



HAL
open science

A new numerical strategy with space-time adaptivity and error control for multi-scale gas discharge simulations

Max Duarte, Zdenek Bonaventura, Marc Massot, Anne Bourdon, Stéphane
Descombes, Thierry Dumont

► **To cite this version:**

Max Duarte, Zdenek Bonaventura, Marc Massot, Anne Bourdon, Stéphane Descombes, et al.. A new numerical strategy with space-time adaptivity and error control for multi-scale gas discharge simulations. *Journal of Computational Physics*, 2011, pp.1-25. 10.1016/j.jcp.2011.07.002 . hal-00573043v1

HAL Id: hal-00573043

<https://hal.science/hal-00573043v1>

Submitted on 2 Mar 2011 (v1), last revised 24 Jul 2011 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A new numerical strategy with space-time adaptivity and error control for multi-scale gas discharge simulations[☆]

Max Duarte^{a,1,*}, Zdeněk Bonaventura^{a,b,2}, Marc Massot^a, Anne Bourdon^a, Stéphane Descombes^c, Thierry Dumont^d

^aLaboratoire EM2C - UPR CNRS 288, Ecole Centrale Paris, Grande Voie des Vignes, 92295 Chatenay-Malabry Cedex, France

^bDepartment of Physical Electronics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

^cLaboratoire J. A. Dieudonné - UMR CNRS 6621, Université de Nice - Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France

^dUniversité de Lyon - Université Lyon 1 - INSA de Lyon, 69621 - Ecole Centrale de Lyon - Institut Camille Jordan, UMR CNRS 5208 - 43 Boulevard du 11 novembre 1918 69622 Villeurbanne Cedex, France

Abstract

This paper presents a new resolution strategy for multi-scale gas discharge simulations based on a second order time adaptive integration and space adaptive multiresolution. A classical fluid model is used to model plasma discharges, considering drift-diffusion equations and electric field computation. The proposed numerical method provides a time-space accuracy control of the solution, and thus, an effective accurate resolution independent of the fastest physical time scale. Important improvement of computational efficiency is achieved whenever the required time steps go beyond standard stability constraints associated with mesh size or source time scales for the resolution of drift-diffusion equations, whereas stability constraint related to dielectric relaxation time scale is respected but with second order precision. Numerical illustrations show that the strategy can be efficiently applied to simulate propagation of highly nonlinear ionizing waves as streamer discharges, as well as highly multi-scale nanosecond repetitively pulsed discharges, describing consistently a broad spectrum of space and time scales as well as different physical scenarios for consecutive discharge/post-discharge phases, out of reach of standard non-adaptive methods.

Keywords: multi-scale discharge, time adaptive integration, space adaptive multiresolution, error control

2000 MSC: 65M08, 65M50, 65Z05, 65G20

1. Introduction

In recent years, plasma discharges at atmospheric pressure have been studied for an increasing list of applications such as chemical and biological decontamination, aerodynamic flow control and combustion [1, 2]. In all these physical configurations, discharges take usually the form of thin plasma filaments driven by highly nonlinear ionizing waves, also called streamers. These ionizing waves occur as a consequence of the high electric field induced by fast variations of net charge density ahead of an electron avalanche with large amplification. Streamer discharge dynamics are mainly governed by the Courant, effective ionization and dielectric relaxation times scales [3], which are usually of the order of $10^{-14} - 10^{-12}$ s, whereas typical time scale of the discharge propagation in centimeter gaps, is about a few tens of nanoseconds. On the other hand, large variation of space scales needs also to be taken into account, since the Debye length at atmospheric pressure can be as small as a few micrometers, while the interelectrode gaps, where discharges propagate, are usually of the order of a few centimeters. As a result, the detailed physics of discharges reveals an important time-space multi-scale character [4, 5].

More complex applications include plasma assisted combustion or flow control, for which the enhancement of the gas flow chemistry or momentum transfer during typical time scales of the flow of $10^{-4} - 10^{-3}$ s, is due to consecutive discharges generated by high frequency (in the kHz range) sinusoidal or pulsed applied voltages [6, 7]. Therefore, during the post-discharge phases of the order of tens of microseconds, not only time scales are very different from those during discharges of a few tens of nanoseconds, but a complete different physics is taking place. Then, to the rapid multi-scale configuration during discharges, we have to add other rather slower multi-scale phenomena in the post-discharge, such as recombination of charged species, heavy-species chemistry, diffusion, gas heating and convection. Therefore, it is very challenging to simulate accurately physics of the plasma/flow interaction due to synergy effects between consecutive discharge/post-discharge phases.

In most numerical models of streamer discharges, the motion of electrons and ions is governed by drift-diffusion equations coupled to Poisson's equation. Early simulation studies were limited to simplified situations where streamer was considered as a cylinder of constant radius [8, 9, 10, 11], in which charged particle densities are assumed to be constant along the radial extension of the streamer: the *1.5D model* approach. Spatio-temporal evolution of charged particle densities is then solved only along one spatial dimension in the direction of propagation, whereas electric field has to be calculated in

^{*}This research was supported by a fundamental project grant from ANR (French National Research Agency - ANR Blancs): *Séchelles* (project leader S. Descombes), and by a DIGITEO RTRA project: *MUSE* (project leader M. Massot). Authors express special thanks to Christian Tenaud (LIMSI-CNRS) for providing the basis of the multiresolution kernel of MR CHORUS, code developed for compressible Navier-Stokes equations (Déclaration d'Invention DI 03760-01).

*Corresponding author

Email addresses: max.duarte@em2c.ecp.fr (Max Duarte), zbona@physics.muni.cz (Zdeněk Bonaventura), marc.massot@em2c.ecp.fr (Marc Massot), anne.bourdon@em2c.ecp.fr (Anne Bourdon), sdescomb@unice.fr (Stéphane Descombes), tdumont@math.univ-lyon1.fr (Thierry Dumont)

¹Ph.D. grant from Mathematics (INSMI) and Engineering (INSIS) Institutes of CNRS and supported by INCA project (National Initiative for Advanced Combustion - CNRS - ONERA - SAFRAN).

²Support of Ecole Centrale Paris is gratefully acknowledged for several month stay of Z. Bonaventura at Laboratory EM2C as visiting Professor.

two dimensions using the so-called *disc method*, based on direct integration of analytical results. Since first 2D streamer simulations using Poisson's equation resolution were performed [12], this approach has been extensively used for streamer simulations in 2D [3, 13, 14, 15, 16, 17, 18] and extended also to 3D [19, 20, 21, 22].

Being aware of the complexity of fully coupled resolution of these modeling equations, a decoupling strategy is usually adopted, which considers an independent and successive numerical resolution of Poisson's equation with fixed charge distribution, and of drift-diffusion equations with fixed electric field during each decoupling time step. These computations might be performed explicitly in time with standard first and eventually second order schemes [23, 24]. In these cases, time steps are limited for the sake of stability by the various characteristic times scales (Courant, ionization, dielectric relaxation), whereas accuracy of simulations is assumed to be given by resolution of the fastest physical time scale. In order to somehow overcome dielectric relaxation limitation, semi-implicit approaches were developed [25, 26, 27], based on a predictive approximation of space charge ahead in time for electric field computation, even though other time scale constraints remain. This gain of stability allows important improvements in terms of computational efficiency but accuracy of simulations becomes rather difficult to quantify.

In this work, a numerical study is conducted in order to build a second order explicit in time decoupling scheme for the resolution of electric field and electron and ion densities. A lower order and embedded method is taken into account to dynamically compute decoupling time steps that guarantee an accurate description with error control of the global physical coupling. At this stage, the only limiting time scale is dielectric relaxation for stability reasons. In a second level, drift-diffusion equations are solved using a Strang second order operator splitting scheme in order to guarantee global order of the strategy [28, 29]. This time integration scheme considers high order dedicated methods during each splitting time step, which is dynamically adapted by an error control procedure [30], in order to overcome time step limitations related to reaction, diffusion and convection phenomena [31].

Both electric field and density resolutions are performed on an adapted mesh obtained by a spatial multiresolution method, based on Harten's pioneering work [32] and further developed in [33]; being aware of the interest of adaptive mesh techniques for spatial multi-scale phenomena with locally steep spatial gradients. In fact, grid adaptive techniques for 2D structured meshes were already used [23, 4, 20] and extensions to 3D have been also proposed [20, 19] for streamer simulations. However, one of the main advantages of multiresolution approach is that it is based on wavelet representation techniques and an error of the spatial approximation can be then mathematically estimated. Consequently, an effective error control is achieved for both time and space resolution of multi-scale phenomena under study.

The performance of the method is first evaluated for a propagating discharge problem with the multi-scale features previously discussed, for which the various simulating parameters are studied. With the physical configuration settled, a 1.5D streamer model is adopted in order to obtain an electric field resolution strategy based on direct computations derived from analytical expressions, suitable for adapted finite volume discretizations [34]. In a second step, a more complex physical configuration is considered for the simulation of repetitively pulsed discharges, for which a time-space adaptive method is required to efficiently overcome high multi-scale features in order to fully describe the

various physical phenomena.

The paper is organized as follows: in section 2, we present the physical configuration and modeling equations. The numerical strategy is presented in section 3, in which the second order adaptive time integration technique is detailed along with the resolution of drift-diffusion equations and electric field, as well as the spatial multiresolution adaptive procedure. Numerical illustration are summarized in section 4 for two configurations given by single propagating and multi-pulsed discharges. We end in the last part with some concluding remarks and prospects on future developments and applications.

2. Model formulation

In this work, we consider positive streamer discharges in air at atmospheric pressure in a point-to-plane geometry, shown in Figure 1. The tip of the anode is placed 1 cm from the planar cathode and the radius of curvature of the anode is $324 \mu\text{m}$.

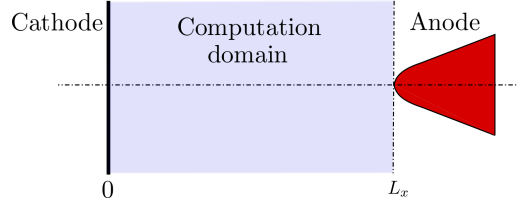


Figure 1: Computational domain for the studied point-to-plane geometry.

The most common and effective model to study streamer dynamics is based on the following drift-diffusion equations for electrons and ions, coupled with Poisson's equation [35, 36]:

$$\left. \begin{aligned} \partial_t n_e - \partial_{\mathbf{x}} \cdot n_e \mathbf{v}_e - \partial_{\mathbf{x}} \cdot (D_e \partial_{\mathbf{x}} n_e) &= n_e \alpha |\mathbf{v}_e| - n_e \eta |\mathbf{v}_e| + n_e n_p \beta_{ep} + n_n \gamma, \\ \partial_t n_p + \partial_{\mathbf{x}} \cdot n_p \mathbf{v}_p - \partial_{\mathbf{x}} \cdot (D_p \partial_{\mathbf{x}} n_p) &= n_e \alpha |\mathbf{v}_e| - n_e n_p \beta_{ep} + n_n n_p \beta_{np}, \\ \partial_t n_n - \partial_{\mathbf{x}} \cdot n_n \mathbf{v}_n - \partial_{\mathbf{x}} \cdot (D_n \partial_{\mathbf{x}} n_n) &= n_e \eta |\mathbf{v}_e| - n_n n_p \beta_{np} - n_n \gamma, \end{aligned} \right\} \quad (1)$$

$$\varepsilon_0 \partial_{\mathbf{x}}^2 V = -q_e (n_p - n_n - n_e), \quad (2)$$

where $\mathbf{x} \in \mathbb{R}^d$, n_i is the density of species i (e: electrons, p: positive ions, n: negative ions), V is the electric potential, $\mathbf{v}_i = \mu_i \mathbf{E}$ (\mathbf{E} being the electric field) is the drift velocity. D_i and μ_i , are diffusion coefficient and absolute value of mobility of charged species i , q_e is the absolute value of electron charge, and ε_0 is permittivity of free space. α is the impact ionization coefficient, η stands for electron attachment on neutral molecules, β_{ep} and β_{np} accounts respectively for electron-positive ion and negative-positive ion recombination, and γ is the detachment coefficient.

