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# Phase transitions of McKean-Vlasov processes in double-wells landscape\*

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## Abstract

We prove under simple assumptions that there exist several phases for the self-stabilizing processes in a non-convex landscape. When the coefficient of diffusion is small, there are at least three stationary measures and when it is sufficiently large, there is a unique one. The non-uniqueness in small noise has already been proved in [Herrmann, Tugaut|2010] when the landscape is symmetric. Here, we will extend it in two directions: when the confining potential  $V$  is not symmetric and when the interacting potential  $F$  is not quadratic. The critical value will also be studied.

**Key words and phrases:** phase transitions ; McKean-Vlasov diffusions ; non-convexity ; stationary measures ; interacting particles systems ; propagation of chaos ; free-energy.

**2000 AMS subject classifications:** primary 60H10; secondary: 60G10, 60J60, 65C05, 82C22

## Introduction

We investigate the phase transitions of the McKean-Vlasov diffusion

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon}B_t - \int_0^t V'(X_s) ds - \int_0^t F' * u_s(X_s) ds \\ u_s = \mathcal{L}(X_s) \end{cases} . \quad (\text{I})$$

Here,  $*$  denotes the convolution. The particularity of this model is the fact that the own law of the process intervenes in the equation. Consequently, it is non-markovian since the past influences the drift through the law  $u_t$ . We remark

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especially  $\mathbb{E}_\mu [f(X_t)] \neq \int_{\mathbb{R}} \mathbb{E}_x [f(X_t)] d\mu(x)$ . We note that  $X_t$  and  $u_t$  depend on  $\epsilon$ . We do not write  $\epsilon$  for simplifying the reading.

The motion of the process is subject to three concurrent forces. The first one is the gradient of the so-called confining potential  $V$ . The second influence is a heat process  $(\sqrt{\epsilon} B_t)_{t \in \mathbb{R}_+}$  which allows the particle to escape from the stable domains of  $V$ . The third term represents the average tension between two processes of law  $u_t$ :  $F' * u_s(X_s(\omega_0)) = \int_{\Omega} F'(X_s(\omega_0) - Y_s(\omega)) d\mathbb{P}(\omega)$  where  $(X_t)_t$  and  $(Y_t)_t$  verify (I) and  $(\Omega, \mathcal{F}, \mathbb{P})$  is the underlying measurable space.

This kind of processes were introduced by McKean, see [McK67, McK66].

The diffusion  $X_t$  in (I) can be seen as one particle in a continuous mean-field system of an infinite number of particles. The mean-field system that we will consider is the continuous random dynamical system

$$\begin{cases} X_t^1 = X_0^1 + \sqrt{\epsilon} B_t^1 - \int_0^t V'(X_s^1) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t F'(X_s^1 - X_s^j) ds \\ \vdots \\ X_t^i = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V'(X_s^i) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t F'(X_s^i - X_s^j) ds \\ \vdots \\ X_t^N = X_0^N + \sqrt{\epsilon} B_t^N - \int_0^t V'(X_s^N) ds - \frac{1}{N} \sum_{j=1}^N \int_0^t F'(X_s^N - X_s^j) ds \end{cases} \quad (\text{II})$$

where the  $N$  brownian motions  $(B_t^i)_{t \in \mathbb{R}_+}$  are independents and the  $N$  random values  $X_0^1, \dots, X_0^N$  are iid and with law  $\mathcal{L}(X_0)$ .

The link between the self-stabilizing process and the mean-field system for  $N$  going to infinity is called the propagation of chaos, see [Szn91] under Lipschitz properties ; [BRTV98] if  $V$  is a constant ; [Mal01] or [Mal03] when both potentials are convex ; [BGV07, DPdH96, DG87] for a sharp estimate ; [CGM08] for a propagation of chaos uniform with respect to the time in the non-uniformly strictly convex case.

As proved in [DG87], the empirical law of the mean-field system satisfies a large deviations principle with a rate function which depends on the law of Diffusion (I). Consequently, the long-time behavior of  $\mathcal{L}(X_t)$  provides some consequences on the exit time for the particle system (II). But, according to [HT10a] and [Tug10b], if  $V$  is even, there are exactly three stationary measures for  $\epsilon$  small enough,  $\mathcal{L}(X_t)$  converges weakly towards a stationary measure and the domain of attraction of each limiting value is not reduced to a point. The study does depend on  $\epsilon$  so we can imagine that there is a phase transition between an area where Diffusion (I) admits exactly three stationary measures and an other one where it admits only one invariant probability.

Let us stress that the previous results about the non-uniqueness of the stationary measures and its consequences ([HT10a, HT10b, HT09, Tug10b]) deal only with the symmetric case. However, if we want to consider a potential which oscillates slowly on the time, we need to study the asymmetrical case.

Let us recall briefly some of the previous results on diffusions like (I). The existence problem can be solved by two different ways. The first one consists in the application of a fixed point theorem, see [BRTV98, HIP08]. The other consists

in a propagation of chaos of a mean-field system, see for example [Mél96].

In [McK67], the author proved that the law of the solution  $u_t$  admits a  $\mathcal{C}^\infty$ -continuous density  $u_t$  with respect to the Lebesgue measure for all  $t > 0$ . Moreover, he provided the following non-linear parabolic partial differential equation:

$$\frac{\partial}{\partial t} u_t = \frac{\partial}{\partial x} \left\{ \frac{\epsilon}{2} \frac{\partial}{\partial x} u_t + u_t (V' + F' * u_t) \right\}. \quad (\text{III})$$

Then we can study Equation (III) by probabilistic methods involving (I) or (II), see [CGM08, Fun84, Mal03]. Reciprocally, equation (III) is a useful tool for describing the stationary measure(s) and the long-time behavior, see [BRTV98, BRV98, Tam84, Tam87, Ver06]. In [HT10a], when  $V$  is symmetric and non-convex, by using (III), it has been proved that Diffusion (I) admits at least three stationary measures under assumptions easy to verify: one is symmetric and the two others are asymmetric. Moreover, Theorem 3.2 in the same article states the thirdness of the stationary measures for  $\epsilon$  small enough if  $V''$  is convex and  $F''$  is constant.

The estimates of the small-noise asymptotic of these three stationary measures are provided in [HT10b, HT09].

In the convex case (including the non-uniformly strictly convex case), Cattiaux, Guillin and Malrieu proceeded a uniform propagation of chaos in [CGM08] and obtained the uniqueness of the stationary measure and the convergence. Nevertheless, according to Proposition 5.17 and Remark 5.18 in [Tug10a], it is impossible to find a general result of uniform propagation of chaos. In the non-convex case and under two restrictions (the center of mass is fixed and  $V''(0) + F''(0) > 0$ ), Carrillo, McCann and Villani provided the convergence, see [CMV03].

The long-time behavior of the law  $u_t$  in the more difficult non-convex case has been the subject of [Tug10b]:  $u_t$  converges weakly towards a stationary measure if the initial entropy is finite. Furthermore, as written before, the basin of attraction of each limiting value is not reduced to a point.

In order to detect the type of regime in which Diffusion (I) is that is to say the uniqueness or the non-uniqueness of the stationary measures, we shall introduce the following free-energy, already used in [Tug10b]:

$$\Upsilon_\epsilon(u) := \int_{\mathbb{R}} \left\{ \frac{\epsilon}{2} \log(u(x)) + V(x) + \frac{1}{2} F * u(x) \right\} u(x) dx$$

for all the probability measures  $u$  which are absolutely continuous with respect to the Lebesgue measure ; with a density also denoted by  $u$ . We can note that  $u_t := \mathcal{L}(X_t)$  satisfies this hypothesis for all  $t > 0$ .

The paper is organized as follows. After giving the assumptions, we provide some results about the existence of stationary measures for self-stabilizing processes even when  $V$  is not symmetric. Particularly, we prove that there is at least one outlying stationary measure around each wells when  $\epsilon$  (resp.  $\alpha$ ) is small (resp. large) enough, with  $F(x) := \frac{\alpha}{2} x^2$ . Also, we state that there is at least

one intermediate stationary measure between each pair of outlying stationary measures. Subsequently, we show that there exists exactly two phases separated by a particular value of  $\epsilon$  (characterized as the zero of a Bessel function): one with a unique stationary measure and one with exactly three ones. Then, we state the uniqueness of the symmetric stationary measure if  $V$  is symmetric by using a uniform propagation of chaos with respect to the time. This permits to extend Theorem 1.11 in [Tug10b] to the case  $\deg(V) = \deg(F)$  and to the asymmetric case. Then, we show the uniqueness of the stationary measure if  $\deg(F) = 2$  if  $\epsilon$  is large enough which implies the existence of a phase transition. Finally, we study several methods for finding the critical value of  $\epsilon$  which separates the uniqueness phase and the non-uniqueness phase.

## Assumptions

We assume the following properties on the confining potential  $V$ :

- (V-1)  $V$  is a polynomial function with  $\deg(V) \geq 4$ .
- (V-2)  $V$  has exactly 3 critical points:  $a_1, a_2, a_3$ . Furthermore,  $V''(a_i) \neq 0$  for all  $1 \leq i \leq 3$ .
- (V-3)  $V(x) \geq C_4 x^4 - C_2 x^2$  for all  $x \in \mathbb{R}$  with  $C_2, C_4 > 0$ .
- (V-4)  $\lim_{x \rightarrow \pm\infty} V''(x) = +\infty$  and  $V''(x) > 0$  for all  $x \notin [a_1; a_3]$ .
- (V-5) Initialization:  $V(0) = 0$ .

Moreover, from now, we assume one of the two following hypotheses:

- (V-6)  $V$  is even and for all  $k \geq 2$ ,  $V^{(2k)}(0) \geq 0$ . We put  $a := a_3$ . Then  $a_1 = -a$  and  $a_2 = 0$ .
- (V-7) There exists  $a, b, c \in \mathbb{R}$  and  $\widehat{V}$  which satisfies (V-1)–(V-6) such that  $V(x) = \widehat{V}(x - c) + ax + b$  for all  $x \in \mathbb{R}$ .

Under (V-6), we know by [HT10a] that (I) admits at least one symmetric stationary measure. We note also that  $V''(a_1) > 0$ ,  $V''(a_3) > 0$  and  $V''(a_2) < 0$ .

Let us present now the properties under the interaction potential  $F$ :

- (F-1)  $F$  is an even polynomial function and  $\deg(F) =: 2n \geq 2$ .
- (F-2)  $F$  and  $F''$  are convex.
- (F-3) Initialization:  $F(0) = 0$ .

Sometimes, we will assume this additional hypothesis:

- (LIN)  $F'$  is linear.

