Splitting in the K-theory localization sequence of number fields

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Abstract - Let p be a rational prime and let F be a number field. Then, for each $i \geq 1$, Quillen's K-theory group $K_{2i}(F)$ is a torsion abelian group, containing the finite subgroup $K_{2i}(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of F. If p is odd or F is nonexceptional or i is even, we give necessary and sufficient conditions for the p-primary component of $K_{2i}(\mathcal{O}_F) \subset K_{2i}(F)$ to split. Our conditions involve coinvariants of twisted p-parts of the p-class groups of certain subfields of the fields $F(\mu_{p^n})$ for $n \in \mathbb{N}$. We also compare our conditions with the weaker condition $Ш^2_S(F, \mathbb{Z}_p(i+1)) = 0$ and give some example.

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1 Introduction and notation

Throughout the paper, p will denote a rational prime. For an abelian group A, set

$$\mathrm{Div}(A) = \text{maximal divisible subgroup of } A$$

$$\mathrm{div}(A) = \{a \in A \,|\, \forall\, n \in \mathbb{N}\ \exists\, a_n \in A : a = na_n\}$$

Then $\mathrm{div}(A)$ is a subgroup of A which is commonly called the subgroup of (infinitely) divisible elements or the subgroup of elements of infinite height of A. We denote by A_p the p-primary part of A and, for $n \in \mathbb{N}$, by $A[p^n]$ the subgroup of elements of A whose order divides p^n.

If R is a ring and j is a natural number, $K_j(R)$ denotes the j-th Quillen's K-group of R. Let F be a number field and i be a positive integer. Thanks to Soulé's results (see [23], Theorem 4.6), Quillen's long exact localization sequence splits into isomorphisms $K_{2i+1}(\mathcal{O}_F) \cong K_{2i+1}(F)$ and short exact sequences of the form

$$0 \longrightarrow K_{2i}(\mathcal{O}_F) \longrightarrow K_{2i}(F) \xrightarrow{\partial_{F,i}} \bigoplus_{v \nmid \infty} K_{2i-1}(k_v) \longrightarrow 0 \tag{1}$$

where $\mathcal{O}_F$ is the ring of integers of F, k_v is the residue field of F at v and the direct sum is taken over the finite places of F. We recall that, thanks to Quillen's and Borel's results, $K_{2i-1}(k_v)$ is cyclic of order $|k_v|^i - 1$ and $K_{2i}(\mathcal{O}_F)$ is a finite group: in particular we always have $\mathrm{Div}(K_{2i}(F)) = 0$ (but in general $\mathrm{div}(K_{2i}(F))$ may be nontrivial).
One can asks for conditions for the exact sequence in (1) to split. As a motivation for this question we quote the following three results:

- if $F = \mathbb{Q}$, Tate showed that the exact sequence (1) splits (see [13], Theorem 11.6);

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- if E is a rational function field in one variable over an arbitrary base field, then the localization sequence for $K_2(E)$ (which is completely analogous to (1)) always splits, thanks to a result of Milnor and Tate (see [12], Theorem 2.3);
- if E is a local field (i.e. a field complete with respect to a discrete valuation whose residue field is finite), then the localization sequence for $K_{2i}(E)$ (which again is completely analogous to (1)) always splits, thanks to a result of Soulé (see [20], Proposition 4).

