



**HAL**  
open science

# Phase transitions of McKean-Vlasov processes in symmetric and asymmetric multi-wells landscape

Julian Tugaut

► **To cite this version:**

Julian Tugaut. Phase transitions of McKean-Vlasov processes in symmetric and asymmetric multi-wells landscape. 2011. hal-00573046v1

**HAL Id: hal-00573046**

**<https://hal.science/hal-00573046v1>**

Preprint submitted on 2 Mar 2011 (v1), last revised 10 Aug 2012 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Phase transitions of McKean-Vlasov processes in symmetric and asymmetric multi-wells landscape\*

Julian Tugaut  
Fakultät für Mathematik  
Universität Bielefeld  
D-33615 Bielefeld  
Germany  
Email: jtugaut@math.uni-bielefeld.de

## Abstract

We prove under simple assumptions that there exist several phases for the self-stabilizing processes in a non-convex landscape. When the coefficient of diffusion is small, there are at least three stationary measures and when it is sufficiently big, there is a unique stationary measure. The non-uniqueness in small noise has already been proved in [Herrmann, Tugaut|2010] when the landscape is symmetric. Here, we will extend it to the symmetric and the asymmetric landscapes with linear or non-linear interaction functions. The critical values will also be examined.

## Résumé

Nous prouvons sous des conditions simples qu'il existe plusieurs phases pour les processus auto-stabilisants dans un paysage non-convexe. Lorsque le coefficient de diffusion est petit, il y a au moins trois mesures stationnaires et lorsqu'il est suffisamment grand, il y en a une seule. La non-unicité à petit bruit a déjà été prouvée dans [Herrmann, Tugaut|2010] lorsque le paysage est symétrique. Ici, nous allons étendre ces résultats pour des potentiels symétriques ou asymétriques avec une fonction d'interaction linéaire ou non. La valeur critique va également être étudiée.

**Key words and phrases:** phase transitions ; McKean-Vlasov stochastic differential equations ; non-convexity ; stationary measures ; fixed point theorem ; interacting particle systems ; propagation of chaos ; free-energy.

**2000 AMS subject classifications:** primary 60H10; secondary: 60G10, 60J60, 65C05, 82C22

---

\*Supported by the DFG-funded CRC 701, Spectral Structures and Topological Methods in Mathematics, at the University of Bielefeld.

## Introduction

We investigate the phase transitions of the following McKean-Vlasov diffusion:

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t V'(X_s) ds - \int_0^t F' * u_s(X_s) ds \\ u_s = \mathcal{L}(X_s) \end{cases} \quad (\text{I})$$

Here,  $*$  denotes the convolution. The particularity of this model is that the own law of the process intervenes in the equation. Consequently, it is non-markovian since the law  $u_t$  depends on the past. We remark especially  $\mathbb{E}_\mu[f(X_t)] \neq \int_{\mathbb{R}} \mathbb{E}_x[f(X_t)] d\mu(x)$ . We note that  $X_t$  and  $u_t$  depend on  $\epsilon$ . We do not write  $\epsilon$  for the comfort reading.

The motion of the process is subject to three concurrent forces. The first one is the gradient of the so-called confining potential  $V$ . The second influence is some heat process  $(\sqrt{\epsilon} B_t)_{t \in \mathbb{R}_+}$  which allows the particle to escape from the stable domains of  $V$ . Let us stress that  $\epsilon$  is not necessary asymptotically small. The third term represents the average tension between two processes of law  $u_t$ . Indeed, we remark:  $F' * u_s(X_s(\omega_0)) = \int_{\mathbb{R}} F'(X_s(\omega_0) - Y_s(\omega)) d\mathbb{P}(\omega)$  where  $(X_t)_t$  and  $(Y_t)_t$  verify (I) and  $(\Omega, \mathcal{F}, \mathbb{P})$  is the underlying measurable space.

This kind of processes were introduced by McKean, see [McK67] or [McK66]. In this paper, we will make some smoothness assumptions on the interaction potential  $F$ . Let just note that there exist some previous works about non-smooth  $F$ . If  $F$  is the Heaviside step function and  $V := 0$ , (I) is the Burgers' equation:

$$dX_t = \sqrt{\epsilon} dB_t - V'(X_t) dt - \mathbb{P}\left\{\omega \in \Omega \mid X_t(\omega) = X_t\right\} dt,$$

see [SV79].

If  $F := \delta_0$  and without confining potential, it is the Oelschläger equation:

$$dX_t = \sqrt{\epsilon} dB_t - V'(X_t) dt - \frac{d}{dx} \Big|_{x=X_t} \mathbb{P}\left\{\omega \in \Omega \mid X_t(\omega) = X_t\right\} dt$$

studied in [Oel85].

The diffusion  $X_t$  in (I) can be seen as one particle in a continuous mean-field system of an infinite number of particles. The mean-field system that we will consider is a continuous random dynamical system like

$$\begin{cases} dX_t^1 = \sqrt{\epsilon} dB_t^1 - V'(X_t^1) dt - \frac{1}{N} \sum_{j=1}^N F'(X_t^1 - X_t^j) dt \\ \vdots \\ dX_t^i = \sqrt{\epsilon} dB_t^i - V'(X_t^i) dt - \frac{1}{N} \sum_{j=1}^N F'(X_t^i - X_t^j) dt \\ \vdots \\ dX_t^N = \sqrt{\epsilon} dB_t^N - V'(X_t^N) dt - \frac{1}{N} \sum_{j=1}^N F'(X_t^N - X_t^j) dt \end{cases} \quad (\text{II})$$

where the  $N$  brownian motions  $(B_t^i)_{t \in \mathbb{R}_+}$  are independents. Mean-field systems have been a lot studied. We will not try to be exhaustiv. These are some references: [DG87] about the large deviations for  $N \rightarrow +\infty$ , [Mél96] under weak assumptions on  $V$  and  $F$ . For applications, see [CDPS10] about social interactions or [CX10] about the stochastic partial differential equations. Let us note that there is also a huge literature about mean-field system in discrete space, in particular the Currie-Weiss.

The link between the self-stabilizing processes and the mean-field system for  $N$  tending to  $+\infty$  is called the propagation of chaos, see [Szn91] under Lipschitz properties ; [BRTV98] if  $V$  is a constant ; [Mal01] or [Mal03] when both potentials are convex ; [BAZ99] for a more precise result ; [BGV07], [DPdH96] or [DG87] for a sharp estimate ; [CGM08] for a uniform result in time in the non-uniformly convex case.

As proved in [DG87], the empirical law of the mean-field system satisfies a large deviations principle with a rate function which depends on the law of Diffusion (I). Consequently, the long-time behavior of  $\mathcal{L}(X_t)$  provides some consequences on the exit time for the particle system (II). But, according to [HT10a] and [Tug10b], there are exactly three stationary measures for  $\epsilon$  small enough,  $\mathcal{L}(X_t)$  converges weakly towards a stationary measure and the domain of attraction of each limiting value is not reduced to a point in simple cases. The study does depend on  $\epsilon$  so we can imagine that there is a phase transition between an area where Diffusion (I) admits exactly three stationary measures and an other where it admits only one.

Let us also note that the previous results about the non-uniqueness of the stationary measures and its consequences ([HT10a], [HT10b], [HT09] and [Tug10b]) deal only with the symmetric case. However, if we want to consider some potential which oscillates slowly on the time, we need to study the case where  $V$  is asymmetric.

Let us recall briefly some of the previous results on diffusions like (I). The existence problem can be solved by two different ways. The first one consists in the application of a fixed point theorem, see [McK67], [BRTV98], [CGM08] or [HIP08] in the non-convex case. The other consists in a propagation of chaos of a mean-field system, see for example [Mél96].

In [McK67], the author proved that the law of the solution  $du_t$  admits a  $\mathcal{C}^\infty$ -continuous density  $u_t$  with respect to the Lebesgue measure since  $t > 0$ . Moreover, he provided the following non-linear parabolic partial differential equation:

$$\frac{\partial}{\partial t} u_t(x) = \frac{\partial}{\partial x} \left\{ \frac{\epsilon}{2} \frac{\partial}{\partial x} u_t(x) + u_t(x) (V'(x) + F' * u_t(x)) \right\}. \quad (\text{III})$$

Then we can study Equation (III) by probabilistic methods involving (I) or (II), see [CGM08], [Fun84], [Mal03]. Reciprocally, equation (I) is a useful tool for describing the stationary measure(s) and the long-time behavior, see [BRTV98], [BRV98], [Tam84], [Tam87] or [Ver06]. In [HT10a], when  $V$  is symmetric and non-convex, by using (III), it has been proved that Diffusion (I) admits at least three stationary measures under assumptions easy to verify: one is symmetric

and the two others are asymmetric. Moreover, Theorem 3.2 in the same article states the thirdness of the stationary measures for  $\epsilon$  small enough if  $V''$  is convex and  $F'$  is linear.

The estimates of the small-noise asymptotic of these three stationary measures are provided in [HT10b] and [HT09]. The limiting values are finite combining of Dirac measures and the convergence rate is polynomial in function of  $\epsilon$ .

In the convex case (including the non-uniformly convex case), Cattiaux, Guillin and Malrieu proceeded a uniform propagation of chaos in [CGM08] and obtained the uniqueness of the stationary measure and the convergence. Nevertheless, according to Proposition 5.17 and Remark 5.18 in [Tug10a], it is impossible to find a general result of uniform propagation of chaos. In the non-convex case and under two strong restrictions (the center of mass is fixed and  $V''(0) + F''(0) > 0$ ), Carrillo, McCann and Villani provided the convergence in [CMV03].

The long-time behavior of the law  $u_t$  in the more difficult non-convex case has been the subject of [Tug10b]:  $u_t$  converges weakly towards a stationary measure if the initial entropy is finite. Furthermore, as written before, the basin of attraction of each limiting value is not reduced to a point.

In order to detect the type of regime Diffusion (I) is in that is to say the uniqueness or the non-uniqueness of the stationary measures, we shall introduce the following free-energy, already used in [Tug10b]:

$$\Upsilon_\epsilon(u) := \int_{\mathbb{R}} \left\{ \frac{\epsilon}{2} \log(u(x)) + V(x) + \frac{1}{2} F * u(x) \right\} u(x) dx \quad (\text{IV})$$

for all measures  $du$  which are absolutely continuous with respect to the Lebesgue measure. We can note that  $du_t$  satisfies this hypothesis since  $t > 0$  where  $du_t$  is the law of a self-stabilizing process, see [HT10a].

The paper is organized as follows. After giving the assumptions, we provide some results about the existence of stationary measures for self-stabilizing processes even when  $V$  is not symmetric or when  $V$  has more than two wells. Particularly, we prove that there is at least one outlying stationary measure around each well since  $\epsilon$  is small enough and  $\alpha$  is big enough with  $F(x) := \frac{\alpha}{2}x^2$ . Also, we state that there is at least one intermediate stationary measure between each pair of outlying stationary measures. In the nonlinear case, we extend the result of [HT10a] and state that there is always at least one stationary measure. Subsequently, in a simple case, we show that there exists exactly two phases separated by a particular value of  $\epsilon$  (characterized as the zero of some Bessel function): one with a unique stationary measure and one with exactly three ones. Then, we state the uniqueness of the symmetric stationary measure if  $V$  is symmetric by using a uniform propagation of chaos with respect to the time. This permit to extend Theorem 1.11 in [Tug10b] to the case  $\deg(V) = \deg(F)$  and to the asymmetric case. Next, we show the uniqueness of the stationary measure if  $\deg(F) = 2$  since  $\epsilon$  is big enough which implies the existence of a phase transition. Finally, we study the critical value when  $V$  is symmetric or not and when  $F'$  is linear or not. We proceed the simulations by five different methods involving the free-energy and some calculus of variations.

## Assumptions

We assume the following properties on the confining potential  $V$ :

- (V-1)  $V$  is a polynomial function with  $\deg(V) \geq 4$ .
- (V-2) The equation  $V'(x) = 0$  admits exactly  $2M + 1$  solutions with  $M \geq 1$ :  $a_1, \dots, a_{2M+1}$ . Furthermore,  $V''(a_i) \neq 0$  for all  $1 \leq i \leq 2M + 1$ .
- (V-3)  $V(x) \geq C_4 x^4 - C_2 x^2$  for all  $x \in \mathbb{R}$  with  $C_2, C_4 > 0$ .
- (V-4)  $\lim_{x \rightarrow \pm\infty} V''(x) = +\infty$  and  $V''(x) > 0$  for all  $x \notin [a_1; a_{2M+1}]$ .
- (V-5) Initialization:  $V(0) = 0$ .

We would like to stress that weaker assumptions could be considered, particularly for the results involving the free-energy. However, all the mathematical difficulties are in the polynomial case and it permits to avoid some technical and tedious computations.

Let us present now the following properties under the interaction potential  $F$ :

- (F-1)  $F$  is an even polynomial function and  $\deg(F) =: 2n \geq 2$ .
- (F-2)  $F$  is convex.
- (F-3) Initialization:  $F(0) = 0$ .

Like for the confining potential  $V$ , we do not need  $F$  to be a polynomial function. Since we will not investigate the self-stabilizing process in the time, we do not need any assumption on the initial law  $du_0$ . However, in the following, we will use some self-stabilizing diffusions and some mean-field systems. Each time this will be necessary, we will admit these assumptions on the initial law  $du_0$ :

- (ES)  $\int_{\mathbb{R}} x^{8q^2} du_0(x) < \infty$  with  $q := \max \left\{ \frac{\deg(V)}{2}, \frac{\deg(F)}{2} \right\}$ .
- (FE) The probability measure  $du_0$  admits a  $\mathcal{C}^\infty$ -continuous density  $u_0$  with respect to the Lebesgue measure. And,  $\int_{\mathbb{R}} u_0(x) \log(u_0(x)) dx < +\infty$ .

Under (ES), we know by Theorem 2.12 in [HIP08] that (I) admits a strong solution. Moreover, we have the following inequality:

$$\max_{1 \leq j \leq 8q^2} \sup_{t \in \mathbb{R}_+} \mathbb{E} \left[ |X_t|^j \right] \leq M_0. \quad (\text{V})$$

We deduce immediately that the family  $(u_t)_{t \in \mathbb{R}_+}$  is tight.

Sometimes, we will use some additional hypotheses:

- (DW)  $V$  is a double-well potential:  $M = 1$ . We call  $a_g := a_1$  and  $a_d := a_3$ .
- (E)  $V$  is even.

- (EDW)  $V$  is an even double-well potential. We call  $a := a_d$ . Then  $a_g = -a$ .
- (HC)  $V$  is even and for all  $k \geq 2$ ,  $V^{(2k)}(0) \geq 0$ .
- (LIN)  $F'$  is linear.

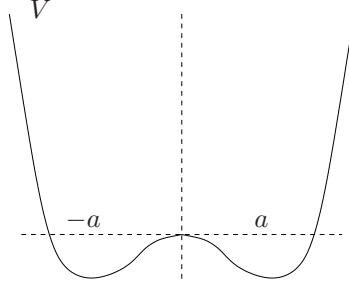


Figure 1: Symmetric landscape  $V$  with two wells  $a$  and  $-a$

Let us note that (HC) implies (EDW). And, under (EDW), we know by [HT10a] that (I) admits at least one symmetric stationary measure. And, if

$$\sum_{p=0}^{2n-2} \frac{|F^{(p+2)}(a)|}{p!} a^p < F''(0) + V''(a), \quad (\text{VI})$$

there are at least three stationary measures:  $u_\epsilon^0$  is symmetric and  $u_\epsilon^+$  and  $u_\epsilon^-$  are asymmetric. Moreover, we know by [Tug10b] that we do not need Inequality (VI) to be true. Furthermore, we know by [HT10b] that there is a unique  $x_0 \geq 0$  such that  $V'(x_0) + \frac{1}{2}F'(2x_0) = 0$  and  $V''(x_0) + \frac{F''(0)+F''(2x_0)}{2} > 0$ . The same paper provides that  $u_\epsilon^0$  converges weakly towards  $\frac{1}{2}\delta_{x_0} + \frac{1}{2}\delta_{-x_0}$  and  $u_\epsilon^\pm$  converges weakly towards  $\delta_{\pm a}$ ; in the small noise limit.

The assumptions (FE), (ES) and (EDW) imply the weak convergence of  $u_t$  towards a stationary measure.

For concluding the introduction, we write the statements of the three main results:

**Multiple stationary measures:** Set  $V$  which verifies the assumptions (V-1)–(V-5). Let  $\alpha > 2 \max_{0 \leq i \leq M} \sup_{y \neq a_{2i+1}} \frac{V(a_{2i+1}) - V(y)}{(a_{2i+1} - y)^2}$  and  $F(x) := \frac{\alpha}{2}x^2$ . Then, for  $\epsilon$  small enough, Diffusion (I) admits at least  $2M + 1$  stationary measures.

**Phase transition:** Set  $V(x) := -\frac{\vartheta_1}{2}x^2 + \sum_{p=2}^q \frac{\vartheta_p}{(2p)!}x^{2p}$  with  $\vartheta_p \geq 0$  for all  $1 \leq p \leq q$  and  $\vartheta_1\vartheta_q > 0$ . By taking  $F(x) := \frac{\alpha}{2}x^2$  with  $\alpha > 0$ , there exists  $\epsilon_c \in \mathbb{R}$  such that:

- For all  $\epsilon \geq \epsilon_c$ , Diffusion (I) admits a unique stationary measure.
- For all  $\epsilon < \epsilon_c$ , Diffusion (I) admits exactly three stationary measures.

Moreover, the critical value  $\epsilon_c$  is the unique solution of the equation:

$$\int_{\mathbb{R}_+} \left( x^2 - \frac{1}{2\alpha} \right) \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx = 0.$$

**Thirdness:** Let  $V$  and  $F$  two potentials which satisfy (V-1)–(V-5) and (F-1)–(F-3). We assume also that  $V''$  and  $F''$  are convex. Then, if  $V''(0) + F''(0) \geq 0$  and  $\deg(V) \geq \deg(F)$ , Diffusion (I) admits exactly three stationary measures for  $\epsilon$  small enough.

## 1 General results on the existence problem

The aim of this section is to extend the results of [HT10a]. Particularly, the existence and the uniqueness of the stationary measure(s) for Diffusion (I) are examined when  $V$  is not symmetric and when it has more than two wells. For doing this, we will use methods similar to the one of [HT10a] and we will derive the existence from the subconvergence's result in [Tug10b].