For concluding the introduction, we write the statements of the two main results:

**Phase transition:** *Let  $V$  which satisfies (V-1)–(V-6). We can write:  $V(x) := -\frac{\vartheta_1}{2}x^2 + \sum_{p=2}^q \frac{\vartheta_p}{(2p)!}x^{2p}$  with  $\vartheta_p \geq 0$  for all  $1 \leq p \leq q$  and  $\vartheta_1\vartheta_q > 0$ . By taking  $F(x) := \frac{\alpha}{2}x^2$  with  $\alpha > 0$ , there exists  $\epsilon_c \in \mathbb{R}$  such that:*

- *For all  $\epsilon \geq \epsilon_c$ , Diffusion (I) admits a unique stationary measure.*
- *For all  $\epsilon < \epsilon_c$ , Diffusion (I) admits exactly three stationary measures.*

*Moreover, the critical value  $\epsilon_c$  is the unique solution of the equation:*

$$\int_{\mathbb{R}_+} \left( x^2 - \frac{1}{2\alpha} \right) \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx = 0.$$

**Thirdness in the small-noise:** *Let  $V$  and  $F$  two potentials which satisfy (V-1)–(V-6) and (F-1)–(F-3). Then, if  $V''(0) + F''(0) \geq 0$  and  $\deg(V) \geq \deg(F)$ , Diffusion (I) admits exactly three stationary measures for  $\epsilon$  small enough.*

**Uniqueness in the large-noise:** *Let  $V$  satisfying (V-1)–(V-5) and (V-7) and  $F(x) := \frac{\alpha}{2}x^2$  with  $\alpha > 0$ . There is a unique stationary measure for  $\epsilon$  large enough.*

## 1 Existence problem of the stationary measures

The aim of this section is to extend the results of [HT10a]. Particularly, the existence of stationary measure(s) for Diffusion (I) is examined when  $V$  is not symmetric. Let us note that all the results in this section are available when  $V$  has more than two wells. The method that we will use is similar to the one in [HT10a].

### 1.1 Linear case

Here, we assume (LIN) that is to say  $F(x) := \frac{\alpha}{2}x^2$  for all  $x \in \mathbb{R}$  with  $\alpha > 0$ . The research of a stationary measure remains *in fine* to a parametrization problem on  $\mathbb{R}$ . Indeed, as noted in Lemma 2.2 in [HT10a],  $u_\epsilon$  is invariant if and only if it verifies

$$u_\epsilon(x) = \frac{\exp \left[ -\frac{2}{\epsilon} \left( V(x) + \alpha \frac{x^2}{2} - \alpha m x \right) \right]}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( V(y) + \alpha \frac{y^2}{2} - \alpha m y \right) \right] dy} \quad \text{and} \quad m = \int_{\mathbb{R}} x u_\epsilon(x) dx. \quad (1.1)$$

We introduce three particular functions which will be used in the following:

**Definition 1.1.** For all  $m \in \mathbb{R}$  and  $\epsilon > 0$ , let us define

$$\begin{aligned} Z_\epsilon(m) &:= \int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \alpha \frac{x^2}{2} - \alpha m x \right) \right] dx, \\ \Psi_\epsilon(m) &:= \int_{\mathbb{R}} (x - m) \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \alpha \frac{x^2}{2} - \alpha m x \right) \right] dx \\ \text{and } \chi_\epsilon(m) &:= \frac{\Psi_\epsilon(m)}{Z_\epsilon(m)}. \end{aligned} \quad (1.2)$$

This function  $\chi_\epsilon$  is exactly the same than the one in [HT10a] but we insist on the fact that the function  $\Psi_\epsilon$  defined in [HT10a] is not the one that we defined here.

Finding a stationary measure to the self-stabilizing diffusion (I) remains to find a zero to the functions  $\Psi_\epsilon$  and  $\chi_\epsilon$ .

Let us recall that - according to the assumption (V-2) - the potential  $V$  has exactly three critical points:  $a_1, a_2, a_3$ . In the following, when we will deal with one of these points, we will use the general notation  $a_0$  without precising if it is a wells or not.

**Proposition 1.2.** Let  $V$  verifying the assumptions (V-1)–(V-5) and  $a_0$  one of its critical points such that  $\alpha + V''(a_0) > 0$ . We assume:

$$\alpha > 2 \sup_{x \neq a_0} \frac{V(a_0) - V(x)}{(a_0 - x)^2}. \quad (1.3)$$

Then, for all  $\delta \in ]0; 1[$ , there exists  $\epsilon_0 > 0$  such that for all  $\epsilon \leq \epsilon_0$ , (I) admits a stationary measure  $u_\epsilon$  satisfying

$$\left| \int_{\mathbb{R}} x u_\epsilon(x) dx - a_0 + \frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))} \epsilon \right| \leq \delta \epsilon.$$

*Proof.* The proof is exactly the same than the one of Proposition 3.1 in [HT10a]. Consequently, we will just sketch it. Let  $\tau > 0$ . Inequality (1.3) allows us to apply Lemma A.3 in [HT10a] to  $f(x) := -2\alpha\tau x$ ,  $n := 1$ ,  $U(x) := V(x) + \frac{\alpha}{2}x^2 - \alpha a_0 x$  and  $\mu := 0$ . By putting  $\tau^0 := \frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))}$ , for all  $\delta \in ]0; 1[$ , we get the following first order approximation:

$$\chi_\epsilon(a_0 - \tau^0(1 \pm \delta)\epsilon) = \pm \delta \frac{V''(a_0)}{\alpha + V''(a_0)} \tau^0 \epsilon + o(\epsilon).$$

Since  $\chi_\epsilon$  is continuous, we deduce that for  $\epsilon$  small enough, there exists  $m$  satisfying  $\chi_\epsilon(m) = 0$  and

$$\left| m - a_0 + \frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))} \epsilon \right| \leq \delta \epsilon.$$

The proof is achieved by considering the measure

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + \alpha\frac{x^2}{2} - \alpha mx\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + \alpha\frac{y^2}{2} - \alpha my\right)\right] dy}.$$

□

**Definition 1.3.** *The measure exhibited in the proof of Proposition 1.2 is called “outlying” around  $a_0$ . We write it  $u_\epsilon^{a_0}$ . However, we do not forget that it is possible a priori that there are several outlying stationary measures around  $a_0$ .*

Now, we can investigate the existence of other stationary measures (which would not be outlying around a wells). Under Conditions (1.3) for  $a_2$  and  $\alpha + V''(a_2) > 0$ , we already know that there exists a stationary measure around  $a_2$ . However, we will see that (1.3) is not necessary for getting the existence of a stationary measure between the wells  $a_1$  and  $a_3$ .

**Proposition 1.4.** *We assume that  $V$  satisfies (1.3) for  $a_1$  and  $a_3$ . Then, there exists  $\delta_0 > 0$  and  $\epsilon_0 > 0$  such that for all  $\epsilon \leq \epsilon_0$ , there exists a stationary measure  $u_\epsilon^{a_1, a_3}$  defined by  $m(\epsilon) \in [a_1 + \delta_0; a_3 - \delta_0]$  in (1.1).*

*Proof.* We introduce the potential  $W_m(x) := V(x) + \frac{\alpha}{2}x^2 - \alpha mx$  for all  $m > 0$ . We note that for all  $x \in \mathbb{R}$ ,  $W_m(x) = W_{a_1}(x) - \alpha(m - a_1)x$  with  $W_{a_1}(x) := V(x) + \frac{\alpha}{2}(x - a_1)^2 - \frac{\alpha}{2}a_1^2$ . Since  $a_1$  satisfies (1.3), the function  $W_{a_1}$  reaches its global minimum in a unique point:  $a_1$ . Let us show now that for  $|\delta|$  sufficiently small, the function  $W_{a_1+\delta}$  verifies the same property.

**Step 1.** Let us prove that each family of points where  $W_{a_1+\delta}$  reaches its global minimum converges towards  $a_1$  for  $\delta$  converging towards 0. For all  $\delta \in [-1; 1] \setminus \{0\}$ , we consider  $x(\delta)$  one point where  $W_{a_1+\delta}$  reaches its global minimum. So  $x(\delta)$  is a critical point of  $W_{a_1+\delta}$  which implies  $W'_{a_1+\delta}(x(\delta)) = 0$  so  $W'_{a_1}(x(\delta)) = \alpha\delta$ . Consequently, for all  $\delta \in [-1; 1] \setminus \{0\}$ ,  $|W'_{a_1}(x(\delta))| \leq \alpha$  then the family  $(x(\delta))_{\delta \in [-1; 1] \setminus \{0\}}$  is bounded. Moreover, by definition, for all  $x \in \mathbb{R}$ ,  $W_{a_1+\delta}(x(\delta)) \leq W_{a_1+\delta}(x)$ . Particularly, for  $x = a_1$ , this implies  $W_{a_1}(x(\delta)) \leq W_{a_1+\delta}(a_1) + \alpha\delta(x(\delta) - a_1)$  which tends towards  $W_{a_1}(a_1)$ . Hence, we get the convergence of  $x(\delta)$  towards  $a_1$  for  $\delta$  converging towards 0.

**Step 2.** Let us prove that the point  $x(\delta)$  is unique for  $\delta$  small enough. As  $W''_{a_1}(a_1) > 0$  according to the assumption (1.3), there exists  $\rho > 0$  such that for all  $x \in [a_1 - \rho; a_1 + \rho]$ ,  $W''_{a_1}(x) > 0$ . For  $\delta$  small enough, we have  $x(\delta) \in [a_1 - \rho; a_1 + \rho]$ . Consequently, there exists  $\delta_1 > 0$  such that for all  $\delta \in [-\delta_1; \delta_1]$ , all the points where  $W_{a_1+\delta}$  reaches its global minimum are in the interval  $[a_1 - \rho; a_1 + \rho]$ . And, as  $W''_{a_1+\delta}(x) = W''_{a_1}(x) > 0$  for all  $x \in [a_1 - \rho; a_1 + \rho]$ , we deduce that the potential  $W_{a_1+\delta}$  reaches its global minimum in a unique point. We call  $x(\delta)$  this unique point. Moreover, for  $\delta$  small enough,  $W''_{a_1+\delta}(x(\delta)) > 0$ .

**Step 3.** Let us show the two inequalities  $\chi_\epsilon(a_1 + \delta) < 0$  and  $\chi_\epsilon(a_1 - \delta) > 0$  for  $\delta > 0$  and  $\epsilon > 0$  small enough. As  $W'_{a_1+\delta}(x(\delta)) = 0$ , we have  $W'_{a_1}(x(\delta)) = \alpha\delta$ .



According to  $W''_{a_1}(a_1) > 0$ , we obtain the following first order approximation:

$$x(\delta) = a_1 + \frac{\alpha}{W''_{a_1}(a_1)} \delta + o(\delta).$$

Consequently:

$$x(\delta) - (a_1 + \delta) = -\frac{V''(a_1)}{\alpha + V''(a_1)} \delta + o(\delta).$$

By recalling that  $W''_{a_1+\delta}(x(\delta)) > 0$  for  $\delta$  small enough, we deduce that there exists  $\delta_2 < \delta_1$  such that for all  $\delta \in ]0; \delta_2[$ , we have  $x(\delta) - (a_1 + \delta) < 0$  and  $x(-\delta) - (a_1 - \delta) > 0$ . By using Lemme A.3 in [HT10a], we get

$$\begin{aligned} \chi_\epsilon(a_1 + \delta) &= x(\delta) - (a_1 + \delta) + o(1) \\ \text{and } \chi_\epsilon(a_1 - \delta) &= x(-\delta) - (a_1 - \delta) + o(1). \end{aligned}$$

Then, there exists  $\epsilon_1 > 0$  such that for all  $0 < \epsilon < \epsilon_1$ , we have  $\chi_\epsilon(a_1 + \delta) < 0$  and  $\chi_\epsilon(a_1 - \delta) > 0$ .