Coming back to the case of a number field F, consider now the exact sequence which (1) induces on p-primary parts, which we shall refer to as the localization sequence for $K_{2i}(F)_p$. The problem of the splitting of the localization sequence for $K_{2i}(F)_p$ was first studied by Banaszak: in one of his papers (see [1], Corollary 1) he claims that, if p is odd, the localization sequence for $K_{2i}(F)_p$ splits if and only if $\mathrm{div}(K_{2i}(F))_p = 0$ (in fact Banaszak's result is stated in terms of $Ш^2_S(F, \mathbb{Z}_p(i+1))$, see Proposition 2). This is obviously a necessary condition, since both the right and the left terms of the localization sequence for $K_{2i}(F)_p$ have trivial subgroup of divisible elements. However the proof of sufficiency seems to be incomplete. It turns out that, for any $i \geq 1$, there is a counterexample, namely there is a number field F and a prime p such that $\mathrm{div}(K_{2i}(F))_p = 0$ but the localization sequence for $K_{2i}(F)_p$ does not split (see Example 2). The counterexample is constructed using Theorem 1, which is our main result and is described in Section 3. We first define certain groups $\Omega^{(p)}_{i,n}$ (see (8)) which are the obstruction to the splitting of the p-part of (1). Then, under the assumption that p is odd or i is even or F is nonexceptional (see Definition 1), the $\Omega^{(p)}_{i,n}$'s are shown to vanish exactly when the groups $(Cl^{S_{i,n}}_{F_{i,n}} \otimes \mu_{p^n}^{\otimes i})_{\mathrm{Gal}(F_{i,n}/F)}$ are trivial. Here $F_{i,n}$ is a particular subfield of the fields $F(\mu_{p^n})$ (see Notation 4), where μ_{p^n} denotes the group of roots of unity in an algebraic closure of F, and $Cl^{S_{i,n}}_{F_{i,n}}$ is the p-split class group of $F_{i,n}$. The techniques used in the proof of Theorem 1 have already been used in [15] and [6]. In many cases, $\Omega^{(p)}_{i,n}$ and $(Cl^{S_{i,n}}_{F_{i,n}} \otimes \mu_{p^n}^{\otimes i})_{\mathrm{Gal}(F_{i,n}/F)}$ are actually both isomorphic to a cohomological kernel, namely $Ш^2_S(F_{i,n}, \mu_{p^n}^{\otimes i})$ (the general case for $p = 2$ requires a different approach). Another approach to the proof of Theorem 1 is possible, following the ideas of [4], Section 3, to describe $\Omega^{(p)}_{i,n}$, but it turns out to be lengthier and more technical.
The difference between our splitting criterion and the condition $\mathrm{div}(K_{2i}(F))_p = 0$ (which is equivalent to the vanishing of the i-th étale wild kernel $Ш^2_S(F, \mathbb{Z}_p(i+1))$) is also analyzed at the end of Section 3. Anyway the condition $\mathrm{div}(K_{2i}(F))_p = 0$ is often also sufficient for the localization sequence for $K_{2i}(F)_p$ to split (for example in the case where $F = \mathbb{Q}$, see Example 1).

2 Localization sequence for continuous Galois cohomology

In this section we are going to translate the problem of the existence of a splitting for the localization sequence for $K_{2i}(F)_p$ in cohomological terms. First of all we recall the following notion.

Definition 1. Let E be a field of characteristic other than 2. Then E is said to be *nonexceptional* if $\mathrm{Gal}(E(\mu_{2^\infty})/E)$ has no element of order 2 (and *exceptional* otherwise). Here μ_{2^∞} is the group of roots of unity whose order is a power of 2 in an algebraic closure of E.

Remark 1. Note that nonexceptional fields have no embeddings in $\mathbb{R}$ (since $\mathbb{R}$ is exceptional and subfields of exceptional fields are exceptional).

As in the preceding section, p denotes a rational prime, i is a positive integer and F is a number field. We are interested in the localization sequence for $K_{2i}(F)_p$, namely

$$0 \longrightarrow K_{2i}(\mathcal{O}_F)_p \longrightarrow K_{2i}(F)_p \xrightarrow{\partial_{F,i}} \bigoplus_{v \nmid p\infty} K_{2i-1}(k_v)_p \longrightarrow 0 \tag{2}$$

which easily follows from (1). This exact sequence has a cohomological counterpart (see also Remark 2), as shown in the next proposition (which can essentially be found [1], Section I, §2-3). In this paper, cohomology is always continuous cohomology of profinite groups, in the sense of Tate ([21]).

