ_ϵ admits a zero, non necessary unique, m_ϵ in the interval $[a_0 - \tau_0(1 + \delta)\epsilon; a_0 - \tau_0(1 - \delta)\epsilon]$. The proof is achieved by considering the associated measure,

$$\mu_{m_\epsilon}^\epsilon(dx) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + \alpha\frac{x^2}{2} - \alpha m_\epsilon x\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + \alpha\frac{y^2}{2} - \alpha m_\epsilon y\right)\right] dy} dx.$$

□

Let us notice that Proposition 1.2 holds also for critical point in which the confining potential does not admit a global minimum. Indeed, Hypothesis (1.2) is immediately satisfied if a_0 is an argument of the global minimum but it can be satisfied if it corresponds to a local and non global minimum or if it corresponds to a global maximum.

For example, let us take $V(x) := \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2}$. The points $a_2 := 0$ and $a_3 := \frac{\sqrt{5}-1}{2}$ respectively correspond to a local maximum and to a local but non global minimum. However, Inequality (1.2) with a_2 is equivalent to $\alpha > \frac{11}{9}$ and Inequality (1.2) with a_3 is equivalent to $\alpha > \frac{3\sqrt{5}+1}{18}$.

The particularity of the proposition is the following. We have to guess the small-noise limit of the measure before being able to prove its existence.

Moreover, we need this limit to be a Dirac measure around a critical point a_0 . Consequently, we talk about *invariant probability around a_0* . However, it has been proved in [HT10b] that a measure of the form $p\delta_{x_1} + (1-p)\delta_{x_2}$ can be the small-noise limit of a family of invariant probabilities. The previous proposition is not adapted to this situation. Furthermore, if a_0 is a critical point of V such that $\alpha + V''(a_0) < 0$, we cannot apply Proposition 1.2.

Next proposition aims to prove the existence of invariant probabilities which cannot be detected by Proposition 1.2. It is written in the case where V has exactly three critical points but we can prove it with any number of critical points.

Proposition 1.3. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M) and (Q). Let us take*

$$\alpha > 2 \max \left\{ \sup_{x \neq a_1} \frac{V(a_1) - V(x)}{(a_1 - x)^2}; \sup_{x \neq a_3} \frac{V(a_3) - V(x)}{(a_3 - x)^2} \right\}.$$

Then, there exists $\delta_0 > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$, there exists an invariant probability μ^ϵ satisfying

$$a_1 + \delta_0 < \int_{\mathbb{R}} x \mu^\epsilon(dx) < a_3 - \delta_0.$$

Proof. Outline of the proof. We again consider the potential $W_m(x) = V(x) + \frac{\alpha}{2}x^2 - \alpha mx$. We then study \mathcal{A}_δ , the set of the points in which $W_{a_1+\delta}$ reaches its global minimum.

In Step 1, we prove $\#\mathcal{A}_\delta = 1$ for sufficiently small δ (positive or negative). We denote $x(\delta)$ the unique point in \mathcal{A}_δ .

In Step 2.1, we prove that $x(\delta) - (a_1 + \delta) < 0$ and $x(-\delta) - (a_1 - \delta) > 0$ if $\delta > 0$ is small enough. In Step 2.2, Lemma A.3 in [HT10a] yields $\chi_\epsilon(a_1 + \delta) < 0$ and $\chi_\epsilon(a_1 - \delta) > 0$ for $\epsilon > 0$ and $\delta > 0$ sufficiently small.

The same holds with a_3 instead of a_1 . Step 3 consists in the application of intermediate value theorem to χ_ϵ on the interval $[a_1 + \delta; a_3 - \delta]$.

Step 1. We again consider the potential

$$W_m(x) = V(x) + \frac{\alpha}{2}x^2 - \alpha mx,$$

for all $m > 0$. We notice that $W_m(x) = W_{a_1}(x) - \alpha(m - a_1)x$. According to the hypotheses, Eq. (1.2) holds for a_1 . Consequently, the function W_{a_1} reaches its global minimum in a unique point and it is a_1 . *We prove in this step that for $|\delta|$ sufficiently small, the function $W_{a_1+\delta}$ reaches its global minimum in a unique point.*

Step 1.1. By \mathcal{A}_δ , we denote the set of the points in which the potential $W_{a_1+\delta}$ reaches its global minimum. Hypotheses (M-3) and (M-5) imply that $\pm\infty \notin \mathcal{A}_\delta$. Let us take $x_\delta \in \mathcal{A}_\delta$ for all $\delta \in [-1; 1]$. It is a critical point of $W_{a_1+\delta}$ so $W'_{a_1+\delta}(x_\delta) = 0$. This implies $W'_{a_1}(x_\delta) = \alpha\delta$. Consequently, for all $\delta \in [-1; 1]$, $|W'_{a_1}(x_\delta)| \leq \alpha$. We deduce the existence of a constant $C > 0$ such that

$$\sup_{-1 \leq \delta \leq 1} \sup_{x_\delta \in \mathcal{A}_\delta} |x_\delta| \leq C.$$

Step 1.2 Let x_δ be an element of \mathcal{A}_δ . By definition of \mathcal{A}_δ , $W_{a_1+\delta}(x_\delta) \leq W_{a_1+\delta}(x)$ for all $x \in \mathbb{R}$. Particularly, with $x = a_1$, we obtain $W_{a_1+\delta}(x_\delta) \leq W_{a_1}(a_1) + \alpha\delta(x_\delta - a_1)$. However, $W_{a_1}(a_1)$ is the global minimum of W_{a_1} . Consequently, we have

$$W_{a_1}(a_1) \leq W_{a_1}(x_\delta) \leq W_{a_1}(a_1) + \alpha\delta(x_\delta - a_1).$$

This implies $|W_{a_1}(x_\delta) - W_{a_1}(a_1)| \leq (C + |a_1|)\delta$ where the constant $C > 0$ has been introduced in Step 1.1. We deduce the convergence of $W_{a_1}(x_\delta)$ to $W_{a_1}(a_1)$. The continuity of W_{a_1} yields

$$\lim_{|\delta| \rightarrow 0} \sup_{x_\delta \in \mathcal{A}_\delta} |x_\delta - a_1| = 0.$$

Step 1.3. As $W_{a_1}''(a_1) = \alpha + V''(a_1) > 0$, there exists $\rho > 0$ such that for all $x \in [a_1 - \rho; a_1 + \rho]$, $W_{a_1}''(x) > 0$. According to Step 1.2, there exists $\delta_1 > 0$ such that $x_\delta \in [a_1 - \rho; a_1 + \rho]$ for all $x_\delta \in \mathcal{A}_\delta$ and for all $\delta \in [-\delta_1; \delta_1]$. However, W_{a_1} is convex on $[a_1 - \rho; a_1 + \rho]$ by construction of $\rho > 0$. This implies that $W_{a_1+\delta}$ reaches its minimum on $[a_1 - \rho; a_1 + \rho]$ in a unique point. Consequently, \mathcal{A}_δ contains a unique point if $|\delta| < \delta_1$. By $x(\delta)$, we denote this unique element.

Step 2. We now prove the two inequalities $\chi_\epsilon(a_1 + \delta) < 0$ and $\chi_\epsilon(a_1 - \delta) > 0$ for $0 < \delta < \delta_1$ and ϵ small enough.

Step 2.1. The equality $W'_{a_1+\delta}(x(\delta)) = 0$ provides $W'_{a_1}(x(\delta)) = \alpha\delta$. Since $W_{a_1}''(a_1) > 0$, we obtain the first order approximation,

$$x(\delta) = a_1 + \frac{\alpha}{W_{a_1}''(a_1)} \delta + o(\delta).$$

Consequently,

$$x(\delta) - (a_1 + \delta) = -\frac{V''(a_1)}{\alpha + V''(a_1)} \delta + o(\delta).$$

By recalling that $V''(a_1) > 0$, we deduce the existence of $\delta_2 < \delta_1$ such that $\pm(x(\pm\delta) - (a_1 \pm \delta)) < 0$ for all $0 < \delta < \delta_2$.

Step 2.2. By applying Lemma A.3 in [HT10a] to $f := 0$, $n := 1$, $U := W_{a_1}$, $G := x \mapsto -\alpha x$ and $\mu := \pm\delta$, we get

$$\begin{aligned} \chi_\epsilon(a_1 + \delta) &= x(\delta) - (a_1 + \delta) + o(1) \\ \text{and } \chi_\epsilon(a_1 - \delta) &= x(-\delta) - (a_1 - \delta) + o(1). \end{aligned}$$

Thus, there exists $\epsilon_1(\delta) > 0$ such that for all $0 < \epsilon < \epsilon_1(\delta)$, $\chi_\epsilon(a_1 + \delta) < 0$ and $\chi_\epsilon(a_1 - \delta) > 0$.

Step 3. By proceeding exactly like in Step 1 and Step 2, there exists $\delta_3 > 0$ such that for all $\delta \in]0; \delta_3[$ and $0 < \epsilon < \epsilon_2(\delta)$, $\chi_\epsilon(a_3 + \delta) < 0$ and $\chi_\epsilon(a_3 - \delta) > 0$. We take $\delta_0 < \min\{\delta_2; \delta_3; \frac{a_3 - a_1}{2}\}$ and $\epsilon_0 := \min\{\epsilon_1(\delta_0); \epsilon_2(\delta_0)\}$. Thus, for all $0 < \epsilon < \epsilon_0$, we have

$$\chi_\epsilon(a_1 + \delta_0) < 0 < \chi_\epsilon(a_3 - \delta_0) \quad \text{and} \quad a_1 + \delta_0 < a_3 - \delta_0.$$

Then, by using the intermediate value theorem, we deduce the existence of $m_\epsilon \in [a_1 + \delta_0; a_3 - \delta_0]$ such that $\chi_\epsilon(m_\epsilon) = 0$. We achieve the proof by considering the associated measure,

$$\mu_{m_\epsilon}^\epsilon(dx) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + \frac{\alpha}{2}x^2 - \alpha m_\epsilon x\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + \frac{\alpha}{2}y^2 - \alpha m_\epsilon y\right)\right] dy} dx.$$

□

Let us remark that this intermediate invariant probability is not necessary unique. Propositions 1.2 and 1.3 yield immediately the following statement (the proof is left to the reader).