**Step 4.** In the same way, there exists  $\delta_3 > 0$  such that for all  $\delta \in ]0; \delta_3[$ , there exists  $\epsilon_2 > 0$  such that for all  $0 < \epsilon < \epsilon_2$ , we have  $\chi_\epsilon(a_3 + \delta) < 0$  and  $\chi_\epsilon(a_3 - \delta) > 0$ . Taking  $\delta_0 < \min \{ \delta_2; \delta_3; \frac{a_3 - a_1}{2} \}$  and  $\epsilon_0 := \min \{ \epsilon_1; \epsilon_2 \}$  is sufficient for achieving the proof. Indeed, for  $\epsilon \leq \epsilon_0$ , we have  $\chi_\epsilon(a_1 + \delta_0) < 0 < \chi_\epsilon(a_3 - \delta_0)$  and  $a_1 + \delta_0 < a_3 - \delta_0$ . Then, by using the theorem of intermediate values, we deduce that there exists  $m(\epsilon) \in [a_1 + \delta_0; a_3 - \delta_0]$  such that  $\chi_\epsilon(m(\epsilon)) = 0$ .  $\square$

**Definition 1.5.** *The measure  $u_\epsilon^{a_1, a_3}$  - non necessary unique - is called intermediate between  $u_\epsilon^{a_1}$  and  $u_\epsilon^{a_3}$ .*

We obtain immediately the following corollary:

**Corollary 1.6.** *Let  $V$  which verifies (V-1)-(V-5). We assume:*

$$\alpha > 2 \max \left\{ \sup_{y \neq a_1} \frac{V(a_1) - V(y)}{(a_1 - y)^2}; \sup_{y \neq a_3} \frac{V(a_3) - V(y)}{(a_3 - y)^2} \right\}. \quad (1.4)$$

*For  $\epsilon$  small enough, Diffusion (I) admits at least three stationary measures.*

*Proof.* Inequality (1.4) implies that each  $a_1$  and  $a_3$  satisfy (1.3). Then, we can apply Proposition 1.4 with  $a_1$  and  $a_3$ . We obtain the existence of  $\delta_1 > 0$  and  $\epsilon_1 > 0$  such that for all  $0 < \delta < \delta_1$  and  $\epsilon < \epsilon_1$ , the function  $\chi_\epsilon$  vanishes at  $m_{1,3}(\epsilon) \in [a_1 + \delta; a_3 - \delta]$ .

Then, we can apply Proposition 1.2 and we know that the system admits one outlying stationary measure  $u_\epsilon^1$  around  $a_1$  and another one around  $a_3$  with means verifying  $m_1(\epsilon), m_3(\epsilon) \notin [a_1 + \delta; a_3 - \delta]$ .

Consequently, we have at least three stationary measures: one is intermediate and two are outlying around the wells.  $\square$

Particularly, Corollary 1.6 can be applied in the synchronized case that is to say if  $\alpha \geq \sup_{x \in \mathbb{R}} -V''(x)$ . Let us remark that these results hold even if the number of critical points is not three.

## 1.2 Nonlinear case

The principal advantage of the linear case is the equivalence of the existence problem of a stationary measure remains and a parametrization problem in dimension 1. In the general case, we can make a parametrization in  $\mathbb{R}^{2n-1}$  with  $\deg(F) = 2n$ . Then, we can not use anymore the intermediate values theorem. Let us note the following result:

**Proposition 1.7.** *Let us assume that  $V$  satisfies (V-1)–(V-5) and  $F$  verifies (F-1)–(F-3). Then, for all  $\epsilon > 0$ , there exists at least one stationary measure.*

*Proof.* The idea consists in proceeding exactly like in Theorem 1.6 in [Tug10b]. By using the fact that the free-energy is nonincreasing and minorated, we prove that a subsequence of the family  $(u_t)_{t \in \mathbb{R}_+}$  converges towards a stationary measure which means that there is always at least one stationary measure, even if  $\deg(F) \geq 4$ . Indeed, Theorem 1.6 in [Tug10b] does not need any symmetric property on  $V$ . Proposition 1.2 in [Tug10b] corresponds to Proposition 2.1 in [CMV03] (which does not require the symmetry) ; Lemma 1.3 just needs the lower-bound  $V(x) \geq C_4 x^4 - C_2 x^2$  and Lemma 1.4 and 1.5 are simple consequences of Proposition 1.2 and Lemma 1.3. Furthermore, the proof of Theorem 1.6 in [Tug10b] only uses the growth property of the two potentials and the fact that they are polynomials.  $\square$

We present now the extension of Theorem 4.6 in [HT10a].

**Theorem 1.8.** *Let  $V$  which verifies the assumptions (V-1)–(V-5) and  $F$  which verifies the assumptions (F-1)–(F-3). We assume also*

$$V(x) + F(x - a_0) > V(a_0) \quad \text{for all } x \neq a_0$$

$$\text{and} \quad \sum_{p=0}^{2n-2} \frac{|F^{(p+2)}(a_0)|}{p!} |a_0|^p < \alpha + V''(a_0).$$

where  $a_0$  is a wells of  $V$ . Then, for  $\epsilon > 0$  small enough, the self-stabilizing equation (I) admits at least one outlying stationary measure around  $a_0$ .

The proof is similar to the one of Theorem 4.6 in [HT10a]. It consists in applying Schauder's fixed point theorem to the system of equations

$$m_k = \frac{\int_{\mathbb{R}} x^k \exp \left[ -\frac{2}{\epsilon} W_m(x) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} W_m(x) \right] dx}$$

$$\text{with } W_m(x) := V(x) + F(x - a_0) + \sum_{p=0}^{2n-1} \frac{(-1)^p}{p!} (m_p - a_0^p) F^{(p)}(x).$$

## 2 Uniqueness and Thirdness

In the previous section, we studied the number of stationary measures for self-stabilizing processes in the small noise. Particularly, we prove that for  $\epsilon$  small

enough, there is non-uniqueness of the stationary measures under easy to verify assumptions. But, we did not study the behavior in the large-noise. Here, we will show that under the assumptions (V-6) and (LIN), there is a phase transition: over a critical value of noise, there is a unique stationary measure and under the same value, there are exactly three ones. Then, we will show the uniqueness of the stationary measure for  $\epsilon$  large enough if  $V$  satisfies (V-7) and  $F$  satisfies (LIN). Then, in the nonlinear case, we will generalize the results about the thirdness problem already studied in [HT10a, HT10b, HT09, Tug10b].

## 2.1 Linear case

In this subsection, we consider:  $F(x) := \frac{\alpha}{2}x^2$  with  $\alpha > 0$ .

### 2.1.1 If $V$ is even

We assume (V-6). Consequently, the potential  $V$  has the form

$$V(x) := -\frac{\vartheta_1}{2}x^2 + \sum_{p=2}^q \frac{\vartheta_p}{(2p)!}x^{2p} \text{ with } \vartheta_p \geq 0 \text{ for all } 1 \leq p \leq q \text{ and } \vartheta_1\vartheta_q > 0.$$

With the two potentials  $V$  and  $F$ , we have the following phase transition result:

**Theorem 2.1.** *There exists  $\epsilon_c \in \mathbb{R}$  such that:*

- For all  $\epsilon \geq \epsilon_c$ , Diffusion (I) admits a unique stationary measure:  $u_\epsilon^0$ .
- For all  $\epsilon < \epsilon_c$ , Diffusion (I) admits exactly three stationary measures:  $u_\epsilon^0, u_\epsilon^+, u_\epsilon^-$  and  $\pm \int_{\mathbb{R}} x u_\epsilon^\pm(x) dx > 0$ .

Moreover, the critical value  $\epsilon_c$  is the unique solution of the equation:

$$\int_{\mathbb{R}_+} \left( x^2 - \frac{1}{2\alpha} \right) \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx = 0. \quad (2.1)$$

*Proof. Step 1.* We note that the expression of the function  $F$  implies:

$$\psi_\epsilon(m) = \int_{\mathbb{R}} (x - m) \exp \left[ \frac{2\alpha mx}{\epsilon} \right] \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2}x^2 \right) \right] dx$$

where  $\psi_\epsilon$  is defined in (1.2). We proceed a series expansion of the function  $m \mapsto \exp \left[ \frac{2\alpha mx}{\epsilon} \right]$  and we get - after using the fact that  $V$  is even:

$$\begin{aligned} \psi_\epsilon(m) &= 2 \sum_{n=0}^{\infty} \frac{I_\epsilon(2n)}{(2n)!} \left( \frac{2\alpha m}{\epsilon} \right)^{2n+1} \left[ \frac{I_\epsilon(2n+2)}{(2n+1)I_\epsilon(2n)} - \frac{\epsilon}{2\alpha} \right] \\ \text{with } I_\epsilon(z) &:= \int_{\mathbb{R}_+} x^z \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2}x^2 \right) \right] dx. \end{aligned} \quad (2.2)$$

**Step 2.** For all  $n \in \mathbb{N}$ , we introduce

$$\gamma_n(\epsilon) := \frac{I_\epsilon(2n+2)}{(2n+1)I_\epsilon(2n)} - \frac{\epsilon}{2\alpha}. \quad (2.3)$$

An integration by parts provides

$$\begin{aligned} (2n+1)I_\epsilon(2n) &= \frac{2}{\epsilon} \int_{\mathbb{R}} (V'(x) + \alpha x) x^{2n+1} \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx \\ &= \frac{2}{\epsilon} \left\{ (\alpha - \vartheta_1) I_\epsilon(2n+2) + \sum_{p=2}^q \frac{\vartheta_p}{(2p-1)!} I_\epsilon(2n+2p) \right\} \end{aligned}$$

after using the particular expression of  $V$ . Consequently, the expression  $\gamma_n(\epsilon)$  becomes

$$\gamma_n(\epsilon) = \frac{\epsilon}{2} \left\{ \alpha - \vartheta_1 + \sum_{p=2}^q \frac{\vartheta_p}{(2p-1)!} \frac{I_\epsilon(2n+2p)}{I_\epsilon(2n+2)} \right\}^{-1} - \frac{\epsilon}{2\alpha}.$$

**Step 3.** We will prove that the sequence  $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$  is nonincreasing for all  $\epsilon > 0$ . It is sufficient to prove that the sequences  $\left( \frac{I_\epsilon(2n+2p)}{I_\epsilon(2n+2)} \right)_{n \in \mathbb{N}}$  are nondecreasing for all  $p \in \llbracket 2; q \rrbracket$ . In a more general way, we will prove that for all  $z > 0$ , the function  $\Xi_z$  defined by  $\Xi_z(x) = \frac{I_\epsilon(x+z)}{I_\epsilon(x)}$  is nondecreasing. The derivation provides:

$$\Xi'_z(x) = \Xi_z(x) \left( \frac{I'_\epsilon(x+z)}{I_\epsilon(x+z)} - \frac{I'_\epsilon(x)}{I_\epsilon(x)} \right).$$

Since  $\Xi_z(x) \geq 0$  for all  $x \in \mathbb{R}_+$  and  $z \in \mathbb{R}_+$ , it remains now to prove that the application  $\zeta(y) := \frac{I'_\epsilon(y)}{I_\epsilon(y)}$  is nondecreasing. We apply the derivation:

$$\begin{aligned} \zeta'(y) &= \frac{I''_\epsilon(y)}{I_\epsilon(y)} - \left( \frac{I'_\epsilon(y)}{I_\epsilon(y)} \right)^2 = \frac{\int_{\mathbb{R}_+} x^y (\log(x))^2 \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx}{\int_{\mathbb{R}_+} x^y \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx} \\ &\quad - \left( \frac{\int_{\mathbb{R}_+} x^y (\log(x)) \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx}{\int_{\mathbb{R}_+} x^y \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx} \right)^2. \end{aligned}$$

By applying the Cauchy-Schwarz inequality, we obtain  $\zeta'(y) \geq 0$  for all  $y > 0$ . Then, we get the previous claim that is to say the sequence  $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$  is nonincreasing for all  $\epsilon > 0$ .