**Lemma 1.1.** *Set an asymmetric confining potential  $V$  which verifies the assumptions (V-1)–(V-5) and an interacting potential  $F$  which verifies (F-1)–(F-3). The self-stabilizing diffusion (I) does not admit a symmetric stationary measure with  $8q^2$ -th moment finite.*

*Proof.* We proceed a *reductio ad absurdum* by assuming that such a measure exists:  $du_\epsilon$ . According to Lemma 2.2 in [HT10a] (in which we did not use any symmetry property), we know that this measure is absolutely continuous with respect to the Lebesgue's measure. We call  $u_\epsilon$  the density. Furthermore, it verifies:

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}(V(x) + F * u_\epsilon(x))\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}(V(y) + F * u_\epsilon(y))\right] dy}.$$

The measure  $u_\epsilon$  being symmetric, the function  $F * u_\epsilon$  is even. Set a point  $x_0 \in \mathbb{R}$  such that  $V(x_0) \neq V(-x_0)$ . Then,  $u_\epsilon(x_0) \neq u_\epsilon(-x_0)$  that is a contradiction.  $\square$

### 1.1 Linear case

In this section, we assume (LIN) that is to say  $F(x) := \frac{\alpha}{2}x^2$  for all  $x$  with  $\alpha > 0$ . The research of a stationary measure remains *in fine* to a parametrization problem on  $\mathbb{R}$ . Indeed, as noted in [HT10a],  $u_\epsilon$  is invariant if and only if it verifies

$$u_\epsilon(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + \alpha\frac{x^2}{2} - \alpha m_1(\epsilon)x\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + \alpha\frac{y^2}{2} - \alpha m_1(\epsilon)y\right)\right] dy} \quad (1.1)$$

$$\text{and } m_1(\epsilon) = \frac{\int_{\mathbb{R}} x \exp\left[-\frac{2}{\epsilon}\left(V(x) + \alpha\frac{x^2}{2} - \alpha m_1(\epsilon)x\right)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + \alpha\frac{y^2}{2} - \alpha m_1(\epsilon)y\right)\right] dy}. \quad (1.2)$$

Obviously, under (E), the measure  $u_\epsilon^0$  defined by (1.1) with  $m_1(\epsilon) = 0$  is symmetric and verifies also (1.2). So, it is invariant and

$$u_\epsilon^0(x) = \frac{\exp\left[-\frac{2}{\epsilon}\left(V(x) + \alpha\frac{x^2}{2}\right)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon}\left(V(y) + \alpha\frac{y^2}{2}\right)\right] dy}.$$



The first moment of a symmetric measure is necessary equal to 0. We deduce immediately the uniqueness of a symmetric stationary measure if the confining potential is symmetric.

It has already been proved in [HT10a] that there exist some other reals  $m_1(\epsilon)$  satisfying (1.1) and (1.2) under (E) for  $\epsilon$  sufficiently small.

For studying in a more general case, we introduce three particular functions:

**Definition 1.2.** For all  $m \in \mathbb{R}$  and  $\epsilon > 0$ , let us define

$$Z_\epsilon(m) := \int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \alpha \frac{x^2}{2} - \alpha m x \right) \right] dx, \quad (1.3)$$

$$\Psi_\epsilon(m) := \int_{\mathbb{R}} (x - m) \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \alpha \frac{x^2}{2} - \alpha m x \right) \right] dx \quad (1.4)$$

$$\text{and } \chi_\epsilon(m) := \frac{\Psi_\epsilon(m)}{Z_\epsilon(m)}. \quad (1.5)$$

This function  $\chi_\epsilon$  is exactly the same than in [HT10a] but we insist on the fact that the function  $\Psi_\epsilon$  defined in [HT10a] and in [Tug10a] is not the one that we defined here.

Finding a stationary measure to the self-stabilizing diffusion (I) with a  $8q^2$ -th finite moment is bounded remains to find a zero to the functions  $\Psi_\epsilon$  and  $\chi_\epsilon$ .

Let us recall that - according to the assumption (V-2) - the potential  $V$  has exactly  $2M + 1$  critical points with  $M \geq 1$ :  $a_1, a_2, \dots, a_{2M+1}$ . In the following, when we will deal with one of these points, we will use the general notation  $a_0$  without precising if it is a wells or a hill.

**Proposition 1.3.** Set  $V$  verifying the assumptions (V-1)–(V-5) and  $a_0$  one of its critical points such that  $\alpha + V''(a_0) > 0$ .

1) We assume:

$$\alpha > 2 \sup_{x \neq a_0} \frac{V(a_0) - V(x)}{(a_0 - x)^2}. \quad (1.6)$$

Then, for all  $\delta \in ]0; 1[$ , there exists  $\epsilon_0 > 0$  such that for all  $\epsilon \leq \epsilon_0$ , the system (1.1)–(1.2) admits a solution  $m_1(\epsilon)$  satisfying

$$\left| m_1(\epsilon) - a_0 + \frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))} \epsilon \right| \leq \delta \epsilon. \quad (1.7)$$

*Proof.* The proof is exactly the same than the one of Proposition 3.1 in [HT10a]. Consequently, we will just sketch it. Set  $\tau > 0$ . Inequality (1.6) allows us to apply Lemma A.3 in [HT10a] to  $f(x) := -2\alpha\tau x$ ,  $n := 1$ ,  $U(x) := V(x) + \frac{\alpha}{2}x^2 - \alpha a_0 x$  and  $\mu := 0$ . By putting  $\tau^0 := \frac{V^{(3)}(a_0)}{4V''(a_0)(\alpha + V''(a_0))}$ , for all  $\delta \in ]0; 1[$ , we get the following first order approximation:

$$\chi_\epsilon(a_0 - \tau^0(1 \pm \delta)\epsilon) = \pm \delta \frac{V''(a_0)}{\alpha + V''(a_0)} \tau^0 \epsilon + o(\epsilon).$$

Since  $\chi_\epsilon$  is continuous, we deduce that for  $\epsilon$  small enough, there exists  $m_1(\epsilon)$  satisfying  $\chi_\epsilon(m_1(\epsilon)) = 0$  and (1.7).  $\square$

**Definition 1.4.** *The stationary measure associated to  $m_1(\epsilon)$  by (1.1) is called "outlying" around  $a_0$ . We write it  $u_\epsilon^{a_0}$  without forgetting that it is possible a priori that there exist several outlying stationary measures around  $a_0$ .*

*By using the same method than the one in the proof of Theorem 2.4 in [HT10b], we can prove easily the convergence of  $u_\epsilon^{a_0}$  towards  $\delta_{a_0}$  in the small noise limit (see [Tug10a]). In the same paper, it is proved that this convergence towards  $\delta_{a_0}$  would imply that  $x \mapsto V(x) + \frac{\alpha}{2}x^2 - \alpha a_0 x$  reaches its global minimum in  $a_0$ . Consequently, if  $\alpha < 2 \sup_{x \neq a_0} \frac{V(a_0) - V(x)}{(x - a_0)^2}$ , it is impossible for a sequence of stationary measures to converge towards  $\delta_{a_0}$  in the small-noise limit.*

**Remark 1.5.** *We obtained the existence of a critical value  $\epsilon_0(V, \alpha)$  under which there exists a stationary measure around  $\delta_{a_0}$ . But, we did not provide any analytic expression of this critical value. Moreover, we did not even prove that this value was finite.*

**Corollary 1.6.** *Let  $V$  which verifies the assumptions (V-1)–(V-5). If  $V$  reaches its global minimum on  $a_0$ , Diffusion (I) admits an outlying stationary measure around  $a_0$  for  $\epsilon$  small enough. Then, there is always at least one stationary measure when the interaction function  $F'$  is linear.*

*Proof.* If  $a_0$  is such that  $\inf_{x \in \mathbb{R}} V(x) \geq V(a_0)$  then  $V''(a_0) \geq 0$ . Hence,  $\alpha + V''(a_0) > 0$  and we get (1.6). We apply Proposition 1.3 which implies the existence of an outlying stationary measure around  $a_0$ .  $\square$

*Particularly, we get the result of Proposition 3.1 in [HT10a] when we consider a double-well symmetric potential  $V$ .*

Now, we can investigate the existence of other stationary measures (which would not be outlying around a wells). Under Conditions (1.6) and  $\alpha + V''(a_{2i}) > 0$ , we already know that there exists a stationary measure around  $a_{2i}$  (for  $1 \leq i \leq M$ ). However, we will see that (1.6) is not necessary for getting the existence between the wells  $a_{2i-1}$  and  $a_{2i+1}$ . We will prove that there exists at least one stationary measure between two outlying stationary measures around the wells  $a_{2i+1}$  and  $a_{2j+1}$  for all  $1 \leq i < j \leq M$ .

**Proposition 1.7.** *Set  $a_0 < b_0$  two critical points of  $V$  satisfying (1.6) and such that  $V''(a_0) > 0$  and  $V''(b_0) > 0$ . Then, there exists  $\delta_0 > 0$  and  $\epsilon_0 > 0$  such that for all  $\epsilon \leq \epsilon_0$ , the system (1.1)–(1.2) admits at least one solution  $m(\epsilon) \in [a_0 + \delta_0; b_0 - \delta_0]$  for  $\epsilon$  small enough. Consequently, there exists a stationary measure  $u_\epsilon^{a_0, b_0}$  defined by  $m(\epsilon)$  with (1.1).*

*Proof.* We introduce the potential  $W_m(x) := V(x) + \frac{\alpha}{2}x^2 - \alpha m x$  for all  $m > 0$ . We note that for all  $x \in \mathbb{R}$   $W_m(x) = W_{a_0}(x) - \alpha(m - a_0)x$  with  $W_{a_0}(x) := V(x) + \frac{\alpha}{2}(x - a_0)^2 - \frac{\alpha}{2}a_0^2$ . Since  $a_0$  satisfies (1.6), the function  $W_{a_0}$  reaches its global minimum in a unique point and it is  $a_0$ . Let us show now that for  $|\delta|$

sufficiently small, the function  $W_{a_0+\delta}$  verifies the same property.

**Step 1.** Let us prove that each family of points where  $W_{a_0+\delta}$  reaches its global minimum converges towards  $a_0$  for  $\delta$  converging towards 0. For all  $\delta \in [-1; 1] \setminus \{0\}$ , we consider  $x(\delta)$  one point where  $W_{a_0+\delta}$  reaches its global minimum. So  $x(\delta)$  is a critical point of  $W_{a_0+\delta}$  which implies  $W'_{a_0+\delta}(x(\delta)) = 0$  so  $W'_{a_0}(x(\delta)) = \alpha\delta$ . Consequently, for all  $\delta \in [-1; 1] \setminus \{0\}$ ,  $|W'_{a_0}(x(\delta))| \leq \alpha$  then the family  $(x(\delta))_{\delta \in [-1; 1] \setminus \{0\}}$  is bounded. Moreover, by definition, for all  $x \in \mathbb{R}$ ,  $W_{a_0+\delta}(x(\delta)) \leq W_{a_0+\delta}(x)$ . Particularly, for  $x = a_0$ , this implies  $W_{a_0}(x(\delta)) \leq W_{a_0+\delta}(a_0) + \alpha\delta(x(\delta) - a_0)$  which tends towards  $W_{a_0}(a_0)$ . Hence, we get the convergence of  $x(\delta)$  towards  $a_0$  for  $\delta$  converging towards 0.

**Step 2.** Let us prove that the point  $x(\delta)$  is unique for  $\delta$  small enough. As  $W''_{a_0}(a_0) > 0$  according to the assumption (1.6), there exists  $\rho > 0$  such that for all  $x \in [a_0 - \rho; a_0 + \rho]$ ,  $W''_{a_0}(x) > 0$ . For  $\delta$  small enough, we have  $x(\delta) \in [a_0 - \rho; a_0 + \rho]$ . Consequently, there exists  $\delta_1 > 0$  such that for all  $\delta \in [-\delta_1; \delta_1]$ , all the points where  $W_{a_0+\delta}$  reaches its global minimum are in the interval  $[a_0 - \rho; a_0 + \rho]$ . And, as  $W''_{a_0+\delta}(x) = W''_{a_0}(x) > 0$  for all  $x \in [a_0 - \rho; a_0 + \rho]$ , we deduce that the potential  $W_{a_0+\delta}$  reaches its global minimum in a unique point. We call  $x(\delta)$  this unique point. Moreover, for  $\delta$  small enough,  $W''_{a_0+\delta}(x(\delta)) > 0$ .

**Step 3.** Let us show the two inequalities  $\chi_\epsilon(a_0 + \delta) < 0$  and  $\chi_\epsilon(a_0 - \delta) > 0$  for  $\delta > 0$  small enough. As  $W'_{a_0+\delta}(x(\delta)) = 0$ , we have  $W'_{a_0}(x(\delta)) = \alpha\delta$ . According to  $W''_{a_0}(a_0) > 0$ , we obtain the following first order approximation:

$$x(\delta) = a_0 + \frac{\alpha}{W''_{a_0}(a_0)} \delta + o(\delta).$$

Consequently:

$$x(\delta) - (a_0 + \delta) = -\frac{V''(a_0)}{\alpha + V''(a_0)} \delta + o(\delta).$$

By recalling that  $W''_{a_0+\delta}(x(\delta)) > 0$  for  $\delta$  small enough, we deduce that there exists  $\delta_2 < \delta_1$  such that for all  $\delta \in ]0; \delta_2[$ , we have  $x(\delta) - (a_0 + \delta) < 0$  and  $x(-\delta) - (a_0 - \delta) > 0$ . By using Lemme A.3 in [HT10a], we get

$$\begin{aligned} \chi_\epsilon(a_0 + \delta) &= x(\delta) - (a_0 + \delta) + o(1) \\ \text{and } \chi_\epsilon(a_0 - \delta) &= x(-\delta) - (a_0 - \delta) + o(1). \end{aligned}$$

Then, there exists  $\epsilon_1 > 0$  such that for all  $0 < \epsilon < \epsilon_1$ , we have  $\chi_\epsilon(a_0 + \delta) < 0$  and  $\chi_\epsilon(a_0 - \delta) > 0$ .

**Step 4.** In the same way, there exists  $\delta_3 > 0$  such that for all  $\delta \in ]0; \delta_3[$ , there exists  $\epsilon_2 > 0$  such that for all  $0 < \epsilon < \epsilon_2$ , we have  $\chi_\epsilon(b_0 + \delta) < 0$  and  $\chi_\epsilon(b_0 - \delta) > 0$ . Taking  $\delta_0 < \min\{\delta_2; \delta_3; \frac{b_0 - a_0}{2}\}$  and  $\epsilon_0 := \min\{\epsilon_1; \epsilon_2\}$  is sufficient for achieving the proof. Indeed, for  $\epsilon \leq \epsilon_0$ , we have  $\chi_\epsilon(a_0 + \delta_0) < 0 < \chi_\epsilon(b_0 - \delta_0)$  and  $a_0 + \delta_0 < b_0 - \delta_0$ . Then, by using the theorem of intermediate values, we deduce that there exists  $m(\epsilon) \in [a_0 + \delta_0; b_0 - \delta_0]$  such that  $\chi_\epsilon(m(\epsilon)) = 0$ .  $\square$

**Definition 1.8.** We assume that for one  $0 \leq i \leq M - 1$  (where  $M$  is defined in the hypothesis (V-2)), there exists one outlying stationary measure  $u_\epsilon^i$  around

the wells  $a_{2i+1}$  (see Definition 1.4) and one outlying stationary measure  $u_\epsilon^{i+1}$  around the wells  $a_{2i+3}$ . We call  $m_i(\epsilon)$  the mean of  $u_\epsilon^i$  and  $m_{i+1}(\epsilon)$  the one of  $u_\epsilon^{i+1}$ . If there exists  $m(\epsilon) \in ]m_i(\epsilon); m_{i+1}(\epsilon)[$  such that  $\chi_\epsilon(m(\epsilon)) = 0$ , we say that the stationary measure associated to  $m(\epsilon)$  is intermediate between  $u_\epsilon^i$  and  $u_\epsilon^{i+1}$  and we call this measure  $u_\epsilon^{i,i+1}$ .

The previous definition does not exclude that an intermediate stationary measure could also be an outlying stationary measure around a wells of  $V$ .

**Corollary 1.9.** *Set  $V$  which verifies (V-1)–(V-5). We assume:*

$$\alpha > 2 \max_{0 \leq i \leq M} \sup_{y \neq a_{2i+1}} \frac{V(a_{2i+1}) - V(y)}{(a_{2i+1} - y)^2}. \quad (1.8)$$

Then, for  $\epsilon$  small enough, the diffusion (I) admits at least  $2M + 1$  stationary measures.

*Proof.* The assumption (V-2) implies  $V''(a_{2i+1}) > 0$  for all  $0 \leq i \leq M$  hence  $V''(a_{2i+1}) + \alpha > 0$ . (1.8) implies that each  $a_{2i+1}$  satisfies (1.6). Then, we can apply Proposition 1.7 with  $a_1$  and  $a_3$ . We obtain the existence of  $\delta_1 > 0$  and  $\epsilon_1 > 0$  such that for all  $0 < \delta < \delta_1$  and  $\epsilon < \epsilon_1$ , the function  $\chi_\epsilon$  vanishes at  $m_{1,3}(\epsilon) \in [a_1 + \delta; a_3 - \delta]$ .

We do the same with each  $a_{2i+1}$  and  $a_{2i+3}$  for  $0 \leq i \leq M - 1$ . Then, we put  $\delta_0 := \min\{\delta_1, \dots, \delta_M\}$  and  $\epsilon_0 := \min\{\epsilon_1, \dots, \epsilon_M\}$ . Consequently, for all  $\epsilon \in ]0; \epsilon_0[$ , the function  $\chi_\epsilon$  vanishes on each interval  $[a_{2i+1} + \delta_0; a_{2i+3} - \delta_0]$ ,  $1 \leq i \leq M - 1$ . We introduce the set  $\mathcal{I} := \bigcup_{i=0}^{M-1} [a_{2i+1} + \delta_0; a_{2i+3} - \delta_0]$ .

Then, we can apply Proposition 1.3 and we know that the system admits one outlying stationary measure  $u_\epsilon^i$  around each wells of  $V$ :  $a_{2i+1}$  for  $0 \leq i \leq M$  with means verifying  $m_i(\epsilon) \notin \mathcal{I}$ .

Consequently, we have at least  $2M + 1$  stationary measures.  $M$  are intermediate and  $M + 1$  are outlying around the wells.  $\square$

Particularly, Corollary 1.9 can be applied in the synchronized case that is to say if  $\alpha \geq \sup_{x \in \mathbb{R}} -V''(x)$ .

## 1.2 Nonlinear case

The principal advantage of the linear case is the fact that the existence problem of a stationary measure remains *in fine* to a parametrization problem in dimension 1. In the general case, we can make a parametrization in  $\mathbb{R}^{2n-1}$  with  $\deg(F) = 2n$ . Then, we can not use anymore the intermediate values theorem.

The aim of this subsection is to extend the results in [HT10a] to the asymmetric case. We will establish the existence of outlying stationary measures around the wells under an assumption similar to Inequality (4.18) in [HT10a].

First, let us explain why there is always at least one stationary measure, even if  $\deg(F) \geq 4$ . For this, we will use the results about the free-energy in [Tug10b].