Notation 1. If T is a finite set of places of F, $G_{F,T}$ denotes the Galois group of the maximal extension F_T of F unramified ouside T and $\mathcal{O}_F^T$ is the ring of T-integers of F. For a field E, G_E denotes the Galois group of a separable algebraic closure of E (and we use the convention $H^j(E, -) := H^j(G_E, -)$ for cohomology). Finally, $S_F^{(p)} = S$ denotes the set of primes above p and ∞ in F.

For a noetherian $\mathbb{Z}[\frac{1}{p}]$-algebra A and a natural number j, let $K_j^{\acute{e}t}(A)$ denote the j-th étale K-theory group of Dwyer and Friedlander (see [3]). For any finite set T containing S, there are functorial maps

$$K_{2i}(\mathcal{O}_F^T)_p \xrightarrow{\nu} K_{2i}^{\acute{e}t}(\mathcal{O}_F^T) \quad \text{and} \quad K_{2i}^{\acute{e}t}(\mathcal{O}_F^T) \xrightarrow{\alpha} H^2(G_{F,T}, \mathbb{Z}_p(i+1))$$

We know that α is an isomorphism (see [3], Remark 8.8). We set $ch = \alpha \circ \nu$ (ch stands for Chern character).

Proposition 1 (Banaszak)**.** *Suppose that p is odd or F is nonexceptional. There is a commutative diagram with exact rows as follows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2i}(\mathcal{O}_F^S)_p & \longrightarrow & K_{2i}(F)_p & \xrightarrow{\partial_{F,i}} & \oplus_{v\nmid p\infty} K_{2i-1}(k_v)_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ch} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(G_{F,S}, \mathbb{Z}_p(i+1)) & \longrightarrow & H^2(F, \mathbb{Z}_p(i+1))_p & \xrightarrow{\partial_{F,i}^c} & \oplus_{v\nmid p\infty} H^1(k_v, \mathbb{Z}_p(i)) & \longrightarrow & 0
\end{array}$$

Moreover vertical maps in the above diagram are surjective and the rightmost is an isomorphism.

Proof. Let T be any finite set of primes T containing S. For the rest of this proof, for any $j \in \mathbb{N}$, set

$$H_T^j(-) := H^j(G_{F,T}, -) \quad \text{and} \quad H^j(-) := H^j(F, -)$$

Then, for any $n \in \mathbb{N}$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2i}(\mathcal{O}_F^S)_p & \longrightarrow & K_{2i}(\mathcal{O}_F^T)_p & \xrightarrow{\partial_{F,i}} & \oplus_{v\in T\smallsetminus S} K_{2i-1}(k_v)_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow & & \\
0 & \longrightarrow & H_S^2(\mathbb{Z}_p(i+1)) & \longrightarrow & H_T^2(\mathbb{Z}_p(i+1)) & \xrightarrow{\partial_{F,i}^c} & \oplus_{v\in T\smallsetminus S} H^1(k_v, \mathbb{Z}_p(i)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & H_S^1(\mathbb{Q}_p/\mathbb{Z}_p(i+1))/\mathrm{Div} & \longrightarrow & H_T^1(\mathbb{Q}_p/\mathbb{Z}_p(i+1))/\mathrm{Div} & \longrightarrow & \oplus_{v\in T\smallsetminus S} H^0(k_v, \mathbb{Q}_p/\mathbb{Z}_p(i)) & \longrightarrow & 0
\end{array}$$

The definition and the exactness of the middle row can be found in [20], Section III and and [18], Section 4. The upper rightmost vertical map is defined by the diagram (note that in general this does *not* coincide with the direct sum of the residue fields Chern characters) and it is bijective. This follows from the surjectivity of the ch's (see [3], Theorem 8.7, for the case p odd and [18], Theorem 0.1, for the case $p = 2$ and F nonexceptional) together with the easy fact that, for any $v \in T \smallsetminus S$, $H^1(k_v, \mathbb{Z}_p(i))$ is cyclic of order $|k_v|^i - 1$ and has therefore the same order as $K_{2i-1}(k_v)_p$ (thanks to Quillen's calculation). As for the maps denoted with δ, they are connecting homomorphisms in the long exact cohomology sequence relative to the exact sequence

$$0 \to \mathbb{Z}_p(j) \to \mathbb{Q}_p(j) \to \mathbb{Q}_p/\mathbb{Z}_p(j) \to 0$$