**Step 4:** In order to prove that  $\gamma_n(\epsilon) < 0$  for  $n$  large enough, we compute  $\psi_\epsilon(a)$ :

$$\begin{aligned} \psi_\epsilon(a) &= e^{\frac{\alpha a^2}{\epsilon}} \int_{\mathbb{R}} (x-a) \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} (x-a)^2 \right) \right] dx \\ &= e^{\frac{\alpha a^2}{\epsilon}} \int_{\mathbb{R}} y \exp \left[ -\frac{2}{\epsilon} \left( V(y+a) + \frac{\alpha}{2} y^2 \right) \right] dy \\ &= e^{\frac{\alpha a^2}{\epsilon}} \int_{\mathbb{R}_+} y e^{-\frac{\alpha}{\epsilon} y^2} \left\{ \exp \left[ -\frac{2}{\epsilon} V(y+a) \right] - \exp \left[ -\frac{2}{\epsilon} V(y-a) \right] \right\} dy. \end{aligned}$$

By using the particular expression of  $V$ , we get:

$$\begin{aligned} V(y+a) - V(y-a) &= 2 \sum_{p=1}^q \frac{V^{(2p)}(0)}{(2p)!} \sum_{k=0}^{p-1} C_{2p}^{2k+1} y^{2k+1} a^{2p-2k-1} \\ &= 2 \sum_{p=2}^q \frac{V^{(2p)}(0)}{(2p)!} \sum_{k=1}^{p-1} C_{2p}^{2k+1} y^{2k+1} a^{2p-2k-1} + 2 \sum_{p=1}^q \frac{V^{(2p)}(0)}{(2p)!} (2p-1) y a^{2p-1}. \end{aligned}$$

The last term is equal to  $V'(a) = 0$ . Since  $V^{(2p)}(0) \geq 0$  for all  $2 \leq p \leq q$ , we deduce that  $V(y+a) \geq V(y-a)$  for all  $y \geq 0$ . Furthermore, since  $\deg(V) > 2$ , we get the inequality  $V(y+a) > V(y-a)$  excepting a finite number of points. Consequently,  $\psi_\epsilon(a) < 0$ . We deduce there exists  $n \in \mathbb{N}$  such that  $\gamma_n(\epsilon) < 0$ .

**Step 5.** We put  $n_\epsilon := \min \{n \mid \gamma_n(\epsilon) \leq 0\}$ . By Step 3, we know that for all  $n \geq n_\epsilon$ , we have  $\gamma_n(\epsilon) \leq 0$  and for all  $n < n_\epsilon$  (if  $n_\epsilon = 0$ ,  $\{n \in \mathbb{N} \mid n < n_\epsilon\} = \emptyset$ ), we have  $\gamma_n(\epsilon) > 0$ . Consequently, we can write:

$$\psi_\epsilon(m) = \sum_{n=0}^{n_\epsilon-1} C_n(\epsilon) m^{2n+1} - \sum_{n=n_\epsilon}^{+\infty} C_n(\epsilon) m^{2n+1} \quad (2.4)$$

with  $C_n(\epsilon) := 2 \frac{I_\epsilon(2n)}{(2n)!} \left(\frac{2\alpha}{\epsilon}\right)^{2n+1} |\gamma_n(\epsilon)| \geq 0$ . We take the derivative then we divide by  $m^{2n_\epsilon}$ :

$$\frac{\psi'_\epsilon(m)}{m^{2n_\epsilon}} = \left\{ \sum_{n=0}^{n_\epsilon-1} (2n+1) C_n(\epsilon) m^{2n-2n_\epsilon} - \sum_{n=n_\epsilon}^{+\infty} (2n+1) C_n(\epsilon) m^{2n-2n_\epsilon} \right\} \quad (2.5)$$

Since the functions  $m \mapsto m^{2n-2n_\epsilon}$  (resp.  $m \mapsto -m^{2n-2n_\epsilon}$ ) are decreasing for all  $n \leq n_\epsilon - 1$  (resp.  $n \geq n_\epsilon$ ), we deduce that  $\psi'_\epsilon$  vanishes at most one time on  $\mathbb{R}_+$  and the same holds for  $\psi_\epsilon$  (by dividing (2.4) by  $m^{2n_\epsilon+1}$ ).

**Step 6.** By (2.4) and (2.5), we deduce that the behavior of the function  $m \mapsto \psi_\epsilon(m)$  depends directly of  $\psi'_\epsilon(0)$ :

- If  $\psi'_\epsilon(0) > 0$  that is to say if  $\gamma_0(\epsilon) > 0$ , then there exists  $x_\epsilon$  such that  $\psi'_\epsilon(x) \geq 0$  on  $[0; x_\epsilon]$  and  $\psi'_\epsilon(x) \leq 0$  on  $[x_\epsilon; +\infty[$ . Since  $\psi_\epsilon(0) = 0$  by symmetry of the potential  $V$ , we deduce that the function  $\psi_\epsilon$  is increasing on  $[0; x_\epsilon]$  then decreasing on  $[x_\epsilon; +\infty[$ . We deduce that there exists  $m_\epsilon > 0$  such that  $\psi_\epsilon(m_\epsilon) = 0$  and it is unique on  $\mathbb{R}_+^*$ . This implies the existence of the two asymmetric stationary measures.
- If  $\psi'_\epsilon(0) \leq 0$  that is to say if  $\gamma_0(\epsilon) \leq 0$ , then  $\gamma_n(\epsilon) \leq 0$  for all  $n \in \mathbb{N}$ . We deduce that the function  $\psi_\epsilon$  is decreasing on  $\mathbb{R}$  that implies the uniqueness of the zero for the function  $\psi_\epsilon$  and consequently the uniqueness of the stationary measure for the self-stabilizing diffusion (I).

**Step 7.** Now, we will investigate on the sign of  $\gamma_0(\epsilon)$ :

$$\begin{aligned}\gamma_0(\epsilon) &:= \frac{I_\epsilon(2)}{I_\epsilon(0)} - \frac{\epsilon}{2\alpha} \\ &= \epsilon \left\{ \frac{\int_{\mathbb{R}_+} y^2 \exp \left[ -\frac{2}{\epsilon} \left( V(\sqrt{\epsilon}y) + \frac{\alpha\epsilon}{2} y^2 \right) \right] dy}{\int_{\mathbb{R}_+} \exp \left[ -\frac{2}{\epsilon} \left( V(\sqrt{\epsilon}y) + \frac{\alpha\epsilon}{2} y^2 \right) \right] dy} - \frac{1}{2\alpha} \right\}\end{aligned}$$

by making the change of variable  $x := \sqrt{\epsilon}y$ . The computation yields

$$\begin{aligned}\gamma_0(\epsilon) &= \epsilon \left\{ \frac{\int_{\mathbb{R}_+} x^2 \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx}{\int_{\mathbb{R}_+} \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx} - \frac{1}{2\alpha} \right\} \\ &=: \epsilon \mathcal{T}_{V,F}(\epsilon).\end{aligned}$$

Since  $\vartheta_p \geq 0$  for all  $1 \leq p \leq q$ , we deduce - after using the Jensen's inequality - that the application  $\mathcal{T}_{V,F}$  is decreasing. Consequently, there exists a unique  $\epsilon_c$  which depends only on the parameters of  $V$  and  $F$  such that  $\gamma_0(\epsilon) = 0$ . Moreover, for all  $\epsilon \geq \epsilon_c$ , we have  $\gamma_0(\epsilon) \leq 0$  which implies the uniqueness of the stationary measure. And, for all  $\epsilon < \epsilon_c$ , we have  $\gamma_0(\epsilon) > 0$  which implies the thirdness of the stationary measures.  $\square$

An immediate consequence is the convergence towards the unique stationary measure if the initial free-energy is finite for all  $\epsilon \geq \epsilon_c(\alpha)$  according to Theorem 1.6 in [Tug10b].

## 2.2 If $V$ is asymmetric

Now, we will show that there is a phase transition even if  $V$  is asymmetric. Indeed, if  $\alpha$  is large enough, we know that there is non-uniqueness of the stationary measures in the small-noise case. We will consider the large-noise one and we will see that the competition between the three different forces in (I) is dominated by the heat process if the coefficient diffusion  $\sqrt{\epsilon}$  is sufficiently large.

**Proposition 2.2.** *We assume that  $V$  satisfies (V-1)–(V-5) and (V-7). For all  $\alpha \geq 0$ , there exists a critical value  $\epsilon_0(\alpha)$  such that for all  $\epsilon > \epsilon_0(\alpha)$ , Diffusion (I) admits a unique stationary measure.*

*Proof.* We recall that the number of stationary measures for Diffusion (I) is the number of roots of the function  $\chi_\epsilon$ :

$$\chi_\epsilon(m) = \frac{\int_{\mathbb{R}} x \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 - \alpha m x \right) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 - \alpha m x \right) \right] dx} - m.$$

According to the hypothesis (V-7), there exists  $\widehat{V}$  which satisfies (V-1)–(V-6) such that  $V(x) = \widehat{V}(x - c) + ax + b$  with  $a, b, c \in \mathbb{R}$ . We apply the change of

variable  $x := y + c$  and we obtain  $\chi_\epsilon(m) = \widehat{\chi}_\epsilon(\widehat{m}) - \frac{a}{\alpha}$  with  $\widehat{m} := m - c - \frac{a}{\alpha} \in \mathbb{R}$  and  $\widehat{\chi}_\epsilon$  is the function defined by

$$\widehat{\chi}_\epsilon(m) := \frac{\int_{\mathbb{R}} y \exp \left[ -\frac{2}{\epsilon} \left( \widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( \widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy} - m.$$

In the same way, we define

$$\begin{aligned} \widehat{\psi}_\epsilon(m) &:= \int_{\mathbb{R}} (y - m) \exp \left[ -\frac{2}{\epsilon} \left( \widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy \\ \text{and } \widehat{Z}_\epsilon(m) &:= \int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( \widehat{V}(y) + \frac{\alpha}{2} y^2 - \alpha m y \right) \right] dy. \end{aligned}$$

We deduce  $\frac{d}{dm} \widehat{\chi}_\epsilon(m) = \frac{\xi_\epsilon(m)}{\widehat{Z}_\epsilon(m)^2}$  with  $\xi_\epsilon(m) := \widehat{\psi}'_\epsilon(m) \widehat{Z}_\epsilon(m) - \widehat{\psi}_\epsilon(m) \widehat{Z}'_\epsilon(m)$ . The function  $\xi_\epsilon$  is even and analytic so it is sufficient to prove that  $\xi_\epsilon^{(2n)}(0) \leq 0$  for all  $n \in \mathbb{N}$  and that at least one of the  $\xi_\epsilon^{(2n)}$  is negative. Indeed, this would imply that  $\widehat{\chi}_\epsilon$  (then  $\chi_\epsilon$ ) is decreasing which means that there is a unique stationary measure. The derivation provides:

$$\begin{aligned} \xi_\epsilon^{(2n)}(0) &= \sum_{k=0}^{2n} C_{2n}^k \widehat{\psi}_\epsilon^{(k+1)}(0) \widehat{Z}_\epsilon^{(2n-k)}(0) - \sum_{k=0}^{2n} C_{2n}^k \widehat{\psi}_\epsilon^{(k)}(0) \widehat{Z}_\epsilon^{(2n+1-k)}(0) \\ &= \widehat{\psi}_\epsilon^{(2n+1)}(0) \widehat{Z}_\epsilon(0) + (1 - 2n) \widehat{\psi}'_\epsilon(0) \widehat{Z}_\epsilon^{(2n)}(0) \\ &\quad + \sum_{k=1}^{n-1} (C_{2n}^{2k} - C_{2n}^{2k+1}) \widehat{\psi}_\epsilon^{(2k+1)}(0) \widehat{Z}_\epsilon^{(2n-2k)}(0). \end{aligned} \tag{2.6}$$