Electric field \mathbf{E} and potential V are related by

$$\mathbf{E} = -\partial_{\mathbf{x}} V, \quad (3)$$

and thus, Poisson's equation (2) is deduced from

$$\varepsilon_0 \partial_{\mathbf{x}} \cdot \mathbf{E} = q_e (n_p - n_n - n_e). \quad (4)$$

All the coefficients of the model are assumed to be functions of the local reduced electric field E/N_{gas} , where E is the electric field magnitude and N_{gas} is the air neutral density. For test studies presented in this paper, transport parameters for air are taken from [37]; detachment and attachment coefficients, respectively from [38] and [39]; and other reaction rates, also from [37].

In simulations of positive streamer discharges in air at atmospheric pressure without any preionization, the photoionization term is crucial to produce seed charges in front of the streamer head and then ensure the streamer propagation [24]. However, in repetitive discharges, [40] and recently [41] have shown that even at low frequency, significant amount of seed charges from previous discharges may be present in the interelectrode gap. In this work, we have neglected the photoionization source term and considered discharge conditions with a preionization background to ensure a stable propagation of the discharge without impacting the main discharge characteristics [15, 40, 18, 42].

3. Construction of the numerical strategy

In this section, we introduce a new numerical technique for multi-scale gas discharge simulations, based on a second order decoupled resolution of electric field and drift-diffusion equations of electrons and ions, with self-adapted decoupling time steps with error control. On the one hand, drift-diffusion equations are solved using a dedicated Strang time operator splitting scheme for multi-scale phenomena. And on the other hand, electric field is computed based on a parallel computing method, specially conceived for the configuration under study in 1.5D geometry. Both resolutions are conducted on a dynamic adaptive mesh using spatial multiresolution transformation with error control of the adapted spatial representation.

3.1. Second order adaptive time integration strategy

Let us write the semi-discretized equations (1) and (4) in the following way just for analysis purposes

$$\left. \begin{aligned} d_t \psi &= \Psi(\psi, \phi), \\ 0 &= \Phi(\psi, \phi), \end{aligned} \right\} \quad (5)$$

for $t > t_0$, where $\psi : \mathbb{R} \rightarrow \mathbb{R}^{N \times m}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}^{N \times d}$, stand respectively for the spatial discretization of (n_e, n_p, n_n) , i.e. $m = 3$, and of \mathbf{E} over N points. Supposing that all functions are sufficiently differentiable in all their variables and using the Taylor expansion of the true solution, one can write after some time Δt from initial time t_0 ,

$$\psi(t_0 + \Delta t) = \psi_0 + \Delta t \Psi(\psi_0, \phi_0) + \frac{\Delta t^2}{2} [\partial_\psi \Psi \Psi + \partial_\phi \Psi d_t \phi]_{t=t_0} + \mathcal{O}(\Delta t^3), \quad (6)$$

with $\psi_0 = \psi(t_0)$, $\phi_0 = \phi(t_0)$.

A second order in time resolution of system (5) must then verify (6) locally for each Δt . However, as it was stated before, solving simultaneously (1) and (2) (or (4)), or equivalently (5), involves important numerical difficulties, considering for instance the different nature of equations (1) and (2) (or (4)). Therefore, a decoupled approach is often used in which one aims at solving drift-diffusion equations and electric field independently; this amounts to solve

$$d_t \tilde{\psi} = \Psi(\tilde{\psi}, \phi^*), \quad t \in]t_0, t_0 + \Delta t], \quad (7)$$

with fixed $\phi^* = \phi(t^*)$, $t^* \in [t_0, t_0 + \Delta t]$ and $\tilde{\psi}(t_0) = \psi_0$.

The most common technique considers $t^* = t_0$, that is, to previously compute electric field at t_0 from $\Phi(\psi_0, \phi_0) = 0$, and then solve (7) with $\phi^* = \phi_0$. This can be interpreted as a standard first order operator splitting method that yields an approximation of order 1, $\tilde{\psi}_1(t)$, of exact solution, $\psi(t)$, based on classical numerical analysis results, obtained by confronting (6) with

$$\tilde{\psi}_1(t_0 + \Delta t) = \psi_0 + \Delta t \Psi(\psi_0, \phi_0) + \frac{\Delta t^2}{2} [\partial_\psi \Psi \Psi]_{t=t_0} + \mathcal{O}(\Delta t^3). \quad (8)$$

Same result follows for $\tilde{\phi}_1(t_0 + \Delta t)$ computed out of $\Phi(\tilde{\psi}_1(t_0 + \Delta t), \tilde{\phi}_1(t_0 + \Delta t)) = 0$ or equivalently, out of its explicit representation $\tilde{\phi}_1(t_0 + \Delta t) = \Upsilon(\tilde{\psi}_1(t_0 + \Delta t))$, assuming Lipschitz condition:

$$\|\Upsilon(\psi) - \Upsilon(\psi^*)\| \leq L \|\psi - \psi^*\|. \quad (9)$$

Considering now any $t^* \in [t_0, t_0 + \Delta t]$ into (7), the only second order solution, $(\tilde{\psi}_2(t), \tilde{\phi}_2(t))$, will be given by resolution of (7) with $\phi^* = \phi_{\frac{1}{2}}$ for $t^* = t_0 + \Delta t/2$, for which

$$\tilde{\psi}_2(t_0 + \Delta t) = \psi_0 + \Delta t \Psi(\psi_0, \phi_{\frac{1}{2}}) + \frac{\Delta t^2}{2} [\partial_\psi \Psi \Psi]_{\psi \rightarrow \psi_0, \phi_{\frac{1}{2}}} + \mathcal{O}(\Delta t^3), \quad (10)$$

where

$$\begin{aligned} \Psi(\psi_0, \phi_{\frac{1}{2}}) &= \Psi\left(\psi_0, \phi\left(t_0 + \frac{\Delta t}{2}\right)\right), \\ &= \Psi\left(\psi_0, \phi_0 + \frac{\Delta t}{2} d_t \phi|_{t=t_0} + \mathcal{O}(\Delta t^2)\right), \\ &= \Psi(\psi_0, \phi_0) + \frac{\Delta t}{2} [\partial_\phi \Psi d_t \phi]_{t=t_0} + \mathcal{O}(\Delta t^2), \end{aligned} \quad (11)$$

and hence,

$$\tilde{\psi}_2(t_0 + \Delta t) = \psi_0 + \Delta t \Psi(\psi_0, \phi_0) + \frac{\Delta t^2}{2} [\partial_\psi \Psi \Psi + \partial_\phi \Psi d_t \phi]_{t=t_0} + \mathcal{O}(\Delta t^3); \quad (12)$$

and

$$\tilde{\phi}_2(t_0 + \Delta t) = \Upsilon(\tilde{\psi}_2(t_0 + \Delta t)). \quad (13)$$

Nevertheless, second order approximation, $\tilde{\psi}_2(t)$, is based on the previous knowledge of $\phi_{\frac{1}{2}} = \phi(t_0 + \Delta t/2)$, and thus, of $\psi(t_0 + \Delta t/2)$. In order to overcome this difficulty, one can solve (7) with $\phi^* = \tilde{\phi}_1(t_0 + \Delta t/2) = \Upsilon(\tilde{\psi}_1(t_0 + \Delta t/2))$, that is, computing first $\tilde{\psi}_1(t_0 + \Delta t/2)$ with first order method. In particular, this does not change previous order estimates as it follows from

$$\begin{aligned} \psi(t_0 + \Delta t) - \tilde{\psi}_2(t_0 + \Delta t) &= \frac{\Delta t^2}{2} [\partial_\phi \Psi d_t(\phi - \tilde{\phi}_1)]_{t=t_0} + \mathcal{O}(\Delta t^3), \\ &= \frac{\Delta t^2}{2} [\partial_\phi \Psi \partial_\psi \Upsilon d_t(\psi - \tilde{\psi}_1)]_{t=t_0} + \mathcal{O}(\Delta t^3), \\ &= \mathcal{O}(\Delta t^3). \end{aligned} \quad (14)$$

Taking into account both methods,

$$\begin{pmatrix} \tilde{\psi}_1(t_0 + \Delta t) \\ \tilde{\phi}_1(t_0 + \Delta t) \end{pmatrix} = \mathcal{T}_1^{\Delta t} \begin{pmatrix} \psi_0 \\ \phi_0 \end{pmatrix}, \quad \begin{pmatrix} \tilde{\psi}_2(t_0 + \Delta t) \\ \tilde{\phi}_2(t_0 + \Delta t) \end{pmatrix} = \mathcal{T}_2^{\Delta t} \begin{pmatrix} \psi_0 \\ \phi_0 \end{pmatrix}, \quad (15)$$

we perform computations with second order scheme $\mathcal{T}_2^{\Delta t}$, which uses embedded and lower order scheme $\mathcal{T}_1^{\Delta t/2}$, as it was previously detailed. An adaptive time step strategy is then implemented in order to control the accuracy of computations. It is based on a local error estimate, dynamically computed at the end of each decoupling time step Δt , given by

$$\|\mathcal{T}_2^{\Delta t}(\psi_0, \phi_0)^t - \mathcal{T}_1^{\Delta t}(\psi_0, \phi_0)^t\| \approx \mathcal{O}(\Delta t^2). \quad (16)$$

Therefore, for a given accuracy tolerance $\eta\mathcal{T}$,

$$\|\mathcal{T}_2^{\Delta t}(\psi_0, \phi_0)^t - \mathcal{T}_1^{\Delta t}(\psi_0, \phi_0)^t\| < \eta\mathcal{T} \quad (17)$$

must be verified in order to accept current computation with Δt , while new time step is calculated by

$$\Delta t^{\text{new}} = \Delta t \sqrt{\frac{\eta\mathcal{T}}{\|\mathcal{T}_2^{\Delta t}(\psi_0, \phi_0)^t - \mathcal{T}_1^{\Delta t}(\psi_0, \phi_0)^t\|}}. \quad (18)$$

3.2. Resolution of drift-diffusion equations

We consider now the numerical resolution of equations (1), that one can write in a general format of convection-reaction-diffusion system of equations

$$\left. \begin{aligned} \partial_t \mathbf{u} - \partial_{\mathbf{x}} (\mathbf{F}(\mathbf{u}) + \mathbf{D}(\mathbf{u}) \partial_{\mathbf{x}} \mathbf{u}) &= \mathbf{f}(\mathbf{u}), \quad t > t_0, \\ \mathbf{u}(t_0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}), \quad t = t_0, \end{aligned} \right\} \quad (19)$$

where $\mathbf{F}, \mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $\mathbf{u} : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^m$, with a tensor of order $d \times d \times m$ as diffusion matrix $\mathbf{D}(\mathbf{u})$. In particular, $\mathbf{u} = (n_e, n_p, n_n)^t$ with $m = 3$ in this study.