($j = i,\, i+1$) and they are bijective (see [21], Proposition 2.3).
Taking direct limits as $T \supset S$ grows, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2i}(\mathcal{O}_F^S)_p & \longrightarrow & K_{2i}(F)_p & \xrightarrow{\partial_{F,i}} & \oplus_{v\nmid p\infty} K_{2i-1}(k_v)_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ch\delta^{-1}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_S^1(\mathbb{Q}_p/\mathbb{Z}_p(i+1))/\mathrm{Div} & \longrightarrow & H^1(\mathbb{Q}_p/\mathbb{Z}_p(i+1))/\mathrm{Div} & \xrightarrow{\partial_{F,i}^c} & \oplus_{v\nmid p\infty} H^0(k_v, \mathbb{Q}_p/\mathbb{Z}_p(i)) & \longrightarrow & 0
\end{array} \tag{3}$$

This is because, for any finite set T containing S,

$$\mathrm{Div}(H^1_T(\mathbb{Q}_p/\mathbb{Z}_p(i+1))) = \mathrm{Div}(H^1(\mathbb{Q}_p/\mathbb{Z}_p(i+1)))$$

(see [19], §4, Lemma 5). Composing again with δ we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{2i}(\mathcal{O}_F^S)_p & \longrightarrow & K_{2i}(F)_p & \xrightarrow{\partial_{F,i}} & \oplus_{v\nmid p\infty} K_{2i-1}(k_v)_p & \longrightarrow & 0 \\ & & \downarrow ch & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^2_S(\mathbb{Z}_p(i+1)) & \longrightarrow & H^2(\mathbb{Z}_p(i+1))_p & \xrightarrow{\partial^c_{F,i}} & \oplus_{v\nmid p\infty} H^1(k_v,\, \mathbb{Z}_p(i)) & \longrightarrow & 0 \end{array}$$

Here again the rightmost map is an isomorphism and the other two vertical maps are surjective. □

Remark 2. The Quillen-Lichtenbaum "conjecture" says that for any finite set of places T containing S,

$$ch : K_{2i}(\mathcal{O}_F^T)_p \longrightarrow H^2(G_{F,T},\, \mathbb{Z}_p(i+1)) \tag{4}$$

is indeed an isomorphism. Tate (see [21], Theorem 5.4) proved this statement for $i = 1$: in this case ch is just the Galois symbol ([3], proof of Theorem 8.2). The general case of the Quillen-Lichtenbaum "conjecture" is a consequence of the Bloch-Kato "conjecture", whose proof has been recently completed in [26] (in that paper the reader will also find suitable references to the work of Voevodsky, Rost, Haesemeyer, Suslin, Joukhovitski, Weibel, ...). Anyway the isomorphism in (4) is not necessary for the proof of Proposition 1 and in fact we shall not use it at all in this paper.

In the following we will be mainly interested in the bottow row of the diagram of Proposition 1, namely the exact sequence

$$0 \to H^2(G_{F,S},\, \mathbb{Z}_p(i+1)) \to H^2(F,\, \mathbb{Z}_p(i+1))_p \to \bigoplus_{v\nmid p\infty} H^1(k_v,\, \mathbb{Z}_p(i)) \to 0 \tag{5}$$

We will refer to it as the localization sequence for $H^2(F,\, \mathbb{Z}_p(i+1))$. The next proposition shows that, even without using the Quillen-Lichtenbaum "conjecture", for our purposes there is no difference between considering (2) or (5).