As seen in the proof of Theorem 2.1, for all  $\alpha \in \mathbb{R}_+$  and  $k \in \mathbb{N}$ , we have

$$\widehat{\psi}_\epsilon^{(2k)}(0) = 2I_\epsilon(2k) \left( \frac{2\alpha}{\epsilon} \right)^{2k+1} (2k+1) \gamma_k(\epsilon) \tag{2.7}$$

where  $I_\epsilon(2k)$  and  $\gamma_k(\epsilon)$  are defined respectively in (2.2) and (2.3). A simple computation provides

$$\widehat{Z}_\epsilon^{(2k)}(0) = 2I_\epsilon(2k) \left( \frac{2\alpha}{\epsilon} \right)^{2k}. \tag{2.8}$$

By combining (2.6), (2.7) and (2.8), we obtain

$$\begin{aligned} \xi_\epsilon^{(2n)}(0) &= 4 \left( \frac{2\alpha}{\epsilon} \right)^{2n+1} \left\{ I_\epsilon(2n) I_\epsilon(0) [(2n+1) \gamma_n(\epsilon) - (2n-1) \gamma_0(\epsilon)] \right. \\ &\quad \left. + \sum_{k=1}^{n-1} C_{2n}^{2k} (4k+1-2n) I_\epsilon(2k) I_\epsilon(2n-2k) \gamma_k(\epsilon) \right\}. \end{aligned}$$

As proved in Theorem 2.1, there exists  $\epsilon_c$  such that for all  $\alpha \geq \alpha_c$ , we have  $\gamma_0(\epsilon) \leq 0$ . As the sequence  $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$  is nonincreasing, we deduce that the sequence  $(|\gamma_n(\epsilon)|)_{n \in \mathbb{N}}$  is nondecreasing. We note that for all  $k > \frac{n}{2}$ ,  $|4k+1-2n| > |2n+1-4k|$ . Consequently, for all  $k > \frac{n}{2}$ ,  $(4k+1-2n)\gamma_k(\epsilon) + (2n+1-4k)\gamma_{n-k}(\epsilon) < 0$ . Also, if  $k = \frac{n}{2}$ , we have  $(4k+1-2n)\gamma_k(\epsilon) = \gamma_k(\epsilon) < 0$ . This implies  $\xi_\epsilon^{(2n)}(0) \leq 0$ .

The function  $\widehat{\chi}'_\epsilon(m)$  is then nonpositive. We deduce that the function  $\widehat{\chi}_\epsilon$  is non-increasing. Moreover, if  $\xi_\epsilon^{(2n)}(0) = 0$  for all  $n \in \mathbb{N}$ , then  $\widehat{\chi}_\epsilon$  would be a constant which is impossible since  $\lim_{m \rightarrow \pm\infty} \widehat{\chi}_\epsilon(m) = \mp\infty$ . Then,  $\widehat{\chi}_\epsilon$  is decreasing on  $\mathbb{R}$ . Consequently, the equation  $\widehat{\chi}_\epsilon(m) - C$  has at most one solution for all  $C \in \mathbb{R}$ . It yields that  $\chi_\epsilon$  has one root or no root. There is at least one stationary measure for all  $\epsilon > 0$  so there exists a critical value  $\epsilon_0(\alpha)$  such that for all  $\epsilon > \epsilon_0(\alpha)$ , Diffusion (I) admits exactly one stationary measure.  $\square$

**Remark 2.3.** *By using a method similar to the one of [Tug10b], we could prove the convergence towards this unique stationary measure.*

Proposition 2.2 implies that there is a phase transition since there is exactly one stationary measure for  $\epsilon$  large enough and several ones for  $\epsilon$  sufficiently small.

### 2.3 Nonlinear case

We assume on this paragraph that  $\deg(F) \geq 4$  and that  $V$  satisfies the assumptions (V-1)–(V-6). We will extend Theorem 3.2 in [HT10a]. One of the difficulty will be the behavior of the symmetric stationary measure(s) in the threshold between the synchronized case and the non-synchronized one that is to say when  $V''(0) + F''(0) = 0$ . For doing this, we will use a particles method involving a propagation of chaos.

**Proposition 2.4.** *Let  $V$  and  $F$  two potentials which satisfy (V-1)–(V-6) and (F-1)–(F-3). Then, for  $\epsilon$  small enough, Diffusion (I) admits a unique symmetric stationary measure.*

*Proof. Step 1.* Since the potential  $V$  is symmetric and verifies  $V^{(4)} \geq 0$ , the inequality  $V''(0) + F''(0) \neq 0$  implies the uniqueness of the stationary measure for  $\epsilon$  small enough according to Theorem 1.2 and Theorem 1.3 in [HT09].

**Step 2.** We consider now the case  $V''(0) + F''(0) = 0$ . Let  $V_0(x) := V(x) + \frac{F''(0)}{2}x^2$  and  $F_0(x) := F(x) - \frac{F''(0)}{2}x^2$ . If the initial law  $u_0$  is symmetric, then using  $V_0$  (resp.  $F_0$ ) instead of  $V$  (resp.  $F$ ) in (I) does not change anything. Let  $u_\epsilon^{(1)}$  and  $u_\epsilon^{(2)}$  two symmetric stationary measures of Diffusion (I). We con-



sider the two mean-field systems:

$$X_t^i = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(X_s^i) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N F_0'(X_s^i - X_s^j) ds$$

and

$$Y_t^i = Y_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(Y_s^i) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N F_0'(Y_s^i - Y_s^j) ds$$

where the  $2N$  initial random values are independents. Furthermore, we assume  $\mathcal{L}(X_0^i) = u_\epsilon^{(1)}$  and  $\mathcal{L}(Y_0^i) = u_\epsilon^{(2)}$ . Also, the  $N$  brownian motions are independents. We define also the self-stabilizing processes which are classical since the two initial laws are invariant probabilities of (I):

$$\overline{X}_t^i = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(\overline{X}_s^i) ds - \int_0^t F_0' * u_\epsilon^{(1)}(\overline{X}_s^i) ds$$

and

$$\overline{Y}_t^{\epsilon, i} = Y_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(\overline{Y}_s^i) ds - \int_0^t F_0' * u_\epsilon^{(2)}(\overline{Y}_s^i) ds.$$

The triangular inequality provides

$$\begin{aligned} \mathbb{E} \left\{ \left| \overline{Y}_t^i - \overline{X}_t^i \right|^2 \right\} &\leq 3\mathbb{E} \left\{ \left| \overline{Y}_t^i - Y_t^i \right|^2 \right\} + 3\mathbb{E} \left\{ \left| X_t^i - \overline{X}_t^i \right|^2 \right\} \\ &+ 3\mathbb{E} \left\{ \left| Y_t^i - X_t^i \right|^2 \right\}. \end{aligned}$$

**Step 3.** A propagation of chaos uniform with respect to the time holds by using the same method than the one in [CGM08]. Consequently, it yields:

$$\mathbb{E} \left\{ \left| X_t^i - \overline{X}_t^i \right|^2 \right\} \leq M N^{-\rho} \quad \text{and} \quad \mathbb{E} \left\{ \left| \overline{Y}_t^i - Y_t^i \right|^2 \right\} \leq M N^{-\rho}$$

with  $\rho \in ]0; 1[$  and  $M > 0$ .

**Step 4.** The results in [BBCG08] permit to obtain the Poincaré inequality in (II) which implies

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left\{ \frac{1}{N} \sum_{j=1}^N \left| X_t^j - Y_t^j \right|^2 \right\} = 0.$$

**Step 5.** By definition of the Wasserstein distance, we have  $\mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2 \leq \mathbb{E} \left\{ \left| \overline{Y}_t^i - \overline{X}_t^i \right|^2 \right\}$  for all  $t \in \mathbb{R}_+$  and for all  $1 \leq i \leq N$ . It yields

$$\mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2 \leq 6MN^{-\rho} + 3\mathbb{E} \left\{ \frac{1}{N} \sum_{j=1}^N \left| X_t^j - Y_t^j \right|^2 \right\}.$$

By taking the limits of  $t$  going to infinity then of  $N$  tending to infinity, we deduce immediatly  $\mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2 = 0$  which implies  $u_\epsilon^{(1)} = u_\epsilon^{(2)}$ .  $\square$

We will now extend Theorem 3.2 in [HT10a] by proving the thirdness of the stationary measures for  $\epsilon$  small enough.

**Theorem 2.5.** *Let  $V$  and  $F$  two potentials which satisfy (V-1)–(V-6) and (F-1)–(F-3). If  $\deg(V) \geq \deg(F)$  and  $V''(0) + F''(0) \geq 0$ , Diffusion (I) admits exactly three stationary measures for  $\epsilon$  small enough.*

*Proof.* If  $V''(0) + F''(0) > 0$  and  $\deg(V) > \deg(F)$ , it is a consequence of Theorem 1.11 in [Tug10b].

**Step 1.** If  $\deg(V) > \deg(F)$ , each family of stationary measures  $(u_\epsilon)_{\epsilon > 0}$  satisfies the condition (H) introduced in [HT10b] which means that the family  $(\int_{\mathbb{R}} x^{2n} u_\epsilon(x) dx)_{\epsilon > 0}$  is bounded where  $2n := \deg(F)$ . We will prove that it is true even if  $\deg(V) = 2n$ . We will proceed *reducto ad absurdum* by assuming the existence of a decreasing sequence  $(\epsilon_k)_{k \in \mathbb{N}}$  which converges towards 0 such that the sequence  $\mu_{2n}(k) := \int_{\mathbb{R}} x^{2n} u_{\epsilon_k}(x) dx$  tends towards  $+\infty$ .

**Step 1.1.** Since  $F$  and  $V$  are two polynomials functions, we can write the  $l$ -th moment of  $u_\epsilon$  in the following form:

$$\mu_l(k) = \frac{\int_{\mathbb{R}} x^l \exp \left[ -\frac{2}{\epsilon_k} \left( \sum_{r=1}^{2n} \mathcal{M}_r(k) x^r \right) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon_k} \left( \sum_{r=1}^{2n} \mathcal{M}_r(k) x^r \right) \right] dx},$$

$$\text{with } \mathcal{M}_r(k) := \frac{1}{r!} \left\{ V^{(r)}(0) + \sum_{j=0}^{2n-r} \frac{(-1)^j}{j!} F^{(j+r)}(0) \mu_j(k) \right\}$$

for all  $1 \leq l \leq 2n$ . The highest moment which intervenes in  $\mathcal{M}_r(k)$  is the one of degree  $2n - r$ .