Corollary 1.4. *With the same hypotheses than the ones of Proposition 1.3, Diffusion (I) admits at least three invariant probabilities if ϵ is sufficiently small.*

Particularly, Corollary 1.4 can be applied in the synchronized case, that is $\alpha \geq \sup_{x \in \mathbb{R}} -V''(x)$. Let us point out that these results hold even if the number of critical points is not three.

Indeed, for α large enough and ϵ small enough, the number of invariant probabilities is at least the number of critical points of V .

1.2 Nonquadratic interacting potential

The main advantage of Hypothesis (Q) is the equivalence between the existence problem of an invariant probability and a fixed-point problem in dimension one. In the general case, we make a parametrisation in \mathbb{R}^{2n-1} with $\deg(F) = 2n$. We cannot use anymore the intermediate value theorem. Let us notice the following statement (see Corollary 2.2 in [Tug11] for a proof).

Proposition 1.5. *We assume that the confining potential V and the interacting potential F satisfy the set of hypotheses (M). Then, for all $\epsilon > 0$, there exists at least one invariant probability.*

We now present the extension of Theorem 4.8 in [HT10a].

Theorem 1.6. *We assume that the confining potential V and the interacting potential F satisfy the set of hypotheses (M). Let a_0 be a localization of a local minimum of V such that*

$$V(x) + F(x - a_0) > V(a_0) \quad \text{for all } x \neq a_0$$

$$\text{and } \sum_{p=0}^{2n-2} \frac{|F^{(p+2)}(a_0)|}{p!} |a_0|^p < \alpha + V''(a_0).$$

Then, for any $\eta > 0$, there exists $\epsilon_0(\eta)$ such that for all $0 < \epsilon < \epsilon_0(\eta)$, Diffusion (I) admits an invariant probability μ^ϵ satisfying

$$\left| \int_{\mathbb{R}} x^k \mu^\epsilon(dx) - a_0^k + \tau_k^0 \epsilon \right| \leq \eta \epsilon \quad \text{for all } 1 \leq k \leq 2n-1$$

$$\text{with } \tau_k^0 := ka_0^{k-1} \frac{a_0 V^{(3)}(a_0) - (k-1)V''(a_0)}{4a_0 V''(a_0) (\alpha + V''(a_0))}.$$

The proof is similar to the one of Theorem 4.8 in [HT10a]. We introduce the application Φ from \mathbb{R}^{2n-1} to \mathbb{R}^{2n-1} defined by

$$\Phi_k(m_1, \dots, m_{2n-1}) = \frac{\int_{\mathbb{R}} x^k \exp \left[-\frac{2}{\epsilon} \left(V(x) + \sum_{p=0}^{2n-1} \frac{(-1)^p}{p!} m_p F^{(p)}(x) \right) \right] dx}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} V(x) + \sum_{p=0}^{2n-1} \frac{(-1)^p}{p!} m_p F^{(p)}(x) \right] dx}.$$

Then, we apply the Schauder fixed-point theorem to Φ on a parallelepiped of the form

$$C(\epsilon) := \prod_{p=1}^{2n-1} [a_0^p - \tau_p^0 \epsilon - \lambda \delta \epsilon; a_0^p - \tau_p^0 \epsilon + \lambda \delta \epsilon]$$

where $\lambda > 0$ is a constant. This result requires more hypotheses than Theorem 2.3 in [Tug11] but it is more precise since we have the first order approximation of the moments.

Let us stress that the second assumption of Theorem 1.6 is not intuitive and is only related to the method used in the proof.

2 Uniqueness and Thirdness

The previous section deals with the invariant probabilities for self-stabilizing processes in the small-noise case. Particularly, there is nonuniqueness of the invariant probabilities under easily checked assumptions. With Hypotheses (M), (E) and (Q), we know that there are exactly three invariant probabilities as ϵ is small enough (see Theorem 3.2 in [HT10a]). However, what happens in the large-noise case has not yet been investigated. In this section, we show that under Assumptions (M), (E) and (Q), there is a phase transition. If ϵ is larger or equal to a critical value ϵ_c , Diffusion (I) admits a unique invariant probability and if ϵ is strictly smaller than ϵ_c , it admits exactly three ones. Then, this result of transition between the uniqueness and the thirdness is partially extended in two directions, when V is not even and when F is not quadratic.

2.1 With quadratic interaction

In this paragraph, we assume that the interacting potential satisfies (Q). In other words, $F(x) := \frac{\alpha}{2}x^2$ with $\alpha > 0$. Thus, as noticed in the previous section, there is a bijection between \mathcal{S}_ϵ , the set of the invariant probabilities of Diffusion (I) and the set of the zeros of the function χ_ϵ defined by

$$\chi_\epsilon(m) = \frac{\int_{\mathbb{R}} x \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx \right) \right] dx}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx \right) \right] dx} - m.$$

However, if this function can easily be used in the small-noise limit thanks to the Laplace method, it is harder to provide a precise study for general $\epsilon > 0$. Consequently, we look at another function from \mathbb{R} to itself,

$$\xi_\epsilon(m) := \int_{\mathbb{R}} (x - m) \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx \right) \right] dx. \quad (2.1)$$

This function ξ_ϵ admits the same zeros than χ_ϵ since ξ_ϵ consists in the product of χ_ϵ with the positive partition function,

$$Z_\epsilon(m) := \int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx \right) \right] dx. \quad (2.2)$$

The method used in this paragraph is the study of the two functions ξ_ϵ and Z_ϵ .

2.1.1 With even confinement

We add another hypothesis in this paragraph. The confining potential V now satisfies Assumption (E). Consequently, we can write

$$V(x) = -\frac{|V''(0)|}{2}x^2 + \sum_{p=2}^q \frac{|V^{(2p)}(0)|}{(2p)!}x^{2p} \text{ with } \deg(V) =: 2q. \quad (2.3)$$

We recall that the unique positive critical point is a and the unique negative one is $-a$. Since V is even, ξ_ϵ is odd and 0 is an obvious zero. The study thus is reduced to the research of the number of zeros of ξ_ϵ on $\mathbb{R}_+^* :=]0; +\infty[$.

Theorem 2.1. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M), (E) and (Q).*

Thus, there exists $\epsilon_c > 0$ such that:

- *For all $\epsilon \geq \epsilon_c$, Diffusion (I) admits a unique invariant probability, μ_0^ϵ . And, μ_0^ϵ is symmetric.*
- *For all $\epsilon < \epsilon_c$, Diffusion (I) admits exactly three invariant probabilities. One is symmetric, μ_0^ϵ and the two other measures μ_+^ϵ and μ_-^ϵ satisfy $\pm \int_{\mathbb{R}} x \mu_\pm^\epsilon(dx) > 0$.*

Moreover, the equation

$$\int_{\mathbb{R}_+} \left(x^2 - \frac{1}{2\alpha} \right) \exp \left[(|V''(0)| - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1} |V^{(2p)}(0)|}{(2p)!} x^{2p} \right] dx = 0 \quad (2.4)$$

admits a unique solution, that is the critical value ϵ_c .

Proof. Outline of the proof. In Step 1, we proceed a series expansion of the function ξ_ϵ . We prove in Step 2 the existence of a critical integer n_ϵ such that for all $n \geq n_\epsilon$, $\xi_\epsilon^{(2n+1)}(0) \leq 0$ and for all $n < n_\epsilon$, $\xi_\epsilon^{(2n+1)}(0) > 0$. We can then write

$$\xi_\epsilon(m) = \sum_{n=0}^{n_\epsilon-1} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n+1)!} m^{2n+1} - \sum_{n=n_\epsilon}^{+\infty} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n+1)!} m^{2n+1}.$$

By using it, we show in Step 3 that Diffusion (I) admits exactly one or three invariant probabilities. Moreover, if there is a unique measure μ_0^ϵ , it is a symmetric one. And, if there are three invariant probabilities, one is symmetric (μ_0^ϵ), one has a positive moment and one has a negative moment.

In Step 4, we establish that uniqueness is equivalent to the inequality $\xi'_\epsilon(0) \leq 0$. We finally prove in Step 5 the existence of a unique critical value ϵ_c such that $\xi'_{\epsilon_c}(0) = 0$. Moreover, the equation $\xi'_\epsilon(0) = 0$ is equivalent to Eq. (2.4).

Step 1. We rewrite the function ξ_ϵ (defined in Eq. (2.1)),

$$\xi_\epsilon(m) = \int_{\mathbb{R}} (x - m) \exp \left[\frac{2\alpha mx}{\epsilon} \right] \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2} x^2 \right) \right] dx.$$

We proceed a series expansion of the function $m \mapsto \exp\left[\frac{2\alpha m x}{\epsilon}\right]$. Due to Hypothesis (E), we obtain

$$\xi_\epsilon(m) = 2 \sum_{n=0}^{\infty} \frac{I_\epsilon(2n)}{(2n)!} \left(\frac{2\alpha m}{\epsilon}\right)^{2n+1} \left[\frac{I_\epsilon(2n+2)}{(2n+1)I_\epsilon(2n)} - \frac{\epsilon}{2\alpha} \right], \quad (2.5)$$

$$\text{with } I_\epsilon(z) := \int_{\mathbb{R}_+} x^z \exp\left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx. \quad (2.6)$$

Step 2. We prove in this step the existence of a critical integer n_ϵ such that for all $n \geq n_\epsilon$, $\xi_\epsilon^{(2n+1)}(0) \leq 0$ and for all $n \leq n_\epsilon - 1$, $\xi_\epsilon^{(2n+1)}(0) > 0$.