We remark that Theorem 1.6 does not need any symmetric property on  $V$ . Indeed, Proposition 1.2 in [Tug10b] corresponds to Proposition 2.1 in [CMV03] (which does not require the symmetry) ; Lemma 1.3 just needs the lower-bound  $V(x) \geq C_4 x^4 - C_2 x^2$  and Lemma 1.4 and 1.5 are simple consequences of Proposition 1.2 and Lemma 1.3. Furthermore, the proof of Theorem 1.6 only uses the growth property of the two potentials and the fact they are polynomials. We deduce the following theorem (the proof is left to the attention of the reader):

**Theorem 1.10.** *Set  $V$  which verifies the assumptions (V-1)–(V-5) and  $F$  which verifies the assumptions (F-1)–(F-3). For all  $\epsilon > 0$ , the self-stabilizing equation (I) admits at least one stationary measure.*

We recall that  $\deg(F) = 2n \leq 2q$ . We write:

$$F(x) = \sum_{k=1}^n \frac{F^{(2k)}(0)}{(2k)!} x^{2k}.$$

Let  $u$  the density of a probability measure with respect to the Lebesgue's measure. We call  $\mu_1, \dots, \mu_{2n-1}$  the first  $2n - 1$  moments of this measure that we design by the same letter  $u$  than the density. We assume that  $u$  admits a  $8q^2$ -th moment finite. Set  $a_0$  a wells of  $V$ . By proceeding exactly like in Subsection 4.3 in [HT10a], we get the following result:

**Proposition 1.11.** *For all  $k \in \mathbb{N}$ , we introduce the function*

$$\varphi_k^{(\epsilon)}(m_1, \dots, m_{2n-1}) := \frac{\int_{\mathbb{R}} x^k \exp\left[-\frac{2}{\epsilon} W_m(x)\right] dx}{\int_{\mathbb{R}} \exp\left[-\frac{2}{\epsilon} W_m(x)\right] dx} \quad \text{with} \quad (1.9)$$

$$W_m(x) := V(x) + F(x - a_0) + \sum_{p=0}^{2n-1} \frac{(-1)^p}{p!} (m_p - a_0^p) F^{(p)}(x). \quad (1.10)$$

*If and only if the vector  $(\mu_1, \dots, \mu_{2n-1})$  is a fixed point of the function  $\Phi^{(\epsilon)} := (\varphi_1^{(\epsilon)}, \dots, \varphi_{2n-1}^{(\epsilon)})$  from  $\mathbb{R}^{2n-1}$  to  $\mathbb{R}^{2n-1}$ , the following is true:  $u$  is a stationary measure for the self-stabilizing process (I).*

We will prove the existence - under simple conditions - of an outlying stationary measure around the wells  $a_0$  by proving there exists a stationary measure whose the first  $2n - 1$  moments are arbitrarily closed to  $a_0, \dots, a_0^{2n-1}$ . For this, we will show that there exists a stable parallelepiped closed to the point  $(a_0, \dots, a_0^{2n-1})$  for the application  $\Phi^{(\epsilon)}$ .

**Theorem 1.12.** *Let  $V$  a confining potential which verifies the assumptions (V-1)–(V-5) and an interaction potential  $F$  which verifies the assumptions (F-1)–(F-3). We admit the following about the wells  $a_0$ :*

$$V(x) + F(x - a_0) > V(a_0) \quad \text{for all } x \neq a_0. \quad (1.11)$$

Let a family  $(\eta_\epsilon)_\epsilon$  verifying  $\lim_{\epsilon \rightarrow 0} \eta_\epsilon = 0$  and  $\lim_{\epsilon \rightarrow 0} \epsilon/\eta_\epsilon = 0$ . Under the condition

$$\sum_{p=0}^{2n-2} \frac{|F^{(p+2)}(a_0)|}{p!} |a_0|^p < \alpha + V''(a_0), \quad (1.12)$$

for all  $\rho > 0$ , the self-stabilizing diffusion (I) admits an outlying stationary measure around  $a_0$  which verifies

$$\left| \int_{\mathbb{R}} x^k u_\epsilon^{a_0}(x) dx - a_0^k \right| \leq \rho \eta_\epsilon \quad \text{for all } 1 \leq k \leq 2n-1.$$

for  $\epsilon$  small enough. We will call  $u_\epsilon^{a_0}$  this measure in the following.

*Proof.* The proof is similar to the one of Theorem 4.6 in [HT10a]. Set  $\lambda > 0$  arbitrarily small. We define the parallelepiped

$$\mathcal{C}(\epsilon) := \prod_{p=1}^{2n-1} \left[ a_0^p - p a_0^{p-1} \lambda \eta_\epsilon ; a_0^p + p a_0^{p-1} \lambda \eta_\epsilon \right].$$

Let  $m \in \mathcal{C}(\epsilon)$  then there exists some coordinates  $(r_p)_{1 \leq p \leq 2n-1}$  which determine  $m$  through the equations  $m_p = a_0^p + r_p \eta_\epsilon$  for all  $1 \leq p \leq 2n-1$ . We can apply Lemma A.4 and Remark A.5 in [HT10a] with  $U(x) := V(x) + F(x - a_0)$ ,  $f(x) := x^k$  and to the parameters  $\mu_p := r_p$  and  $G_p(x) := \frac{(-1)^p}{p!} F^{(p)}(x)$  in (1.9) and (1.10). We obtain:

$$\varphi_k^{(\epsilon)}(m) - a_0^k = -\eta_\epsilon \frac{k a_0^{k-1}}{\alpha + V''(a_0)} \sum_{p=1}^{2n-1} \frac{(-1)^p r_p}{p!} F^{(p+1)}(a_0) + o(\eta_\epsilon).$$

Condition (1.12) implies

$$\begin{aligned} \left| \varphi_k^{(\epsilon)}(m) - a_0^k \right| &\leq \eta_\epsilon \lambda \frac{k |a_0|^{k-1}}{\alpha + V''(a_0)} \sum_{p=1}^{2n-1} \frac{|F^{(p+1)}(a_0)|}{p!} p |a_0|^{p-1} + o(\eta_\epsilon) \\ &\leq \eta_\epsilon k |a_0|^{k-1} \lambda \xi + o(\eta_\epsilon) \quad \xi \in ]0; 1[. \end{aligned}$$

Since this inequality is uniform with respect to the coordinates  $(r_p)_{1 \leq p \leq 2n-1}$ , we have  $|\varphi_k^{(\epsilon)}(m) - a_0^k| \leq k |a_0|^{k-1} \lambda \eta_\epsilon$  for  $\epsilon$  small enough, that means  $\Phi^{(\epsilon)}(m) \in \mathcal{C}(\epsilon)$ .  $\mathcal{C}(\epsilon)$  is closed, convex and bounded in the space  $\mathbb{R}^{2n-1}$  - which is a Banach. Hence, the continuity of the function  $\Phi^{(\epsilon)}$  implies that the closure of  $\Phi^{(\epsilon)}(\mathcal{C}(\epsilon))$  is a compact.

The application of Schauder's theorem (Proposition 4.1 in [HT10a]) provides the existence of a fixed point for  $\Phi^{(\epsilon)}$  in the parallelepiped  $\mathcal{C}(\epsilon)$ . We deduce the existence of a stationary measure associated to the density

$$u_{\epsilon, m}(x) := \frac{\exp \left[ -\frac{2}{\epsilon} W_m(x) \right]}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} W_m(z) \right] dz}.$$

The proof is finite by taking  $\lambda \leq \min \left\{ \frac{\rho}{p |a_0|^{p-1}} ; 1 \leq p \leq 2n-1 \right\}$ .  $\square$

By using the same method than the one in the proof of Proposition 4.1 in [HT10b], we can prove easily the convergence of  $u_\epsilon^{a_0}$  towards  $\delta_{a_0}$  in the small noise limit (see [Tug10a]). In the same paper, it is proved that this convergence towards  $\delta_{a_0}$  would imply that  $x \mapsto V(x) + F(x - a_0)$  reaches its global minimum in  $a_0$ . Consequently, if there exists  $x \in \mathbb{R}$  such that  $V(x) + F(x - a_0) < V(a_0)$ , it is impossible for a sequence of stationary measures to converge towards  $\delta_{a_0}$  in the small noise limit.

**Corollary 1.13.** *If  $V$  reaches its global minimum in a point  $a_0$  which satisfies (1.12), Diffusion (I) admits an outlying stationary measure around  $a_0$  for  $\epsilon$  sufficiently small.*

*Proof.* If  $a_0$  is a critical point where  $V$  reaches its global minimum, then  $V(x) \geq V(a_0)$  for all  $x \neq a_0$ . Since  $F(x - a_0) > 0$  for all  $x \neq a_0$ , the point  $a_0$  verifies the condition (1.11) and we can apply Theorem 1.12 if (1.12) is true.  $\square$

As previously in the linear case, we find Theorem 4.6 in [HT10a] as a consequence. We can precise this result by using the same method than the one of Lemma 4.7 and Theorem 4.8 in [HT10a]:

**Theorem 1.14.** *Set  $V$  which verifies the assumptions (V-1)–(V-5) and  $F$  which verifies the assumptions (F-1)–(F-3). Let a wells  $a_0$  which satisfies (1.11) and (1.12). For all  $1 \leq k \leq 2n - 1$ , we define*

$$\tau_k^0 := ka_0^{k-1} \frac{a_0 V^{(3)}(a_0) - (k-1)V''(a_0)}{4a_0 V''(a_0) (\alpha + V''(a_0))}.$$

*Then, for all  $\rho > 0$ , there exists  $\epsilon_0$  such that Diffusion (I) admits a stationary measure  $u_\epsilon^{a_0}$  which verifies*

$$\left| \int_{\mathbb{R}} x^k u_\epsilon^{a_0}(x) dx - \left( a_0^k - \tau_k^0 \epsilon \right) \right| \leq \rho \epsilon, \quad 1 \leq k \leq 2n - 1, \quad \epsilon \leq \epsilon_0.$$

## 2 Uniqueness and Thirdness

In the previous section, we studied the number of stationary measures for self-stabilizing processes in the small noise. Particularly, we prove that *for  $\epsilon$  small enough*, there is non-uniqueness of the stationary measures under easy to verify assumptions. But, we did not study the behavior in the "non-small noise". Here, we will exhibit a simple case under which there is a phase transition: over some critical value of noise, there is a unique stationary measure and under the same value, there are exactly three ones. Then, in the more difficult nonlinear case, we will generalize the results about the thirdness problem already studied in [HT10a], [HT10b], [HT09] and [Tug10b]. We will end this section by proving some phase transition for the asymmetric landscape ; by making a study in the small noise and another in the big noise.

## 2.1 Under (HC) and (LIN)

Here, we will deal with potentials  $V$  and  $F$  which have this form:

$$F(x) := \frac{\alpha}{2}x^2 \text{ with } \alpha > 0 \text{ and}$$

$$V(x) := -\frac{\vartheta_1}{2}x^2 + \sum_{p=2}^q \frac{\vartheta_p}{(2p)!}x^{2p} \text{ with } \vartheta_p \geq 0 \text{ for all } 1 \leq p \leq q \text{ and } \vartheta_1\vartheta_q > 0.$$

With these two potentials, we have the following phase transition result:

**Theorem 2.1.** *There exists  $\epsilon_c \in \mathbb{R}$  such that:*

- For all  $\epsilon \geq \epsilon_c$ , Diffusion (I) admits a unique stationary measure:  $u_\epsilon^0$ .
- For all  $\epsilon < \epsilon_c$ , Diffusion (I) admits exactly three stationary measures:  $u_\epsilon^0$ ,  $u_\epsilon^+$  and  $u_\epsilon^-$  with  $\pm \int_{\mathbb{R}} xu_\epsilon^\pm(x)dx > 0$ .

Moreover, the critical value  $\epsilon_c$  is the unique solution of the equation:

$$\frac{\int_{\mathbb{R}_+} x^2 \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx}{\int_{\mathbb{R}_+} \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx} = \frac{1}{2\alpha}. \quad (2.1)$$

*Proof. Step 1.* We note that the expression of the function  $F$  implies:

$$\psi_\epsilon(m) = \int_{\mathbb{R}} (x - m) \exp \left[ \frac{2\alpha mx}{\epsilon} \right] \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2}x^2 \right) \right] dx.$$

We proceed a series expansion of the function  $m \mapsto \exp \left[ \frac{2\alpha mx}{\epsilon} \right]$  and we get - after using the fact that  $V$  is even:

$$\begin{aligned} \psi_\epsilon(m) &= 2 \sum_{n=0}^{\infty} \frac{I_\epsilon(2n)}{(2n)!} \left( \frac{2\alpha m}{\epsilon} \right)^{2n+1} \left[ \frac{I_\epsilon(2n+2)}{(2n+1)I_\epsilon(2n)} - \frac{\epsilon}{2\alpha} \right] \\ \text{with } I_\epsilon(z) &:= \int_{\mathbb{R}_+} x^z \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2}x^2 \right) \right] dx. \end{aligned} \quad (2.2)$$

**Step 2.** For all  $n \in \mathbb{N}$ , we introduce

$$\gamma_n(\epsilon) := \frac{I_\epsilon(2n+2)}{(2n+1)I_\epsilon(2n)} - \frac{\epsilon}{2\alpha}. \quad (2.3)$$

An integration by parts provides

$$\begin{aligned} (2n+1)I_\epsilon(2n) &= \frac{2}{\epsilon} \int_{\mathbb{R}} (V'(x) + \alpha x) x^{2n+1} \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2}x^2 \right) \right] dx \\ &= \frac{2}{\epsilon} \left\{ (\alpha - \vartheta_1) I_\epsilon(2n+2) + \sum_{p=2}^q \frac{\vartheta_p}{(2p-1)!} I_\epsilon(2n+2p) \right\} \end{aligned}$$



after using the particular expression of  $V$ . Consequently, the expression  $\gamma_n(\epsilon)$  becomes

$$\gamma_\epsilon(n) = \frac{\epsilon}{2} \left\{ \alpha - \vartheta_1 + \sum_{p=2}^q \frac{\vartheta_p}{(2p-1)!} \frac{I_\epsilon(2n+2p)}{I_\epsilon(2n+2)} \right\}^{-1} - \frac{\epsilon}{2\alpha}.$$

**Step 3.** We will prove that the sequence  $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$  is nonincreasing for all  $\epsilon$ . It is sufficient to prove that the sequences  $\frac{I_\epsilon(2n+2p)}{I_\epsilon(2n+2)}$  are nondecreasing. In a more general way, we will prove that for all  $z > 0$ , the function  $\Xi_z$  defined by  $\Xi_z(x) = \frac{I_\epsilon(x+z)}{I_\epsilon(x)}$  is nondecreasing. The derivation provides:

$$\Xi'_z(x) = \Xi_z(x) \left( \frac{I'_\epsilon(x+z)}{I_\epsilon(x+z)} - \frac{I'_\epsilon(x)}{I_\epsilon(x)} \right).$$

Since  $\Xi_z(x) \geq 0$  for all  $x \in \mathbb{R}_+$  and  $z \in \mathbb{R}_+$ , it remains now to prove that the application  $\zeta(y) := \frac{I'_\epsilon(y)}{I_\epsilon(y)}$  is nondecreasing. We apply the derivation:

$$\begin{aligned} \zeta'(y) &= \frac{I''_\epsilon(y)}{I_\epsilon(y)} - \left( \frac{I'_\epsilon(y)}{I_\epsilon(y)} \right)^2 \\ &= \frac{\int_{\mathbb{R}_+} x^y (\log(x))^2 \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx}{\int_{\mathbb{R}_+} x^y \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx} \\ &\quad - \left( \frac{\int_{\mathbb{R}_+} x^y (\log(x)) \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx}{\int_{\mathbb{R}_+} x^y \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 \right) \right] dx} \right)^2. \end{aligned}$$

By applying the Cauchy-Schwarz inequality, we obtain  $\zeta'(y) \geq 0$  for all  $y > 0$ . Then, we get the previous claim that is to say the sequence  $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$  is nonincreasing for all  $\epsilon > 0$ .

**Step 4:** In order to prove that  $\gamma_n(\epsilon) < 0$  for  $n$  big enough, we compute  $\psi_\epsilon(a)$ :

$$\begin{aligned} \psi_\epsilon(a) &= e^{\frac{\alpha a^2}{\epsilon}} \int_{\mathbb{R}} (x-a) \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} (x-a)^2 \right) \right] dx \\ &= e^{\frac{\alpha a^2}{\epsilon}} \int_{\mathbb{R}} y \exp \left[ -\frac{2}{\epsilon} \left( V(y+a) + \frac{\alpha}{2} y^2 \right) \right] dy \\ &= e^{\frac{\alpha a^2}{\epsilon}} \int_{\mathbb{R}_+} y e^{-\frac{\alpha}{\epsilon} y^2} \left\{ \exp \left[ -\frac{2}{\epsilon} V(y+a) \right] - \exp \left[ -\frac{2}{\epsilon} V(y-a) \right] \right\} dy. \end{aligned}$$

By using the particular expression of  $V$ , we get:

$$\begin{aligned} V(y+a) - V(y-a) &= 2 \sum_{p=1}^q \frac{V^{(2p)}(0)}{(2p)!} \sum_{k=0}^{p-1} C_{2p}^{2k+1} y^{2k+1} a^{2p-2k-1} \\ &= 2 \sum_{p=2}^q \frac{V^{(2p)}(0)}{(2p)!} \sum_{k=1}^{p-1} C_{2p}^{2k+1} y^{2k+1} a^{2p-2k-1} + 2 \sum_{p=1}^q \frac{V^{(2p)}(0)}{(2p)!} (2p-1) y a^{2p-1}. \end{aligned}$$

The last term is equal to  $V'(a) = 0$ . Since  $V^{(2p)}(0) \geq 0$  for all  $2 \leq p \leq q$ , we deduce that  $V(y+a) \geq V(y-a)$  for all  $y \geq 0$ . Furthermore, since  $\deg(V) > 2$ , we get the inequality  $V(y+a) > V(y-a)$  excepting a finite number of points. Consequently,  $\psi_\epsilon(a) < 0$ . We deduce there exists  $n \in \mathbb{N}$  such that  $\gamma_n(\epsilon) < 0$ .

**Step 5.** We put  $n_\epsilon := \min \{n \mid \gamma_n(\epsilon) \leq 0\}$ . By Step 3, we deduce that for all  $n \geq n_\epsilon$ , we have  $\gamma_n(\epsilon) \leq 0$  and for all  $n < n_\epsilon$  (if  $n_\epsilon = 0$ ,  $\{n \in \mathbb{N} \mid n < n_\epsilon\} = \emptyset$ ), we have  $\gamma_n(\epsilon) > 0$ . Consequently, we can write:

$$\psi_\epsilon(m) = \sum_{n=0}^{n_\epsilon-1} C_n(\epsilon) m^{2n+1} - \sum_{n=n_\epsilon}^{+\infty} C_n(\epsilon) m^{2n+1} \quad (2.4)$$

with  $C_n(\epsilon) := 2 \frac{I_\epsilon(2n)}{(2n)!} \left(\frac{2\alpha}{\epsilon}\right)^{2n+1} |\gamma_n(\epsilon)| \geq 0$ . We take the derivative then we divide by  $m^{2n_\epsilon}$ :

$$\frac{\psi'_\epsilon(m)}{m^{2n_\epsilon}} = \left\{ \sum_{n=0}^{n_\epsilon-1} (2n+1) C_n(\epsilon) m^{2n-2n_\epsilon} - \sum_{n=n_\epsilon}^{+\infty} (2n+1) C_n(\epsilon) m^{2n-2n_\epsilon} \right\} \quad (2.5)$$

Since the functions  $m \mapsto m^{2n-2n_\epsilon}$  are decreasing for all  $n \leq n_\epsilon - 1$  and since the functions  $m \mapsto -m^{2n-2n_\epsilon}$  are nonincreasing for all  $n \geq n_\epsilon$ , we deduce that  $\psi'_\epsilon$  vanishes at most one time on  $\mathbb{R}_+$  and the same holds for  $\psi_\epsilon$  (by dividing by  $m^{2n_\epsilon+1}$  in (2.4)).