System (19) corresponds to problem (7) for a fixed electric field, and it is solved during decoupling Δt into \mathcal{T}_2 (or \mathcal{T}_1) scheme, using a Strang time operator scheme with dedicated high order time integrators on a dynamic adaptive mesh, based on a strategy introduced in [28]. This resolution is briefly detailed in what follows.

3.2.1. Time operator splitting

An operator splitting procedure allows to consider dedicated solvers for the reaction part which is decoupled from other physical phenomena like convection, diffusion or both, for which there also exist dedicated numerical methods. These dedicated methods chosen for each subsystem are then responsible for dealing with the fast scales associated with each one of them, in a separate manner, while the reconstruction of the global solution by the splitting scheme should guarantee an accurate description with error control of the global physical coupling, without being related to the stability constraints of the numerical resolution of each subsystem.

Considering problem (19) and in order to remain consistent with second order \mathcal{T}_2 scheme, a second order Strang scheme is implemented [43]

$$\mathcal{S}^{\Delta t_s}(\mathbf{u}_0) = \mathcal{R}^{\Delta t_s/2} \mathcal{D}^{\Delta t_s/2} \mathcal{C}^{\Delta t_s} \mathcal{D}^{\Delta t_s/2} \mathcal{R}^{\Delta t_s/2}(\mathbf{u}_0), \quad (20)$$

where operators \mathcal{R} , \mathcal{D} , \mathcal{C} indicate respectively the independent resolution of reaction, diffusion and convection problems with splitting time step, Δt_s , taken inside overall decoupling time step, $\Delta t_s \leq \Delta t$. Usually, for propagating reaction waves where for instance, the speed of propagation is much slower than some of the chemical scales, the fastest scales are not directly related to the global physics of the phenomenon, and thus, larger splitting time steps might be considered [28, 29]. Nevertheless, order reductions may then appear due to short-life transients associated to fast variables and in these cases, it has been proved in [44] that better performances are expected while ending the splitting scheme by operator \mathcal{R} or in a more general case, the part involving the fastest time scales of the phenomenon.

Resolution of (19) should be precise enough to guarantee theoretical estimates given in Section 3.1. Therefore, an adaptive splitting time step strategy, based on a local error estimate at the end of each splitting Δt_s , is also implemented in order to control the accuracy of computations [31]. In this context, a second, embedded and lower order Strang splitting method $\tilde{\mathcal{S}}^{\Delta t_s}$ was developed by [30], which allows to dynamically calculate a local error estimate that should verify

$$\|\mathcal{S}^{\Delta t_s}(\mathbf{u}_0) - \tilde{\mathcal{S}}^{\Delta t_s}(\mathbf{u}_0)\| \approx \mathcal{O}(\Delta t_s^2) < \eta_{\text{split}}, \quad (21)$$

in order to accept current computation with Δt_s , and thus, new splitting time step is given by

$$\Delta t_s^{\text{new}} = \min \left(\Delta t_s \sqrt{\frac{\eta_{\text{split}}}{\|\mathcal{S}^{\Delta t_s}(\mathbf{u}_0) - \tilde{\mathcal{S}}^{\Delta t_s}(\mathbf{u}_0)\|}}, t_0 + \Delta t - \hat{t} \right), \quad (22)$$

with $\eta_{\text{split}} \leq \eta_{\mathcal{T}}$ and $\hat{t} = \sum_i \Delta t_{s_i}$ while $\hat{t} \in]t_0, t_0 + \Delta t]$.

The choice of suitable time integration methods to approximate numerically \mathcal{R} , \mathcal{D} and \mathcal{C} during each Δt_s is mandatory not only to guarantee the theoretical framework of the numerical analysis but also to take advantage of particular features of each independent subproblem. A new operator splitting for reaction-diffusion systems was recently introduced [28, 29], which considers on the one hand, a high fifth order, A -stable, L -stable method like Radau5 [45], based on implicit Runge-Kutta schemes for stiff ODEs, that solves with a local cell by cell approach the reaction term: a system of stiff ODEs without spatial coupling in a splitting context. And on the other hand, another high fourth order method like ROCK4 [46], based on explicit stabilized Runge-Kutta schemes which features extended stability domains along the negative real axis, very appropriate for diffusion problems because of the usual predominance of negative real eigenvalues. Both methods incorporate adaptive time integration tools, similar to (18) and (22), in order to control accuracy for given η_{Radau5} and η_{ROCK4} , chosen such that $\eta_{\text{Radau5}} < \eta_{\text{split}}$ and $\eta_{\text{ROCK4}} < \eta_{\text{split}}$.

An explicit high order in time and space one step monotonicity preserving scheme OSMP [47] is used as convective scheme. It combines monotonicity preserving constraints for non-monotone data to avoid extrema clipping, with TVD features to prevent spurious oscillations around discontinuities or sharp spatial gradients. Classical CFL stability restrictions are though imposed during each splitting time step Δt_s . Notice that the overall combination of explicit treatment of spatial phenomena as convection and diffusion, with local implicit integration of stiff reaction implies important savings in computing time and memory resources. For the reaction, local treatment plus adaptive time stepping

allow to discriminate cells of high reactive activity in the neighborhood of the localized wavefront, saving as a consequence an enormous quantity of integration time [29].

3.2.2. Mesh refinement technique

Regarding problem (19), we are concerned with the propagation of reacting wavefronts, hence important reactive activity as well as steep spatial gradients are localized phenomena. This implies that if we consider the resolution of reactive problem, a considerable amount of computing time is spent on nodes that are practically at (partial) equilibrium. Moreover, there is no need to represent these quasi-stationary regions with the same spatial discretization needed to describe the reaction front, so that drift and diffusion problems might also be solved over a smaller number of nodes. An adapted mesh obtained by a multiresolution process which discriminates the various space scales of the phenomenon, turns out to be a very convenient solution to overcome these difficulties [28, 48]. Furthermore, in plasma applications, resolution of Poisson's equation takes usually $\sim 80\%$ of computing time; thus, important savings are achieved with mesh adaptive techniques, as a consequence of strong reduction of computing cells.

In practice, if one considers a set of nested spatial grids from the coarsest to the finest one, a multiresolution transformation allows to represent a discretized function as values on the coarsest grid plus a series of local estimates at all other levels of such nested grids. These estimates correspond to the wavelet coefficients of a wavelet decomposition obtained by inter-level transformations, and retain the information on local regularity when going from a coarse to a finer grid. Hence, the main idea is to use the decay of the wavelet coefficients to obtain information on local regularity of the solution: lower wavelet coefficients are associated to local regular spatial configurations and vice-versa. The basis of this strategy is presented in the following; for further details on adaptive multiresolution techniques, we refer to the books of [49] and [50].

3.2.3. Basis of multiresolution representation

To simplify the presentation let us consider nested finite volume discretizations of (19) with only one component, $m = 1$. For $l = 0, 1, \dots, L$ from the coarsest to the finest grid, we have then regular disjoint partitions (cells) $(\Omega_\gamma)_{\gamma \in S_l}$ of an open subset $\Omega \subset \mathbb{R}^d$, such that each Ω_γ , $\gamma \in S_l$, is the union of a finite number of cells Ω_μ , $\mu \in S_{l+1}$, and thus, S_l and S_{l+1} are consecutive embedded grids. We denote $\mathbf{U}_l := (u_\gamma)_{\gamma \in S_l}$ as the representation of u on the grid S_l where u_γ represents the cell-average of $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ in Ω_γ ,

$$u_\gamma := |\Omega_\gamma|^{-1} \int_{\Omega_\gamma} u(t, \mathbf{x}) \, d\mathbf{x}. \quad (23)$$

Data at different levels of discretization are related by two inter-level transformations which are defined as follows:

1. The *projection* operator P_{l-1}^l , which maps \mathbf{U}_l to \mathbf{U}_{l-1} . It is obtained through exact averages computed at the finer level by

$$u_\gamma = |\Omega_\gamma|^{-1} \sum_{|\mu|=|\gamma|+1, \Omega_\mu \subset \Omega_\gamma} |\Omega_\mu| u_\mu, \quad (24)$$

where $|\gamma| := l$ if $\gamma \in S_l$. As far as grids are nested, this projection operator is *exact* and *unique* [49].

2. The *prediction* operator P_l^{l-1} , which maps \mathbf{U}_{l-1} to an approximation $\hat{\mathbf{U}}_l$ of \mathbf{U}_l . There is an infinite number of choices to define P_l^{l-1} , but we impose at least two basic constraints [33]:
- (a) The prediction is local, i.e., \hat{u}_μ for a given Ω_μ depends on a set of values u_γ in a finite stencil R_μ surrounding Ω_μ , where $|\mu| = |\gamma| + 1$.
 - (b) The prediction is consistent with the projection in the sense that

$$|\Omega_\gamma|u_\gamma = \sum_{|\mu|=|\gamma|+1, \Omega_\mu \subset \Omega_\gamma} |\Omega_\mu| \hat{u}_\mu; \quad (25)$$

$$\text{i.e., } P_{l-1}^l \circ P_l^{l-1} = Id.$$

With these operators, we define for each cell Ω_μ the prediction error or *detail* as the difference between the exact and predicted values:

$$d_\mu := u_\mu - \hat{u}_\mu, \quad (26)$$

or in terms of inter-lever operations:

$$d_\mu = u_\mu - P_{|\mu|}^{|\mu|-1} \circ P_{|\mu|-1}^{|\mu|} u_\mu; \quad (27)$$

and then, construct a *detail vector* \mathbf{D}_l as shown in [33] in order to get a one-to-one correspondence from expressions (26) and (25):

$$\mathbf{U}_l \longleftrightarrow (\mathbf{U}_{l-1}, \mathbf{D}_l). \quad (28)$$

Hence, by iteration of this decomposition, we finally obtain a multi-scale representation of \mathbf{U}_L in terms of $\mathbf{M}_L = (\mathbf{U}_0, \mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}_L)$:

$$\mathcal{M} : \mathbf{U}_L \longmapsto \mathbf{M}_L, \quad (29)$$

where the details computed with (27) stand for the wavelet coefficients in a wavelet basis.

One of the main interests of carrying on such a wavelet decomposition is that this new representation defines a whole set of regularity estimators all over the spatial domain and thus, a data compression might be achieved by deleting cells whose detail verifies

$$|d_\mu| < \varepsilon_l, \quad l = |\mu|, \quad \varepsilon_l = 2^{\frac{d}{2}(l-L)} \eta_{\text{MR}}, \quad (30)$$

where η_{MR} is the threshold value for the finest level L .

An important theoretical result is that if we denote by $\mathbf{V}_L^n := (v_\lambda^n)_{\lambda \in \mathcal{S}_L}$, the solution fully computed on the finest grid, and denote by \mathbf{U}_L^n , the solution reconstructed on the finest grid that used adaptive multiresolution (keeping in mind that the time integration was really performed on a compressed representation of \mathbf{U}^n); then, for a fixed time $T = n\Delta t$, it can be shown that [32, 33]

$$\|\mathbf{U}_L^n - \mathbf{V}_L^n\|_{L^2} \propto n\eta_{\text{MR}}. \quad (31)$$

3.3. Computation of electric field

In this part, we are concerned with the resolution of electric field according to \mathcal{T}_2 (or \mathcal{T}_1) scheme at some fixed time for a given distribution of charges (n_e, n_p, n_n) considering 1.5D model. Computation is also performed on the adapted mesh obtained by multiresolution analysis. A description of the resolution is presented in what follows.