Definition 2. A subgroup B of an abelian group A is *pure* if $nA \cap B = nB$ for each $n \in \mathbb{N}$.

Proposition 2. *Suppose that p is odd or F is nonexceptional. Then the localization sequence for $H^2(F,\, \mathbb{Z}_p(i+1))$ splits if and only the localization sequence for $K_{2i}(F)_p$ splits.*

Proof. A diagram chasing in the diagram of Proposition 1 shows that if the localization sequence for $K_{2i}(F)_p$ splits then the localization sequence for $H^2(F,\, \mathbb{Z}_p(i+1))$ splits. As for the converse, we can assume $p \neq 2$ because if $p = 2$, then F is totally imaginary and the ch's are isomorphisms by [18], Theorem 0.1[1]. Banaszak (see [1], Proposition 2) proved that the natural map

$$\nu : K_{2i}(\mathcal{O}_F)_p \longrightarrow K^{\acute{e}t}_{2i}(\mathcal{O}_F[\tfrac{1}{p}]) \tag{6}$$

is split surjective. We are going to prove the analogous result for the map

$$\nu : K_{2i}(F)_p \longrightarrow K^{\acute{e}t}_{2i}(F)_p \tag{7}$$

with the same strategy as Banaszak, taking into account that the groups involved are no longer finite (but still torsion). First of all, for each $n \in \mathbb{N}$, there is a commutative diagram

$$\begin{array}{ccccc} K_{2i+1}(F,\, \mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & K_{2i}(F)[p^n] & \longrightarrow & 0 \\ \downarrow & & \downarrow \nu & & \\ K^{\acute{e}t}_{2i+1}(F,\, \mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & K^{\acute{e}t}_{2i}(F)[p^n] & \longrightarrow & 0 \end{array}$$

[1]It is likely that there exists a proof of the proposition for $p = 2$ and F nonexceptional which does not use the results in [18].

with exact rows and surjective vertical maps (see [1], Diagram 1.6). This implies that the kernel C_i of the map ν in (7) is a pure subgroup of $K_{2i}(F)_p$. Moreover C_i is finite since it is easily seen to coincide with the kernel of the map in (6) (see [1], Remark 7). Hence Theorem 7 of [9] tells us that the map in (7) is split. By definition of the maps ch, this proves that the map

$$K_{2i}(F)_p \longrightarrow H^2(F,\, \mathbb{Z}_p(i+1))_p$$

in the diagram of Proposition 1 splits. The proof of the proposition is then achieved by a diagram chasing in the diagram of Proposition 1. □

3 Main result

We are going to describe the obstruction to the existence of a splitting for the localization sequence for $H^2(F, \mathbb{Z}_p(i+1))$ in terms of coinvariants of twisted p-parts of the class groups of certain subfields of the fields $F(\mu_{p^n})$. For a field E, we denote by $\mu_{p^n}(E)$ the group of p^n-th roots of unity in an algebraic closure of E (the reference to E in $\mu_{p^n}(E)$ will be often omitted).

Notation 2. For typographical convenience, we set

$$\Omega_{F,i,n}^{(p)} = \Omega_{i,n}^{(p)} = \left(H^2(G_{F,\,S},\, \mathbb{Z}_p(i+1)) \cap p^n H^2(F,\, \mathbb{Z}_p(i+1))_p\right) \Big/ p^n H^2(G_{F,\,S},\, \mathbb{Z}_p(i+1)) \tag{8}$$

Remark 3. Note that, from the definition of $\Omega_{i,n}^{(p)}$, we have $\Omega_{i,0}^{(p)} = 0$.

The following lemma shows that the $\Omega_{i,n}^{(p)}$'s are the obstructions to the existence of a splitting for the localization sequence for $H^2(F, \mathbb{Z}_p(i+1))$.