**Step 1.2.** Let us recall that  $\mu_{2n}(k)$  tends towards  $+\infty$ . If all the  $\mathcal{M}_r(k)$  were bounded then - since the coefficient  $\mathcal{M}_{2n}(k)$  is positive -  $\mu_{2l}(k)$  would be bounded for all  $l \in \mathbb{N}$ . We deduce that there exists an index  $r$  such that  $\mathcal{M}_r(k)$  tends towards  $+\infty$  or  $-\infty$ . We introduce the sequence

$$\eta_r(k) := \mathcal{M}_r(k) (\mu_{2n}(k))^{-\left(1 - \frac{r}{2n}\right)}$$

for all  $1 \leq r \leq 2n$ . The change of variable  $x := (\mu_{2n}(k))^{\frac{1}{2n}} y$  provides

$$\frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}} = \frac{\int_{\mathbb{R}} y^l \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy}{\int_{\mathbb{R}} \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy}.$$

**Step 1.3.** The Jensen inequality implies the existence of  $C > 0$  such that  $|\mathcal{M}_r(k)| \leq C \mu_{2n}(k)^{\frac{2n-r}{2n}}$  for all  $1 \leq r \leq 2n - 1$ . We obtain the following higher-bound:  $|\eta_r(k)| \leq C$ . By considering a subsequence of  $(\epsilon_k)_k$  (we continue to write it  $\epsilon_k$  for simplicity), we get the convergence of  $\eta_r(k)$  towards  $\eta_r$ . Besides, the quantity  $\frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}}$  tends towards 0 for all  $1 \leq l \leq 2n$ . Consequently, we

have the following limit:

$$\frac{(-1)^{2n-l}}{(2n-l)!} F^{(2n)}(0) \lim_{k \rightarrow +\infty} \frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}} = \eta_l \quad (2.9)$$

for all  $1 \leq l \leq 2n$ .

**Step 1.4.** According to Lemma A.4 in [Tug10b], we can extract a subsequence (we continue to write it  $\epsilon_k$  for simplicity) such that

$$\lim_{k \rightarrow +\infty} \frac{\int_{\mathbb{R}} y^l \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy}{\int_{\mathbb{R}} \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy} = \sum_{s=1}^q p_s A_s^l$$

where  $A_1 < \dots < A_q$  are the locations of the global minimum of the polynomial function  $U_0(x) := \sum_{j=1}^{2n} \eta_j x^j + \frac{V^{(2n)}(0)}{(2n)!} x^{2n}$  and  $p_1 + \dots + p_q = 1$  with  $p_i \geq 0$  for all  $1 \leq i \leq q$ . By using (2.9), we obtain:

$$\eta_l = \frac{(-1)^{2n-l}}{(2n-l)!} F^{(2n)}(0) \sum_{s=1}^q p_s A_s^l.$$

**Step 1.5.** Then, we have  $U_0(x) = \frac{F^{(2n)}(0)}{(2n)!} \sum_{s=1}^q p_s (x - A_s)^{2n} + \frac{V^{(2n)}(0)}{(2n)!} x^{2n}$ . By definition,  $U_0'(A_q) = 0$ . If  $q \geq 2$ , since  $A_q - A_i > 0$  for all  $1 \leq i \leq q-1$ , it yields  $A_q < 0$ . By the same way, we have  $A_1 > 0$ . This is impossible. If  $q = 1$ , we have immediately  $A_1 = 0$  then  $\eta_{2n} = 0$  which implies  $F^{(2n)}(0) = 0$ . This is also impossible.

**Step 1.6.** We deduce that each family of stationary measures satisfies the condition (H). According to Lemma 3.2 and Proposition 3.3 in [HT10b], this implies that each family of stationary measures admits an adherence value in the small-noise limit. Furthermore, since  $F''(0) + V''(0) \geq 0$ , Proposition 3.7 and Remark 3.8 in [HT10b] imply that there are exactly three possible adherence values:  $\delta_0$ ,  $\delta_a$  and  $\delta_{-a}$  with  $-a = a_1$ .

**Step 2.** According to Theorem 4.5 in [HT10a], Diffusion (I) admits at least one symmetric stationary measure. Theorem 2.4 implies the uniqueness of the symmetric stationary measure for  $\epsilon$  small enough. Corollary 1.9 in [Tug10b] provides the existence of at least two asymmetric stationary measures. It remains now to prove that there are exactly two asymmetric stationary measures for  $\epsilon$  small enough. We proceed exactly like in Theorem 1.11 in [Tug10b] by using the rate of convergence method from [HT09] (Theorem 1.5) and we obtain the thirdness of the stationary measures for  $\epsilon$  small enough.  $\square$

### 3 Simulations of the phase transitions

We saw in Theorem 2.1 that there exists a continuous transition between two phases (uniqueness and thirdness of the stationary measures) when  $V$  satisfies (V-1)–(V-6) and  $F(x) := \frac{\alpha}{2} x^2$  with  $\alpha > 0$ . Before studying the general case, we provide some example with the potential  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ .

**Example 3.1.** Let us choose  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$  and  $F(x) := \frac{\alpha}{2}x^2$ . We will call  $\epsilon_c(\alpha)$  the critical value which corresponds to the phase transition. According to Theorem 2.1, for all  $\alpha > 0$ ,  $\epsilon_c(\alpha)$  is defined as the solution of (2.1) which is equivalent to

$$\int_{\mathbb{R}_+} \left\{ x^2 - \frac{1}{2\alpha} \right\} \exp \left[ (1 - \alpha) x^2 - \frac{\epsilon}{2} x^4 \right] dx = 0.$$

By making the change of variable  $z := \sqrt{\frac{1}{2\alpha}} x$ , we get:

$$\mathbb{E} \left\{ \xi(\epsilon_c(\alpha), \alpha, X) \right\} = 0 \quad \text{with} \quad \xi(x, y, z) := (z^2 - 1) \exp \left[ \frac{z^2}{2y} - \frac{x}{8y^2} z^4 \right]$$

and  $\mathcal{L}(X) = \mathcal{N}(0, 1)$ . In order to simulate  $\epsilon_c(\alpha)$ , we choose  $r > 0$  and  $N \in \mathbb{N}^*$ . We want  $N$  large and  $r$  small. We take  $N$  random variables independents and identically distributed by the law  $\mathcal{N}(0, 1)$ :  $(X_i)_{1 \leq i \leq N}$ . The Monte-Carlo method provides the following estimation of  $\epsilon_c(\alpha)$ :

$$\epsilon_c^{r, N}(\alpha) := r \min \left\{ p \in \mathbb{N}^* \mid \sum_{i=1}^N \xi(pr, \alpha, X_i) < 0 \right\}.$$

We get this inequality after using the weak law of the large numbers:

$$\lim_{N \rightarrow +\infty} \epsilon_c^{r, N}(\alpha) - r \leq \epsilon_c(\alpha) \leq \lim_{N \rightarrow +\infty} \epsilon_c^{r, N}(\alpha).$$

We take  $N := 2 \times 10^5$  and  $r := 10^{-4}$  then we obtain the following curve of the critical value:

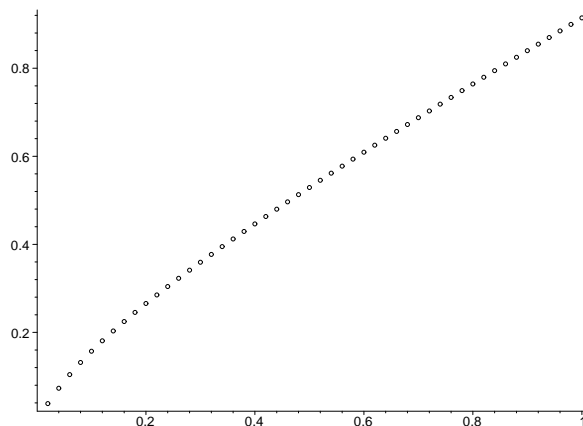


Figure 1: Critical value  $\epsilon_c(\alpha)$

We remark that it is increasing. And, it seems to be almost linear sufficiently far from 0. In fact, simple computations provide:

$$\lim_{\alpha \rightarrow +\infty} \frac{\epsilon_c(\alpha)}{\alpha} = \frac{2}{3} \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \frac{\epsilon_c(\alpha)}{\alpha} = 2.$$

### 3.1 Under hypothesis (V-6) in the non-linear case

In the previous results about the non-uniqueness of the stationary measures, we assumed that  $\epsilon$  is small enough. We can wonder if there exists a threshold over which there is uniqueness of the invariant probability.

In the simple case with  $\deg(V) = 4$  and  $\deg(F) = 2$ , we just see that we can, as predicted by Theorem 2.1. We will now deal with a non-linear interaction function  $F'$ . As there is always a symmetric stationary measure, the study remains to find the critical value  $\epsilon_c$  such that Diffusion (I) admits at least two stationary measures if  $\epsilon < \epsilon_c$ .

#### 3.1.1 Low-energy method

We know by Proposition 2.4 that there is a unique symmetric stationary measure for  $\epsilon$  small enough. We introduce  $\epsilon_c^{(0)}$  the largest value (maybe equal to  $+\infty$ ) such that there is a unique symmetric stationary measure under this value. For  $\epsilon < \epsilon_c^{(0)}$ , let  $u_\epsilon^0$  this unique symmetric stationary measure.

**Proposition 3.2.** *Let a potential  $V$  which verifies (V-1)–(V-6) and a potential  $F$  which verifies (F-1)–(F-3). We have the following lower-bound of the critical value:*

$$\epsilon_c \geq \sup \left\{ \epsilon_0 \in ]0, \epsilon_c^{(0)}[ \mid \forall \epsilon \in ]0; \epsilon_0[, \inf_{u \in \mathcal{M}_{8q^2}} \Upsilon_\epsilon(u) < \Upsilon_\epsilon(u_\epsilon^0) \right\} \quad (3.1)$$

where  $\mathcal{M}_{8q^2}$  is the set of all the measures  $u$  which admits a  $\mathcal{C}^\infty$ -continuous density  $u$  such that  $\int_{\mathbb{R}} x^{8q^2} u(x) dx < +\infty$ , with  $2q := \max\{\deg(V), \deg(F)\}$ .

*Proof.* Let  $\epsilon_0 \in ]0; \epsilon_c^{(0)}[$  such that  $\inf_{u \in \mathcal{M}_{8q^2}} \Upsilon_\epsilon(u) < \Upsilon_\epsilon(u_\epsilon^0)$  for all  $\epsilon \in ]0; \epsilon_0[$ .

We use then the same argument than the one in the proof of Corollary 1.9 in [Tug10b]. Let  $\epsilon \in ]0; \epsilon_0[$ . Since  $\inf_{u \in \mathcal{M}_{8q^2}} \Upsilon_\epsilon(u) < \Upsilon_\epsilon(u_\epsilon^0)$ , there exists  $v_\epsilon \in \mathcal{M}_{8q^2}$

such that  $\Upsilon_\epsilon(v_\epsilon) < \Upsilon_\epsilon(u_\epsilon^0)$  and  $v_\epsilon \neq u_\epsilon^0$ . We consider now the self-stabilizing process  $(X_t)_{t \in \mathbb{R}_+}$  starting with  $X_0$  which has the law  $v_\epsilon$ . We call  $u_t$  the law of  $X_t$  for all  $t \in \mathbb{R}_+$ . By Theorem 1.6 in [Tug10b], we know that a subsequence of  $u_t$  converges towards a stationary measure  $u_\infty$  when  $t$  tends to  $+\infty$ . Proposition 2.5 in [Tug10b] implies  $\Upsilon_\epsilon(u_\infty) \leq \liminf_{t \rightarrow +\infty} \Upsilon_\epsilon(u_t) \leq \Upsilon_\epsilon(v_\epsilon) < \Upsilon_\epsilon(u_\epsilon^0)$ .