Step 2.1. For all $n \in \mathbb{N}$, we introduce the quantity

$$\gamma_n(\epsilon) := \frac{I_\epsilon(2n+2)}{(2n+1)I_\epsilon(2n)} - \frac{\epsilon}{2\alpha}. \quad (2.7)$$

An integration by parts provides

$$\begin{aligned} (2n+1)I_\epsilon(2n) &= \frac{2}{\epsilon} \int_{\mathbb{R}} (V'(x) + \alpha x) x^{2n+1} \exp\left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx \\ &= \frac{2}{\epsilon} \left\{ (\alpha - |V''(0)|) I_\epsilon(2n+2) + \sum_{p=2}^q \frac{|V^{(2p)}(0)|}{(2p-1)!} I_\epsilon(2n+2p) \right\} \end{aligned}$$

after using Eq. (2.3). Consequently, the term $\gamma_n(\epsilon)$ becomes

$$\gamma_\epsilon(n) = \frac{\epsilon}{2} \left\{ \alpha - |V''(0)| + \sum_{p=2}^q \frac{|V^{(2p)}(0)|}{(2p-1)!} \frac{I_\epsilon(2n+2p)}{I_\epsilon(2n+2)} \right\}^{-1} - \frac{\epsilon}{2\alpha}.$$

Step 2.2. The aim here is to prove that for all $\epsilon > 0$, the sequence $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$ is nonincreasing.

It is sufficient to prove that the sequences $\left(\frac{I_\epsilon(2n+2p)}{I_\epsilon(2n+2)}\right)_{n \in \mathbb{N}}$ are nondecreasing for all $p \in \llbracket 2; q \rrbracket$. In a more general way, we state that the function $\Xi_z := x \mapsto \frac{I_\epsilon(x+z)}{I_\epsilon(x)}$ is nondecreasing for all $z > 0$. We proceed the derivation of Ξ_z ,

$$\Xi'_z(x) = \Xi_z(x) \left(\frac{I'_\epsilon(x+z)}{I_\epsilon(x+z)} - \frac{I'_\epsilon(x)}{I_\epsilon(x)} \right).$$

Since $\Xi_z(x) \geq 0$ for all $x \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, it now remains to prove that the application $\zeta := \frac{I'_\epsilon}{I_\epsilon}$ is nondecreasing. We apply the derivation.

$$\begin{aligned} \zeta'(y) &= \frac{I''_\epsilon(y)}{I_\epsilon(y)} - \left(\frac{I'_\epsilon(y)}{I_\epsilon(y)} \right)^2 = \frac{\int_{\mathbb{R}_+} x^y (\log(x))^2 \exp\left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx}{\int_{\mathbb{R}_+} x^y \exp\left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx} \\ &\quad - \left(\frac{\int_{\mathbb{R}_+} x^y (\log(x)) \exp\left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx}{\int_{\mathbb{R}_+} x^y \exp\left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx} \right)^2. \end{aligned}$$

By applying the Cauchy-Schwarz inequality, we obtain $\zeta'(y) \geq 0$ for all $y > 0$. Then, the claim holds, that is the sequence $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$ is nonincreasing for all $\epsilon > 0$.

Step 2.3. We here prove that $\gamma_n(\epsilon) < 0$ for n large enough. For doing this, it is sufficient to find a real x_0 such that $\xi_\epsilon(x_0) < 0$. We compute $\xi_\epsilon(a)$.

$$\begin{aligned}\xi_\epsilon(a) &= e^{\frac{\alpha a^2}{\epsilon}} \int_{\mathbb{R}} (x-a) \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2} (x-a)^2 \right) \right] dx \\ &= e^{\frac{\alpha a^2}{\epsilon}} \int_0^{+\infty} y e^{-\frac{\alpha}{\epsilon} y^2} \left\{ \exp \left[-\frac{2}{\epsilon} V(y+a) \right] - \exp \left[-\frac{2}{\epsilon} V(y-a) \right] \right\} dy.\end{aligned}$$

By using Assumption (E), we easily prove that $V(y+a) - V(y-a) > 0$ for all $y > 0$. Consequently, $\xi_\epsilon(a) < 0$. We deduce the existence of $n \in \mathbb{N}$ such that $\gamma_n(\epsilon) < 0$. We put $n_\epsilon := \min \left\{ n \mid \gamma_n(\epsilon) \leq 0 \right\}$. Since $\gamma_n(\epsilon)$ and $\xi_\epsilon^{(2n+1)}(0)$ have the same sign, we deduce that for all $n \geq n_\epsilon$, $\xi_\epsilon^{(2n+1)}(0)$ is nonpositive and for all $n < n_\epsilon$, $\xi_\epsilon^{(2n+1)}(0)$ is positive.

Step 3. Consequently, we can write

$$\xi_\epsilon(m) = \sum_{n=0}^{n_\epsilon-1} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n+1)!} m^{2n+1} - \sum_{n=n_\epsilon}^{+\infty} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n+1)!} m^{2n+1}. \quad (2.8)$$

By making factorization with $m^{2n_\epsilon+1}$, we obtain

$$\xi_\epsilon(m) = m^{2n_\epsilon+1} \left\{ \sum_{n=0}^{n_\epsilon-1} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n+1)!} m^{2n-2n_\epsilon} - \sum_{n=n_\epsilon}^{+\infty} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n+1)!} m^{2n-2n_\epsilon} \right\}.$$

Since the functions $m \mapsto m^{2n-2n_\epsilon}$ (resp. $m \mapsto -m^{2n-2n_\epsilon}$) are decreasing for all $n \leq n_\epsilon - 1$ (resp. $n \geq n_\epsilon$), we deduce that ξ_ϵ admits at most one positive zero. The function ξ_ϵ is odd so its admits exactly one or three zeros on \mathbb{R} . This means that Diffusion (I) admits exactly one or three invariant probabilities.

Step 4. We here prove that the uniqueness of the invariant probability directly is related to the sign of $\xi_\epsilon'(0)$. We take the derivative of Eq. (2.8) then we make a factorization by m^{2n_ϵ} .

$$\xi_\epsilon'(m) = m^{2n_\epsilon} \left\{ \sum_{n=0}^{n_\epsilon-1} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n)!} m^{2n-2n_\epsilon} - \sum_{n=n_\epsilon}^{+\infty} \frac{|\xi_\epsilon^{(2n+1)}(0)|}{(2n)!} m^{2n-2n_\epsilon} \right\}.$$

By the same argument than the one in previous step, we deduce that ξ_ϵ' vanishes at most one time on the set of the positive reals. As it tends to $-\infty$ when m goes to infinity, this means that ξ_ϵ' is either always nonpositive either positive then nonpositive. Consequently, the behavior of the function ξ_ϵ is related to the sign of $\xi_\epsilon'(0)$.

- If $\xi'_\epsilon(0) \leq 0$, the function ξ_ϵ is nonincreasing on \mathbb{R}_+ . Since $\xi_\epsilon(0) = 0$, the function ξ_ϵ does not vanish on \mathbb{R}_+^* so Diffusion (I) admits a unique invariant probability and this measure is symmetric.
- If $\xi'_\epsilon(0) > 0$, the function ξ_ϵ is first increasing then nonincreasing. Since, $\xi_\epsilon(0) = 0$, it implies the existence of $m_\epsilon > 0$ such that $\xi_\epsilon(m_\epsilon) = 0$. There is thus an invariant probability μ_+^ϵ associated to m_ϵ and since m_ϵ is the moment of μ_+^ϵ , we have $\int_{\mathbb{R}} x \mu_+^\epsilon(dx) > 0$. Furthermore, the measure μ_-^ϵ associated to $-m_\epsilon$ satisfies $\int_{\mathbb{R}} x \mu_-^\epsilon(dx) < 0$.

Step 5. We now investigate on the sign of $\xi'_\epsilon(0)$. By Eq. (2.5), $\xi'_\epsilon(0)$ and $\gamma_0(\epsilon)$ have the same sign. We recall that

$$\gamma_0(\epsilon) = \frac{I_\epsilon(2)}{I_\epsilon(0)} - \frac{\epsilon}{2\alpha} = \frac{\int_{\mathbb{R}_+} x^2 \exp\left[-\frac{2}{\epsilon}\left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx}{\int_{\mathbb{R}_+} \exp\left[-\frac{2}{\epsilon}\left(V(x) + \frac{\alpha}{2}x^2\right)\right] dx} - \frac{\epsilon}{2\alpha}.$$

The change of variable $x := \sqrt{\epsilon}y$ yields

$$\gamma_0(\epsilon) = \epsilon \left\{ \frac{\int_{\mathbb{R}_+} y^2 \exp\left[(|V''(0)| - \alpha)y^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1} |V^{(2p)}(0)|}{(2p)!} y^{2p} \right] dy}{\int_{\mathbb{R}_+} \exp\left[(|V''(0)| - \alpha)y^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1} |V^{(2p)}(0)|}{(2p)!} y^{2p} \right] dy} - \frac{1}{2\alpha} \right\} \\ =: \epsilon \mathcal{T}_{\alpha, V}(\epsilon).$$

By using the Jensen inequality, we prove that the function $\epsilon \mapsto \mathcal{T}_{\alpha, V}(\epsilon)$ is decreasing. Consequently, there exists a unique ϵ_c such that $\gamma_0(\epsilon_c) = 0$. Moreover, for all $\epsilon \geq \epsilon_c$, we have $\gamma_0(\epsilon) \leq 0$ which implies the uniqueness of the invariant probability. And, for all $\epsilon < \epsilon_c$, we have $\gamma_0(\epsilon) > 0$ which implies the thirdness of the invariant probabilities.

Finally, we remark that the sign of $\mathcal{T}_{\alpha, V}(\epsilon)$ is the same than the one of

$$\int_{\mathbb{R}_+} \left(x^2 - \frac{1}{2\alpha} \right) \exp\left[(|V''(0)| - \alpha)x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1} |V^{(2p)}(0)|}{(2p)!} x^{2p} \right] dx,$$

which achieves the proof. \square

Let us remark that the function $|V''(0)| \mapsto \mathcal{T}_{\alpha, V}(\epsilon)$ is nondecreasing and that the functions $V^{(2p)}(0) \mapsto \mathcal{T}_{\alpha, V}(\epsilon)$ are nonincreasing for all $2 \leq p \leq q$. We immediately deduce that the function $V''(0) \mapsto \epsilon_c$ is nondecreasing and that the functions $|V^{(2p)}(0)| \mapsto \epsilon_c$ are nonincreasing for all $2 \leq p \leq q$.