3.3.1. Discretization of computation domain

According to Figure 1, computational domain is limited by planar cathode at $x = 0$ and the tip of hyperbolic anode at $x = L_x$. Anode is not included in the domain. We consider streamers of fixed radius R_s along the axis of symmetry. Computational domain is divided into n_x cells of different size corresponding to the multiresolution adapted mesh, with faces x_f^i , where $i \in [0, n_x]$ and cell centers x_c^j , where $j \in [1, n_x]$. Face x_f^0 corresponds to the position of the cathode and $x_f^{n_x}$ corresponds to the position of the tip of the anode. Therefore for each cell x_c^i , there is its left face x_f^{i-1} , and its right face x_f^i . For each cell x_c^j we define a width $w_j = x_f^j - x_f^{j-1}$ (see Figure 2).

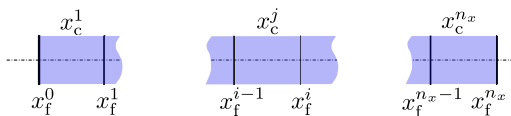


Figure 2: Definition of the grid: cell centers located at x_c^j , cell faces located at x_f^i . Domain is bounded by faces x_f^0 (cathode) and $x_f^{n_x}$ (tip of anode).

3.3.2. Resolution of electric field in the 1.5D model

To determine the electric field in the streamer during its propagation, the space charge of the streamer is considered as a set of finite cylinders of width w_j , bounded by cell faces x_f^{j-1} and x_f^j . As the computational domain is bounded by conducting electrodes of fixed potential, each volume charge ρ_j creates infinite series of image charges. Then the principle of superposition is used to sum individual contributions from all the cylindrical space charges in the domain, their image charges, and the Laplacian electric field (computed based on classical results [51]). An advantage of this approach dwells in the fact that electric field contributions from individual cylinders can be expressed analytically in a simple form and determination of the electric field in each point of the domain can be done in parallel.

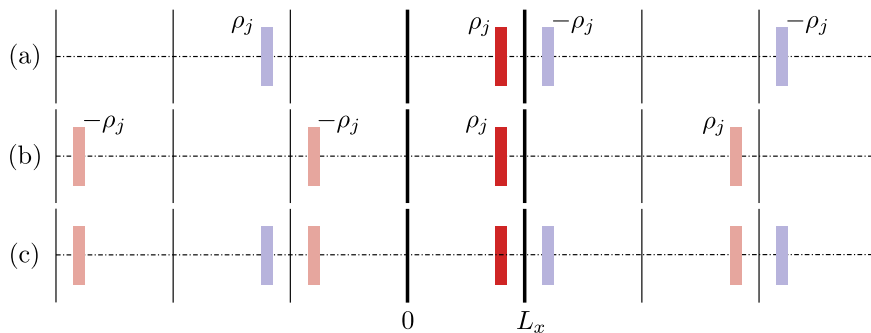


Figure 3: Image charges up to third order: (a) charge ρ_j is first mirrored behind the anode ($x = L_x$), (b) charge ρ_j is first mirrored behind the cathode ($x = 0$), (c) charge ρ_j and its images.

For simplicity, planar geometry of electrodes is assumed for charge mirroring. In such

a case, a mirrored cylinder charge conserves its geometrical shape and value of charge as shown in Figure 3. For volume charge ρ_j centered at x_c^j , there exist image charges of the first order with charge $-\rho_j$ at $x = 2L_x - x_c^j$ mirrored through the anode, see Figure 3a, and at $x = -x_c^j$ mirrored through the cathode, see Figure 3b. And for each of these image charges there exist higher order image charges of opposite signs and so forth. All the image charges of ρ_j up to order three are depicted on Figure 3c.

Integrating the generalized Coulomb's law [52] and using principle of superposition, we find that the cylinder charges of cells $j \in [1, n_x]$ of width w_j , radius R_s , charged with densities ρ_j , and the Laplacian electric field $E_L(x_f^i)$ at x_f^i [51], create the electric field E at position x_f^i as follows:

$$E(x_f^i) = E_L(x_f^i) + \sum_{j=1}^{n_x} s \frac{\rho_j w_j}{2\varepsilon_0} \left(1 - \frac{w_j + 2h_{i,j}}{\sqrt{h_{i,j}^2 + R_s^2} + \sqrt{(h_{i,j} + w_j)^2 + R_s^2}} \right), \quad (32)$$

where

$$h_{i,j} = \begin{cases} x_f^i - x_f^j & \text{for } i \geq j \\ x_f^{j-1} - x_f^i & \text{for } i < j \end{cases} \quad \text{and} \quad s = \begin{cases} +1 & \text{for } i \geq j \\ -1 & \text{for } i < j \end{cases}$$

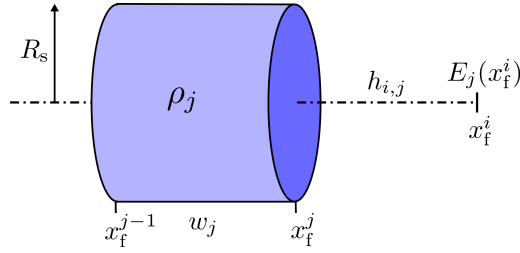


Figure 4: Charged cylinder considered to compute the electric field in the 1.5D model.

Positive sign of s accounts for electric field calculated on the right from the position of cylinder and vice versa. The same formula applies for image charges, but appropriate sign of the charge has to be carefully taken into account according to Figure 3. In particular, in a shared memory computing environment, a straightforward parallelization is accomplished for equation (32), in which each core solves successively the electric field on one single position x_f^i , and where neither synchronization stages nor data exchange are needed among nodes.

Note that for $R_s \rightarrow \infty$ (infinite plane charges), Equation (32) gives exact electric field

$$E_{\text{inf}} = \sum_{j=1}^{n_x} s \frac{\rho_j w_j}{2\varepsilon_0}.$$

For finite radius R_s the solution (32) is valid only on the axis of the discharge, but when applied to a discharge of a small radius, the electric field will vary only negligibly over the cross section of the discharge. This approach is expected to be more accurate for any finite radius than any discretization of Poisson's equation [8].

4. Numerical results

In this section, we present numerical illustrations of the proposed numerical strategy for simulations of positive streamers using 1.5D model in point-to-plane geometry. First, we will consider discharge propagation in constant applied voltage for which different features of the numerical strategy, e.g., error estimates, data compression values and computing time, are discussed and allows one to select simulation parameters. Then, the potential of the method is fully exploited for a more complex configuration of repetitive discharges generated by high frequency pulsed applied voltages followed by long time scale relaxation, for which a complete physical description of discharge and post-discharge phases is achieved.

4.1. Propagation of positive streamer with constant applied voltage

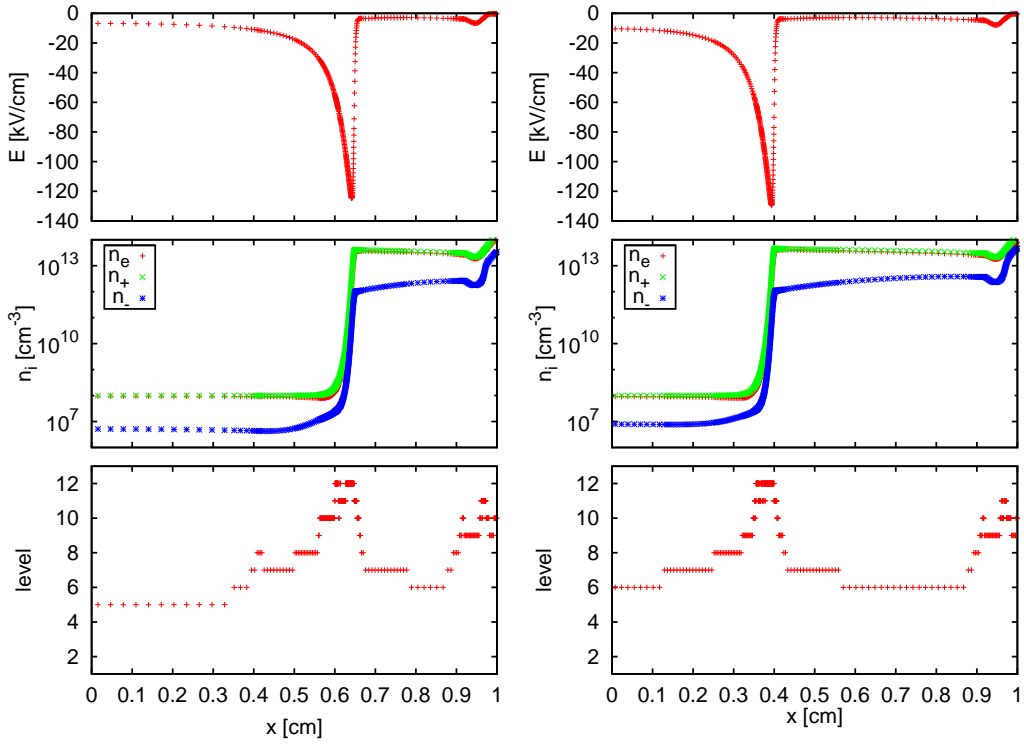


Figure 5: Positive streamer propagation at $t = 6$ ns (left) and $t = 10$ ns (right). Top: electric field; middle: charged species density; and bottom: grid levels. Finest grid: 4096, $\eta_T = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}$.

We consider a point-to-plane geometry with a 1 cm gap between the point electrode and the plane, and a constant applied voltage of 13 kV. For all following simulations, discharge is initiated by placing a neutral plasma cloud with a Gaussian distribution close to the point electrode. Initial distributions of electrons and ions are then given by

$$n_{e,p}(x)|_{t=0} = n_{\text{max}} \exp\left(-\frac{(x-c)^2}{w^2}\right) + n_0, \quad n_n = 0, \quad (33)$$

where $w = 0.027$ cm, $c = 1$ cm, $n_{\max} = 10^{14}$ cm $^{-3}$, and preionization $n_0 = 10^8$ cm $^{-3}$. There are no negative ions as initial condition. Streamer radius is set to $R_s = 0.05$ cm to have a typical electric field magnitude in the streamer head of 120 kV/cm [53]. Homogeneous Neumann boundary conditions were considered for drift-diffusion equations.

Two instances of discharge propagation are shown in Figure 5, for 12 nested grids equivalent to 4096 cells on the finest grid, $L = 12$, and threshold values of $\eta_{\mathcal{T}} = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}$; spatial refinement takes place only where it is required. Fine tolerances were chosen in all cases for the solvers, $\eta_{\text{Radau5}} = \eta_{\text{ROCK4}} = 10^{-7}$, to guarantee accurate integrations. For all the simulation cases, detail in each cell is taken as the minimum of the details computed according to (27) for each variable, where the prediction operator is a polynomial interpolation of order 3, performed on normalized $\log \mathbf{u}$ of the density variables in order to properly discriminate the streamer heads from the high ionized plasma channel; this scale guarantees a correct spatial representation of the phenomenon as seen in Figure 5 for the density profiles.