**Step 6.** By (2.4) and (2.5), we deduce that the behavior of the function  $m \mapsto \psi_\epsilon(m)$  depends directly of  $\psi'_\epsilon(0)$ :

- If  $\psi'_\epsilon(0) > 0$  that is to say if  $\gamma_0(\epsilon) > 0$ , then there exists  $x_\epsilon$  such that  $\psi'_\epsilon(x) \geq 0$  on  $[0; x_\epsilon]$  and  $\psi'_\epsilon(x) \leq 0$  on  $[x_\epsilon; +\infty[$ . Since  $\psi_\epsilon(0) = 0$  by symmetry of the potential  $V$ , we deduce that the function  $\psi_\epsilon$  is increasing on  $[0; x_\epsilon]$  then decreasing on  $[x_\epsilon; +\infty[$ . We deduce that there exists  $m_\epsilon > 0$  such that  $\psi_\epsilon(m_\epsilon) = 0$  and it is unique on  $\mathbb{R}_+$ . This implies the existence of the two asymmetric stationary measures.
- If  $\psi'_\epsilon(0) \leq 0$  that is to say if  $\gamma_0(\epsilon) \leq 0$ , then  $\gamma_n(\epsilon) \leq 0$  for all  $n \in \mathbb{N}$ . We deduce that the function  $\psi_\epsilon$  is decreasing on  $\mathbb{R}$  that implies the uniqueness of the zero for the function  $\psi_\epsilon$  and consequently the uniqueness of the stationary measure for the self-stabilizing diffusion (I).

**Step 7.** Now, we will investigate on the sign of  $\gamma_0(\epsilon)$ :

$$\begin{aligned} \gamma_0(\epsilon) &:= \frac{I_\epsilon(2)}{I_\epsilon(0)} - \frac{\epsilon}{2\alpha} \\ &= \epsilon \left\{ \frac{\int_{\mathbb{R}_+} y^2 \exp \left[ -\frac{2}{\epsilon} \left( V(\sqrt{\epsilon}y) + \frac{\alpha\epsilon}{2} y^2 \right) \right] dy}{\int_{\mathbb{R}_+} \exp \left[ -\frac{2}{\epsilon} \left( V(\sqrt{\epsilon}y) + \frac{\alpha\epsilon}{2} y^2 \right) \right] dy} - \frac{1}{2\alpha} \right\} \end{aligned}$$

by making the change of variable  $x := \sqrt{\epsilon}y$ . The computation yields

$$\begin{aligned} \gamma_0(\epsilon) &= \epsilon \left\{ \frac{\int_{\mathbb{R}_+} x^2 \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx}{\int_{\mathbb{R}_+} \exp \left[ (\vartheta_1 - \alpha) x^2 - \sum_{p=2}^q \frac{2\epsilon^{p-1}\vartheta_p}{(2p)!} x^{2p} \right] dx} - \frac{1}{2\alpha} \right\} \\ &=: \epsilon \mathcal{T}_{V,F}(\epsilon). \end{aligned}$$

Since  $\vartheta_p \geq 0$  for all  $1 \leq p \leq q$ , we deduce - after using the Jensen's inequality - that the application  $\mathcal{T}_{V,F}$  is decreasing. Consequently, there exists a unique  $\epsilon_c$  which depends only on the parameters of  $V$  and  $F$  such that  $\gamma_0(\epsilon) = 0$ . Moreover, for all  $\epsilon \geq \epsilon_c$ , we have  $\gamma_0(\epsilon) \leq 0$  which implies the uniqueness of the stationary measure. And, for all  $\epsilon < \epsilon_c$ , we have  $\gamma_0(\epsilon) > 0$  which implies the thirdness of the stationary measures.  $\square$

An immediate consequence is the convergence towards  $u_\epsilon^0$  if the initial free-energy is finite for all  $\epsilon \geq \epsilon_c(\alpha)$ . Indeed, according to Theorem 1.6 in [Tug10b], there is convergence towards one stationary measure.

## 2.2 Under (E) and $\deg(F) \geq 4$

We take an even potential  $V$  but we do not assume (HC). We will extend Theorem 3.2 in [HT10a] which deals with  $\deg(F) = 2$ . We assume that  $V''$  and  $F''$  are convex. One of the difficulty will be the behavior of the symmetric stationary measure(s) in the threshold between the synchronized case and the non-synchronized one that is to say when  $F''(0) + V''(0) = 0$ . For doing this, we will use a particles method involving a propagation of chaos.

**Proposition 2.2.** *Let  $V$  and  $F$  two potentials which satisfy (V-1)–(V-5) and (F-1)–(F-3). We assume that  $V''$  and  $F''$  are convex. Then, for  $\epsilon$  small enough, Diffusion (I) admits a unique symmetric stationary measure.*

*Proof. Step 1.* Since the potential  $V$  is symmetric and verifies  $V^{(4)} \geq 0$ , the inequality  $V''(0) + F''(0) \neq 0$  implies the uniqueness of the stationary measure for  $\epsilon$  small enough according to Theorem 7.3 and Theorem 7.4 in [HT09].

**Step 2.** We consider now the case  $V''(0) + F''(0) = 0$ . Let  $V_0(x) := V(x) + \frac{F''(0)}{2}x^2$  and  $F_0(x) := F(x) - \frac{F''(0)}{2}x^2$ . If the initial law  $u_0$  is symmetric, then using  $V_0$  and  $F_0$  instead of  $V$  and  $F$  in the equation (I) does not change anything.

Let  $u_\epsilon^{(1)}$  and  $u_\epsilon^{(2)}$  two symmetric stationary measures of Diffusion (I). We con-

sider the two following mean-field systems:

$$X_t^{i,N} = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(X_s^{i,N}) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N F_0'(X_s^{i,N} - X_s^{j,N}) ds$$

and  $Y_t^{i,N} = Y_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(Y_s^{i,N}) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N F_0'(Y_s^{i,N} - Y_s^{j,N}) ds$

where the  $2N$  initial random values are independents. Furthermore, we assume  $\mathcal{L}(X_0^i) = u_\epsilon^{(1)}$  and  $\mathcal{L}(Y_0^i) = u_\epsilon^{(2)}$ . Also, the  $N$  brownian motions are independents. We put  $\mu_t^{N,1} := (X_t^{1,N}, \dots, X_t^{N,N})^T$  and  $\mu_t^{N,2} := (Y_t^{1,N}, \dots, Y_t^{N,N})^T$ . We define also the self-stabilizing processes (which are classical since the two initial laws are stationary):

$$\overline{X}_t^i = X_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(\overline{X}_s^i) ds - \int_0^t F_0' * u_\epsilon^{(1)}(\overline{X}_s^i) ds$$

and  $\overline{Y}_t^{i,\epsilon} = Y_0^i + \sqrt{\epsilon} B_t^i - \int_0^t V_0'(\overline{Y}_s^i) ds - \int_0^t F_0' * u_\epsilon^{(2)}(\overline{Y}_s^i) ds$ .

The triangular inequality provides

$$\begin{aligned} \mathbb{E} \left\{ \left| \overline{Y}_t^i - \overline{X}_t^i \right|^2 \right\} &\leq 3\mathbb{E} \left\{ \left| \overline{Y}_t^i - Y_t^{i,N} \right|^2 \right\} + 3\mathbb{E} \left\{ \left| X_t^{i,N} - \overline{X}_t^i \right|^2 \right\} \\ &+ 3\mathbb{E} \left\{ \left| Y_t^{i,N} - X_t^{i,N} \right|^2 \right\}. \end{aligned} \quad (2.6)$$

**Step 3.** By definition, for all  $1 \leq i \leq N$ , we have

$$\begin{aligned} X_t^{i,N} - \overline{X}_t^i &= - \int_0^t \left\{ V_0'(X_s^{i,N}) - V_0'(\overline{X}_s^i) \right\} ds \\ &- \int_0^t \left\{ \frac{1}{N} \sum_{j=1}^N F_0'(X_s^{i,N} - X_s^{j,N}) - F_0' * u_\epsilon^{(1)}(\overline{X}_s^i) \right\} ds. \end{aligned}$$

**Step 3.1.** We apply the Itô formula to  $X_t^{i,N} - \overline{X}_t^i$  with the function  $x \mapsto x^2$  and by putting  $\xi_i(t) := \left| X_t^{i,N} - \overline{X}_t^i \right|^2$ , we obtain:

$$d\xi_i(t) = -2\Delta_1(i, t) dt - \frac{2}{N} \Delta_2(i, t) dt$$

with  $\Delta_1(i, t) := \left( X_t^{i,N} - \overline{X}_t^i \right) \left( V_0'(X_t^{i,N}) - V_0'(\overline{X}_t^i) \right)$

and  $\Delta_2(i, t) := \left( X_t^{i,N} - \overline{X}_t^i \right) \sum_{j=1}^N \left[ F_0'(X_t^{i,N} - X_t^{j,N}) - F_0' * u_\epsilon^{(1)}(\overline{X}_t^i) \right]$ .

By taking the sum on all the index  $1 \leq i \leq N$ , it yields

$$d \sum_{i=1}^N \xi_i(t) = -2\Delta_1(t)dt - \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N \left( \Delta_2(i, j, t) + \Delta_3(i, j, t) \right) dt$$

$$\text{with } \Delta_1(t) := \sum_{i=1}^N \Delta_1(i, t),$$

$$\Delta_2(i, j, t) := \left( F'_0(X_t^{i,N} - X_t^{j,N}) - F'_0(\overline{X}_t^i - \overline{X}_t^j) \right) \left( X_t^{i,N} - \overline{X}_t^i \right)$$

$$\text{et } \Delta_3(i, j, t) := \left( F'_0(\overline{X}_t^i - \overline{X}_t^j) - F'_0 * u_\epsilon^{(1)} \left( \overline{X}_t^i \right) \right) \left( X_t^{i,N} - \overline{X}_t^i \right).$$

**Step 3.2.** Since  $F_0$  is even and its derivative  $F'_0$  is convex on  $\mathbb{R}_+$  (because  $F''$  is even and convex), we have the inequality  $(x - y) F'_0(x - y) \geq \lambda (x - y)^{2k_0} \geq 0$  where  $\lambda > 0$  and  $k_0 := \min \{ k \geq 2 \mid F^{(2k)}(0) > 0 \}$ . However,  $\Delta_2(i, j, t) + \Delta_2(j, i, t) \geq 0$ . Indeed:

$$\begin{aligned} & \Delta_2(i, j, t) + \Delta_2(j, i, t) \\ &= \left( F'_0(X_t^{i,N} - X_t^{j,N}) - F'_0(\overline{X}_t^i - \overline{X}_t^j) \right) \times \left\{ \left( X_t^{i,N} - \overline{X}_t^i \right) - \left( X_t^{j,N} - \overline{X}_t^j \right) \right\} \\ &= \left( F'_0(X_t^{i,N} - X_t^{j,N}) - F'_0(\overline{X}_t^i - \overline{X}_t^j) \right) \times \left\{ \left( X_t^{i,N} - X_t^{j,N} \right) - \left( \overline{X}_t^i - \overline{X}_t^j \right) \right\} \\ &\geq \lambda \left| \left( X_t^{i,N} - X_t^{j,N} \right) - \left( \overline{X}_t^i - \overline{X}_t^j \right) \right|^{2k_0} \geq 0. \end{aligned}$$

Consequently

$$\mathbb{E} \left\{ \sum_{i=1}^N \sum_{j=1}^N \Delta_2(i, j, t) \right\} = \mathbb{E} \left\{ \sum_{1 \leq i < j \leq N} \left( \Delta_2(i, j, t) + \Delta_2(j, i, t) \right) \right\} \geq 0. \quad (2.7)$$

**Step 3.3.** Since  $V''$  (then  $V_0''$ ) is convex, we deduce that  $V_0^{(2l_0)}(0) > 0$  where the integer  $l_0$  is defined by  $l_0 := \min \{ l \geq 2 \mid V_0^{(2l)}(0) \neq 0 \} < \infty$ . Then there exists  $\lambda_0 > 0$  such that

$$(x - y) (V'_0(x) - V'_0(y)) \geq \lambda_0 (x - y)^{2l_0}.$$

This implies

$$-2 \sum_{i=1}^N \Delta_1(i, t) \leq -2\lambda_0 \sum_{i=1}^N \xi_i(t)^{l_0}. \quad (2.8)$$

**Step 3.4.** Now, we will deal with the double sum containing  $\Delta_3(i, j, t)$ . We apply the Cauchy-Schwarz inequality:

$$-\mathbb{E} \left[ \sum_{j=1}^N \Delta_3(i, j, t) \right] \leq \left\{ \mathbb{E} \left[ \left| X_t^{i,N} - \overline{X}_t^i \right|^2 \right] \right\}^{\frac{1}{2}} \left\{ \sum_{j=1}^N \sum_{k=1}^N \mathbb{E} [\rho_j \rho_k] \right\}^{\frac{1}{2}}$$

$$\text{with } \rho_j := F'_0(\overline{X}_t^i - \overline{X}_t^j) - F'_0 * u_\epsilon^{(1)} \left( \overline{X}_t^{\epsilon, i} \right).$$

By conditioning with respect to  $\overline{X}_t^i$  then to  $\overline{X}_t^j$ , we obtain that  $\mathbb{E}[\rho_j \rho_k] = 0$  for  $j \neq k$ . Consequently, it yields

$$-\mathbb{E} \left[ \sum_{j=1}^N \Delta_3(i, j, t) \right] \leq \sqrt{N \mathbb{E}[\xi_i(t)]} \left\{ \mathbb{E} \left[ \left| F_0'(X_t - Y_t) - F_0' * u_\epsilon^{(1)}(X_t) \right|^2 \right] \right\}^{\frac{1}{2}}$$

where  $X_t$  and  $Y_t$  are two independent random values with law  $u_\epsilon^{(1)}$ .  $F_0'$  is a polynomial function with degree  $2n-1$  according to the hypothesis (F-1). Since the  $8q^2$ -th moment of  $u_\epsilon^{(1)}$  is finite, we deduce that there exists  $C > 0$  such that

$$-\mathbb{E} \left[ \sum_{j=1}^N \Delta_3(i, j, t) \right] \leq C \sqrt{N \mathbb{E}[\xi_i(t)]}. \quad (2.9)$$

**Step 3.5.** By combining (2.7), (2.8) and (2.9), we obtain

$$\frac{d}{dt} \sum_{i=1}^N \mathbb{E}[\xi_i(t)] \leq 2 \sum_{i=1}^N \left\{ -\lambda_0 \mathbb{E}[\xi_i(t)^{l_0}] + \frac{C}{\sqrt{N}} \sqrt{\mathbb{E}[\xi_i(t)]} \right\}.$$

But, we can permute the particles so  $X_t^{i,N} - \overline{X}_t^i$  and  $X_t^{j,N} - \overline{X}_t^j$  have the same law. Then, for all  $1 \leq i \leq N$ , we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E}\{\xi_i(t)\} &\leq -2\lambda_0 \mathbb{E}\{\xi_i(t)^{l_0}\} + \frac{2C}{\sqrt{N}} \sqrt{\mathbb{E}[\xi_i(t)]} \\ &\leq -2\sqrt{\mathbb{E}[\xi_i(t)]} \left( \lambda_0 \mathbb{E}\{\xi_i(t)\}^{l_0 - \frac{1}{2}} - \frac{C}{\sqrt{N}} \right). \end{aligned}$$

Since  $\xi_i(0) = 0$ , we obtain  $\xi_i(t) \leq M N^{-\frac{1}{2l_0-1}}$  where  $M$  is a constant which depends only of  $u_\epsilon^{(1)}$ .

**Step 3.6** We deduce the two following inequalities:

$$\mathbb{E} \left\{ \left| X_t^{i,N} - \overline{X}_t^i \right|^2 \right\} \leq M_1 N^{-\frac{1}{2l_0-1}} \quad (2.10)$$

$$\text{and } \mathbb{E} \left\{ \left| \overline{Y}_t^i - Y_t^{i,N} \right|^2 \right\} \leq M_2 N^{-\frac{1}{2l_0-1}}. \quad (2.11)$$

**Step 4.** For all  $t \geq 0$ , by definition of the stationary measures  $u_\epsilon^{(1)}$  and  $u_\epsilon^{(2)}$ , we have the following inequality

$$\mathbb{E} \left\{ \left| \overline{Y}_t^i - \overline{X}_t^i \right|^2 \right\} \geq \mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2$$

where  $\mathbb{W}_2(u; v)$  denotes the Wasserstein distance between the probability measures  $u$  and  $v$ . Consequently, for all  $t \geq 0$ , (2.6), (2.10) and (2.11) imply

$$\mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2 \leq C_0 N^{-\frac{1}{2l_0-1}} + 3 \mathbb{E} \left\{ \left| Y_t^{i,N} - X_t^{i,N} \right|^2 \right\}.$$

This inequality is true for all  $1 \leq i \leq N$  so we have:

$$\begin{aligned} \mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2 &\leq C_0 N^{-\frac{1}{2t_0-1}} + \frac{3}{N} \sum_{i=1}^N \mathbb{E} \left\{ \left| Y_t^{i,N} - X_t^{i,N} \right|^2 \right\} \\ &\leq C_0 N^{-\frac{1}{2t_0-1}} + 3 \mathbb{E} \left\{ \left\| \mu_t^{N,1} - \mu_t^{N,2} \right\|_N^2 \right\} \end{aligned} \quad (2.12)$$

where  $\|X\|_N^2 := \frac{1}{N} \sum_{i=1}^N x_i^2$ .

**Step 4.1.** According to [BBCG08], if  $U \in \mathcal{C}^2 \left( \mathbb{R}^N, \mathbb{R} \right)$ , the probability measure  $d\mu(X) := \frac{\exp\left[-\frac{2N}{\epsilon}U(X)\right]}{\int_{\mathbb{R}} \exp\left[-\frac{2N}{\epsilon}U(Y)\right]dY} dX$  satisfies Poincaré inequality if there exist two constants  $\tau \in ]0; 1[$  and  $C > 0$  such that

$$\liminf_{\|X\| \rightarrow +\infty} \left( \frac{2N}{\epsilon} \Delta U - \tau \|\nabla U\|^2 \right) (X) < -C. \quad (2.13)$$

**Step 4.2.** We can prove that Inequality (2.13) is verified with the meta-potential  $\Upsilon^N(x_1, \dots, x_N) := \frac{1}{N} \sum_{j=1}^N V_0(x_j) + \frac{1}{2N^2} \sum_{j=1}^N \sum_{i=1}^N F_0(x_i - x_j)$  by using the fact that  $V$  and  $F$  are polynomial functions, see Lemma 5.8 in [Tug10a] for details.