We do not know anything about the monotonicity of the function $\alpha \mapsto \epsilon_c$. However, in the following, numerical estimations in a simple example establish that this function seems to be nondecreasing, see Figure 1, page 23.

2.1.2 With asymmetric confinement

We now look at the asymmetric case. We assume that the confining potential V admits properties similar to (E), that is Hypothesis (O). Here, there exist \widehat{V} satisfying (E) and three constants λ , μ and γ such that

$$V(x) = \widehat{V}(x - \lambda) + \mu x + \gamma.$$

Let us remark that all the non-convex polynomial functions with degree equal to 4 satisfy (E) or (O).

According to Section 1, we already know that Diffusion (I) admits several invariant probabilities if $\alpha := F''(0)$ is large enough and if ϵ is sufficiently small. The aim of this paragraph is to prove that it admits a unique invariant probability if the diffusion coefficient is large enough. In other words, we show that the dynamic of the diffusion is dominated by the heat process if the noise $\sqrt{\epsilon}$ is sufficiently large.

The method consists in using the results from the above paragraph. We recall the existence of a bijection between \mathcal{S}_ϵ and the set of zeros of the function χ_ϵ defined by

$$\chi_\epsilon(m) = \frac{\int_{\mathbb{R}} x \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2} x^2 - \alpha m x \right) \right] dx}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2} x^2 - \alpha m x \right) \right] dx} - m.$$

We apply the change of variable $x := y + \lambda$ and we obtain

$$\chi_\epsilon(m) = \widehat{\chi}_\epsilon(\widehat{m}) - \frac{\mu}{\alpha}$$

with $\widehat{m} := m - \lambda - \frac{\mu}{\alpha} \in \mathbb{R}$. The function $\widehat{\chi}_\epsilon$ here is the function defined by

$$\widehat{\chi}_\epsilon(m) := \frac{\int_{\mathbb{R}} y \exp \left[-\frac{2}{\epsilon} \left(\widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(\widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy} - m. \quad (2.9)$$

We also introduce

$$\widehat{\xi}_\epsilon(m) := \int_{\mathbb{R}} (y - m) \exp \left[-\frac{2}{\epsilon} \left(\widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy \quad (2.10)$$

$$\text{and } \widehat{Z}_\epsilon(m) := \int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(\widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy. \quad (2.11)$$

Before giving the main statement of the paragraph, let us give some technical results.

Definition 2.2. For all $k \in \mathbb{N}$, by $\widehat{I}_\epsilon(2k)$ and $\widehat{\gamma}_k(\epsilon)$, we denote the quantities

$$\widehat{I}_\epsilon(2k) := \int_{\mathbb{R}_+} x^{2k} \exp \left[-\frac{2}{\epsilon} \left(\widehat{V}(x) + \frac{\alpha}{2} x^2 \right) \right] dx$$

$$\text{and } \widehat{\gamma}_k(\epsilon) := \frac{\widehat{I}_\epsilon(2k+2)}{(2k+1)\widehat{I}_\epsilon(2k)} - \frac{\epsilon}{2\alpha}.$$

The next statement is just computational. We do not write the proof.

Lemma 2.3. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M), (O) and (Q). For all $k \in \mathbb{N}$, we have*

$$\widehat{\xi}_\epsilon^{(2k+1)}(0) = 2\widehat{I}_\epsilon(2k) \left(\frac{2\alpha}{\epsilon}\right)^{2k+1} (2k+1)\widehat{\gamma}_k(\epsilon) \quad (2.12)$$

$$\text{and } \widehat{Z}_\epsilon^{(2k)}(0) = 2\widehat{I}_\epsilon(2k) \left(\frac{2\alpha}{\epsilon}\right)^{2k}. \quad (2.13)$$

We now give the main result of the paragraph.

Proposition 2.4. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M), (O) and (Q).*

Thus, for all $\alpha \geq 0$, there exists a critical value $\epsilon_0(\alpha)$ such that for all $\epsilon > \epsilon_0(\alpha)$, Diffusion (I) admits a unique invariant probability.

Proof. Since there is a bijection between \mathcal{S}_ϵ and the set of the roots of the function χ_ϵ , the proof is based on an analytical study of χ_ϵ . However, $\chi_\epsilon(m) = \widehat{\chi}_\epsilon(\widehat{m}) - C_1$ with $\widehat{m} := m - C_2 \in \mathbb{R}$ where C_1 and C_2 are constants and $\widehat{\chi}_\epsilon$ has been defined in Eq. (2.9). Consequently, it is sufficient to prove that $\widehat{\chi}_\epsilon$ is decreasing for ϵ large enough.

Step 1. *We first compute the derivative of $\widehat{\chi}_\epsilon$,*

$$\frac{d}{dm} \widehat{\chi}_\epsilon(m) = \frac{\tau_\epsilon(m)}{\widehat{Z}_\epsilon(m)^2},$$

with $\tau_\epsilon(m) := \widehat{\xi}_\epsilon'(m)\widehat{Z}_\epsilon(m) - \widehat{\xi}_\epsilon(m)\widehat{Z}_\epsilon'(m)$. The function τ_ϵ is even and analytic so it is sufficient to prove that $\tau_\epsilon^{(2n)}(0)$ is nonnegative for all $n \in \mathbb{N}$.

Step 2. *We here compute the derivatives of τ_ϵ ,*

$$\begin{aligned} \tau_\epsilon^{(2n)}(0) &= \sum_{k=0}^{2n} C_{2n}^k \widehat{\xi}_\epsilon^{(k+1)}(0) \widehat{Z}_\epsilon^{(2n-k)}(0) - \sum_{k=0}^{2n} C_{2n}^k \widehat{\xi}_\epsilon^{(k)}(0) \widehat{Z}_\epsilon^{(2n+1-k)}(0) \\ &= \widehat{\xi}_\epsilon^{(2n+1)}(0) \widehat{Z}_\epsilon(0) - 2n \widehat{\xi}_\epsilon'(0) \widehat{Z}_\epsilon^{(2n)}(0) \\ &\quad + \sum_{k=1}^{n-1} (C_{2n}^{2k} - C_{2n}^{2k+1}) \widehat{\xi}_\epsilon^{(2k+1)}(0) \widehat{Z}_\epsilon^{(2n-2k)}(0). \end{aligned} \quad (2.14)$$

By applying Lemma 2.3, we obtain

$$\begin{aligned} \tau_\epsilon^{(2n)}(0) &= 4 \left(\frac{2\alpha}{\epsilon}\right)^{2n+1} \left\{ \widehat{I}_\epsilon(2n) \widehat{I}_\epsilon(0) [(2n+1)\widehat{\gamma}_n(\epsilon) - 2n\widehat{\gamma}_0(\epsilon)] \right. \\ &\quad \left. + \sum_{k=1}^{n-1} C_{2n}^{2k} (4k+2-2n) \widehat{I}_\epsilon(2k) \widehat{I}_\epsilon(2n-2k) \widehat{\gamma}_k(\epsilon) \right\}. \end{aligned}$$

Step 3. As proved in Step 5 of the proof of Theorem 2.1, there exists ϵ_c such that for all $\epsilon \geq \epsilon_c$, we have $\widehat{\gamma}_0(\epsilon) < 0$. From now on, we take $\epsilon > \epsilon_c$. It has been proved in Step 2 of the proof of Theorem 2.1 that the sequence $(\widehat{\gamma}_n(\epsilon))_{n \in \mathbb{N}}$ is nonincreasing. Consequently, $(|\widehat{\gamma}_n(\epsilon)|)_{n \in \mathbb{N}}$ is nondecreasing.

Step 4. We notice that if $k > \frac{n}{2}$, we have $2k \geq n + 1$ then $4k \geq 2n + 2$. This implies $|4k + 2 - 2n| - |2n + 2 - 4k| = 4k + 2 - 2n - (4k - 2 - 2n) = 4 > 0$. We then remark $(4k + 2 - 2n)\widehat{\gamma}_k(\epsilon) + (2n + 2 - 4k)\widehat{\gamma}_{n-k}(\epsilon) = |4k + 2 - 2n|\widehat{\gamma}_k(\epsilon) - |2n + 2 - 4k|\widehat{\gamma}_{n-k}(\epsilon)$. This quantity is negative due to the monotonicity of the sequence $(\widehat{\gamma}_n(\epsilon))_{n \in \mathbb{N}}$. And, if $k = \frac{n}{2}$, we have $(4k + 2 - 2n)\widehat{\gamma}_k(\epsilon) = 2\widehat{\gamma}_k(\epsilon) < 0$. This implies $\tau_\epsilon^{(2n)}(0) \leq 0$ for all $n \geq 0$.

Step 5. The function $\widehat{\chi}_\epsilon(m)$ is then nonpositive. We deduce that $\widehat{\chi}_\epsilon$ is nonincreasing. Moreover, $\widehat{\chi}_\epsilon$ is not a constant function since it is odd and $\widehat{\chi}_\epsilon(\widehat{a}) < 0$ where \widehat{a} is the unique positive critical point of \widehat{V} . Thus, at least one of the $\tau_\epsilon^{(2n)}(0)$ is negative. Hence, $\widehat{\chi}_\epsilon$ is decreasing on \mathbb{R} . \square

Proposition 1.2 tells us that Diffusion (I) admits several invariant probabilities if α is large enough and ϵ sufficiently small. However, Proposition 2.4 shows that the self-stabilizing diffusion admits a unique invariant probability if the noise ϵ is large enough. It means that there exists a phase transition between the uniqueness and the nonuniqueness. *However, the uniqueness (or not) of this phase transition is still an open question.*

Indeed, we do not have a complete picture like in the even case. The difficulty here is the fact that the function ξ_ϵ is not odd so the knowledge of $\xi_\epsilon^{(2n)}(0)$ for all $n \in \mathbb{N}$ is not sufficient. And, the series expansions of the functions χ_ϵ and $\widehat{\chi}_\epsilon$ are not immediate.