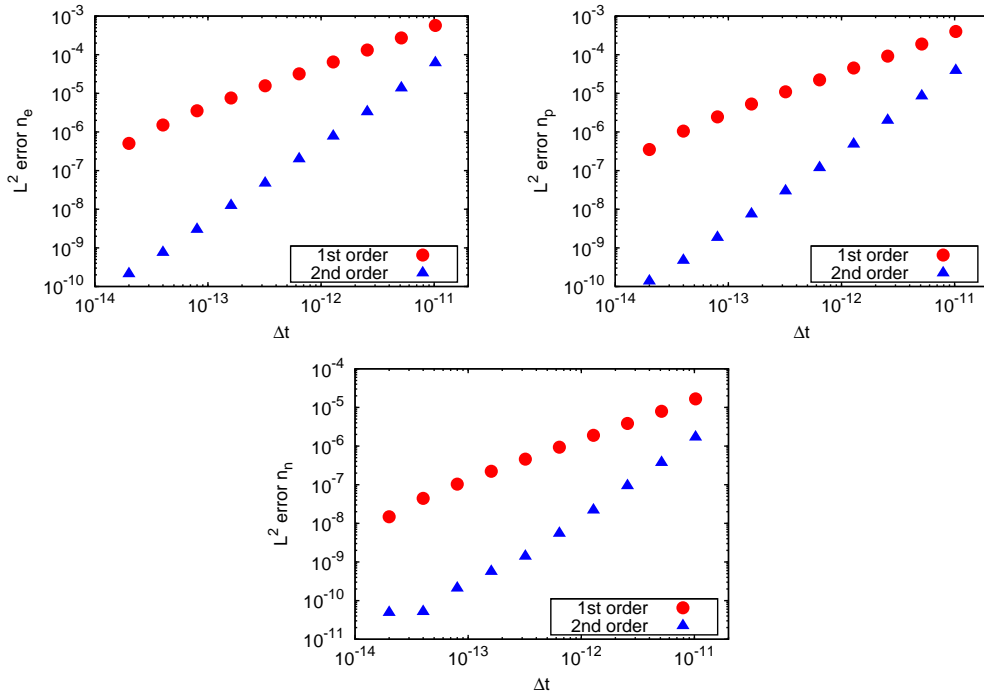


Figure 6: L^2 errors between reference and \mathcal{T}_1 (first order) and \mathcal{T}_2 (second order) solutions for various decoupling time steps Δt on an uniform grid of 4096 cells. Top: electron (left) and positive ions (right); and bottom: negative ions.

In order to perform an analysis of the numerical results, we define as *reference* solution a fine resolution with \mathcal{T}_2 scheme that considers a fixed decoupling time step, $\Delta t = 10^{-14}$ s and an uniform grid of 4096 cells. For this *reference* solution, the memory requirements are acceptable and the simulation is still feasible, but it requires about 14 days of real simulation time on an AMD Opteron 6136 Processor cluster, while running the electric

field computation in parallel on 16 CPU cores. In this case, the direct integration of the electric field represents 80% of total CPU time (about 3.2 s) per time step.

First of all, we must verify previous order estimates for \mathcal{T}_1 and \mathcal{T}_2 schemes given in section (3.1). We consider as initial condition, reference solution at $t = 10$ ns. In order to only evaluate errors coming from the decoupling techniques, \mathcal{T}_1 and \mathcal{T}_2 , we consider a fine splitting time step, $\Delta t_s = 10^{-14}$ s, to solve drift-diffusion problem (1) and an uniform grid; then, we solve (5) with both schemes for several decoupling time steps Δt_i , and calculate L^2 error between first/second order and reference solutions after $t = 2^{10}\Delta t_s = 1.024 \times 10^{-11}$ s. Figure 6 shows results with $\Delta t_i = 2^i \Delta t_s$, where $i \in [1, 10]$, which clearly verify first and second order in time for \mathcal{T}_1 and \mathcal{T}_2 schemes, respectively, and prove important gains in accuracy for same time steps.

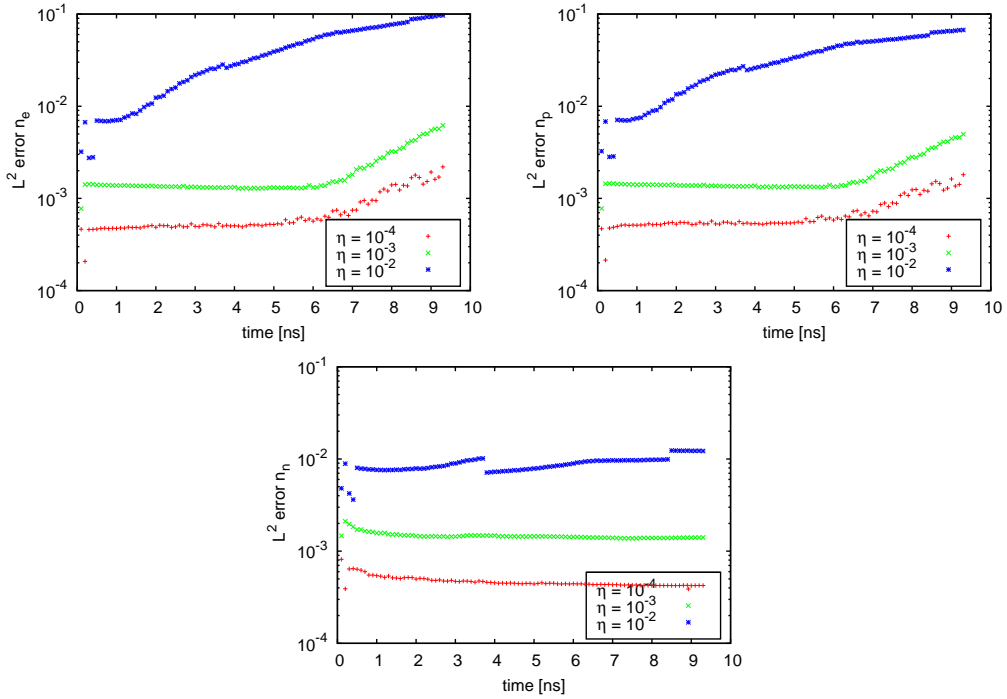


Figure 7: Time evolution of L^2 errors between reference and adapted solutions with $\eta = \eta_{\mathcal{T}} = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}, 10^{-3}$, and 10^{-2} , and 4096 cells corresponding to finest discretization. Top: electron (left) and positive ions (right); and bottom: negative ions.

Figure 7 shows time evolution of L^2 error between time-space adapted and reference solutions for various threshold values, $\eta_{\mathcal{T}} = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}, 10^{-3}$, and 10^{-2} for each variable. These are rather approximations of the error since reference and adapted solutions are not evaluated exactly at the same time, and therefore, they are sometimes slightly shifted of about $\sim 10^{-14} - 10^{-13}$ s. In these tests, decoupling time steps Δt were limited by dielectric relaxation time step, Δt_{DR} , after noticing an important amount of rejections of computed time steps according to (18), whenever $\Delta t \gtrsim 1.5 \times \Delta t_{DR}$. Otherwise, Δt is dynamically chosen in order to locally satisfy the required accuracy, but

does not show important variations considering the steady propagating phenomenon.

In Figure 8, we can see the adapted grid corresponding to each configuration. Representations of electric field and densities show that for $\eta\tau = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-2}$, the streamer front propagates faster than in the reference case, with a slightly higher peak of electric field in the front. On the other hand, for $\eta\tau = \eta_{\text{split}} = \eta_{\text{MR}} \leq 10^{-3}$, we observe a quite good agreement between adapted and reference resolutions.

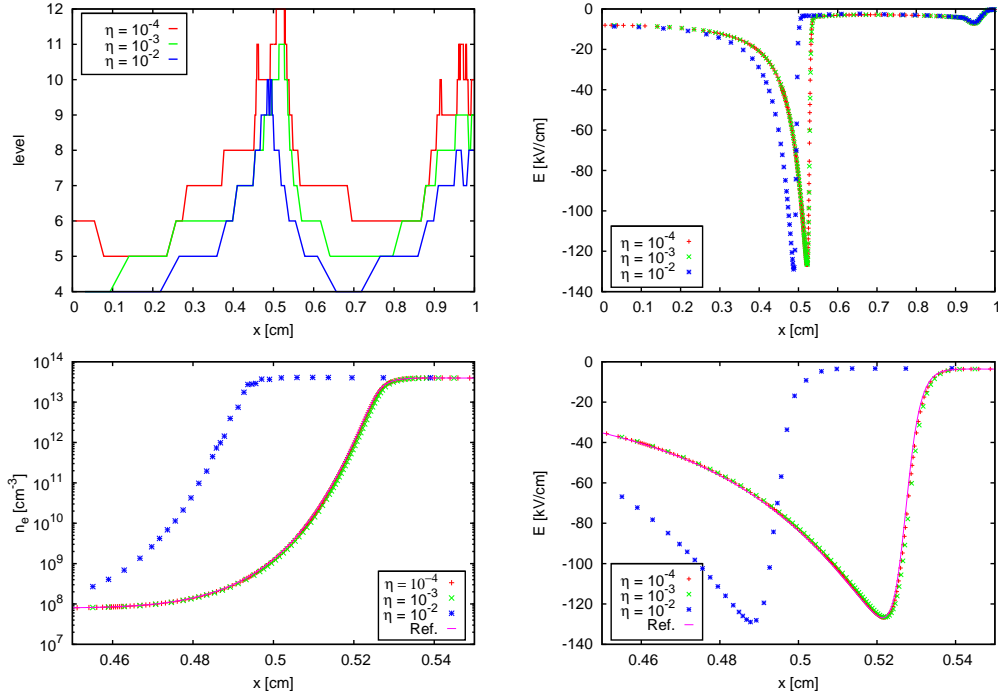


Figure 8: Top: adapted grids (left) and electric fields (right) at $t = 8$ ns corresponding to 4096 cells as finest discretization, and $\eta = \eta\tau = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}$, 10^{-3} , and 10^{-2} . Bottom: zoom on electron distributions (left) and electric field (right) for same parameters and reference solution.

We consider now an accurate enough resolution $\eta\tau = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}$ and investigate the influence of the number of grids, that is, the finest spatial discretization at level L that is taken into account. Figure 9 shows the adapted grids for $L = 10$, 11 and 12, respectively equivalent to 1024, 2048 and 4096 cells in the finest grid; and a close-up of the corresponding electric fields in the discharge head at $t = 8$ ns. We see that for this level of tolerance, the streamer front propagates slightly slower than the reference case for $L = 10$, whereas $L = 11$ give already good resolutions compared to the reference solution and to $L = 12$; in particular, higher L would need lower tolerances in order to retain regions at the finest level; this is already the case for $L = 13$ (equivalent to 8192 cells). Therefore, a finest level $L = 11$ with 2048 cells seems to be an appropriate choice for this level of accuracy.