**Step 4.3.** Consequently, we deduce that  $\mu_t^{N,1}$  and  $\mu_t^{N,2}$  converge weakly towards the unique stationary measure of the mean-field system  $d\mathcal{X}_t = \sqrt{\epsilon} d\mathcal{B}_t - N \nabla \Upsilon^N(\mathcal{X}_t) dt$  when  $t$  tends to  $+\infty$ . And, since the  $N$ -dimensional brownian motion is the same in the definition of  $\mu_t^{N,1}$  and in the one of  $\mu_t^{N,2}$ , it yields:

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left\{ \left\| \mu_t^{N,1} - \mu_t^{N,2} \right\|_N^2 \right\} = 0.$$

**Step 4.4.** Inequality (2.12) is uniform with respect to the time so by taking the limit when  $t$  tends to infinity, we obtain  $\mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2 \leq C_0 N^{-\frac{1}{2t_0-1}}$ . With  $N$  tending to  $+\infty$ , we deduce immediatly:

$$\mathbb{W}_2 \left( u_\epsilon^{(1)}; u_\epsilon^{(2)} \right)^2 = 0$$

which implies  $u_\epsilon^{(1)} = u_\epsilon^{(2)}$ . □

**Remark 2.3. 1)** *The uniqueness of the symmetric stationary measure is true for all  $\epsilon > 0$  when  $V''(0) + F''(0) = 0$ . The same result holds for all  $\epsilon > 0$  if  $V''(0) + F''(0) > 0$  by the same way.*

**2)** *The central tool of the proof was a result of uniform propagation of chaos. However, we did not use it to the first diffusion (I) but to a diffusion with the same dynamic if the initial law is symmetric. Consequently, we call it a "half-uniform propagation of chaos".*

**3)** *We could prove the convergence by using the half-uniform propagation of*

chaos when the initial law is symmetric under the inequality  $V''(0) + F''(0) \geq 0$ . The proof is faster than the one in [Tug10b] and we obtain the rate of convergence (polynomial if  $V''(0) + F''(0) = 0$ , exponential if  $V''(0) + F''(0) > 0$ , see [Tug10a]). But this method is not available in the general case.

**Theorem 2.4.** *Let  $V$  and  $F$  two potentials which satisfy (V-1)–(V-5) and (F-1)–(F-3). If  $V''(0) + F''(0) \geq 0$ ,  $V$  is even and  $\deg(V) \geq \deg(F)$ , then Diffusion (I) admits exactly three stationary measures for  $\epsilon$  small enough.*

*Proof.* If  $V''(0) + F''(0) > 0$  and  $\deg(V) > \deg(F)$ , it is a consequence of Theorem 1.11 in [Tug10b].

**Step 1.** If  $\deg(V) > \deg(F)$ , each family of stationary measures  $(u_\epsilon)_{\epsilon > 0}$  satisfies the condition (H) introduced in [HT10b] which means that the family  $(\int_{\mathbb{R}} x^{2n} u_\epsilon(x) dx)_{\epsilon > 0}$  is bounded where  $2n := \deg(F)$ . We will prove that it is true even if  $\deg(V) = 2n$ . We will proceed a *reductio ad absurdum* by assuming the existence of a decreasing sequence  $(\epsilon_k)_{k \in \mathbb{N}}$  which converges towards 0 such that the sequence  $\mu_{2n}(k) := \int_{\mathbb{R}} x^{2n} u_{\epsilon_k}(x) dx$  tends towards  $+\infty$ .

**Step 1.1.** Since  $F$  and  $V$  are two polynomials functions, we can write the  $l$ -th moment of  $u_\epsilon$  in the following form:

$$\mu_l(k) = \frac{\int_{\mathbb{R}} x^l \exp \left[ -\frac{2}{\epsilon_k} \left( \sum_{r=1}^{2n} \mathcal{M}_r(k) x^r \right) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon_k} \left( \sum_{r=1}^{2n} \mathcal{M}_r(k) x^r \right) \right] dx},$$

$$\text{with } \mathcal{M}_r(k) := \frac{1}{r!} \left\{ V^{(r)}(0) + \sum_{j=0}^{2n-r} \frac{(-1)^j}{j!} F^{(j+r)}(0) \mu_j(k) \right\}$$

for all  $1 \leq l \leq 2n$ . The highest moment which intervenes in  $\mathcal{M}_r(k)$  is the one of degree  $2n - r$ .

**Step 1.2.** Let us recall that  $\mu_{2n}(k)$  tends towards  $+\infty$ . If all the  $\mathcal{M}_r(k)$  were bounded then - since the coefficient  $\mathcal{M}_{2n}(k)$  is positive -  $\mu_{2l}(k)$  would be bounded for all  $l \in \mathbb{N}$ . We deduce that there exist some index  $r$  such that  $\mathcal{M}_r(k)$  tends towards  $+\infty$  or  $-\infty$ . We introduce the sequence

$$\eta_r(k) := \mathcal{M}_r(k) (\mu_{2n}(k))^{-\left(1 - \frac{r}{2n}\right)}$$

for all  $1 \leq r \leq 2n$ . The change of variable  $x := (\mu_{2n}(k))^{\frac{1}{2n}} y$  provides

$$\frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}} = \frac{\int_{\mathbb{R}} y^l \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy}{\int_{\mathbb{R}} \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy}.$$

**Step 1.3.** The Jensen inequality implies the existence of  $C > 0$  such that  $|\mathcal{M}_r(k)| \leq C \mu_{2n}(k)^{\frac{2n-r}{2n}}$  for all  $1 \leq r \leq 2n - 1$ . We obtain the following higher-bound:  $|\eta_r(k)| \leq C$ . By considering a subsequence of  $(\epsilon_k)_k$  (we continue to write it  $\epsilon_k$  for simplicity), we get the convergence of  $\eta_r(k)$  towards  $\eta_r$ . Besides,



the quantity  $\frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}}$  tends to 0 for all  $1 \leq l \leq 2n$ . Consequently, we have the following limit:

$$\frac{(-1)^{2n-l}}{(2n-l)!} F^{(2n)}(0) \lim_{k \rightarrow +\infty} \frac{\mu_l(k)}{\mu_{2n}(k)^{\frac{l}{2n}}} = \eta_l \quad (2.14)$$

for all  $1 \leq l \leq 2n$ .

**Step 1.4.** According to Lemma A.4 in [Tug10b], we can extract a subsequence (we continue to write it  $\epsilon_k$  for simplicity) such that

$$\lim_{k \rightarrow +\infty} \frac{\int_{\mathbb{R}} y^l \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy}{\int_{\mathbb{R}} \exp \left[ -\frac{2\mu_{2n}(k)}{\epsilon_k} \left( \sum_{r=1}^{2n} \eta_r(k) y^r \right) \right] dy} = \sum_{s=1}^q p_s A_s^l$$

where  $A_1 < \dots < A_q$  are the locations of the global minimum of the polynomial function  $U_0(x) := \sum_{j=1}^{2n} \eta_j x^j$  and  $p_1 + \dots + p_q = 1$  with  $p_i \geq 0$  for all  $1 \leq i \leq q$ . By using (2.14), we obtain:

$$\eta_l = \frac{(-1)^{2n-l}}{(2n-l)!} F^{(2n)}(0) \sum_{s=1}^q p_s A_s^l.$$

**Step 1.5.** Then, we have  $U_0(x) = \frac{F^{(2n)}(0)}{(2n)!} \sum_{s=1}^q p_s (x - A_s)^{2n} + \frac{V^{(2n)}(0)}{(2n)} x^{2n}$ . By definition,  $U_0'(A_q) = 0$ . If  $q \geq 2$ , since  $A_q - A_i > 0$  for all  $1 \leq i \leq q-1$ , it yields  $A_q < 0$ . By the same way, we have  $A_1 > 0$ . This is impossible. If  $q = 1$ , we have immediately  $A_1 = 0$  then  $\eta_{2n} = 0$  which implies  $F^{(2n)}(0) = 0$ . This is also impossible.

**Step 1.6.** We deduce that each family of stationary measures satisfies the condition (H). According to Lemma 3.2 and Proposition 3.3 in [HT10b], this implies that each family of stationary measures admits an adherence value in the small-noise limit. Furthermore, since  $F''(0) + V''(0) \geq 0$ , Proposition 3.7 and Remark 3.8 in the same article imply that there are exactly three possible adherence values:  $\delta_0$ ,  $\delta_a$  and  $\delta_{-a}$ .

**Step 2.** According to Theorem 4.5 in [HT10a], Diffusion (I) admits at least one symmetric stationary measure. Theorem 2.2 implies the uniqueness of the symmetric stationary measure for  $\epsilon$  small enough. Corollary 1.9 in [Tug10b] provides the existence of at least two asymmetric stationary measures. It remains now to prove there are exactly two asymmetric stationary measures for  $\epsilon$  small enough. We proceed exactly like in Theorem 1.11 in [Tug10b] by using the rate of convergence method from [HT09] and we obtain the thirdness of the stationary measures for  $\epsilon$  small enough.  $\square$

### 2.3 If $V$ is asymmetric

Now, we will extend Theorem 3.2 in [HT10a] to the asymmetric landscape. Also, by studying the number of stationary measures in small noise and in big noise,

we will show that there is a phase transition if  $F'$  is linear. We will deal with the two following potentials:

$$V(x) = \frac{x^4}{4} + \frac{\gamma}{3}x^3 - \frac{\rho}{2}x^2 \quad \text{with } \rho > 0 \quad \text{and } \gamma \geq 0$$

$$\text{and } F(x) = \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4 \quad \text{with } \alpha, \beta \geq 0 \quad \text{and } \alpha + \beta > 0.$$

We assume  $V^{(4)} = 6$  and  $V^{(3)}(0) \geq 0$  but with simple transformations, we can obtain all the cases. We have  $a_g = \frac{-\gamma - \sqrt{\gamma^2 + 4\rho}}{2}$  and  $a_d = \frac{-\gamma + \sqrt{\gamma^2 + 4\rho}}{2}$ .

The underlying idea of the existence results in [HT10a] and in Proposition 1.3, Proposition 1.7, Theorem 1.12 and Theorem 1.14 is to study the asymptotic analysis in the small-noise limit of the stationary measures, to find the possible limiting values and then to prove that the function  $\Phi^{(\epsilon)}$  defined on Proposition 1.11 admits a fixed point close to the moments vector of this adherence value. We will proceed the same method when the confining potential  $V$  is not even. Consequently, we give the following lemma which is directly linked to the possible limiting values (which are always finite combining of Dirac measures):

**Lemma 2.5.** *For all  $0 < \alpha \leq \alpha_c^{(1)} := 2 \sup_{x \neq a_d} \frac{V(a_d) - V(x)}{(x - a_d)^2} = \frac{\gamma^2 + 3\gamma\sqrt{\gamma^2 + 4\rho}}{18}$  and for all  $\alpha \geq \alpha_c^{(2)} := 2 \sup_{x \neq 0} \frac{-V(x)}{x^2} = \rho + \frac{2}{9}\gamma^2$ , the following system does not have any solution:*

$$W'(A_1) = W'(A_2) = 0 \quad \text{and} \quad \inf_{x \in \mathbb{R}} W(x) = W(A_1) = W(A_2) \quad (2.15)$$

with  $W := V + F * (p\delta_{A_1} + (1-p)\delta_{A_2})$ ,  $A_1 < A_2$  and  $p \in ]0, 1[$ .

And, for all  $\alpha \in ]\alpha_c^{(1)}; \alpha_c^{(2)}[$ , (2.15) admits exactly one solution:

$$A_1 = -\frac{\gamma}{3} - \sqrt{\frac{\gamma^2 + 3\rho}{3} - \alpha} < 0, \quad A_2 = -\frac{\gamma}{3} + \sqrt{\frac{\gamma^2 + 3\rho}{3} - \alpha} > 0$$

$$\text{and } p = 1 - \frac{\gamma(2\gamma^2 + 9\rho)}{9\alpha\sqrt{3\gamma^2 + 9\rho} - 9\alpha}.$$

*Proof.* Let us solve the equation (2.15). We have:

$$W(x) - W(A_1) = \frac{W^{(4)}(A_1)}{24}(x - A_1)^4 + \frac{W^{(3)}(A_1)}{6}(x - A_1)^3$$

$$+ \frac{W''(A_1)}{2}(x - A_1)^2$$

We get  $\frac{W^{(4)}(A_1)}{24}(x - A_1)^2 + \frac{W^{(3)}(A_1)}{6}(x - A_1) + \frac{W''(A_1)}{2} \geq 0$  for all  $x \in \mathbb{R}$  since  $\inf_{x \in \mathbb{R}} W(x) = W(A_1)$ . As  $W(A_2) = W(A_1)$ , the equation  $\frac{W^{(4)}(A_1)}{24}Y^2 + \frac{W^{(3)}(A_1)}{6}Y + \frac{W''(A_1)}{2} = 0$  admits one solution so the discriminant is equal to 0. We obtain  $(W^{(3)}(A_1))^2 - 3W''(A_1)W^{(4)}(A_1) = 0$ . A simple computation

provides  $A_1 = -\frac{\gamma}{3} \pm \sqrt{\frac{\gamma^2+3\rho}{3}} - \alpha$ . We can define this quantity if and only if  $\alpha \leq \frac{\gamma^2+3\rho}{3}$ .

The same holds for  $A_2$ . So, we obtain directly  $A_1 = -\frac{\gamma}{3} - \sqrt{\frac{\gamma^2+3\rho}{3}} - \alpha$  and  $A_2 = -\frac{\gamma}{3} + \sqrt{\frac{\gamma^2+3\rho}{3}} - \alpha$  because  $A_1 < A_2$ . Since  $A_1 \neq A_2$ , we deduce immediatly  $\alpha < \frac{\gamma^2+3\rho}{3}$ . As  $W'(A_2) = 0$ , it yields  $V'(A_2) + p\frac{\alpha}{2}(A_2 - A_1) = 0$  then  $V'(A_2) < 0$  which implies  $-\frac{\gamma}{3} + \sqrt{\frac{\gamma^2+3\rho}{3}} - \alpha = A_2 < a_d = \frac{-\gamma + \sqrt{\gamma^2+4\rho}}{2}$ . Consequently  $\alpha > \frac{\gamma^2+3\rho\sqrt{\gamma^2+4\rho}}{18} = \alpha_c^{(1)}$ . Then, as claimed in the statement of the lemma, (2.15) does not have any solution if  $\alpha \leq \alpha_c^{(1)}$ . Moreover, like for  $A_2$ , we have  $V'(A_1) > 0$  so  $A_1 > a_g$ . Consequently,  $a_g < A_1 < 0 < A_2 < a_d$ .  $A_2 > 0$  is equivalent to  $\alpha < \rho + \frac{2}{9}\gamma^2 = \alpha_c^{(2)}$ . We deduce that (2.15) does not admit a solution if  $\alpha \geq \alpha_c^{(2)}$ . Now, by assuming  $\alpha \in ]\alpha_c^{(1)}; \alpha_c^{(2)}[$ , we have the unique solution:  $A_1 = -\frac{\gamma}{3} - \sqrt{\frac{\gamma^2+3\rho}{3}} - \alpha$ ,  $A_2 = -\frac{\gamma}{3} + \sqrt{\frac{\gamma^2+3\rho}{3}} - \alpha$  and  $p = -\frac{2V'(A_2)}{\alpha(A_2 - A_1)} = 1 - \frac{\gamma(2\gamma^2+9\rho)}{9\alpha\sqrt{3\gamma^2+9\rho-9\alpha}}$ . This achieves the proof.  $\square$

Now, we are able to provide the small-noise description of the set of the stationary measures. We will use some similar computations than the ones in the proof of Theorem 3.2 in [HT10a].

**Theorem 2.6.** *Let  $V$  a potential which satisfies (V-1)–(V-5) and  $F(x) := \frac{\alpha}{2}x^2$ . For all  $\alpha > 0$ , there exists  $\epsilon_c(\alpha) > 0$  such that for all  $\epsilon < \epsilon_c(\alpha)$ , Diffusion (I) admits exactly*

- one stationary measure if  $\alpha \leq \alpha_c := \sup_{x \in \mathbb{R}} \frac{V(x) - V(a_d)}{(x - a_d)^2}$ .
- three stationary measures if  $\alpha > \alpha_c$ .

*Proof. Step 1.* By proceeding exactly like in Step 1 in the proof of Theorem 2.4, we can prove that each family of stationary measures admits a limiting value. Furthermore, according to the method in [HT10b] (Proposition 3.5, Theorem 3.6, Proposition 3.7), these adherence values have two possible forms:

- $u_0 = \delta_{A_0}$  such that  $V'(A_0) = 0$  and  $V + F * \delta_{A_0}$  reaches its global minimum in  $A_0$ .
- $u_0 = p\delta_{A_1} + (1-p)\delta_{A_2}$  with  $p \in ]0; 1[$  and  $A_1 < A_2$  verifies (2.15).

**Step 2.** We will enumerate the number of adherence values. We split the study into three different cases:

**Step 2.1.** If  $\alpha \leq \alpha_c^{(1)} = \alpha_c$ , the adherence values are necessary Dirac measures according to Lemma 2.5. Also, if  $\alpha$  is under the critical value  $\alpha_c$ , then  $\delta_{a_d}$  is not a possible adherence value. Indeed, by definition of this critical value,  $\alpha \leq \alpha_c$  implies that  $V(a_d)$  is not the global minimum of the potential  $V + F * \delta_{a_d}$ . So

there is a unique adherence value.

**Step 2.2.** If  $\alpha \geq \alpha_c^{(2)}$ , Lemma 2.5 implies that each limiting value is a Dirac measure. By making a proof similar to the one of Theorem 2.4 in [HT10b] for the outlying stationary measure defined by Proposition 1.3 with  $a_0 = 0$ , with  $a_0 = a_g$  then with  $a_0 = a_d$ , we deduce that  $\delta_{a_0}$  is a limiting value of a family of stationary measures in the small-noise limit for each  $a_0 \in \{a_g, a_d, 0\}$ .

**Step 2.3.** If  $\alpha_c^{(1)} < \alpha < \alpha_c^{(2)}$ , in the same way, we can prove that  $\delta_{a_d}$  and  $\delta_{a_g}$  are limiting values but  $\delta_0$  is not. However, according to Lemma 2.5,  $p\delta_{A_1} + (1-p)\delta_{A_2}$  is a limiting value where  $(A_1, A_2, p)$  is the unique solution of (2.15).