2.2 With nonquadratic interaction

This paragraph deals with the thirdness of the invariant probabilities for Diffusion (I) in the small-noise case when the interacting potential is not quadratic and when the confining potential is even. It has been proved in [HT10a] (Theorem 3.2) that there are exactly three invariant probabilities if F is quadratic and if V is even and such that $V^{(4)} \geq 0$ for ϵ sufficiently small. This statement has been extended in [Tug10] (Theorem 1.11). There are exactly three invariant probabilities if the degree of the confining potential V is strictly more than the one of the interacting potential F and if $V''(0) + F''(0) > 0$. We aim to extend to the cases $\deg(V) = \deg(F)$ and $V''(0) + F''(0) = 0$.

The difficulty which occurs in the synchronized but non-strictly synchronized case, that is $V''(0) + F''(0) = 0$, is related to the symmetric invariant probability. The method which has been used in [HT10b, HT09, Tug10] in order to obtain the uniqueness of the symmetric invariant probability consists in providing the first order approximations of the first $\deg(F) - 1$ moments of any symmetric invariant probability in order to confine these moments in a parallelepiped \mathcal{K}_ϵ with ridges dominated by ϵ . Then, we construct a function which acts on these moments, which is equal to 0 if and only if the moments characterize an invari-

ant probability and which is a bijection from \mathcal{K}_ϵ to its image. However, if the quantity $V''(0) + F''(0)$ is equal to 0, we cannot confine the moments in such a parallelepiped, see Theorem 1.4 in [HT09]. Here, in order to solve the case $V''(0) + F''(0) = 0$, we transfer the excedent of convexity from F to V then we apply Theorem 2.4 in [CMV03]. Let us notice that we can also use a coupling method.

The difficulty related to the assumption $\deg(V) = \deg(F)$ is linked to an initial and sufficient condition which is used in [HT10b, Tug10] for proving the convergence in the small-noise limit of any sequence of invariant probabilities $(\mu^{\epsilon_k})_k$ where $(\epsilon_k)_k$ is a decreasing sequence which converges to 0 as k goes to infinity. Let us recall this initial condition.

(H) The family $\{\int_{\mathbb{R}} x^{2n} \mu^\epsilon(dx), \epsilon > 0\}$ is bounded for any family of invariant probabilities $\{\mu^\epsilon; \epsilon > 0\}$, with $2n := \deg(F)$.

Condition (H) is satisfied if $\deg(V) > \deg(F)$, see Step 3.2 in the proof of Theorem 1.11 in [Tug10]. We extend it to $\deg(V) = \deg(F)$.

Let us first prove the uniqueness of the symmetric invariant probability for sufficiently small ϵ .

Proposition 2.5. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M) and (E).*

Thus, for ϵ small enough, Diffusion (I) admits a unique symmetric invariant probability.

Proof. Step 1. Let us first notice that the uniqueness of the symmetric invariant probability has already been pointed out in [HT10a] if F is quadratic. From now on, we assume that it is not quadratic. Consequently, $F''(x) > F''(0)$ for all $x \neq 0$.

Step 2. We now define the two potentials $V_0(x) := V(x) + \frac{F''(0)}{2}x^2$ and $F_0(x) := F(x) - \frac{F''(0)}{2}x^2$. Since the initial law μ_0 is symmetric, the law μ_t is symmetric for all $t \geq 0$. Consequently, the first moment $\int_{\mathbb{R}} x \mu_t(dx)$ is equal to 0 for all $t \geq 0$. This implies that Diffusion (I) is the solution of the stochastic differential equation

$$\begin{cases} Y_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t V'_0(Y_s) ds - \int_0^t F'_0 * \nu_s(Y_s) ds, \\ \nu_s = \mathcal{L}(Y_s). \end{cases} \quad (2.15)$$

Step 3. By construction, V_0 and F_0 are convex. Moreover, $V''_0(x) > 0$ and $F''_0(x) > 0$ if $x \neq 0$. We apply Theorem 2.4 in [CMV03] and we obtain the uniqueness of the invariant probability for Diffusion (2.15) which means that there is a unique symmetric invariant probability for Diffusion (I). \square

We now extend Theorem 3.2 in [HT10a] by proving the thirdness of the invariant probabilities for ϵ small enough. Before it, we need to show that Condition (H) holds.

Proposition 2.6. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M) and (E). If $\deg(V) \geq \deg(F) =: 2n$,*

Condition (H) is satisfied, that is the family $\{\int_{\mathbb{R}} x^{2n} \mu^\epsilon(dx); \epsilon > 0\}$ is bounded for any family of invariant probabilities $\{\mu^\epsilon; \epsilon > 0\}$.

Proof. It has already been proved when $\deg(V) > \deg(F)$. It remains then to prove it when $\deg(V) = \deg(F) = 2n$. We proceed a *reductio ad absurdum* by assuming the existence of a decreasing sequence $(\epsilon_k)_{k \in \mathbb{N}}$ which converges to 0 and of a family of invariant probability measures $\{\mu^{\epsilon_k}; k \in \mathbb{N}\}$ such that the sequence $\mu_{2n}(k) := \int_{\mathbb{R}} x^{2n} \mu^{\epsilon_k}(dx)$ tends to infinity as k goes to infinity.

Step 1. Since F and V are two polynomial functions, we can write the l th moment of μ^ϵ in the following form:

$$\mu_l(k) = \frac{\int_{\mathbb{R}} x^l \exp\left[-\frac{2}{\epsilon_k} \left(\sum_{r=1}^{2n} \mathcal{M}_r(k) x^r\right)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon_k} \left(\sum_{r=1}^{2n} \mathcal{M}_r(k) x^r\right)\right] dx},$$

$$\text{with } \mathcal{M}_r(k) := \frac{1}{r!} \left\{ V^{(r)}(0) + \sum_{j=0}^{2n-r} \frac{(-1)^j}{j!} F^{(j+r)}(0) \mu_j(k) \right\}$$

for all $1 \leq l \leq 2n$. Let us notice that the highest moment which intervenes in $\mathcal{M}_r(k)$ is the one of degree $2n - r$.

Step 2. We prove in this step the existence of an index r such that $|\mathcal{M}_r(k)|$ tends to infinity. Indeed, if such an index does not exist, we can extract a subsequence of $(\epsilon_k)_k$ (we continue to write it $(\epsilon_k)_k$ for simplifying) such that each $\mathcal{M}_l(k)$ converges, for $1 \leq l \leq 2n$. Then, due to the positivity of $\mu_{2n}(k) = \frac{V^{(2n)}(0) + F^{(2n)}(0)}{(2n)!}$, the application of Lemma A.4 in [Tug10] provides the existence of a subsequence of $\mu_{2n}(k)$ which converges to a real as k goes to infinity. This is impossible, since $(\mu_{2n}(k))$ goes to infinity as k goes to infinity.

Step 3. We now introduce the sequences

$$\eta_r(k) := \mathcal{M}_r(k) (\mu_{2n}(k))^{-\left(1 - \frac{r}{2n}\right)},$$

for all $1 \leq r \leq 2n$. The change of variable $x := (\mu_{2n}(k))^{\frac{1}{2n}} y$ provides

$$\frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}} = \frac{\int_{\mathbb{R}} y^l \exp\left[-\frac{2\mu_{2n}(k)}{\epsilon_k} \left(\sum_{r=1}^{2n} \eta_r(k) y^r\right)\right] dy}{\int_{\mathbb{R}} \exp\left[-\frac{2\mu_{2n}(k)}{\epsilon_k} \left(\sum_{r=1}^{2n} \eta_r(k) y^r\right)\right] dy}.$$

Step 4. By recalling that the highest moment which intervenes in $\mathcal{M}_r(k)$ is the one of degree $2n - r$, the Jensen inequality implies the existence of $C > 0$ such that

$$|\mathcal{M}_r(k)| \leq C \mu_{2n}(k)^{\frac{2n-r}{2n}},$$

for all $1 \leq r \leq 2n$. We obtain the higher-bound $|\eta_r(k)| \leq C$. By considering a subsequence of $(\epsilon_k)_k$ (we continue to write it ϵ_k for simplifying), we get the convergence of $\eta_r(k)$ to some η_r , for all $1 \leq r \leq 2n$.

Step 5. Besides, the quantity $\mu_l(k) \mu_{2n}(k)^{-\frac{l+1}{2n}}$ tends to 0 as k goes to infinity,

for all $1 \leq l \leq 2n$. Indeed, $\mu_l(k)\mu_{2n}(k)^{-\frac{l}{2n}}$ is bounded and $\mu_{2n}(k)^{\frac{1}{2n}}$ goes to infinity as k goes to infinity. This implies the convergence to 0 as k goes to infinity of

$$\frac{1}{r!} \left\{ V^{(r)}(0) + \sum_{j=0}^{2n-r-1} \frac{(-1)^j}{j!} F^{(j+r)}(0) \mu_j(k) \right\} (\mu_{2n}(k))^{-\left(1-\frac{r}{2n}\right)},$$

for all $1 \leq r \leq 2n - 1$. Consequently, we have the limit

$$\frac{(-1)^{2n-l}}{(2n-l)!} F^{(2n)}(0) \lim_{k \rightarrow +\infty} \frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}} = \eta_l, \quad (2.16)$$

for each $1 \leq l \leq 2n - 1$. Moreover, $\eta_{2n} = \frac{V^{(2n)}(0) + F^{(2n)}(0)}{(2n)!}$.