Table 1 summarizes the number of cells in the adapted grid ($\#AG$) at time $t = 8$ ns, and the corresponding data compression (DC) defined as the percentage of active cells

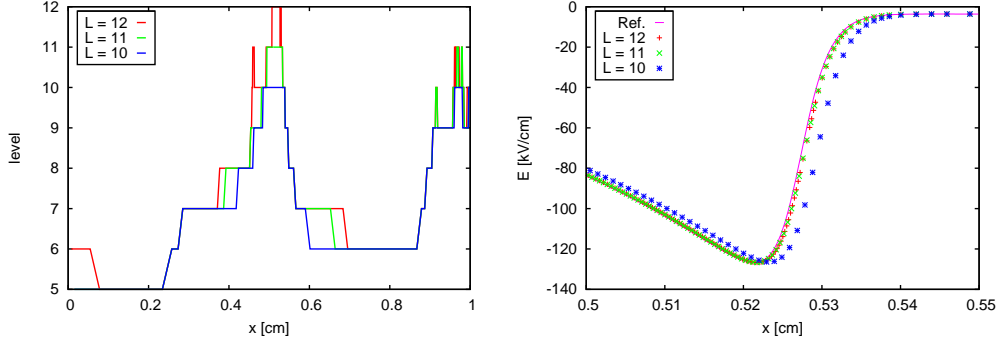


Figure 9: Adapted grids (left) and electric fields (right) at $t = 8$ ns, for finest spatial discretization $L = 10, 11$ and 12 , $\eta_{\mathcal{T}} = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}$, and reference solution.

with respect to the equivalent number of cells for the finest discretization, in this case 2048 for $L = 11$. For this propagating case, data compression remains of the same order during time simulation interval. The CPU computing times correspond to a time domain of study of $t \in [0, 10]$ ns computed by one sole CPU core. If we consider for example total computing time for $L = 11$ and tolerances $\eta_{\mathcal{T}} = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}$, it is ~ 44 times less expensive with respect to a resolution on an uniform grid with 2048 cells and $\eta_{\mathcal{T}} = \eta_{\text{split}} = 10^{-4}$ (CPU time of 8552 s). This is quite reasonable, taking into account that computing time for electric field resolution is proportional to at least $\mathcal{O}(N^2)$ for N computing cells, after (32).

Table 1: Number of cells in the adapted grid ($\#AG$) and data compression (DC) at time $t = 8$ ns, CPU computing time for $t \in [0, 10]$ ns, $L = 11$, and various tolerances $\eta = \eta_{\mathcal{T}} = \eta_{\text{split}} = \eta_{\text{MR}}$.

η	$\#AG$	$DC\%$	CPU(s)
10^{-6}	724	35.35	1360
10^{-5}	421	20.56	517
10^{-4}	263	12.84	193
10^{-3}	138	6.74	66
10^{-2}	70	3.42	24

In conclusion, in this section we have shown that the numerical strategy developed can be efficiently applied to simulate the propagation of highly nonlinear ionizing waves as streamer discharges. An important reduction of computing time results from significant data compression with still accurate resolutions. In addition, this study allows to properly tune the various simulation parameters in order to guarantee fine resolution of more complex configurations, based on the time-space accuracy control capabilities of the method.

4.2. Simulation of multi-pulsed discharges

In this section, we analyze the performance of the proposed numerical strategy on simulation of nanosecond repetitively pulsed discharges [6, 54]. Applied voltage profile for this type of discharges is a high voltage *pulse* followed by zero voltage *relaxation* phase. Typical pulse duration is $\sim 10^{-8}$ s, while relaxation phase takes over $\sim 10^{-4}$ s. The detailed experimental study of these discharges in air has shown that the cumulative effect of repeated pulsing achieves a steady-state behavior [54]. In following illustrations, we choose a pulse duration of $T_p = 15$ ns, which is approximately equal to the time that is needed for the discharge to cross the interelectrode gap. The rise time considers the time needed to go from zero to maximum voltage and it is set to $T_r = 2$ ns. Pulse repetition period is set to $T_P = 10^{-4}$ s, equal to 10 kHz of repetition frequency, a typical value used in experiments [6]. We model voltage pulse P by using sigmoid functions

$$P(t, s, r, p) = 1 - \sigma(-t, -s, r) - \sigma(t, s + p, r), \quad (34)$$

with

$$\sigma(t, s, r) = \frac{1}{1 + \exp(-8(t - s)/r)}, \quad (35)$$

for time t , where s indicates when pulse starts; r , the rise time; and p , pulse duration; $t, s, r, p \in [0, T_P]$. With maximum applied voltage V_{\max} , applied voltage $V(t)$ is computed by

$$V(t) = V_{\max} \cdot P\left(t - \left\lfloor \frac{t}{T_P} \right\rfloor \cdot T_P, T_r, T_r, T_p\right). \quad (36)$$

In this section, we assume as initial condition a homogeneous preionization consisting only of ions with density 10^9 cm^{-3} ; electrons are considered to have a low homogeneous background of 10^1 cm^{-3} . This initial configuration is based on a rough guess of the final steady-state value after some initial discharges [40, 41].

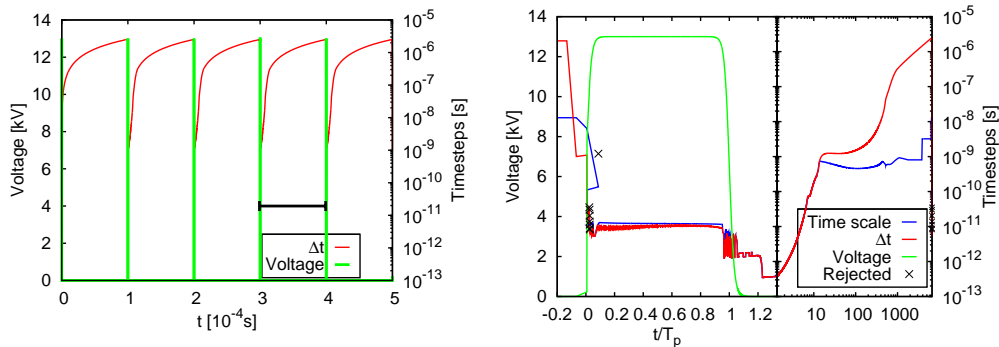


Figure 10: Time evolution of applied voltage and time steps for multi-pulse simulation for first six pulses (left) and the fourth one (right) with subsequent relaxation. Rejected time steps are marked with black crosses, while minimum time scales correspond to blue line.

We set tolerances to $\eta_{\mathcal{T}} = \eta_{\text{split}} = \eta_{\text{MR}} = 10^{-4}$ and consider $L = 11$ grid levels, equivalent to 2048 cells in the finest grid. Figure 10 shows time evolution of decoupling time

steps and applied voltage for the first six pulses, even though simulation was performed for 100 pulses, that is $t \in [0, 10^{-2}]$ s. This simulation took over 8h44m while running electric field computation in parallel on 6 CPU cores of the same AMD Opteron 6136 Processor cluster; this gives an average of 5.24 minutes per pulse period. Second figure corresponds to fourth pulse for which the steady state of the periodic phenomenon was already reached and almost the same numerical performance is reproduced during the rest of computations. Time steps are about $\sim 10^{-11}$ s during pulses, then increase from $\sim 10^{-12}$ s up to about $\sim 10^{-6}$ s during a period ~ 6000 times longer, and they are computed according to required tolerance. Solving this problem for such different scales with a constant time step is out of question and even a standard strategy that considers minimum of all time scales would limit considerably the efficiency of the method as it is shown in the representation. In this particular case, dielectric relaxation is the governing time scale during discharge as in the previous case with constant applied voltage, whereas the post-discharge phase is sometimes ruled by diffusive or convective CFL, or by ionization time scale, with all security factors and CFL conditions set to one in Figure 10.

Computation is initialized with a time step included in the pulse duration. Nevertheless, after each relaxation phase, since the new time step is computed based on the previous one according to (18), this new time step will surely skip the next pulse. In order to avoid this, each time we get into a new period, that is $\lfloor t/T_P \rfloor$ changes, we initialize time step with $\Delta t = 0.5T_r = 1$ ns; this time step is obviously rejected as seen in Figure 10, as well as the next ones, until we are able to retrieve the right dynamics of the phenomenon for the required accuracy tolerance. No other intervention is needed neither for modeling parameters nor for numerical solvers in order to automatically adapt time step to describe the various time scales of the phenomenon within a prescribed accuracy.

Figure 11 represents time evolution of data compression which ranges from $\sim 2\%$ up to $\sim 16\%$ during each pulse period. Regarding only electric field resolution with the same time integration strategy, grid adaptation involves resolutions ~ 39 to ~ 2500 times faster, based on a really rough estimate for $\mathcal{O}(N^2)$ operations.

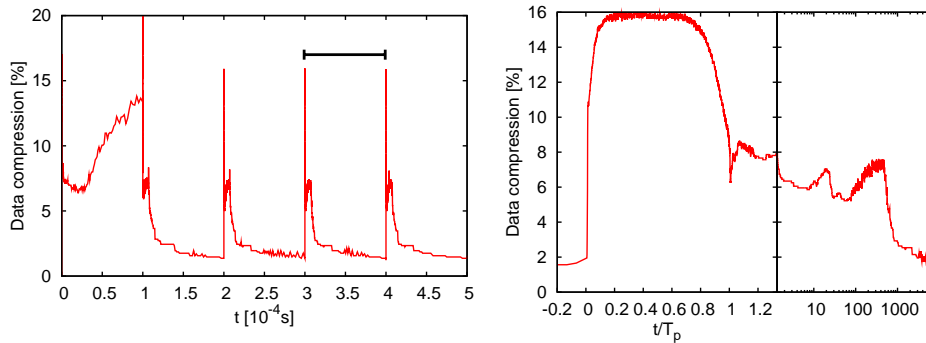


Figure 11: Time evolution of data compression for multi-pulse simulation for first six pulses (left) and the fourth one (right) with subsequent relaxation.

In this last part, we will describe physical details of the studied phenomenon obtained by simulation, in order to evaluate the numerical performance of the strategy for the

different multi-scale phases. Figure 12 presents the discharge dynamics for the first period. First, we observe at $t = 10$ ns after the beginning of the pulse, the propagation of a positive streamer in the gap. In section 4.1, a preionization of positive ions and electrons was used to ensure the positive streamer propagation. In this section, seed electrons ahead of the streamer front are created as the front propagates by detachment of negative ions present initially. We note that at 15 ns, which corresponds to almost the end of the plateau before the decrease of the applied voltage, the discharge has crossed ~ 0.75 cm of the 1 cm gap. As a consequence, during the voltage decrease and at the beginning of the relaxation phase where the applied voltage is zero, there is a remaining space charge and steep gradients of charged species densities in the gap. Then for $t = 50$ ns, Figure 12 shows that the electric field in the discharge is almost equal to zero except in a small area where steep gradients of the electric field with peak values of 30 V/cm are observed. We have checked that this area corresponds to the location of the streamer head at the end of its propagation. We note that in the post-discharge, electrons are attaching (timescale of the order of 100 ns) and then at $t = 50$ ns, the density of positive ions is almost equal to the density of negative ions in the whole gap. At $t = 99972$ ns, the densities of charged species have significantly decreased due to charged species recombination. However, it is interesting to note that the location of the previous streamer head can still be observed at the same location as at $t = 50$ ns, but with much smaller gradients of charged species densities and a very small electric field. This final state is the initial condition of the second pulse with a non-uniform axial preionization with positive and negative ions.

After a few repetitive pulses, we have observed that the discharge dynamics reached a steady-state behavior as observed in the experiments. To show the characteristics of the discharge when the steady-state is reached, Figure 13 shows the discharge dynamics of the 100th period. The sequence of images is the same as in Figure 12. At the end of the 99th pulse, we have observed that the axial distribution of charged species in the gap is uniform and that the level of preionization is $5 \times 10^{10} \text{ cm}^{-3}$ positive and negative ions and 10^4 cm^{-3} electrons. We note that 10 ns after the beginning of the 100th pulse the propagation of the discharge is faster than for the first pulse. This faster propagation is mostly due to the higher preionization level of positive and negative ions in the gap in comparison of the first voltage pulse. We note that for the 100th pulse, 15 ns after the beginning of the pulse the discharge has almost crossed the interelectrode gap and then during the relaxation phase, there is no remaining space charge in the whole gap. Consequently, 50 ns after the beginning of the 100th pulse, axial distributions of all charged species are uniform. As already observed for the first pulse, at 50 ns after the beginning of the voltage pulse most electrons have attached and then, the density of positive ions is almost equal to the density of negative ions in the whole gap. We note that the corresponding electric field distribution is not uniform at 50 ns, but no steep gradients are observed as for the first voltage pulse. At $t = 999998$ ns, that is to say at the end of the 100th period, we note that a very low electric field is obtained in the gap. An axially uniform distribution of charges is obtained with $5 \times 10^{10} \text{ cm}^{-3}$ for positive and negative ions and 10^4 cm^{-3} for electrons, which was the initial condition of the 100th pulse, and then this demonstrates the existence of the steady-state behavior of nanosecond repetitively pulsed discharges.