**Step 3.** If  $\alpha \leq \alpha_c^{(1)}$  or if  $\alpha \geq \alpha_c^{(2)}$ , we can apply the method in [HT09] with the rate of convergence since  $V''$  and  $F''$  are convex because we have  $\alpha + V''(a_0) > 0$  for each  $a_0$  such that  $\delta_{a_0}$  is an adherence value, by definition of the critical values  $\alpha_c^{(1)}$  and  $\alpha_c^{(2)}$ . We deduce that there is a unique stationary measure for  $\epsilon$  small enough if  $\alpha \leq \alpha_c^{(1)}$  and there are exactly three stationary measures for  $\epsilon$  small enough if  $\alpha \geq \alpha_c^{(2)}$ . Similarly, if  $\alpha \in ]\alpha_c^{(1)}; \alpha_c^{(2)}[$ , we can prove that there are exactly one outlying stationary measure around  $\delta_{a_d}$  and an other one around  $\delta_{a_g}$  for  $\epsilon$  small enough.

**Step 4.** It remains to prove that there exist  $\delta > 0$  and  $\epsilon_0 > 0$  such that the function  $\psi_\epsilon$  has a unique zero on the interval  $[pA_1 + (1-p)A_2 - \delta; pA_1 + (1-p)A_2 + \delta]$  for all  $\epsilon < \epsilon_0$ . We will proceed like in Step 5 of the proof of Theorem 3.2 in [HT10a]. Let us prove that  $\psi_\epsilon$  is increasing on  $[pA_1 + (1-p)A_2 - \delta; pA_1 + (1-p)A_2 + \delta]$  where  $\psi_\epsilon$  is defined in Definition 1.2. We put  $m_0 := pA_1 + (1-p)A_2$ . Since  $(A_1, A_2, p)$  is a solution of (2.15),  $W_0(x) := V(x) + \frac{\alpha}{2}x^2 - \alpha m_0 x$  reaches a unique global minimum on  $\mathbb{R}_+$  for  $x = A_2 = -\frac{\gamma}{3} + \sqrt{\frac{\gamma^2 + 3\rho}{3} - \alpha}$ . Let us fix some small  $\delta > 0$  (we shall precise it in the following). Taking the derivative of  $\psi_\epsilon$ , we get

$$\psi'_\epsilon(m) = \frac{2\alpha}{\epsilon} \int_{\mathbb{R}} \left( x^2 - mx - \frac{\epsilon}{2\alpha} \right) \exp \left[ -\frac{2}{\epsilon} W_m(x) \right] dx$$

with  $W_m(x) := W_0(x) - \alpha(m - m_0)x$ . By using the symmetry property of  $x \mapsto \frac{x^4}{4} + \frac{\alpha - \rho}{2}x^2$  and the upper bound  $m \leq \delta$ , we obtain

$$\begin{aligned} \psi'_\epsilon(m) &= \frac{2\alpha}{\epsilon} \int_0^\infty \left( x^2 - \frac{\epsilon}{2\alpha} \right) \cosh \left( \frac{2\alpha m x}{\epsilon} + \frac{\gamma}{3} x^3 \right) e^{-\frac{2}{\epsilon} \left( \frac{x^4}{4} + \frac{\alpha - \rho}{2} x^2 \right)} dx \\ &\quad - \frac{2\alpha}{\epsilon} \int_0^\infty m x \sinh \left( \frac{2\alpha m x}{\epsilon} + \frac{\gamma}{3} x^3 \right) e^{-\frac{2}{\epsilon} \left( \frac{x^4}{4} + \frac{\alpha - \rho}{2} x^2 \right)} dx \\ &\geq \frac{\alpha}{\epsilon} \int_0^\infty P_\delta(x) e^{-\frac{2}{\epsilon} W_m(x)} dx \quad \text{with } P_\delta(x) := x^2 - (m_0 + \delta)x - \frac{\epsilon}{\alpha}. \end{aligned}$$

By definition of  $m_0$ , we have  $A_2 > m_0$ . In the following, we consider  $\delta < \frac{A_2 - m_0}{4}$  and  $\epsilon < \frac{\alpha}{8}(A_2^2 - m_0^2)$ . We split the preceding integral into two parts: the first integral  $I_1$  concerns the support  $[0, \frac{A_2 + m_0}{2}]$  and the second integral  $I_2$  the complementary support  $[\frac{A_2 + m_0}{2}, \infty[$ . We get  $\psi'_\epsilon(m) \geq \frac{\alpha}{\epsilon} (I_1 + I_2)$ .

Since the positive root of the polynomial function  $P_\delta$  satisfies

$$x_+ = \frac{1}{2} \left( m_0 + \delta \pm \sqrt{(m_0 + \delta)^2 + \frac{4\epsilon}{\alpha}} \right) < \frac{A_2 + m_0}{2},$$

the polynomial is positive on the interval  $[\frac{A_2+m_0}{2}, \infty[$  and can be lower bounded by  $P_\delta(\frac{A_2+m_0}{2}) > 0$ . Lemma A.3 in [HT10a] implies the existence of some constant  $C > 0$  leading to the following estimate as  $\epsilon \rightarrow 0$ :

$$\begin{aligned} I_2 &\geq P_\delta \left( \frac{A_2 + m_0}{2} \right) \int_{\frac{A_2+m_0}{2}}^{A_2+1} e^{-\frac{2}{\epsilon} W_m(x)} dx \\ &\geq CP_\delta \left( \frac{A_2 + m_0}{2} \right) \sqrt{\epsilon} e^{-\frac{2}{\epsilon} \alpha \delta (A_2+1)} e^{-\frac{2}{\epsilon} W_0(A_2)} \end{aligned}$$

Let us finally focus our attention to the lower bound of the integral term  $I_1$ . Since the minimum value of  $P_\delta$  is  $-\frac{(m_0+\delta)^2}{4} - \frac{\epsilon}{\alpha}$ , we have

$$\begin{aligned} I_1 &\geq - \left( \frac{(m_0 + \delta)^2}{4} + \frac{\epsilon}{\alpha} \right) \int_0^{\frac{A_2+m_0}{2}} e^{\frac{2\alpha(m-m_0)x}{\epsilon}} e^{-\frac{2}{\epsilon} W_0(x)} dx \\ &\geq - \frac{A_2 + m_0}{2} \left( \frac{(m_0 + \delta)^2}{4} + \frac{\epsilon}{\alpha} \right) e^{\frac{2}{\epsilon} \alpha \delta (A_2+1)} e^{-2\frac{\omega}{\epsilon}} \end{aligned}$$

with  $\omega := \inf_{z \in [0, \frac{A_2+m_0}{2}]} W_0(z)$ . For  $\delta > 0$  small enough,  $\omega - 2\alpha\delta(A_2 + 1) > W_0(A_2)$ . Consequently the negative lower bound of  $I_1$  is negligible with respect to the positive lower bound of  $I_2$  as  $\epsilon$  becomes small. We deduce that there exists  $\epsilon_0$  such that  $\psi'_\epsilon(m) > 0$  for all  $m \in [m_0 - \delta, m_0 + \delta]$  and  $\epsilon \leq \epsilon_0$ .  $\square$

**Remark 2.7.** *If  $F$  is not linear and if  $\alpha > \rho + \frac{\gamma^2}{3}$  (the synchronized case), we can prove the thirdness of the stationary measures for  $\epsilon$  small enough. Indeed, in this case, there are exactly three limiting values and the local uniqueness of each of these values is directly obtained by the method of the rate of convergence.*

After studying the small-noise case, we will consider the big-noise one and we will see that the competition between the three different forces in (I) is dominated by the heat process if the coefficient diffusion  $\sqrt{\epsilon}$  is sufficiently big.

**Proposition 2.8.** *For all  $\alpha \geq 0$ , there exists a critical value  $\epsilon_0(\alpha)$  such that for all  $\epsilon > \epsilon_0(\alpha)$ , Diffusion (I) admits a unique stationary measure.*

*Proof.* We recall that the number of stationary measures for Diffusion (I) is the number of roots of the function  $\chi_\epsilon$ :

$$\chi_\epsilon(m) = \frac{\int_{\mathbb{R}} x \exp \left[ -\frac{2}{\epsilon} \left( \frac{x^4}{4} + \frac{\gamma}{3} x^3 + \frac{\alpha-\rho}{2} x^2 - \alpha m x \right) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( \frac{x^4}{4} + \frac{\gamma}{3} x^3 + \frac{\alpha-\rho}{2} x^2 - \alpha m x \right) \right] dx} - m.$$

We apply the change of variable  $x := y - \frac{\gamma}{3}$  and we obtain:

$$\chi_\epsilon(m) = \tilde{\chi}_\epsilon(\hat{m}) - \frac{2\gamma^3}{27\alpha} + \frac{\alpha - \rho}{3\alpha} \gamma$$

with  $\widehat{m} := m - \frac{2\gamma^3}{27\alpha} + \frac{\alpha-\rho}{3\alpha}\gamma \in \mathbb{R}$  and  $\widetilde{\chi}_\epsilon$  is the function defined by

$$\widetilde{\chi}_\epsilon(m) := \frac{\int_{\mathbb{R}} y \exp \left[ -\frac{2}{\epsilon} \left( \frac{y^4}{4} + \frac{\alpha - (\rho + \frac{\gamma^2}{3})}{2} y^2 - \alpha m y \right) \right] dy}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( \frac{y^4}{4} + \frac{\alpha - (\rho + \frac{\gamma^2}{3})}{2} y^2 - \alpha m y \right) \right] dy} - m.$$

In the same way, we define

$$\begin{aligned} \widetilde{\psi}_\epsilon(m) &:= \int_{\mathbb{R}} (y - m) \exp \left[ -\frac{2}{\epsilon} \left( \frac{y^4}{4} + \frac{\alpha - (\rho + \frac{\gamma^2}{3})}{2} y^2 - \alpha m y \right) \right] dy \\ \text{and } \widetilde{Z}_\epsilon(m) &:= \int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( \frac{y^4}{4} + \frac{\alpha - (\rho + \frac{\gamma^2}{3})}{2} y^2 - \alpha m y \right) \right] dy. \end{aligned}$$

We deduce that  $\frac{d}{dm} \widetilde{\chi}_\epsilon(m) = \frac{\xi_\epsilon(m)}{\widetilde{Z}_\epsilon(m)^2}$  with  $\xi_\epsilon(m) := \widetilde{\psi}'_\epsilon(m) \widetilde{Z}_\epsilon(m) - \widetilde{\psi}_\epsilon(m) \widetilde{Z}'_\epsilon(m)$ .

The function  $\xi_\epsilon$  is even and analytic so it is sufficient to prove that  $\xi_\epsilon^{(2n)}(0) \leq 0$  for all  $n \in \mathbb{N}$ . The derivation provides:

$$\begin{aligned} \xi_\epsilon^{(2n)}(0) &= \sum_{k=0}^{2n} C_{2n}^k \widetilde{\psi}_\epsilon^{(k+1)}(0) \widetilde{Z}_\epsilon^{(2n-k)}(0) - \sum_{k=0}^{2n} C_{2n}^k \widetilde{\psi}_\epsilon^{(k)}(0) \widetilde{Z}_\epsilon^{(2n+1-k)}(0) \\ &= \widetilde{\psi}_\epsilon^{(2n+1)}(0) \widetilde{Z}_\epsilon(0) + (1 - 2n) \widetilde{\psi}'_\epsilon(0) \widetilde{Z}_\epsilon^{(2n)}(0) \\ &\quad + \sum_{k=1}^{n-1} (C_{2n}^{2k} - C_{2n}^{2k+1}) \widetilde{\psi}_\epsilon^{(2k+1)}(0) \widetilde{Z}_\epsilon^{(2n-2k)}(0). \end{aligned} \quad (2.16)$$

As seen in the proof of Theorem 2.1, for all  $\alpha \in \mathbb{R}_+$  and  $k \in \mathbb{N}$ , we have

$$\widetilde{\psi}_\epsilon^{(2k)}(0) = 2I_\epsilon(2k) \left( \frac{2\alpha}{\epsilon} \right)^{2k+1} (2k+1) \gamma_k(\epsilon) \quad (2.17)$$

where  $I_\epsilon(2k)$  and  $\gamma_k(\epsilon)$  are defined respectively in (2.2) and (2.3). A simple computation provides

$$\widetilde{Z}_\epsilon^{(2k)}(0) = 2I_\epsilon(2k) \left( \frac{2\alpha}{\epsilon} \right)^{2k}. \quad (2.18)$$

By combining (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned} \xi_\epsilon^{(2n)}(0) &= 4 \left( \frac{2\alpha}{\epsilon} \right)^{2n+1} \left\{ I_\epsilon(2n) I_\epsilon(0) [(2n+1) \gamma_n(\epsilon) - (2n-1) \gamma_0(\epsilon)] \right. \\ &\quad \left. + \sum_{k=1}^{n-1} C_{2n}^{2k} (4k+1-2n) I_\epsilon(2k) I_\epsilon(2n-2k) \gamma_k(\epsilon) \right\}. \end{aligned}$$

As proved in Theorem 2.1, there exists  $\epsilon_c$  such that for all  $\alpha \geq \alpha_c$ , we have  $\gamma_0(\epsilon) \leq 0$ . As the sequence  $(\gamma_n(\epsilon))_{n \in \mathbb{N}}$  is nonincreasing, we deduce that the sequence  $(|\gamma_n(\epsilon)|)_{n \in \mathbb{N}}$  is nondecreasing. We note that for all  $k > \frac{n}{2}$ ,  $|4k + 1 - 2n| > |2n + 1 - 4k|$ . Consequently, for all  $k > \frac{n}{2}$ ,  $(4k + 1 - 2n)\gamma_k(\epsilon) + (2n + 1 - 4k)\gamma_{n-k}(\epsilon) < 0$ . Also, if  $k = \frac{n}{2}$ , we have  $(4k + 1 - 2n)\gamma_k(\epsilon) = \gamma_k(\epsilon) < 0$ . This implies  $\xi_\epsilon^{(2n)}(0) \leq 0$ .

The function  $\tilde{\chi}'_\epsilon(m)$  is then nonpositive. We deduce that the function  $\tilde{\chi}_\epsilon$  is nondecreasing then the equation  $\tilde{\chi}_\epsilon(m) - C$  has at more one solution for all  $C \in \mathbb{R}$ . This is particularly true for  $C = \frac{2\gamma^3}{27\alpha} - \frac{\alpha - \rho}{3\alpha}\gamma$ . It yields that  $\chi_\epsilon$  has zero or one root. There is at least one stationary measure for all  $\epsilon > 0$  so there exists a critical value  $\epsilon_0(\alpha)$  such that for all  $\epsilon > \epsilon_0(\alpha)$ , Diffusion (I) admits exactly one stationary measure.  $\square$

**Remark 2.9.** *By using a method similar to the one of [Tug10b], we could prove the convergence towards this unique stationary measure.*

Theorem 2.6 and Proposition 2.8 imply that there is a phase transition since there is exactly one stationary measure for  $\epsilon$  big enough and exactly three ones for  $\epsilon$  small enough ; for  $\alpha \geq \alpha_c$ .

According to Theorem 2.1, Theorem 2.6 and Proposition 2.8, it is reasonable to conjecture the following statement:

**Conjecture:** *Let  $V$  a confining potential which verifies (V-1)–(V-5) and an interacting potential  $F$  which satisfies (F-1)–(F-3). We assume also that  $V''$  and  $F''$  are convex. Then, there exists  $\epsilon_c(V, F) \in \mathbb{R}_+ \cup \{+\infty\}$  such that Diffusion (I) admits exactly one stationary measure for all  $\epsilon > \epsilon_c(V, F)$  and admits exactly three stationary measures for all  $\epsilon < \epsilon_c(V, F)$ .*

Let us remark that the following simulations tie in the previous conjecture.

### 3 Simulations of the phase transitions

We saw in Theorem 2.1 that there exists a continuous transition between two phases (uniqueness and thirdness of the stationary measures) when  $V$  is symmetric,  $F'$  is linear and  $V^{(2k)}$  is convex for all  $k \geq 1$ . Before studying the general case, we provide some example with the most famous confining potential of the literature:  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ . Before, we will show briefly that the knowledge of this critical value implies the one of all the critical values with  $\deg(V) = 4$ . Indeed, by applying the change of variable  $x := \frac{x}{\sqrt{\vartheta_1}}$  in (2.1), we obtain:

**Lemma 3.1.** *Let us consider  $V(x) := \frac{\vartheta_2}{4}x^4 - \frac{\vartheta_1}{2}x^2$  with  $\vartheta_1 > 0$  and  $\vartheta_2 > 0$ . We call  $\epsilon_c(\alpha, \vartheta_1, \vartheta_2)$  the phase transition of Diffusion (I) with the potentials  $V$  and  $F(x) := \frac{\alpha}{2}x^2$ . We have*

$$\epsilon_c(\alpha, \vartheta_1, \vartheta_2) = \frac{\vartheta_2}{\vartheta_1} \epsilon_c \left( \frac{\alpha}{\vartheta_1}, 1, 1 \right).$$

**Example 3.2.** Let us choose  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$  and  $F(x) := \frac{\alpha}{2}x^2$ . We will call  $\epsilon_c(\alpha)$  the critical value which corresponds to the phase transition. We can remark that some computation provides  $\epsilon_c(1) = \left(\frac{2\Gamma(\frac{3}{4})}{\pi}\right)^2$  where  $\Gamma$  is the Euler's function. According to Theorem 2.1, for all  $\alpha > 0$ ,  $\epsilon_c(\alpha)$  is defined as the solution of (2.1) which is equivalent to

$$\int_{\mathbb{R}_+} \left\{ x^2 - \frac{1}{2\alpha} \right\} \exp \left[ (1 - \alpha) x^2 - \frac{\epsilon}{2} x^4 \right] dx = 0.$$

By making the change of variable  $z := \sqrt{\frac{1}{2\alpha}} x$ , we get:

$$\mathbb{E} \left\{ \xi(\epsilon_c(\alpha), \alpha, X) \right\} = 0 \quad \text{with} \quad \xi(x, y, z) := (z^2 - 1) \exp \left[ \frac{z^2}{2y} - \frac{x}{8y^2} z^4 \right]$$

and  $\mathcal{L}(X) = \mathcal{N}(0, 1)$ . In order to simulate  $\epsilon_c(\alpha)$ , we choose  $r > 0$  and  $N \in \mathbb{N}^*$ . We want  $N$  big and  $r$  small.