Step 6. According to Lemma A.4 in [Tug10], we can extract a subsequence (we continue to write it $(\epsilon_k)_k$ for simplifying) such that

$$\lim_{k \rightarrow +\infty} \frac{\int_{\mathbb{R}} y^l \exp \left[-\frac{2\mu_{2n}(k)}{\epsilon_k} \left(\sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy}{\int_{\mathbb{R}} \exp \left[-\frac{2\mu_{2n}(k)}{\epsilon_k} \left(\sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy} = \sum_{s=1}^q p_s A_s^l,$$

where $A_1 < \dots < A_q$ are the localizations of the global minimum of the polynomial function $U_0(x) := \sum_{j=1}^{2n} \eta_j x^j$ and $p_1 + \dots + p_q = 1$ with $p_i \geq 0$ for all $1 \leq i \leq q$. Hence, we obtain

$$\lim_{k \rightarrow +\infty} \frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}} = \sum_{s=1}^q p_s A_s^l. \quad (2.17)$$

Step 7. By combining Eq. (2.16) and Eq. (2.17), we obtain

$$\eta_l = \frac{(-1)^{2n-l}}{(2n-l)!} F^{(2n)}(0) \sum_{s=1}^q p_s A_s^l,$$

for all $1 \leq l \leq 2n - 1$. Thus, for all $x \in \mathbb{R}$, we have

$$U_0(x) = \frac{F^{(2n)}(0)}{(2n)!} \sum_{s=1}^q p_s (x - A_s)^{2n} + \frac{V^{(2n)}(0)}{(2n)} x^{2n}.$$

Step 8. By definition, $U_0'(A_q) = 0$. If $q \geq 2$, since $A_q - A_i$ is positive for all $1 \leq i \leq q - 1$, it implies $A_q < 0$. By the same argument, we have $A_1 > 0$. This is impossible.

If $q = 1$, the equality $U_0'(A_1) = 0$ implies

$$\frac{F^{(2n)}(0)}{(2n-1)!} (A_1 - A_1)^{2n-1} + \frac{V^{(2n)}(0)}{(2n-1)} A_1^{2n-1} = 0,$$

so $A_1 = 0$. By using Eq. (2.17) with $l := 2n$, we obtain $\lim_{k \rightarrow \infty} 1 = 0$. This is absurd. \square

Let us remark that we cannot extend this method to the case $\deg(V) < \deg(F)$. Indeed, we can proceed similarly from Step 1 to Step 7 but in Step 8, we need $\frac{V^{(2n)}(0)}{(2n)!}$ to be present in order to obtain $A_1 = 0$. However, if $\deg(V) < \deg(F)$, in Step 5, this term disappears by taking the limit $k \rightarrow +\infty$. Then, it is impossible to conclude immediately to an absurdity.

We now give the claimed thirdness result.

Theorem 2.7. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M) and (E). If $\deg(V) \geq \deg(F) =: 2n$ and if $V''(0) + F''(0) \geq 0$, Diffusion (I) admits exactly three invariant probabilities for ϵ small enough.*

Proof. Step 1. According to Proposition 2.6, Condition (H) is satisfied. We apply Lemma 3.2 and Proposition 3.3 in [HT10b] thus each family $\{\mu^\epsilon; \epsilon > 0\}$ of invariant probabilities of Diffusion (I) admits an adherence value. Furthermore, since $F''(0) + V''(0) \geq 0$, Proposition 3.7 and Remark 3.8 in [HT10b] imply that there are exactly three possible adherence values, δ_0 , δ_a and δ_{-a} .

Step 2. According to Theorem 4.5 in [HT10a], Diffusion (I) admits a symmetric invariant probability. Theorem 2.5 implies the uniqueness of this symmetric invariant probability for ϵ small enough.

Step 3. Corollary 1.9 in [Tug10] provides the existence of at least two asymmetric invariant probabilities. It now remains to prove that there are exactly two asymmetric invariant probabilities for ϵ small enough. We know by Step 1 that the possible adherence values of these asymmetric invariant probabilities are δ_a and δ_{-a} . We proceed exactly like in Theorem 1.11 in [Tug10] by using the rate of convergence method from [HT09] (Theorem 1.5) and we obtain the thirdness of the invariant probabilities for ϵ small enough. \square

3 Numerical estimations of the phase transitions

It has been proved in previous section that under easily checked hypotheses, there exist several phases, one where Diffusion (I) admits a unique invariant probability and one where it admits several ones.

We have a more complete picture if the confining potential V and the interacting potential F satisfy the hypotheses (M), (E) and (Q).

In this section, we aim to provide simulations of the critical values of ϵ which separate the different phases. Let us begin with the simplest case, an example which illustrates Theorem 2.1.

3.1 With quadratic interaction and even confinement

Let us choose $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\alpha}{2}x^2$ with $\alpha > 0$. By $\epsilon_c(\alpha)$, we denote the critical value pointed out in Theorem 2.1 and which corresponds to the phase transition. For all $\alpha > 0$, $\epsilon_c(\alpha)$ is defined as the solution of Eq. (2.4),

that here is

$$\int_{\mathbb{R}_+} \left\{ x^2 - \frac{1}{2\alpha} \right\} \exp \left[(1 - \alpha) x^2 - \frac{\epsilon}{2} x^4 \right] dx = 0.$$

By making the change of variable $z := \sqrt{\frac{1}{2\alpha}} x$, we obtain

$$\mathbb{E} \left\{ \Lambda(\epsilon_c(\alpha), \alpha, X) \right\} = 0 \quad \text{with} \quad \Lambda(x, y, z) := (z^2 - 1) \exp \left[\frac{z^2}{2y} - \frac{x}{8y^2} z^4 \right]$$

and $\mathcal{L}(X) = \mathcal{N}(0, 1)$. In order to approximate $\epsilon_c(\alpha)$, we choose $r > 0$ and $N \in \mathbb{N}^*$. We take N independent and identically distributed random variables with common law $\mathcal{N}(0, 1)$, $(X_i)_{1 \leq i \leq N}$. The Monte-Carlo method gives the following estimation of $\epsilon_c(\alpha)$:

$$\epsilon_c^{r,N}(\alpha) := r \min \left\{ p \in \mathbb{N}^* \mid \sum_{i=1}^N \Lambda(pr, \alpha, X_i) < 0 \right\}.$$

After using the weak law of the large numbers, we get the inequalities

$$\lim_{N \rightarrow +\infty} \epsilon_c^{r,N}(\alpha) - r \leq \epsilon_c(\alpha) \leq \lim_{N \rightarrow +\infty} \epsilon_c^{r,N}(\alpha).$$

We take $N := 2 \times 10^5$ and $r := 10^{-4}$ thus we obtain the following curve.

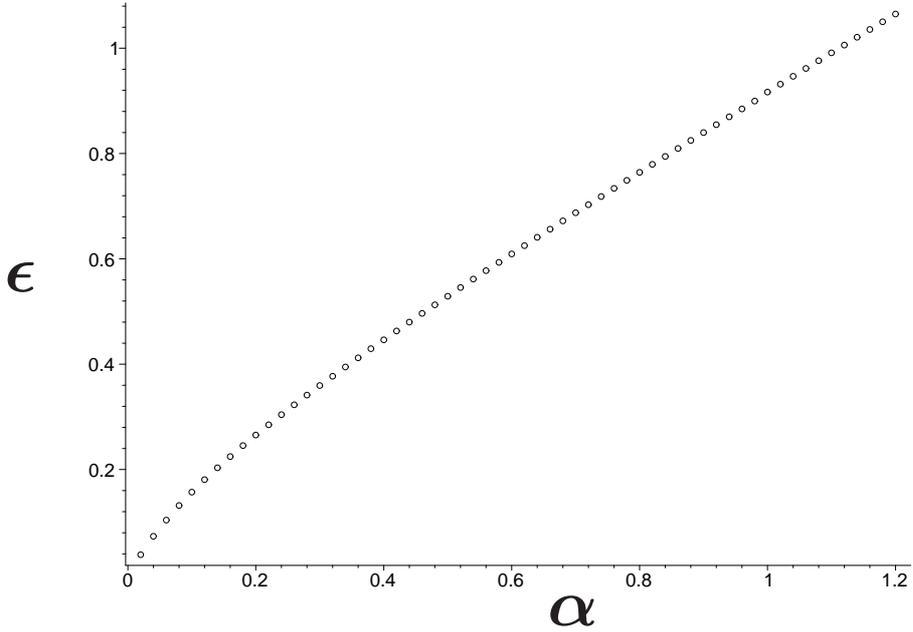


Figure 1: Critical value, $\epsilon_c(\alpha)$.

We remark that it is increasing. And, it seems to be almost linear sufficiently far from 0. In fact, simple computations provide

$$\lim_{\alpha \rightarrow +\infty} \frac{\epsilon_c(\alpha)}{\alpha} = \frac{2}{3} \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \frac{\epsilon_c(\alpha)}{\alpha} = 2.$$

Let us notice that this estimation is easy since it only requires the knowledge of the second derivative of the function ξ_ϵ at the point 0.

3.2 With nonquadratic interaction and even confinement

We now deal with a nonquadratic interacting potential F . There exists a symmetric invariant probability. Hence, the study consists in finding the critical value ϵ_c such that Diffusion (I) admits another invariant probability if $\epsilon < \epsilon_c$.

In order to simplify, we restrict ourselves to the simplest case. From now on, we take

$$V(x) := \frac{x^4}{4} - \frac{x^2}{2} \quad \text{and} \quad F(x) := \frac{\beta}{4}x^4 \quad \text{with } \beta > 0.$$

By Subsection 4.2 in [HT10a], there is a unique symmetric invariant probability μ_0^ϵ for all $\epsilon > 0$ and

$$\mu_0^\epsilon(dx) = Z^{-1} \exp \left[-\frac{2}{\epsilon} \left(\frac{1+\beta}{4}x^4 + \frac{3\beta m_2 - 1}{2}x^2 \right) \right] dx,$$

where

$$m_2 = \frac{\int_{\mathbb{R}} x^2 \exp \left[-\frac{2}{\epsilon} \left(\frac{1+\beta}{4}x^4 + \frac{3\beta m_2 - 1}{2}x^2 \right) \right] dx}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(\frac{1+\beta}{4}x^4 + \frac{3\beta m_2 - 1}{2}x^2 \right) \right] dx}$$

is unique. A simple computation provides

$$\Upsilon_\epsilon(\mu_0^\epsilon) = -\frac{\epsilon}{2} \log \left\{ \int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(U_\beta(x) + \frac{3\beta}{2}m_2x^2 \right) \right] dx \right\} - \frac{3\beta}{4}m_2^2,$$

with $U_\beta(x) := \frac{1+\beta}{4}x^4 - \frac{x^2}{2}$.