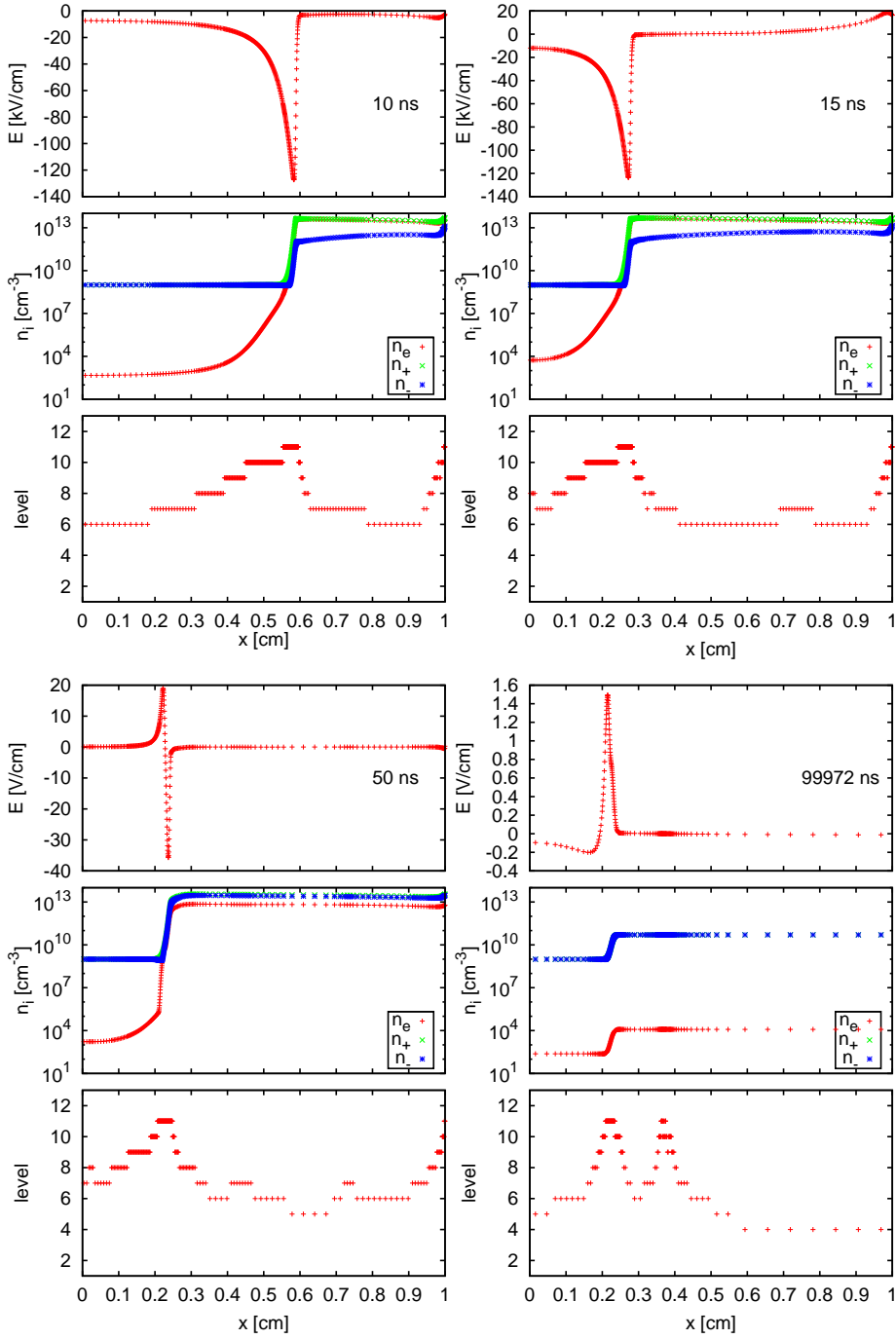


Figure 12: First period of pulsed discharge. Top: propagation of discharge in the domain at $t = 10$ ns after the beginning of the pulse (left); and at $t = 15$ ns (right). Bottom: relaxation on the short time scale $t = 50$ ns; and end of the relaxation phase after $t = 99972$ ns (right).

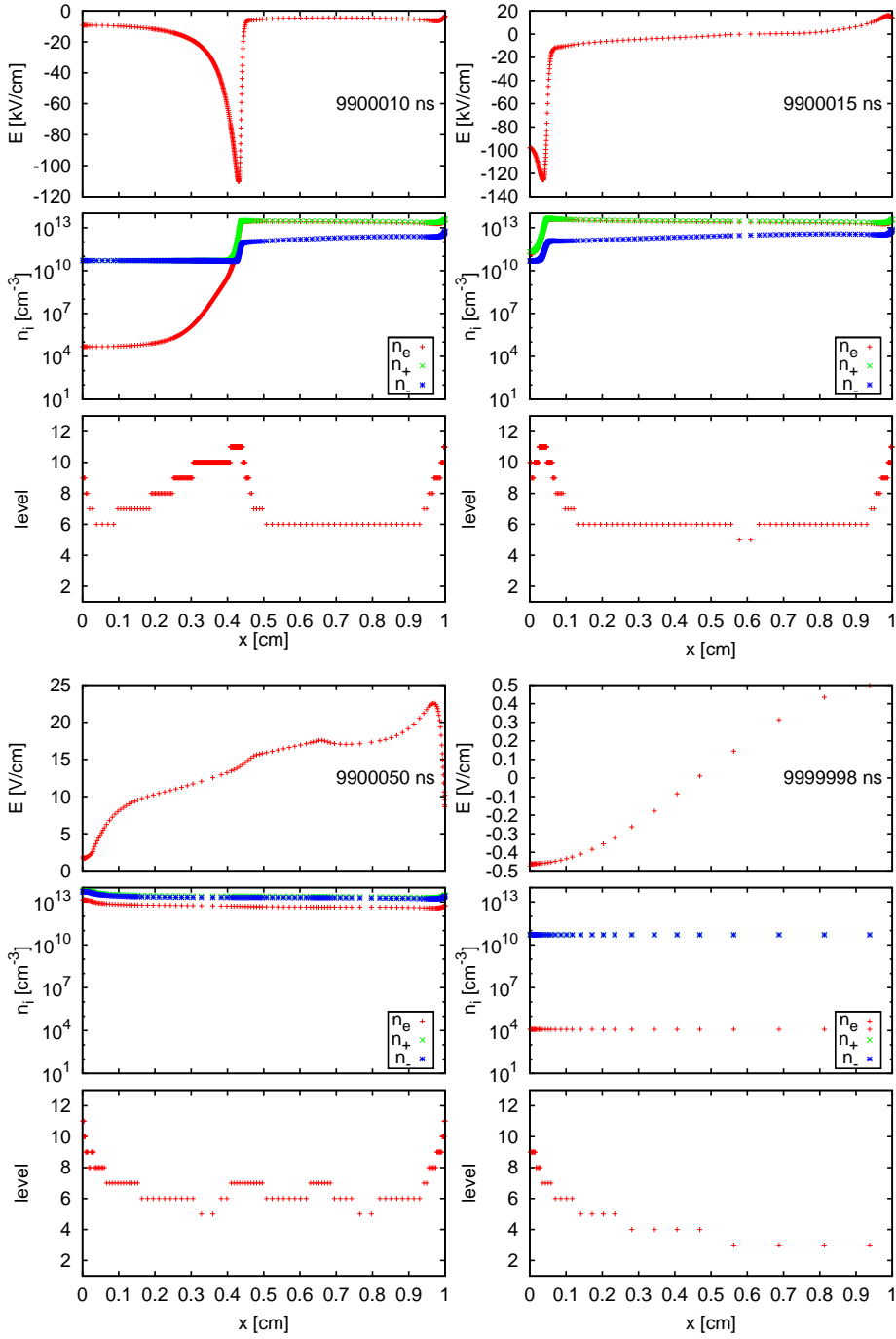


Figure 13: Steady state of pulsed discharge (last period). Top: propagation of discharge in the domain at $t = 9900010$ ns after the beginning of the pulse (left); and at $t = 9900015$ ns (right). Bottom: relaxation on the short time scale $t = 9900050$ ns; and end of the relaxation phase after $t = 9999998$ ns (right).

5. Conclusions

The present work proposes a new numerical strategy for multi-scale discharge simulations. It is based on an adaptive second order time integration strategy that allows to discriminate time scales related features of the phenomena, given a required level of accuracy of computations. Compared to a standard procedure for which accuracy is guaranteed by considering time steps of the order of the fastest scale, the control error approach implies on the one hand, an effective accurate resolution independent of the fastest physical time scale, and on the other hand, an important improvement of computational efficiency whenever the required time steps go beyond standard stability constraints. The latter is a direct consequence of the self-adapting time step strategy for the resolution of drift-diffusion equations which considers splitting time steps not limited by stability constraints for reaction, diffusion and convection phenomena. So far, global decoupling time steps are limited by dielectric relaxation stability constraint but with second order accuracy; nevertheless, we have demonstrated that time steps are rather chosen based on an accuracy criterion. Besides, if a technique such as a semi-implicit approach is implemented, the same ideas of the proposed strategy remain valid.

An adaptive multiresolution technique was also proposed in order to provide error control of the spatial adapted representation. Numerical results have proven a natural coupling between time and space accuracy requirements and how the set of time-space accuracy tolerances tunes the precise description of the time-space multi-scale phenomenon. As a consequence, numerical results on time-space adaptation strategy for multi-pulsed discharge configurations prove first, that this kind of multi-scale phenomena, previously out of reach, can be successfully simulated with conventional computing resources; and second, that a consistent physical description is achieved for a broad spectrum of space and time scales as well as different physical scenarios.

In this work, we focused on a 1.5D model, which in terms of spatial representation implies a 1D configuration, in order to first evaluate the numerical performance of the strategy. However, the dimension of the problem will only have an influence on the computational efficiency measurements but not on any space-time accuracy or stability aspects. At this stage of development, the same numerical strategy can be coupled to a multi-dimensional Poisson's equation solver, even for adapted grid configurations as developed recently in [23, 4, 20]. Finally, an important amount of work is still in progress concerning programming features such as data structures, optimized routines and parallelization strategies. On the other hand, numerical analysis of theoretical aspects is also underway to extend and further improve the proposed numerical strategy. These issues constitute particular topics of our current research.

References

References

- [1] E. M. van Veldhuizen (Ed.), *Electrical Discharges for Environmental Purposes: Fundamentals and Applications*, Nova Science, New York, 2000.
- [2] A. Fridman, A. Chirokov, A. Gutsol, Non-thermal atmospheric pressure discharges, *J. Phys. D: Appl. Phys.* 38 (2005) R1–R24.
- [3] P. A. Vitello, B. M. Penetrante, J. N. Bardsley, Simulation of negative-streamer dynamics in nitrogen, *Phys. Rev. E* 49 (1994) 5574–5598.