We take  $N$  random variables independents and identically distributed by the law  $\mathcal{N}(0, 1)$ :  $(X_i)_{1 \leq i \leq N}$ . The Monte-Carlo's method provides the following estimation of  $\epsilon_c(\alpha)$ :

$$\epsilon_c^{r,N}(\alpha) := r \min \left\{ p \in \mathbb{N}^* \mid \sum_{i=1}^N \xi(pr, \alpha, X_i) < 0 \right\}.$$

We get this inequality after using the weak law of the large numbers:

$$\lim_{N \rightarrow +\infty} \epsilon_c^{r,N}(\alpha) - r \leq \epsilon_c(\alpha) \leq \lim_{N \rightarrow +\infty} \epsilon_c^{r,N}(\alpha).$$

We take  $N := 2 \times 10^5$  and  $r := 10^{-4}$  then we obtain the following curve of the critical value:

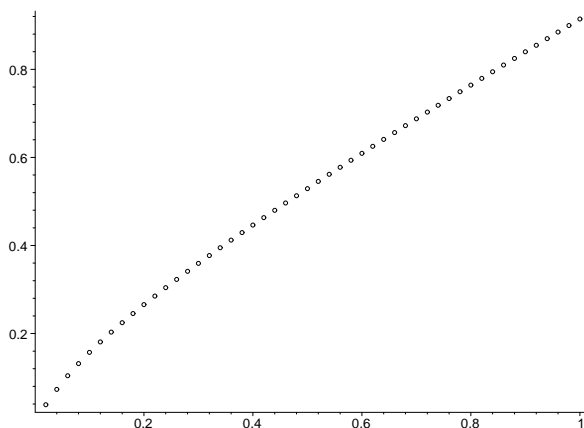


Figure 2: Critical value  $\epsilon_c(\alpha)$



We remark that it is increasing. And, it seems to be almost linear sufficiently far from 0. In fact, simple computations provide:

$$\lim_{\alpha \rightarrow +\infty} \frac{\epsilon_c(\alpha)}{\alpha} = \frac{2}{3} \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \frac{\epsilon_c(\alpha)}{\alpha} = 2.$$

### 3.1 Under (E)

In the previous results about the existence of the stationary measures (Proposition 1.3, Corollary 1.6, Proposition 1.7, Corollary 1.9 and Theorem 3.2 in [HT10a]), we assumed that  $\epsilon$  is small enough. We can wonder if there exists a threshold over which these results are false. Furthermore, if such a critical value exists, can we compute it?

In the simple case with  $\deg(V) = 4$  and  $\deg(F) = 2$ , we just see that we can, as predicted by Theorem 2.1. We will now deal with a non-linear interaction function  $F'$ . We still assume that  $V''$  is convex. As there is always a symmetric stationary measure, the study remains to find the critical value  $\epsilon_c$  such that Diffusion (I) admits an outlying stationary measure. By definition, it is:

$$\epsilon_c := \sup \left\{ \epsilon_0 > 0 \mid \forall \epsilon \in ]0; \epsilon_0[, \exists (m_1(\epsilon), m_2(\epsilon), m_3(\epsilon)) \in \mathbb{R}^3 \text{ with} \right. \\ \left. - \infty < m_1(\epsilon) < m_2(\epsilon) < m_3(\epsilon) < +\infty \text{ and } \psi_\epsilon(m_i(\epsilon)) = 0 \right\}.$$

A simple computation provides  $\psi_\epsilon(y) < 0$  for all  $y \geq a$  where  $a$  is the right wells of the confining potential  $V$ . Furthermore, by using the symmetry of the function  $\psi_\epsilon$ , we get:

$$\epsilon_c = \sup \left\{ \epsilon_0 > 0 \mid \forall \epsilon \in ]0; \epsilon_0[, \exists m(\epsilon) \in ]0; a[ \text{ such that } \psi_\epsilon(m(\epsilon)) = 0 \right\}.$$

In opposite with the results of the simplest case, we will only provide some lower-bound.

#### 3.1.1 Low-energy method

We know by Proposition 2.2 that there is a unique stationary measure for  $\epsilon$  small enough. We introduce  $\epsilon_c^{(0)}$  the biggest value (maybe equal to  $+\infty$ ) such that there is a unique symmetric stationary measure under this value. For  $\epsilon < \epsilon_c^{(0)}$ , set  $u_\epsilon^0$  this unique symmetric stationary measure.

**Proposition 3.3.** *Set a potential  $V$  which verifies (V-1)–(V-5) and a potential  $F$  which verifies (F-1)–(F-3). We have the following lower-bound of the critical value:*

$$\epsilon_c \geq \sup \left\{ \epsilon_0 \in ]0, \epsilon_c^{(0)}[ \mid \forall \epsilon \in ]0; \epsilon_0[, \inf_{u \in \mathcal{M}_{8q^2}, u \neq u_\epsilon^0} \Upsilon_\epsilon(u) < \Upsilon_\epsilon(u_\epsilon^0) \right\} \quad (3.1)$$

where  $\mathcal{M}_{8q^2}$  is the set of all the measures  $du$  which admits a  $C^\infty$ -continuous density  $u$  such that  $\int_{\mathbb{R}} x^{8q^2} u(x) dx < +\infty$ .

*Proof.* Set  $\epsilon_0 \in ]0; \epsilon_c^{(0)}[$  such that  $\inf_{u \in \mathcal{M}_{8q^2}, u \neq u_\epsilon^0} \Upsilon_\epsilon(u) < \Upsilon_\epsilon(u_\epsilon^0)$  for all  $\epsilon \in ]0; \epsilon_0[$ .

We use then the same argument than the one in the proof of Corollary 1.9 in [Tug10b]. Set  $\epsilon \in ]0; \epsilon_0[$ . Since  $\inf_{u \in \mathcal{M}_{8q^2}, u \neq u_\epsilon^0} \Upsilon_\epsilon(u) < \Upsilon_\epsilon(u_\epsilon^0)$ , there exists

$v_\epsilon \in \mathcal{M}_{8q^2}$  such that  $\Upsilon_\epsilon(v_\epsilon) < \Upsilon_\epsilon(u_\epsilon^0)$  and  $v_\epsilon \neq u_\epsilon^0$ . We consider now the self-stabilizing process  $(X_t^\epsilon)_{t \in \mathbb{R}_+}$  starting with  $\mathcal{L}(X_0^\epsilon) = v_\epsilon$ . We call  $u_t^\epsilon$  the law of  $X_t^\epsilon$  for all  $t \in \mathbb{R}_+$ . By Theorem 1.6 in [Tug10b], we know that a subsequence of  $u_t^\epsilon$  converges towards a stationary measure  $u_\epsilon$  when  $t$  tends to  $+\infty$ . Proposition 2.5 in [Tug10b] implies  $\Upsilon_\epsilon(u_\epsilon) \leq \liminf_{t \rightarrow +\infty} \Upsilon_\epsilon(u_t^\epsilon) \leq \Upsilon_\epsilon(v_\epsilon) < \Upsilon_\epsilon(u_\epsilon^0)$ . Since  $\Upsilon_\epsilon(u_\epsilon) < \Upsilon_\epsilon(u_\epsilon^0)$ , we deduce that the stationary measure  $u_\epsilon$  is not the symmetric one.

Consequently, for all  $\epsilon < \epsilon_0$ , Diffusion (I) admits at least two stationary measures. Hence, it admits at least three by using an argument of symmetry. We deduce that  $\epsilon_0 \leq \epsilon_c$ .  $\square$

This result may seem hard to apply since the question still remains to the study of an application in a space different from  $\mathbb{R}$ . We will see in the simplest case that there is in fact an obvious candidate for  $v_\epsilon$ . It is the one used in Corollary 1.9 in [Tug10b] for proving the existence of asymmetric stationary measures:

$$v_\epsilon^{a_0}(x) := Z^{-1} \exp \left[ -\frac{2}{\epsilon} (V(x) + F(x - a_0)) \right]$$

where  $a_0 \in \mathbb{R}$  satisfies  $V(a_0) = \inf_{x \in \mathbb{R}} V(x)$ . Indeed,  $v_\epsilon^{a_0}$  is closed to the outlying stationary measure  $u_\epsilon^{a_0}$  since the two measures converge towards  $\delta_{a_0}$  in the small-noise limit. By using a method similar to the one of Corollary 1.9 in [Tug10b], we get the existence of  $\epsilon_0 > 0$  such that for all  $\epsilon \in ]0; \epsilon_0[$ ,  $\Upsilon_\epsilon(v_\epsilon^{a_0}) < \Upsilon_\epsilon(u_\epsilon^0)$ .

Finally, the simulation is the following:

1. We obtain one vector  $(m_2^0, \dots, m_{2n-2}^0) \in \mathbb{R}_+^{n-1}$  such that the vector  $m^0 := (0, m_2(\epsilon), 0, \dots, m_{2n-2}^0, 0) \in \mathbb{R}^{2n-1}$  is a fixed point of  $\Phi^{(\epsilon)}$  defined in Proposition 1.11 by using the Newton's method in  $\mathbb{R}^{n-1}$ .
2. We use the Monte carlo method for making an approximation of the two free-energy:  $\Upsilon_\epsilon(u_\epsilon^{m^0})$  and  $\Upsilon_\epsilon(v_\epsilon^{a_0})$ .
3. We proceed these computations for  $\epsilon := \frac{j}{M} \times \epsilon_0$  for all  $1 \leq j \leq M$  with  $\epsilon_0 > 0$ . Each time we note a change of sign for the quantity  $\Upsilon_\epsilon(u_\epsilon^{m^0}) - \Upsilon_\epsilon(v_\epsilon^{a_0})$ , we deduce that there is a phase transition.

We consider the following example:

**Example 3.4.** Set  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$  and  $F(x) := \frac{\beta}{4}x^4 + \frac{\alpha}{2}x^2$  with  $\beta > 0$  and  $\alpha \geq 0$ . We call  $\epsilon_c(\alpha, \beta)$  the critical value which corresponds to the transition between  $\{\Upsilon_\epsilon(u_\epsilon^0) < \Upsilon_\epsilon(v_\epsilon^1)\}$  and  $\{\Upsilon_\epsilon(u_\epsilon^0) > \Upsilon_\epsilon(v_\epsilon^1)\}$ . By Subsection 4.2 in

[HT10a], there is a unique symmetric stationary measure  $u_\epsilon^0$  for all  $\epsilon > 0$  such that:

$$u_\epsilon^0(x) = Z^{-1} \exp \left[ -\frac{2}{\epsilon} \left( \frac{1+\beta}{4} x^4 + \frac{\alpha + 3\beta m_2 - 1}{2} x^2 \right) \right]$$

where  $m_2$  is the only positive real which satisfies

$$m_2 = \frac{\int_{\mathbb{R}} x^2 \exp \left[ -\frac{2}{\epsilon} \left( \frac{1+\beta}{4} x^4 + \frac{\alpha + 3\beta m_2 - 1}{2} x^2 \right) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( \frac{1+\beta}{4} x^4 + \frac{\alpha + 3\beta m_2 - 1}{2} x^2 \right) \right] dx}.$$

A simple computation provides then

$$\Upsilon_\epsilon(u_\epsilon^0) = -\frac{\epsilon}{2} \log \left\{ \int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} \left( U_{\alpha,\beta}(x) + \frac{3\beta}{2} m_2 x^2 \right) \right] dx \right\} - \frac{3\beta}{4} m_2^2$$

with  $U_{\alpha,\beta}(x) := \frac{1+\beta}{4} x^4 + \frac{\alpha-1}{2} x^2$ .

In the other hand, we have:

$$\Upsilon_\epsilon(v_\epsilon^1) = -\frac{\epsilon}{2} \log \left\{ \int_{\mathbb{R}} e^{-\frac{\epsilon}{2}(V(x)+F(x-1)-F(1))} dx \right\} + (\alpha + \beta)n_1 - \frac{3\beta}{2}n_2$$

$$+ \beta n_3 - \frac{\alpha}{2}n_1^2 - \beta n_1 n_3 + \frac{3\beta}{4}n_2^2$$

with  $n_k := \frac{\int_{\mathbb{R}} x^k \exp \left[ -\frac{2}{\epsilon} (V(x) + F(x-1)) \right] dx}{\int_{\mathbb{R}} \exp \left[ -\frac{2}{\epsilon} (V(x) + F(x-1)) \right] dx}$ .

We take  $N := 2 \times 10^5$  and we get the following surface:

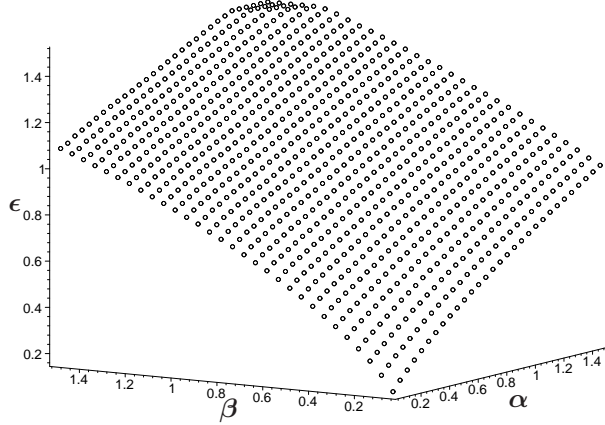


Figure 3: Critical value  $\epsilon_c(\alpha, \beta)$

**Remark 3.5.** Both simulations of Figure 3, Figure 4 and Figure 5 are saturated by the hyperplan ( $\epsilon = 1.5$ ) in order to compare the different methods.

We remark on Figure 3 that the provided value is increasing in the two variables and seems linear sufficiently far from (0, 0).

### 3.1.2 Convexity of the free-energy method

Now, we will point out the particular link between the function  $\psi_\epsilon$  and the free-energy of the system.

Since the function  $F'$  is linear, we know that any stationary measure of Diffusion (I) has the following form:

$$u_\epsilon^{(m)} = Z^{-1} \exp \left[ -\frac{2}{\epsilon} \left( V(x) + \frac{\alpha}{2} x^2 - \alpha m x \right) \right].$$

For all  $\epsilon > 0$ , we introduce the function  $\tau_\epsilon(m) := \Upsilon_\epsilon(u_\epsilon^{(m)})$  for all  $m \in \mathbb{R}$ .

**Lemma 3.6.** *For all  $\epsilon > 0$  and  $m \in \mathbb{R}$ , the following equality holds:*

$$\tau'_\epsilon(m) = -\frac{\alpha \psi_\epsilon(m)}{Z_\epsilon(m)} \text{Var} \left( u_\epsilon^{(m)} \right). \quad (3.2)$$

*Proof.* By definition, we have:

$$\begin{aligned} \tau_\epsilon(m) &= \int_{\mathbb{R}} \left( \frac{\epsilon}{2} \log \left( u_\epsilon^{(m)}(x) \right) + V(x) + \frac{1}{2} F * u_\epsilon^{(m)}(x) \right) u_\epsilon^{(m)}(x) dx \\ &= -\frac{\epsilon}{2} \log(Z_\epsilon(m)) + \alpha m \int_{\mathbb{R}} x u_\epsilon^{(m)}(x) dx - \frac{\alpha}{2} \left( \int_{\mathbb{R}} x u_\epsilon^{(m)}(x) dx \right)^2. \end{aligned}$$

By remarking that  $\int_{\mathbb{R}} x u_\epsilon^{(m)}(x) dx = \frac{\psi_\epsilon(m) + m}{Z_\epsilon(m)}$ , we get:

$$\tau_\epsilon(m) = -\frac{\epsilon}{2} \log \left[ Z_\epsilon(m) e^{-\frac{\alpha}{2} m^2} \right] - \frac{\alpha \psi_\epsilon(m)^2}{2 Z_\epsilon(m)^2}.$$

The derivation of this equality provides

$$\begin{aligned} \tau'_\epsilon(m) &= -\frac{\alpha \psi_\epsilon(m)}{Z_\epsilon(m)} \left( 1 + \frac{\psi'_\epsilon(m)}{Z_\epsilon(m)} - \frac{\psi_\epsilon(m) Z'_\epsilon(m)}{Z_\epsilon(m)^2} \right) \\ &= -\frac{\alpha \psi_\epsilon(m)}{Z_\epsilon(m)} \text{Var} \left( u_\epsilon^{(m)} \right) \end{aligned}$$

□

Consequently,  $\tau'_\epsilon(m) \psi_\epsilon(m) = -\frac{\alpha \text{Var}(u_\epsilon^{(m)})}{Z_\epsilon(m)} \psi_\epsilon(m) \leq 0$ . We deduce that the behavior of the function  $\tau_\epsilon$  is directly linked to the sign of  $\psi_\epsilon$ .

**Proposition 3.7.** *Set a confining potential  $V$  which verifies (V-1)–(V-5) and an interacting potential  $F(x) := \frac{\alpha}{2} x^2$  with  $\alpha > 0$ . Under (HC), the critical value  $\epsilon_c$  is the only zero of the function  $\epsilon \mapsto \frac{d^2}{dm^2} \Upsilon_\epsilon(u_\epsilon^m) \Big|_{m=0}$ .*

*Furthermore, for all  $\epsilon < \epsilon_c$ , we have  $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon(u_\epsilon^-) < \Upsilon_\epsilon(u_\epsilon^0)$ .*

*Proof.* By taking the derivative of (3.2), we get:

$$\tau_\epsilon''(m) = -\frac{\alpha \text{Var}(u_\epsilon^m)}{Z_\epsilon(m)} \psi_\epsilon'(m) - \psi_\epsilon(m) \frac{d}{dm} \left[ \frac{\alpha \text{Var}(u_\epsilon^m)}{Z_\epsilon(m)} \right].$$

As  $\psi_\epsilon(0) = 0$ , we obtain:  $\tau_\epsilon''(0) = -\frac{\alpha \text{Var}(u_\epsilon^0)}{Z_\epsilon(0)} \psi_\epsilon'(0)$ . We recall that  $\psi_\epsilon'(0) = \gamma_0(\epsilon)$  (defined in the proof of Theorem 2.1). According to Theorem 2.1, the critical value of  $\epsilon_c$  is the only solution of  $\gamma_0(\epsilon) = 0$  consequently there is only one value of  $\epsilon$  such that  $\frac{d^2}{dm^2} \Upsilon_\epsilon(u_\epsilon^m) \Big|_{m=0} = 0$  and this value is  $\epsilon_c$ .

When  $\epsilon < \epsilon_c$ , there are exactly three stationary measures:  $u_\epsilon^+$ ,  $u_\epsilon^-$  and  $u_\epsilon^0$ . The symmetry implies directly  $\Upsilon_\epsilon(u_\epsilon^+) = \Upsilon_\epsilon(u_\epsilon^-)$ . By definition of the two outlying stationary measures,  $\int_{\mathbb{R}} x u_\epsilon^+(x) dx$  is the only positive zero of the function  $\psi_\epsilon$  and consequently the only positive critical point of the function  $\tau_\epsilon$  according to Lemma 3.6. Since  $\epsilon < \epsilon_c$ ,  $\frac{d^2}{dm^2} \Upsilon_\epsilon(u_\epsilon^m) \Big|_{m=0} < 0$  that implies  $\tau_\epsilon$  reaches its global minimum in  $\int_{\mathbb{R}} x u_\epsilon^+(x) dx$  and in  $\int_{\mathbb{R}} x u_\epsilon^-(x) dx$ . Also,  $\tau_\epsilon$  reaches one local maximum in  $0 = \int_{\mathbb{R}} x u_\epsilon^0(x) dx$ . Immediatly, we get  $\Upsilon_\epsilon(u_\epsilon^\pm) < \Upsilon_\epsilon(u_\epsilon^0)$ .  $\square$

According to (3.1), after recalling that  $\epsilon_c^{(0)}$  is the critical value under which there is a unique symmetric stationary measure, we get this new lower-bound:

$$\epsilon_c \geq \sup \left\{ \epsilon_0 \in ]0, \epsilon_c^{(0)}[ \mid \forall \epsilon \in ]0; \epsilon_0[, \min \text{Sp}(J_\epsilon(m_0)) < 0 \right\} \quad (3.3)$$

where  $J_\epsilon(m)$  is the Hessian matrix of the function  $\tau_\epsilon$  and  $m^0$  is the only point in  $\mathbb{R}^{2n-1}$  such that  $u_\epsilon^{m^0}$  is a symmetric stationary measure.