Since  $\Upsilon_\epsilon(u_\infty) < \Upsilon_\epsilon(u_\epsilon^0)$ , we deduce that the stationary measure  $u_\infty$  is not the symmetric one. Consequently, for all  $\epsilon < \epsilon_0$ , Diffusion (I) admits at least two stationary measures. We deduce that  $\epsilon_0 \leq \epsilon_c$ .  $\square$

This result may seem hard to apply since the question still remains to the study of an application in a space different from  $\mathbb{R}$ . There is in fact an obvious candidate for  $v_\epsilon$ . It is the one used in Corollary 1.9 in [Tug10b] for proving the existence of asymmetric stationary measures:

$$v_\epsilon^a(x) := Z^{-1} \exp \left[ -\frac{2}{\epsilon} (V(x) + F(x - a)) \right]$$

where  $a \in \mathbb{R}_+$  satisfies  $V(a) = \inf_{x \in \mathbb{R}} V(x)$ . Indeed,  $v_\epsilon^a$  is closed to the outlying stationary measure  $u_\epsilon^a$  since the two measures converge towards  $\delta_a$  in the small-noise limit. By using a method similar to the one of Corollary 1.9 in [Tug10b], we get the existence of  $\hat{\epsilon}_0 > 0$  such that for all  $\epsilon \in ]0; \hat{\epsilon}_0[$ ,  $\Upsilon_\epsilon(v_\epsilon^a) < \Upsilon_\epsilon(u_\epsilon^0)$ .

We consider the following example:

**Example 3.3.** Let  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$  and  $F(x) := \frac{\beta}{4}x^4 + \frac{\alpha}{2}x^2$  with  $\beta > 0$  and  $\alpha \geq 0$ . We call  $\epsilon_c(\alpha, \beta)$  the critical value which corresponds to the transition between  $\{\Upsilon_\epsilon(u_\epsilon^0) < \Upsilon_\epsilon(v_\epsilon^1)\}$  and  $\{\Upsilon_\epsilon(u_\epsilon^0) > \Upsilon_\epsilon(v_\epsilon^1)\}$ . By Subsection 4.2 in [HT10a], there is a unique symmetric stationary measure  $u_\epsilon^0$  for all  $\epsilon > 0$ :

$$u_\epsilon^0(x) = Z^{-1} \exp \left[ -\frac{2}{\epsilon} \left( \frac{1+\beta}{4}x^4 + \frac{\alpha+3\beta m_2-1}{2}x^2 \right) \right]$$

$$\text{where } m_2 = \frac{\int_{\mathbb{R}} x^2 \exp \left[ -\frac{2}{\epsilon} \left( \frac{1+\beta}{4}x^4 + \frac{\alpha+3\beta m_2-1}{2}x^2 \right) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( \frac{1+\beta}{4}x^4 + \frac{\alpha+3\beta m_2-1}{2}x^2 \right) \right] dx}$$

is unique. A simple computation provides then

$$\Upsilon_\epsilon(u_\epsilon^0) = -\frac{\epsilon}{2} \log \left\{ \int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( U_{\alpha,\beta}(x) + \frac{3\beta}{2}m_2x^2 \right) \right] dx \right\} - \frac{3\beta}{4}m_2^2$$

with  $U_{\alpha,\beta}(x) := \frac{1+\beta}{4}x^4 + \frac{\alpha-1}{2}x^2$ . In the other hand, we have:

$$\Upsilon_\epsilon(v_\epsilon^1) = -\frac{\epsilon}{2} \log \left\{ \int_{\mathbb{R}} e^{-\frac{\epsilon}{2}(V(x)+F(x-1)-F(1))} dx \right\} + (\alpha + \beta)n_1 - \frac{3\beta}{2}n_2$$

$$+ \beta n_3 - \frac{\alpha}{2}n_1^2 - \beta n_1 n_3 + \frac{3\beta}{4}n_2^2$$

$$\text{with } n_k := \frac{\int_{\mathbb{R}} x^k \exp \left[ -\frac{2}{\epsilon} (V(x) + F(x-1)) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} (V(x) + F(x-1)) \right] dx}.$$

We take  $N := 2 \times 10^5$  and we obtain the following surface:

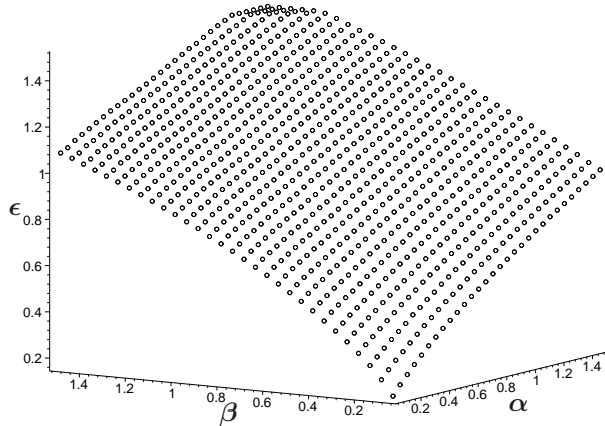


Figure 2: Critical value  $\epsilon_c(\alpha, \beta)$

**Remark 3.4.** Both simulations of Figure 2, Figure 3 and Figure 4 are saturated by the hyperplan ( $\epsilon = 1.5$ ) in order to compare the different methods.

We remark on Figure 2 that the provided value is increasing in the two variables and seems linear sufficiently far from  $(0, 0)$ .

### 3.1.2 Convexity of the free-energy method

Now, we will point out the particular link between the function  $\psi_\epsilon$  and the free-energy of the system.

If the function  $F'$  is linear, we know that any stationary measure of Diffusion (I) has the following form:

$$u_\epsilon^{(m)} = Z^{-1} \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 - \alpha m x \right) \right].$$

For all  $\epsilon > 0$ , we introduce the function  $\tau_\epsilon(m) := \Upsilon_\epsilon(u_\epsilon^{(m)})$  for all  $m \in \mathbb{R}$ .

**Lemma 3.5.** For all  $\epsilon > 0$  and  $m \in \mathbb{R}$ , the following equality holds:

$$\tau'_\epsilon(m) = -\frac{\alpha \psi_\epsilon(m)}{Z_\epsilon(m)} \text{Var} \left( u_\epsilon^{(m)} \right). \quad (3.2)$$

*Proof.* By definition, we have:

$$\begin{aligned} \tau_\epsilon(m) &= \int_{\mathbb{R}} \left( \frac{\epsilon}{2} \log \left( u_\epsilon^{(m)}(x) \right) + V(x) + \frac{1}{2} F * u_\epsilon^{(m)}(x) \right) u_\epsilon^{(m)}(x) dx \\ &= -\frac{\epsilon}{2} \log(Z_\epsilon(m)) + \alpha m \int_{\mathbb{R}} x u_\epsilon^{(m)}(x) dx - \frac{\alpha}{2} \left( \int_{\mathbb{R}} x u_\epsilon^{(m)}(x) dx \right)^2. \end{aligned}$$

By remarking that  $\int_{\mathbb{R}} x u_\epsilon^{(m)}(x) dx = \frac{\psi_\epsilon(m) + m}{Z_\epsilon(m)}$ , we get:

$$\tau_\epsilon(m) = -\frac{\epsilon}{2} \log \left[ Z_\epsilon(m) e^{-\frac{\alpha}{\epsilon} m^2} \right] - \frac{\alpha \psi_\epsilon(m)^2}{2 Z_\epsilon(m)^2}.$$

The derivation of this equality provides

$$\tau'_\epsilon(m) = -\frac{\alpha \psi_\epsilon(m)}{Z_\epsilon(m)} \left( 1 + \frac{\psi'_\epsilon(m)}{Z_\epsilon(m)} - \frac{\psi_\epsilon(m) Z'_\epsilon(m)}{Z_\epsilon(m)^2} \right) = -\frac{\alpha \psi_\epsilon(m)}{Z_\epsilon(m)} \text{Var} \left( u_\epsilon^{(m)} \right).$$

□

Consequently,  $\tau'_\epsilon(m) \psi_\epsilon(m) = -\frac{\alpha \text{Var}(u_\epsilon^{(m)})}{Z_\epsilon(m)} \psi_\epsilon(m) \leq 0$ . We deduce that the behavior of the function  $\tau_\epsilon$  is directly linked to the sign of  $\psi_\epsilon$ .

**Proposition 3.6.** Let a confining potential  $V$  which verifies (V-1)–(V-6) and an interacting potential  $F(x) := \frac{\alpha}{2} x^2$  with  $\alpha > 0$ . The critical value  $\epsilon_c$  is the only zero of the function  $\epsilon \mapsto \frac{d^2}{dm^2} \Upsilon_\epsilon(u_\epsilon^m) \Big|_{m=0}$ .

Furthermore, for all  $\epsilon < \epsilon_c$ , we have  $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon(u_\epsilon^-) < \Upsilon_\epsilon(u_\epsilon^0)$ .

*Proof.* By taking the derivative in (3.2), we get:

$$\tau_\epsilon''(m) = -\frac{\alpha \text{Var}(u_\epsilon^m)}{Z_\epsilon(m)} \psi_\epsilon'(m) - \psi_\epsilon(m) \frac{d}{dm} \left[ \frac{\alpha \text{Var}(u_\epsilon^m)}{Z_\epsilon(m)} \right].$$

As  $\psi_\epsilon(0) = 0$ , we obtain:  $\tau_\epsilon''(0) = -\frac{\alpha \text{Var}(u_\epsilon^0)}{Z_\epsilon(0)} \psi_\epsilon'(0)$ . We recall that  $\psi_\epsilon'(0) = \gamma_0(\epsilon)$  (defined in the proof of Theorem 2.1). According to Theorem 2.1, the critical value of  $\epsilon_c$  is the only solution of  $\gamma_0(\epsilon) = 0$  consequently there is only one value of  $\epsilon$  such that  $\frac{d^2}{dm^2} \Upsilon_\epsilon(u_\epsilon^m) \Big|_{m=0} = 0$  and this value is  $\epsilon_c$ .

When  $\epsilon < \epsilon_c$ , there are exactly three stationary measures:  $u_\epsilon^+$ ,  $u_\epsilon^-$  and  $u_\epsilon^0$ . The symmetry implies directly  $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon(u_\epsilon^-)$ . By definition of the two outlying stationary measures,  $\int_{\mathbb{R}} x u_\epsilon^+(x) dx$  is the only positive zero of the function  $\psi_\epsilon$  and consequently the only positive critical point of the function  $\tau_\epsilon$  according to Lemma 3.5. Since  $\epsilon < \epsilon_c$ ,  $\frac{d^2}{dm^2} \Upsilon_\epsilon(u_\epsilon^m) \Big|_{m=0} < 0$  that implies  $\tau_\epsilon$  reaches its global minimum in  $\int_{\mathbb{R}} x u_\epsilon^+(x) dx$  and in  $\int_{\mathbb{R}} x u_\epsilon^-(x) dx$ . Also,  $\tau_\epsilon$  reaches one local maximum in  $0 = \int_{\mathbb{R}} x u_\epsilon^0(x) dx$ . Immediatly, we get  $\Upsilon_\epsilon(u_\epsilon^\pm) < \Upsilon_\epsilon(u_\epsilon^0)$ .  $\square$

According to (3.1), after recalling that  $\epsilon_c^{(0)}$  is the critical value under which there is a unique symmetric stationary measure, we get this new lower-bound:

$$\epsilon_c \geq \sup \left\{ \epsilon_0 \in ]0, \epsilon_c^{(0)}[ \mid \forall \epsilon \in ]0, \epsilon_0[, \min \text{Sp}(J_\epsilon(m^0)) < 0 \right\}$$

where  $J_\epsilon(m)$  is the Hessian matrix of the function from  $\mathbb{R}^{2n-1}$  to  $\mathbb{R}$ :  $m \mapsto \Upsilon_\epsilon(u_\epsilon^m)$  and  $m^0$  is the only point in  $\mathbb{R}^{2n-1}$  such that  $u_\epsilon^{m^0}$  is the unique symmetric stationary measure. Here, we defined  $u_\epsilon^m$  by:

$$u_\epsilon^m(x) = \frac{\exp \left[ -\frac{2}{\epsilon} W_m(x) \right]}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} W_m(y) \right] dy}$$

with  $W_m(x) = V(x) + F(x) + \sum_{p=0}^{2n-1} \frac{(-1)^p}{p!} m_p F^{(p)}(x)$

However, the computations are so tedious that it is better to use the previous method. Also, we will derive this method for a simpler one.