Let us recall that we did not prove that there is a unique phase transition. However, in the following, when we use the expression “critical value”, we implicitly refer to

$$\epsilon_c(\beta) := \min \{ \epsilon_0 \mid \forall 0 < \epsilon < \epsilon_0, \# \mathcal{S}_\epsilon > 1 \},$$

the smallest critical value.

3.2.1 Low-energy method

We know that the free-energy is nonincreasing along the trajectory $(\mu_t)_t$ so if there exists a probability measure with free-energy less than the one of μ_0^ϵ , we immediately deduce that there is another invariant probability. We can use this method in order to minorate the critical value.

Proposition 3.1. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M) and (E). We have the following lower-bound of the critical value:*

$$\epsilon_c(\beta) \geq \sup \left\{ \epsilon_0 > 0 \mid \forall \epsilon \in]0; \epsilon_0[, \inf_{\mu \in \mathcal{M}_{8q^2}} \Upsilon_\epsilon(\mu) < \Upsilon_\epsilon(\mu_0^\epsilon) \right\} \quad (3.1)$$

where \mathcal{M}_{8q^2} is the set of the probability measures μ absolutely continuous with respect to the Lebesgue measure with density u such that $\int_{\mathbb{R}} x^{8q^2} u(x) dx < +\infty$.

This result may seem hard to apply since the question still consists in studying an application in a space which dimension is infinity. However, there is an obvious candidate to test. It has been used in Corollary 1.9 in [Tug10] for proving the existence of asymmetric invariant probabilities. We define the probability measure

$$\nu_a^\epsilon(dx) := Z_\epsilon^{-1} \exp \left[-\frac{2}{\epsilon} (V(x) + F(x-a)) \right] dx$$

where $a \in \mathbb{R}_+$ is the localization of the global minimum of V . Indeed, ν_a^ϵ is close to δ_a in the small-noise limit. By using a method similar to the one of Corollary 1.9 in [Tug10], we get the existence of $\hat{\epsilon}_0 > 0$ such that for all $\epsilon \in]0; \hat{\epsilon}_0[$, $\Upsilon_\epsilon(\nu_a^\epsilon) < \Upsilon_\epsilon(\mu_0^\epsilon)$.

Example 3.2. *We take $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\beta}{4}x^4$ with $\beta > 0$. By $\epsilon_c^1(\beta)$, we denote the critical value which corresponds to the transition between $\{\Upsilon_\epsilon(\mu_0^\epsilon) < \Upsilon_\epsilon(\nu_1^\epsilon)\}$ and $\{\Upsilon_\epsilon(\mu_0^\epsilon) > \Upsilon_\epsilon(\nu_1^\epsilon)\}$. We recall*

$$\Upsilon_\epsilon(\mu_0^\epsilon) = -\frac{\epsilon}{2} \log \left\{ \int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(U_\beta(x) + \frac{3\beta}{2} m_2 x^2 \right) \right] dx \right\} - \frac{3\beta}{4} m_2^2,$$

with $U_\beta(x) := \frac{1+\beta}{4}x^4 - \frac{x^2}{2}$. In the other hand, a simple computation leads to

$$\begin{aligned} \Upsilon_\epsilon(\nu_1^\epsilon) &= -\frac{\epsilon}{2} \log \left\{ \int_{\mathbb{R}} e^{-\frac{2}{\epsilon}(V(x)+F(x-1)-F(1))} dx \right\} + \beta n_1 - \frac{3\beta}{2} n_2 \\ &\quad + \beta n_3 - \beta n_1 n_3 + \frac{3\beta}{4} n_2^2, \end{aligned}$$

$$\text{with } n_k := \frac{\int_{\mathbb{R}} x^k \exp \left[-\frac{2}{\epsilon} (V(x) + F(x-1)) \right] dx}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} (V(x) + F(x-1)) \right] dx}.$$

We proceed a simulation similar to the one of Subsection 3.1 with $N := 2 \times 10^5$ and we obtain Figure 2, see page 28 (the points which correspond to $\epsilon_c^1(\beta)$ described in this method are the circles).

Let us remark that we can do the same with $F(x) := \frac{\beta}{4}x^4 + \frac{\alpha}{2}x^2$.

3.2.2 Convexity of the free-energy method

Now, we point out the particular link between the function ξ_ϵ (defined in Eq. (2.1)) and the free-energy of the measures μ_m^ϵ (defined in Eq. (1.1)). Let us temporarily look at the quadratic interaction case.

Lemma 3.3. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M), (E) and (Q).*

Thus for all $\epsilon > 0$ and $m \in \mathbb{R}$, the following equality holds.

$$\frac{d}{dm} \Upsilon_\epsilon(\mu_m^\epsilon) = -\frac{F''(0) \xi_\epsilon(m)}{Z_\epsilon(m)} \text{Var}(\mu_m^\epsilon), \quad (3.2)$$

$$\text{with } \mu_m^\epsilon(dx) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + \frac{F''(0)}{2}x^2 - F''(0)mx\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + \frac{F''(0)}{2}y^2 - F''(0)my\right)\right] dy} dx.$$

Proof. We remark that μ_m^ϵ is absolutely continuous with respect to the Lebesgue measure. By u_m^ϵ , we denote its density. Thus, we have

$$\begin{aligned} \Upsilon_\epsilon(\mu_m^\epsilon) &= \int_{\mathbb{R}} \left(\frac{\epsilon}{2} \log(u_m^\epsilon(x)) + V(x) + \frac{1}{2} F * u_m^\epsilon(x) \right) u_m^\epsilon(x) dx \\ &= -\frac{\epsilon}{2} \log(Z_\epsilon(m)) + F''(0)m \int_{\mathbb{R}} x u_m^\epsilon(x) dx - \frac{F''(0)}{2} \left(\int_{\mathbb{R}} x u_m^\epsilon(x) dx \right)^2. \end{aligned}$$

By noting that $\int_{\mathbb{R}} x u_m^\epsilon(x) dx = \frac{\xi_\epsilon(m) + m}{Z_\epsilon(m)}$, we get

$$\Upsilon_\epsilon(\mu_m^\epsilon) = -\frac{\epsilon}{2} \log\left[Z_\epsilon(m) e^{-\frac{F''(0)}{\epsilon} m^2}\right] - \frac{F''(0)}{2} \frac{\xi_\epsilon(m)^2}{Z_\epsilon(m)^2}.$$

The derivation of this equality provides the result. \square

Consequently, $\xi_\epsilon(m) \frac{d}{dm} \Upsilon_\epsilon(\mu_m^\epsilon) = -\frac{F''(0) \text{Var}(\mu_m^\epsilon)}{Z_\epsilon(m)} \xi_\epsilon(m)^2 \leq 0$. We deduce that the behavior of the function $m \mapsto \Upsilon_\epsilon(\mu_m^\epsilon)$ is directly linked to the sign of ξ_ϵ . We now provide a link between the critical value and the second derivative of the function $m \mapsto \Upsilon_\epsilon(\mu_m^\epsilon)$ in $m = 0$.

Proposition 3.4. *We assume that the confining potential V and the interacting potential F satisfy the hypotheses (M), (E) and (Q).*

Thus, the critical value ϵ_c is the only zero of the function $\epsilon \mapsto \frac{d^2}{dm^2} \Upsilon_\epsilon(\mu_m^\epsilon) \Big|_{m=0}$.

Furthermore, for all $\epsilon < \epsilon_c$, we have $\Upsilon_\epsilon(\mu_+^\epsilon) = \Upsilon_\epsilon(\mu_-^\epsilon) < \Upsilon_\epsilon(\mu_0^\epsilon)$ where μ_\pm^ϵ are the two asymmetric invariant probabilities.

Proof. By taking the derivative in Eq. (3.2), we get

$$\frac{d^2}{dm^2} \Upsilon_\epsilon(\mu_m^\epsilon) = -\frac{\alpha \text{Var}(\mu_m^\epsilon)}{Z_\epsilon(m)} \xi_\epsilon'(m) - \xi_\epsilon(m) \frac{d}{dm} \left[\frac{\alpha \text{Var}(\mu_m^\epsilon)}{Z_\epsilon(m)} \right].$$

As $\xi_\epsilon(0) = 0$, we obtain

$$\frac{d^2}{dm^2} \Upsilon_\epsilon(\mu_m^\epsilon) \Big|_{m=0} = -\frac{\alpha \operatorname{Var}(\mu_0^\epsilon)}{Z_\epsilon(0)} \xi_\epsilon'(0).$$

We recall that $\xi_\epsilon'(0) = \gamma_0(\epsilon)$ (defined in the proof of Theorem 2.1). According to Theorem 2.1, the critical value of ϵ_c is the only solution of $\gamma_0(\epsilon) = 0$. Consequently there is only one value of ϵ such that $\frac{d^2}{dm^2} \Upsilon_\epsilon(\mu_m^\epsilon) \Big|_{m=0} = 0$ and this value is ϵ_c .

When $\epsilon < \epsilon_c$, there are exactly three invariant probabilities, μ_+^ϵ , μ_-^ϵ and μ_0^ϵ . The symmetry directly implies $\Upsilon_\epsilon(\mu_+^\epsilon) = \Upsilon_\epsilon(\mu_-^\epsilon)$.