However, the computations are so tedious that it is better to use the previous method. Also, we will derive this method for a simpler one.

### 3.1.3 Stability of the free-energy method

We will use here an other method in the particular example  $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$  and  $F(x) := \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4$  but it can be computed for more general potentials. We provide some simulation of an *a priori* other critical value. We define  $\tilde{\epsilon}_c(\alpha, \beta)$  as the transition phase between the stability and the instability of the symmetric stationary measure for the free-energy.

**Definition 3.8.** *We will say that  $u_\epsilon^0$  is stable for  $\Upsilon_\epsilon$  if for each function  $f \in \mathcal{L}_2(u_\epsilon^0)$  which verifies  $\int_{\mathbb{R}} f(x) u_\epsilon^0(x) dx = 0$  and  $C \geq f(x) \geq -C$  with  $C > 0$ , the application  $\zeta_\epsilon(\delta) := \Upsilon_\epsilon(u_\epsilon^0 + \delta f \times u_\epsilon^0)$  admits a local maximum in  $\delta = 0$ . We will call  $\tilde{\epsilon}_c^+(\alpha, \beta)$  the value such that for all  $\epsilon > \tilde{\epsilon}_c^+(\alpha, \beta)$ ,  $u_\epsilon^0$  is stable for  $\Upsilon_\epsilon$  and  $\tilde{\epsilon}_c^-(\alpha, \beta)$  the value such that for all  $\epsilon < \tilde{\epsilon}_c^-(\alpha, \beta)$ ,  $u_\epsilon^0$  is not stable for  $\Upsilon_\epsilon$ .*

**Remark 3.9.** *It corresponds to some variationnal calculus. Intuitively, since  $V''$  is convex, the landscape  $V$  has two wells and one hill. We expect a similar result for the free-energy  $\Upsilon_\epsilon$ .*

Since  $u_\epsilon^0$  is a stationary measure, we get directly  $\zeta'_\epsilon(0) = 0$ . It remains now to compute the second derivative:

$$\begin{aligned} \zeta''_\epsilon(0) &= \frac{\epsilon}{4} \int_{\mathbb{R}} f_-(x)^2 u_\epsilon^0(x) dx + \frac{\epsilon}{4} \int_{\mathbb{R}} f_+(x)^2 u_\epsilon^0(x) dx - \frac{\alpha}{2} \left( \int_{\mathbb{R}} x f_-(x) u_\epsilon^0(x) dx \right)^2 \\ &\quad - \beta \left( \int_{\mathbb{R}} x f_-(x) u_\epsilon^0(x) dx \right) \left( \int_{\mathbb{R}} x^3 f_-(x) u_\epsilon^0(x) dx \right) \end{aligned}$$

with  $f_-(x) := \frac{f(x)-f(-x)}{2}$  and  $f_+(x) := \frac{f(x)+f(-x)}{2}$ . The Cauchy-Schwarz inequality implies

$$\zeta''_\epsilon(0) \geq \left( \frac{\epsilon}{4} - \frac{\alpha}{2} m_2(\epsilon) - \beta \sqrt{m_2(\epsilon) m_6(\epsilon)} \right) \int_{\mathbb{R}} f(x)^2 u_\epsilon^0(x) dx$$

where  $m_2(\epsilon)$  (resp.  $m_6(\epsilon)$ ) is the second (resp. sixth) moment of  $u_\epsilon^0$ . Consequently, the inequality

$$\frac{2\alpha}{\epsilon} m_2(\epsilon) + \frac{4\beta}{\epsilon} \sqrt{m_2(\epsilon) m_6(\epsilon)} \leq 1$$

implies  $\epsilon \geq \tilde{\epsilon}_c^+(\alpha, \beta)$ . This will permit us to simulate  $\tilde{\epsilon}_c^+(\alpha, \beta)$ . Let us note that if  $\beta = 0$ , we recover the exact representation of the critical value which separates the domain of uniqueness and the one of thirdness of the stationary measure(s).

The method of stability derives from the one of convexity since the idea of the stability around  $u_\epsilon^0$  consists in choosing a particular direction and computing the second derivative in this direction. Also, we note that an infinitesimal change on the direction  $x \mapsto x$  corresponds to an infinitesimal change of the moments  $m_1$  and  $m_3$ . Besides, we know that the first diagonal plays a particular role in the mean-field systems and consequently in the self-stabilizing processes as noted in [CGM08].

We consider the sequence of bounded functions  $(f_n)_{n \in \mathbb{N}}$  with  $f_n(x) := x \mathbb{1}_{[-n, n]}$ . We remark that  $f_n \in \mathcal{L}^2(u_\epsilon^0)$ . For all  $n \in \mathbb{N}$ , we can define  $\zeta_\epsilon$  on an interval containing 0 in its interior. Consequently,  $\zeta''_\epsilon(0)$  has a sense for all  $n \geq 0$  (we do not write the dependence in  $n$  in  $\zeta''_\epsilon(0)$  for simplifying the reading). Since  $f_n$  converges towards  $x \mapsto x$  in  $\mathcal{L}^2(u_\epsilon^0)$ , it yields:

$$\lim_{n \rightarrow +\infty} \zeta''_\epsilon(0) = m_2(\epsilon) \left( \frac{\epsilon}{4} - \frac{\alpha}{2} m_2(\epsilon) - \beta m_4(\epsilon) \right).$$

We deduce that the inequality  $\frac{2\alpha}{\epsilon} m_2(\epsilon) + \frac{4\beta}{\epsilon} m_4(\epsilon) > 1$  implies  $\epsilon < \tilde{\epsilon}_c^-(\alpha, \beta)$ . This will permit us to simulate  $\tilde{\epsilon}_c^-(\alpha, \beta)$ .

By taking  $N := 2 \times 10^5$ , we get the following simulations of the surfaces of  $\tilde{\epsilon}_c^-(\alpha, \beta)$  and  $\tilde{\epsilon}_c^+(\alpha, \beta)$ :

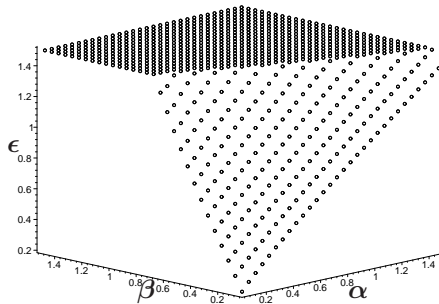


Figure 4:  $\tilde{\epsilon}_c^+(\alpha, \beta)$

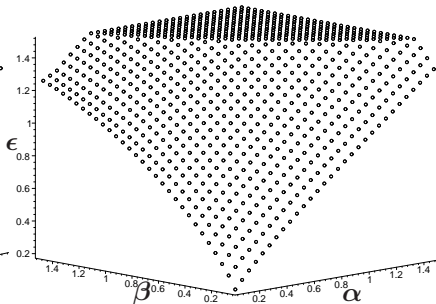


Figure 5:  $\tilde{\epsilon}_c^-(\alpha, \beta)$

**Remark 3.10.** *Let us note that it would also have been possible to compute with  $x \mapsto x^{2n+1}$  for  $n \geq 1$ .*

### 3.1.4 Comparison of the methods

Under (E) and when  $\deg(F) \geq 4$ , three methods have been examined.

The first one is the low-energy method which consists in providing one symmetric stationary measure  $u_\epsilon^0$  and one asymmetric measure with lower free-energy than  $u_\epsilon^0$ . The main advantage of this method is the simplicity of computation. But, the critical value provided is far from the phase transition. Indeed, when  $\epsilon$  is closed to the critical value  $\epsilon_c$  and less than this value, the asymmetric stationary measures are closed to the symmetric one(s) for continuity arguments. Thanks to this method, we see that the critical value is not asymptotically small.

The method which uses the convexity of the free-energy (IV) around a symmetric stationary measure seems to be the most precise because it needs only a local knowledge of the free-energy around this measure and this is exactly in this sense that we can separate the two phases in the case described in Theorem 2.1. However, even if  $F(x) := \frac{\beta}{4}x^4$ , the computations are too tedious.

The last method consists in the stability or the instability of the free-energy around a symmetric stationary measure. Since the direction  $x \mapsto x$  plays a special role in the study of the self-stabilizing processes, we do not study the stability in all the directions but in the special one. Figure 5 and Figure 3 prove that the estimation provided by the low-energy method is less than the one provided by the instability of the free-energy around  $u_\epsilon^0$ . In the other hand, the stability method provides a larger value but by construction, it is reasonable to assume that this is larger than the critical value.

We could use another method: the fixed point method. It consists to find  $\epsilon$  small enough such that the function  $\Phi^{(\epsilon)}$  admits a fixed point which does not correspond to an even measure. Nevertheless, this method is the worst. It is theoretically based on Theorem 1.12 and this statement needs Inequality (1.12) which corresponds to  $\beta < \frac{1}{6}$  in the chosen example. So it is not adapted.

### 3.2 If $V$ is asymmetric

We will only simulate with  $F(x) := \frac{\alpha}{2}x^2$ . It remains to study the number of roots of a function from  $\mathbb{R}$  to  $\mathbb{R}$ . However, it is not as simple as under (E). Indeed, when it is symmetric, there is an immediate and obvious solution then the number of stationary measures is directly linked to the second derivative in this obvious solution (0). Here, there is not any obvious solution so we need to know the whole trajectory of the function. Consequently, the simulation is the following:

We compute  $\chi_\epsilon(a_g + \frac{k}{M}(a_d - a_g))$  for all  $1 \leq k \leq M - 1$  by using the Monte Carlo method. Then, we count the number of change of sign without forgetting that  $\chi_\epsilon(a_g) > 0$  and  $\chi_\epsilon(a_d) < 0$ .

We apply this simulation to

$$V(x) := \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} \quad \text{and} \quad F(x) := \frac{\alpha}{2}x^2$$

for  $\alpha := j \times 0.01$  for  $1 \leq j \leq 120$  and  $\epsilon := i \times 0.005$  for  $1 \leq i \leq 60$ . By taking  $N := 2 \times 10^5$  and  $M := 250$ , we obtain:

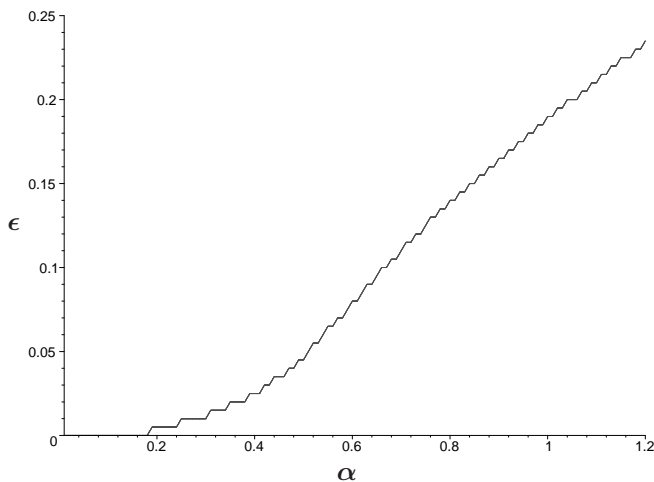


Figure 6:  $\epsilon_c(\alpha)$  before smoothing

This figure corresponds to the raw data of the critical value. We observe that the critical value is unique and increasing.

However, the lack of precision and the slow increasing of the critical value  $\epsilon_c(\alpha)$  in function of  $\alpha$  implies that the curve seems jerky. In order to make some regularization of the datas, we use two times the following modification:

$$\begin{aligned} (\mathcal{R}(X))(j) &:= \frac{X(j-1) + 2 \times X(j) + X(j+1)}{4} \quad \text{for all } 2 \leq j \leq 119, \\ (\mathcal{R}(X))(1) &:= X(1) \quad \text{and} \quad (\mathcal{R}(X))(120) := X(120) \end{aligned}$$



with  $X(j) := \epsilon_c(j \times 0.01)$ . Consequently, the jumps (which are inevitable since the step of the precision is 0.005) are erased. We get the following figure:

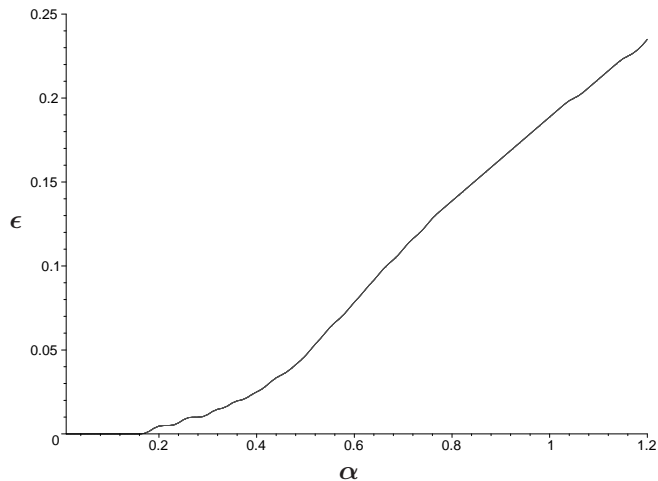


Figure 7:  $\epsilon_c(\alpha)$  after smoothing

Let us observe that this curve contradicts the result of Theorem 2.6 since we would have non-uniqueness of the stationary measures for small  $\epsilon$  when  $\alpha = 0.19 < \alpha_c = \frac{1+3\sqrt{5}}{18}$ . However, by taking  $N := 1.2 \times 10^6$  instead of  $2 \times 10^5$ , the precision increases and we get the uniqueness. There is still a mistake due to the imprecision for  $\alpha = 0.2$  and it is why we choose to present the curve with  $N := 2 \times 10^5$  since the global behavior of the critical value is respected.

The reason for which the mistakes persist is systemic. Indeed, in the other cases, the mistakes change directly the value simulated so we have a good control. Here, a little change of sign in comparison to one of the real values computed can change completely the number of change of sign for the trajectory and consequently the control on the critical value is weaker.

**Thanks:** *The most of the ideas of this work have been found while I was at the Institut Élie Cartan in nancy. And so I wanted to mention that I would not have been able to write it without the hospitality I received from the beginning especially from Samuel Herrmann.*

*I also would like to thank Florent Malrieu for one of the remarks he made to me on Wednesday 23th June 2010 and which provided me the ideas of the convexity method and consequently of the stability/instability one.*

*Finalemnt, un très grand merci à Manue et à Sandra pour tout.*

## References

- [BAZ99] G. Ben Arous and O. Zeitouni. Increasing propagation of chaos for mean field models. *Ann. Inst. H. Poincaré Probab. Statist.*, 35(1):85–102, 1999.
- [BBCG08] Dominique Bakry, Franck Barthe, Patrick Cattiaux, and Arnaud Guillin. A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Electron. Commun. Probab.*, 13:60–66, 2008.
- [BGV07] François Bolley, Arnaud Guillin, and Cédric Villani. Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields*, 137(3-4):541–593, 2007.
- [BRTV98] S. Benachour, B. Roynette, D. Talay, and P. Vallois. Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos. *Stochastic Process. Appl.*, 75(2):173–201, 1998.
- [BRV98] S. Benachour, B. Roynette, and P. Vallois. Nonlinear self-stabilizing processes. II. Convergence to invariant probability. *Stochastic Process. Appl.*, 75(2):203–224, 1998.
- [CDPS10] Francesca Collet, Paolo Dai Pra, and Elena Sartori. A simple mean field model for social interactions: dynamics, fluctuations, criticality. *J. Stat. Phys.*, 139(5):820–858, 2010.
- [CGM08] P. Cattiaux, A. Guillin, and F. Malrieu. Probabilistic approach for granular media equations in the non-uniformly convex case. *Probab. Theory Related Fields*, 140(1-2):19–40, 2008.
- [CMV03] José A. Carrillo, Robert J. McCann, and Cédric Villani. Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoamericana*, 19(3):971–1018, 2003.
- [CX10] Dan Crisan and Jie Xiong. Approximate McKean-Vlasov representations for a class of SPDEs. *Stochastics*, 82(1-3):53–68, 2010.
- [DG87] D A. Dawson and J. Gärtner. Large deviations from the McKean-Vlasov limit for weakly interacting diffusions. *Stochastics*, 20(4):247–308, 1987.
- [DPdH96] Paolo Dai Pra and Frank den Hollander. McKean-Vlasov limit for interacting random processes in random media. *J. Statist. Phys.*, 84(3-4):735–772, 1996.
- [Fun84] Tadahisa Funaki. A certain class of diffusion processes associated with nonlinear parabolic equations. *Z. Wahrsch. Verw. Gebiete*, 67(3):331–348, 1984.
- [HIP08] Samuel Herrmann, Peter Imkeller, and Dierk Peithmann. Large deviations and a Kramers’ type law for self-stabilizing diffusions. *Ann. Appl. Probab.*, 18(4):1379–1423, 2008.
- [HT09] S. Herrmann and J. Tugaut. Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small noise limit. *Prépublications de l’Institut Élie Cartan*, 2009.
- [HT10a] S. Herrmann and J. Tugaut. Non-uniqueness of stationary measures for self-stabilizing processes. *Stochastic Process. Appl.*, 120(7):1215–1246, 2010.

- [HT10b] S. Herrmann and J. Tugaut. Stationary measures for self-stabilizing processes: asymptotic analysis in the small noise limit. *Electron. J. Probab.*, 15:2087–2116, 2010.
- [Mal01] F. Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.*, 95(1):109–132, 2001.
- [Mal03] Florent Malrieu. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.*, 13(2):540–560, 2003.
- [McK66] H. P. McKean, Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, 56:1907–1911, 1966.
- [McK67] H. P. McKean, Jr. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57. Air Force Office Sci. Res., Arlington, Va., 1967.
- [Mél96] Sylvie Méléard. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 42–95. Springer, Berlin, 1996.
- [Oel85] Karl Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 69(2):279–322, 1985.
- [SV79] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.
- [Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.
- [Tam84] Yozo Tamura. On asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 31(1):195–221, 1984.
- [Tam87] Yozo Tamura. Free energy and the convergence of distributions of diffusion processes of McKean type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):443–484, 1987.
- [Tug10a] J. Tugaut. *Processus autostabilisants dans un paysage multi-puits*. PhD thesis, Université Henri Poincaré, Nancy, 2010.
- [Tug10b] J. Tugaut. Convergence to the equilibria for self-stabilizing processes in double well landscape. *Prépublication*, 2010.
- [Ver06] A. Yu. Veretennikov. On ergodic measures for McKean-Vlasov stochastic equations. In *Monte Carlo and quasi-Monte Carlo methods 2004*, pages 471–486. Springer, Berlin, 2006.