### 3.1.3 Stability of the free-energy method

We will use here another method in the particular example  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$  and  $F(x) := \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4$  but it can be computed for more general potentials. We provide some simulation of an *a priori* other critical value. We define  $\tilde{\epsilon}_c(\alpha, \beta)$  as the transition phase between the stability and the instability of the symmetric stationary measure for the free-energy.

**Definition 3.7.** We will say that  $u_\epsilon^0$  is stable for  $\Upsilon_\epsilon$  if for each function  $f \in \mathcal{L}_2(u_\epsilon^0)$  which verifies  $\int_{\mathbb{R}} f(x) u_\epsilon^0(x) dx = 0$  and  $C \geq f(x) \geq -C$  with  $C > 0$ , the



application  $\zeta_\epsilon(\delta) := \Upsilon_\epsilon(u_\epsilon^0 + \delta f \times u_\epsilon^0)$  admits a local maximum in  $\delta = 0$ . We will call  $\tilde{\epsilon}_c^+(\alpha, \beta)$  the value such that for all  $\epsilon > \tilde{\epsilon}_c^+(\alpha, \beta)$ ,  $u_\epsilon^0$  is stable for  $\Upsilon_\epsilon$  and  $\tilde{\epsilon}_c^-(\alpha, \beta)$  the value such that for all  $\epsilon < \tilde{\epsilon}_c^-(\alpha, \beta)$ ,  $u_\epsilon^0$  is not stable for  $\Upsilon_\epsilon$ .

**Remark 3.8.** It corresponds to a variational calculus. Intuitively, since the second derivative of the confining potential  $V$  is convex, we expect a similar result for the free-energy  $\Upsilon_\epsilon$ .

Since  $u_\epsilon^0$  is a stationary measure, we get directly  $\zeta'_\epsilon(0) = 0$ . It remains now to compute the second derivative:

$$\begin{aligned} \zeta''_\epsilon(0) &= \frac{\epsilon}{4} \int_{\mathbb{R}} f_-(x)^2 u_\epsilon^0(x) dx + \frac{\epsilon}{4} \int_{\mathbb{R}} f_+(x)^2 u_\epsilon^0(x) dx - \frac{\alpha}{2} \left( \int_{\mathbb{R}} x f_-(x) u_\epsilon^0(x) dx \right)^2 \\ &\quad - \beta \left( \int_{\mathbb{R}} x f_-(x) u_\epsilon^0(x) dx \right) \left( \int_{\mathbb{R}} x^3 f_-(x) u_\epsilon^0(x) dx \right) \end{aligned}$$

with  $f_-(x) := \frac{f(x) - f(-x)}{2}$  and  $f_+(x) := \frac{f(x) + f(-x)}{2}$ . The Cauchy-Schwarz inequality implies

$$\zeta''_\epsilon(0) \geq \left( \frac{\epsilon}{4} - \frac{\alpha}{2} m_2(\epsilon) - \beta \sqrt{m_2(\epsilon) m_6(\epsilon)} \right) \int_{\mathbb{R}} f(x)^2 u_\epsilon^0(x) dx$$

where  $m_2(\epsilon)$  (resp.  $m_6(\epsilon)$ ) is the second (resp. sixth) moment of  $u_\epsilon^0$ . Consequently, the inequality

$$\frac{2\alpha}{\epsilon} m_2(\epsilon) + \frac{4\beta}{\epsilon} \sqrt{m_2(\epsilon) m_6(\epsilon)} \leq 1$$

implies  $\epsilon \geq \tilde{\epsilon}_c^+(\alpha, \beta)$ . This will permit us to simulate  $\tilde{\epsilon}_c^+(\alpha, \beta)$ . Let us note that if  $\beta = 0$ , we recover the implicit equation which defines the critical value  $\epsilon_c(\alpha)$  exhibited in Theorem 2.1.

The method of stability derives from the one of convexity since the idea of the stability around  $u_\epsilon^0$  consists in choosing a particular direction and computing the second derivative in this direction.

We consider the sequence of bounded functions  $(f_n)_{n \in \mathbb{N}}$  with  $f_n(x) := x \mathbb{1}_{[-n, n]}$ . We remark that  $f_n \in \mathcal{L}^2(u_\epsilon^0)$ . For all  $n \in \mathbb{N}$ , we can define  $\zeta_\epsilon$  on an interval containing 0 in its interior. Consequently,  $\zeta''_\epsilon(0)$  has a sense for all  $n \geq 0$  (we do not write the dependence in  $n$  in  $\zeta''_\epsilon(0)$  for simplifying the reading). Since  $f_n$  converges towards  $x \mapsto x$  in  $\mathcal{L}^2(u_\epsilon^0)$ , it yields:

$$\lim_{n \rightarrow +\infty} \zeta''_\epsilon(0) = m_2(\epsilon) \left( \frac{\epsilon}{4} - \frac{\alpha}{2} m_2(\epsilon) - \beta m_4(\epsilon) \right).$$

We deduce that the inequality  $\frac{2\alpha}{\epsilon} m_2(\epsilon) + \frac{4\beta}{\epsilon} m_4(\epsilon) > 1$  implies  $\epsilon < \tilde{\epsilon}_c^-(\alpha, \beta)$ . This will permit us to simulate  $\tilde{\epsilon}_c^-(\alpha, \beta)$ .

By taking  $N := 2 \times 10^5$ , we get the following simulations of the surfaces of  $\tilde{\epsilon}_c^-(\alpha, \beta)$  and  $\tilde{\epsilon}_c^+(\alpha, \beta)$ :

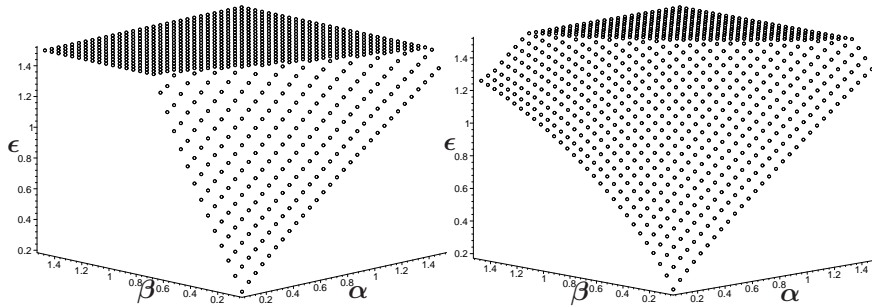


Figure 3:  $\tilde{\epsilon}_c^+(\alpha, \beta)$

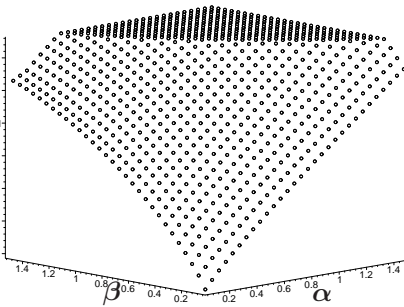


Figure 4:  $\tilde{\epsilon}_c^-(\alpha, \beta)$

**Remark 3.9.** *Let us note that it would also have been possible to compute with  $x \mapsto x^{2n+1}$  for  $n \geq 1$ .*

### 3.1.4 Comparison of the methods

If  $V$  is even and if  $\deg(F) \geq 4$ , three methods have been examined.

The first one is the low-energy method which consists in providing one symmetric stationary measure  $u_\epsilon^0$  and one asymmetric measure with lower free-energy than  $u_\epsilon^0$ . The main advantage of this method is the simplicity of computation. But, the critical value provided is far from the phase transition. Thanks to this method, we see that the critical value is not asymptotically small.

The method which uses the convexity of the free-energy around a symmetric stationary measure seems to be the most precise because it needs only a local knowledge of the free-energy around this measure and this is exactly in this sense that we can separate the two phases in the case described in Theorem 2.1. However, even if  $F(x) := \frac{\beta}{4}x^4$ , the computations are too tedious.

The last method consists in the stability or the instability of the free-energy around a symmetric stationary measure. Since the direction  $x \mapsto x$  plays a special role in the study of the self-stabilizing processes, we do not study the stability in all the directions but in this special one. Figure 3 and Figure 4 prove that the estimation provided by the low-energy method is less than the one provided by the instability of the free-energy around  $u_\epsilon^0$ . In the other hand, the stability method provides a larger value but by construction, it is reasonable to assume that this is larger than the critical value.

## 3.2 If $V$ is asymmetric

We will only simulate with  $F(x) := \frac{\alpha}{2}x^2$ . It remains to study the number of roots of a function from  $\mathbb{R}$  to  $\mathbb{R}$ . However, it is not as simple as if  $V$  was even. Indeed, when it is symmetric, there is an immediate and obvious solution - and this is 0 - then the number of stationary measures is directly linked to the second derivative in this obvious solution. Here, there is not any obvious solution so we need to know the whole trajectory of the function. Consequently, the simulation

is the following. We compute  $\chi_\epsilon(a_g + \frac{k}{M}(a_d - a_g))$  for all  $1 \leq k \leq M - 1$  by using the Monte Carlo method. Then, we count the number of changes of sign. We apply this simulation to  $V(x) := \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2}$  and  $F(x) := \frac{\alpha}{2}x^2$  for  $\alpha := j \times 0.01$  for  $1 \leq j \leq 120$  and  $\epsilon := i \times 0.005$  for  $1 \leq i \leq 60$ . By taking  $N := 2 \times 10^5$  and  $M := 250$ , we obtain:

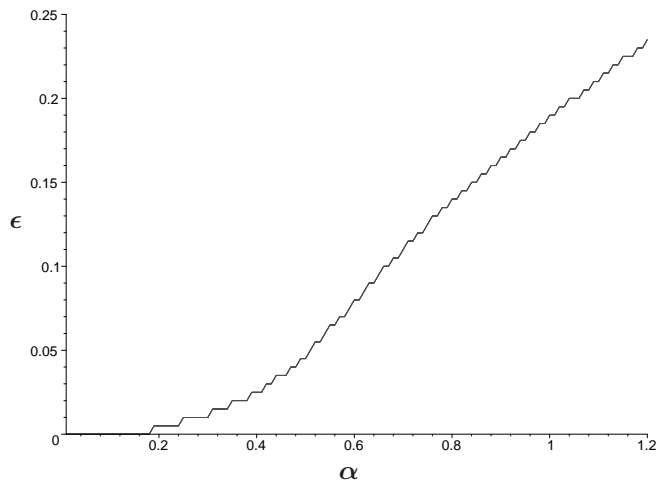


Figure 5:  $\epsilon_c(\alpha)$

The lack of precision and the slow increasing of the critical value  $\epsilon_c(\alpha)$  in function of  $\alpha$  implies that the curve seems jerky.

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