- [4] T. Unfer, J.-P. Boeuf, F. Rogier, F. Thivet, Multi-scale gas discharge simulations using asynchronous adaptive mesh refinement, *Computer Physics Communications* 181 (2) (2010) 247 – 258.
- [5] U. Ebert, F. Brau, G. Derks, W. Hundsdorfer, C.-Y. Kao, C. Li, A. Luque, B. Meulenbroek, S. Nijdam, V. Ratushnaya, L. Schäfer, S. Tanveer, Multiple scales in streamer discharges, with an emphasis on moving boundary approximations, *Nonlinearity* 24 (1) (2011) C1–C26.
- [6] G. Pilla, D. Galley, D. Lacoste, F. Lacas, D. Veynante, C. Laux, Stabilization of a turbulent premixed flame using a nanosecond repetitively pulsed plasma, *IEEE Trans. Plasma Sci.* 34 (6, Part 1) (2006) 2471–2477.
- [7] D. F. Opaits, M. N. Shneider, R. B. Miles, A. V. Likhanskii, S. O. Macheret, Surface charge in dielectric barrier discharge plasma actuators, *Phys. Plasmas* 15 (7) (2008) 073505.
- [8] A. Davies, F. Jones, C. Evans, Electrical breakdown of gases - Spatio-temporal growth of ionization in fields distorted by space charge, *Proc. Royal Soc. London S. A-Math. and Phys. Scien.* 281 (1385) (1964) 164–183.
- [9] A. Davies, C. Davies, C. Evans, Computer simulation of rapidly developing gaseous discharges, *Proc. of the Institution of Electrical Engineers-London* 118 (6) (1971) 816–823.
- [10] I. Abbas, P. Bayle, A critical analysis of ionizing wave-propagation mechanisms in breakdown, *J. Phys. D: Appl. Phys.* 13 (6) (1980) 1055–1068.
- [11] R. Morrow, Theory of negative corona in oxygen, *Phys. Rev. A* 32 (3) (1985) 1799–1809.
- [12] S. K. Dhali, P. F. Williams, Two-dimensional studies of streamers in gases, *J. Appl. Phys.* 62 (1987) 4696–4707.
- [13] N. Y. Babaeva, G. V. Naidis, Dynamics of positive and negative streamers in air in weak uniform electric fields, *IEEE Trans. Plasma Sci.* 25 (1997) 375–379.
- [14] A. A. Kulikovskiy, The role of photoionization in positive streamer dynamics, *J. Phys. D: Appl. Phys.* 33 (2000) 1514–1524.
- [15] S. V. Pancheshnyi, S. M. Starikovskaia, A. Y. Starikovskii, Role of photoionization processes in propagation of cathode-directed streamer, *J. Phys. D: Appl. Phys.* 34 (2001) 105–115.
- [16] M. Arrayás, U. Ebert, W. Hundsdorfer, Spontaneous branching of anode-directed streamers between planar electrodes, *Phys. Rev. Lett.* 88 (2002) 174502(R).
- [17] S. Celestin, Z. Bonaventura, B. Zeghondy, A. Bourdon, P. Segur, The use of the ghost fluid method for Poisson’s equation to simulate streamer propagation in point-to-plane and point-to-point geometries, *J. Phys. D: Appl. Phys.* 42 (6) (2009) 065203.
- [18] A. Bourdon, Z. Bonaventura, S. Celestin, Influence of the pre-ionization background and simulation of the optical emission of a streamer discharge in preheated air at atmospheric pressure between two point electrodes, *Plasma Sources Sci. Technol.* 19 (3) (2010) 034012.
- [19] D. S. Nikandrov, R. R. Arslanbekov, V. I. Kolobov, Streamer simulations with dynamically adaptive cartesian mesh, *IEEE Trans. Plasma Sci.* 36 (4) (2008) 932–933.
- [20] S. Pancheshnyi, P. Segur, J. Capeillere, A. Bourdon, Numerical simulation of filamentary discharges with parallel adaptive mesh refinement, *J. Comp. Phys.* 227 (13) (2008) 6574–6590.
- [21] A. Luque, U. Ebert, W. Hundsdorfer, Interaction of streamer discharges in air and other oxygen-nitrogen mixtures, *Phys. Rev. Lett.* 101 (7) (2008) 075005. doi:10.1103/PhysRevLett.101.075005.
- [22] L. Papageorgiou, A. C. Metaxas, G. E. Georghiou, Three-dimensional numerical modelling of gas discharges at atmospheric pressure incorporating photoionization phenomena, *J. Phys. D: Appl. Phys.* 44 (4) (2011) 045203.
- [23] C. Montijn, W. Hundsdorfer, U. Ebert, An adaptive grid refinement strategy for the simulation of negative streamers, *J. Comput. Phys.* 219 (2) (2006) 801–835.
- [24] A. Bourdon, V. P. Pasko, N. Y. Liu, S. Celestin, P. Segur, E. Marode, Efficient models for photoionization produced by non-thermal gas discharges in air based on radiative transfer and the Helmholtz equations, *Plasma Sources Sci. Technol.* 16 (3) (2007) 656–678.
- [25] P. L. G. Ventzek, T. J. Sommerer, R. J. Hoekstra, M. J. Kushner, Twodimensional hybrid model of inductively coupled plasma sources for etching, *Appl. Phys. Lett.* 63 (5) (1993) 605 – 607.
- [26] P. Colella, M. R. Dorr, D. D. Wake, A conservative finite difference method for the numerical solution of plasma fluid equations, *J. Comput. Phys.* 149 (1) (1999) 168 – 193.
- [27] G. J. M. Hagelaar, G. M. W. Kroesen, Speeding up fluid models for gas discharges by implicit treatment of the electron energy source term, *J. Comp. Phys.* 159 (1) (2000) 1–12.
- [28] M. Duarte, M. Massot, S. Descombes, C. Tenaud, T. Dumont, V. Louvet, F. Laurent, New resolution strategy for multi-scale reaction waves using time operator splitting, space adaptive multiresolution and dedicated high order implicit/explicit time integrators., Submitted to *SIAM J. Sci. Comput.* (2010) available on HAL (<http://hal.archives-ouvertes.fr/hal-00457731>).
- [29] T. Dumont, M. Duarte, S. Descombes, M.-A. Dronne, M. Massot, V. Louvet, Simulation of human

- ischemic stroke in realistic 3D geometry: A numerical strategy, Submitted to *Bulletin of Math. Biology* (2010) available on HAL (<http://hal.archives-ouvertes.fr/hal-00546223>).
- [30] S. Descombes, M. Duarte, T. Dumont, V. Louvet, M. Massot, Adaptive time splitting method for multi-scale evolutionary PDEs., *Confluentes Mathematici* (2011) (to appear).
 - [31] M. Duarte, M. Massot, S. Descombes, T. Dumont, Adaptive time-space algorithms for the simulation of multi-scale reaction waves, *Proceedings of FVCA6* (2011) (to appear).
 - [32] A. Harten, Multiresolution algorithms for the numerical solution of hyperbolic conservation laws, *Comm. Pure and Applied Math.* 48 (1995) 1305–1342.
 - [33] A. Cohen, S. Kaber, S. Müller, M. Postel, Fully adaptive multiresolution finite volume schemes for conservation laws, *Mathematics of Comp.* 72 (2003) 183–225.
 - [34] D. Bessieres, J. Paillol, A. Bourdon, P. Segur, E. Marode, A new one-dimensional moving mesh method applied to the simulation of streamer discharges, *J. Phys. D: Appl. Phys.* 40 (21) (2007) 6559–6570.
 - [35] N. Y. Babaeva, G. V. Naidis, Two-dimensional modelling of positive streamer dynamics in non-uniform electric fields in air, *J. Phys. D: Appl. Phys.* 29 (1996) 2423–2431.
 - [36] A. A. Kulikovskiy, Positive streamer between parallel plate electrodes in atmospheric pressure air, *J. Phys. D: Appl. Phys.* 30 (1997) 441–450.
 - [37] R. Morrow, J. J. Lowke, Streamer propagation in air, *J. Phys. D: Appl. Phys.* 30 (1997) 614–627.
 - [38] M. S. Benilov, G. V. Naidis, Modelling of low-current discharges in atmospheric-pressure air taking account of non-equilibrium effects, *J. Phys. D: Appl. Phys.* 36 (15) (2003) 1834–1841.
 - [39] I. A. Kossyi, A. Y. Kostinsky, A. A. Matveyev, V. P. Silakov, Kinetic scheme of the non-equilibrium discharge in nitrogen-oxygen mixtures, *Plasma Sources Sci. Technol.* 1 (3) (1992) 207–220.
 - [40] S. V. Pancheshnyi, Role of electronegative gas admixtures in streamer start, propagation and branching phenomena, *Plasma Sources Sci. Technol.* 14 (4) (2005) 645–653.
 - [41] G. Wormeester, S. Pancheshnyi, A. Luque, S. Nijdam, U. Ebert, Probing photo-ionization: simulations of positive streamers in varying N₂:O₂ mixtures, *J. Phys. D: Appl. Phys.* 43 (50) (2010) 505201.
 - [42] S. Celestin, Study of the streamer dynamics in air at atmospheric pressure, PhD thesis, Ecole Centrale Paris, France, 2008.
 - [43] G. Strang, On the construction and comparison of difference schemes, *SIAM J. Numer. Anal.* 5 (1968) 506–517.
 - [44] S. Descombes, M. Massot, Operator splitting for nonlinear reaction-diffusion systems with an entropic structure: Singular perturbation and order reduction, *Numer. Math.* 97 (4) (2004) 667–698.
 - [45] E. Hairer, G. Wanner, *Solving ordinary differential equations II*, 2nd Edition, Springer-Verlag, Berlin, 1996, Stiff and differential-algebraic problems.
 - [46] A. Abdulle, Fourth order Chebyshev methods with recurrence relation, *SIAM J. Sci. Comput.* 23 (2002) 2041–2054.
 - [47] V. Daru, C. Tenaud, High order one-step monotonicity-preserving schemes for unsteady compressible flow calculations, *J. Comput. Phys.* 193 (2) (2004) 563 – 594.
 - [48] M. Duarte, M. Massot, S. Descombes, C. Tenaud, T. Dumont, V. Louvet, F. Laurent, New resolution strategy for multi-scale reaction waves using time operator splitting and space adaptive multiresolution: Application to human ischemic stroke., *ESAIM Proceedings* (2011) (to appear).
 - [49] A. Cohen, *Wavelet methods in numerical analysis*, Vol. 7, Elsevier, Amsterdam, 2000.
 - [50] S. Müller, *Adaptive multiscale schemes for conservation laws*, Vol. 27, Springer-Verlag, 2003.
 - [51] C. F. Eyring, S. S. Mackeown, R. A. Millikan, Fields currents from points, *Phys. Rev.* 31 (5) (1928) 900–909.
 - [52] J. D. Jackson, *Classical Electrodynamics*, 3rd Edition, John Wiley and Sons, Inc., 1999, page 26, equation (1.5).
 - [53] A. A. Kulikovskiy, Positive streamer in a weak field in air: A moving avalanche-to-streamer transition, *Phys. Rev. E* 57 (6) (1998) 7066–7074.
 - [54] D. Z. Pai, D. A. Lacoste, C. O. Laux, Transitions between corona, glow, and spark regimes of nanosecond repetitively pulsed discharges in air at atmospheric pressure, *J. Appl. Phys.* 107 (9) (2010) 093303.