By definition of the two asymmetric invariant probabilities, $\int_{\mathbb{R}} x \mu_+^\epsilon(dx)$ is the only positive zero of the function ξ_ϵ and consequently the only positive critical point of the function $m \mapsto \Upsilon_\epsilon(\mu_m^\epsilon)$ according to Lemma 3.3. Since $\epsilon < \epsilon_c$, $\frac{d^2}{dm^2} \Upsilon_\epsilon(\mu_m^\epsilon) \Big|_{m=0} < 0$ which implies that the function $m \mapsto \Upsilon_\epsilon(\mu_m^\epsilon)$ reaches its global minimum in $\int_{\mathbb{R}} x \mu_+^\epsilon(dx)$ and in $\int_{\mathbb{R}} x \mu_-^\epsilon(dx)$. Also, $m \mapsto \Upsilon_\epsilon(\mu_m^\epsilon)$ reaches a local maximum in $0 = \int_{\mathbb{R}} x \mu_0^\epsilon(dx)$. Immediately, we get $\Upsilon_\epsilon(\mu_\pm^\epsilon) < \Upsilon_\epsilon(\mu_0^\epsilon)$. \square

We now return to the nonquadratic interaction case.

Example 3.5. We take $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $F(x) := \frac{\beta}{4}x^4$ with $\beta > 0$. According to Eq. (3.1), we get this new lower-bound on the critical value:

$$\epsilon_c(\beta) \geq \sup \left\{ \epsilon_0 > 0 \mid \forall \epsilon \in]0; \epsilon_0[, \min \operatorname{Sp}(J_\epsilon(m^0)) < 0 \right\},$$

where $J_\epsilon(m)$ is the Hessian matrix of the function $m := (m_1, m_2, m_3) \mapsto \Upsilon_\epsilon(\mu_m^\epsilon)$ and $m^0 := (0, m_2^0, 0)$ is the only point in \mathbb{R}^3 such that $\mu_{m^0}^\epsilon$ is the unique symmetric invariant probability. Here, we defined μ_m^ϵ by

$$\mu_m^\epsilon(dx) = \frac{\exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\beta}{4}x^4 - \beta m_1 x^3 + \frac{3\beta}{2}m_2 x^2 - \beta m_3 x \right) \right]}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(V(y) + \frac{\beta}{4}y^4 - \beta m_1 y^3 + \frac{3\beta}{2}m_2 y^2 - \beta m_3 y \right) \right] dy} dx.$$

For all $k \geq 0$, by m_{2k} we denote the $2k$ th order moment of $\mu_{m^0}^\epsilon$. After tedious computations, we find the three eigenvalues of the matrix $J_\epsilon(m^0)$. The first one is

$$\lambda_1(\epsilon) := \frac{9\beta^2}{4\epsilon} (m_4 - m_2^2) \left(1 + \frac{3\beta}{\epsilon} (m_4 - m_2^2) \right)$$

which is positive for all $\epsilon > 0$. The two other eigenvalues $\lambda_2(\epsilon)$ and $\lambda_3(\epsilon)$ are the two solutions of the equation

$$X^2 - (m_2 + m_6) \left(1 - \frac{4\beta}{\epsilon} m_4 \right) X + C(\epsilon) = 0,$$

$$\text{with } C(\epsilon) := m_2 m_6 \left(1 - \frac{4\beta}{\epsilon} m_4 \right)^2 - \left(m_4 - \frac{2\beta}{\epsilon} m_2 m_6 - \frac{2\beta}{\epsilon} m_4^2 \right)^2.$$

One of the eigenvalue $\lambda_2(\epsilon)$ or $\lambda_3(\epsilon)$ is negative if and only if one of the two quantities $1 - \frac{4\beta}{\epsilon}m_4$ or $C(\epsilon)$ is negative. Consequently, we can characterize the critical value ϵ_c by looking only at the second derivative of the function $m := (m_1, m_2, m_3) \mapsto \Upsilon_\epsilon(u_\epsilon^m)$ in $(0, m_2, 0)$. This local method thus provides

$$\epsilon_c(\beta) \geq \epsilon_c^2(\beta) := \min \left\{ \epsilon_0 > 0 \mid \forall 0 < \epsilon < \epsilon_0, \min \left\{ C(\epsilon); 1 - \frac{4\beta}{\epsilon}m_4 \right\} < 0 \right\}.$$

We proceed a simulation similar to the one of Subsection 3.1 with $N := 2 \times 10^5$ and we obtain the following curve.

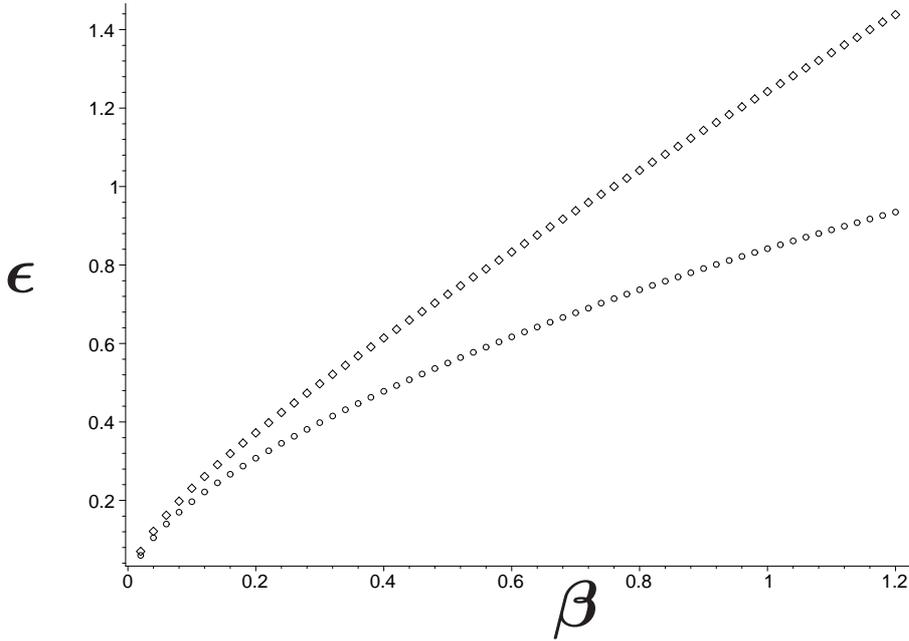


Figure 2: Critical values, $\epsilon_c^1(\beta)$ and $\epsilon_c^2(\beta)$.

The points corresponding to the method described in this paragraph are the diamonds. The circles denote the points of the previous method.

Let us compare these two simulations. Both are based on the same observation. If there are several invariant probabilities, the symmetric one is not a minimizer of the free-energy functional, for sufficiently small ϵ .

The second method, based on the Hessian of this free-energy functional, is local. Hence, it is more precise. However, the computations are tedious, even in the simplest example, $F(x) := \frac{\beta}{4}x^4$.

3.3 With asymmetric confinement

In this paragraph, the confining potential V does not satisfy (E). However, we provide the numerical estimations with a quadratic interacting potential,

$F(x) := \frac{\alpha}{2}x^2$ with $\alpha > 0$.

It remains to study the number of roots of the function χ_ϵ , defined by

$$\chi_\epsilon(m) = \frac{\int_{\mathbb{R}} x \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx \right) \right] dx}{\int_{\mathbb{R}} \exp \left[-\frac{2}{\epsilon} \left(V(x) + \frac{\alpha}{2}x^2 - \alpha mx \right) \right] dx} - m.$$

However, it is not as simple as if V is even. Indeed, when the confining potential V is even, there is an obvious solution, that is 0, then the number of invariant probabilities directly is linked to the second derivative in this obvious solution. Here, there is not any obvious solution so we need to know the whole trajectory of the function.

The simulation is the following. We compute $\chi_\epsilon \left(a_1 + \frac{k}{M}(a_3 - a_1) \right)$ for all $0 \leq k \leq M$ by using the Monte Carlo method. Then, we count the number of changes of sign. If there is a unique change of sign, we conclude that we are over the critical value because there is uniqueness of the invariant probability measure. If there are several changes of sign, we deduce that we are under the critical value.

We apply this approximation to $V(x) := \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2}$ and $F(x) := \frac{\alpha}{2}x^2$ for $\alpha := j \times 0.01$ with $1 \leq j \leq 120$ and $\epsilon := i \times 0.005$ with $1 \leq i \leq 60$. We here take $M := 250$. We proceed a simulation similar to the one of Subsection 3.1 with $N := 2 \times 10^5$ and we obtain the following curve.

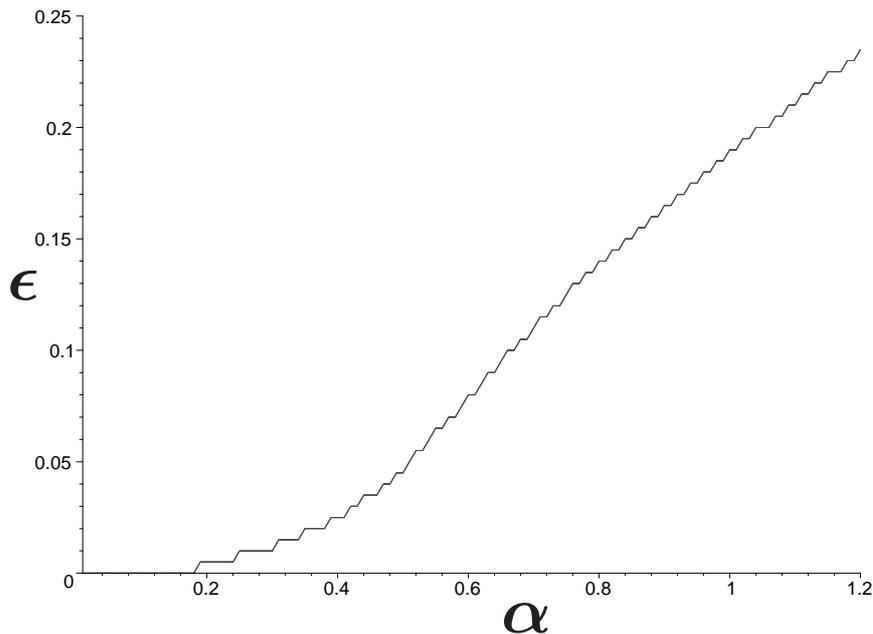


Figure 3: Critical value, $\epsilon_c(\alpha)$.

Let us remark that there is uniqueness of the invariant probability if the interaction is sufficiently small. Indeed, a simple study of the small-noise limit

of the function χ_ϵ implies that χ_ϵ admits a unique zero if α and ϵ are small enough.

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