Lemma 1. *The localization sequence for $H^2(F,\, \mathbb{Z}_p(i+1))$ splits if and only if for every $n \in \mathbb{N}$ we have $\Omega_{i,n}^{(p)} = 0$.*

Proof. Thanks to Theorem 5 in [9] and the fact that any direct summand of an abelian group is pure, we have the following equivalences

$$\begin{aligned} \Omega_{i,n}^{(p)} = 0 \quad \forall\, n \in \mathbb{N} \quad &\Leftrightarrow \quad H^2(G_{F,\,S},\, \mathbb{Z}_p(i+1)) \text{ is pure in } H^2(F,\, \mathbb{Z}_p(i+1))_p \\ &\Leftrightarrow \quad \text{the localization sequence for } H^2(F,\, \mathbb{Z}_p(i+1)) \text{ splits.} \end{aligned}$$

□

We will make use of the following notation (see [24]).

Notation 3. Let E be any field. If M is a G_E-module, we denote by $E(M)$ the fixed field of the kernel of the homomorphism $G_E \to \mathrm{Aut}(M)$ induced by the action of $\mathrm{Gal}(\overline{E}/E)$ on M.

For each $n \in \mathbb{N}$, we now introduce the subfield $F_{i,n}$ of $F(\mu_{p^n})$ which will be relevant for us. Such subfields has been used for the first time by Weibel in [24].

Notation 4. Let F be a number field. For $n \in \mathbb{N}$ and $i \geq 1$, set $F_{i,n} = F(\mu_{p^n}^{\otimes i})$ (note that $F_{i,0} = F$ for any $i \geq 1$). We will also use the notation $\Gamma_{i,n} = \mathrm{Gal}(F_{i,n}/F)$. If w is a place in $F_{i,n}$, then denote by $(k_{i,n})_w$ the residue field of $F_{i,n}$ at w (thus $(k_{i,0})_w = k_w$). Finally, let $S_{i,n}$ be the set of primes of $F_{i,n}$ above p and ∞ (thus $S_{i,0} = S$) and let $G_{S_{i,n}}$ denote the Galois group of the maximal extension of $F_{i,n}$ unramified outside $S_{i,n}$ (thus $G_{S_{i,0}} = G_S = G_{F,\,S}$). Of course, in all this notation, a reference to p should appear but the context should prevent any misunderstanding.

Lemma 2. *Suppose that p is an odd prime or F is nonexceptional. For every $n, m \in \mathbb{N}$, there are isomorphisms of $\Gamma_{i,m}$-modules*

$$H^2(G_{S_{i,m}},\, \mathbb{Z}_p(i))/p^n \cong H^2(G_{S_{i,m}},\, \mu_{p^n}^{\otimes i}) \quad \textit{and} \quad H^2(F_{i,m},\, \mathbb{Z}_p(i))/p^n \cong H^2(F_{i,m},\, \mu_{p^n}^{\otimes i})$$

Proof. The first isomorphism come from the cohomology sequence corresponding to the exact sequence

$$0 \to \mathbb{Z}_p(i) \xrightarrow{p^n} \mathbb{Z}_p(i) \longrightarrow \mu_{p^n}^{\otimes i} \to 0 \tag{9}$$

In fact $H^3(G_{S_{i,m}}, \mathbb{Z}_p(i)) = 0$ since $G_{S_{i,m}}$ has p-cohomological dimension less or equal to 2 (see [14], (8.3.17)): this is true even in the case $p = 2$ because in that case $F_{i,m}$ has to be nonexceptional and therefore it has no real embeddings.
Similarly, consider the exact sequence

$$0 \to \mu_{p^n}^{\otimes i} \xrightarrow{p^n} \mathbb{Q}_p/\mathbb{Z}_p(i) \longrightarrow \mathbb{Q}_p\mathbb{Z}_p(i) \to 0$$

Since $H^3(F_{i,m}, \mu_{p^n}^{\otimes i}) = 0$ because $G_{F_{i,m}}$ has p-cohomological dimension less or equal to 2 as before, we deduce that $H^2(F_{i,m}, \mathbb{Q}_p/\mathbb{Z}_p(i))$ is divisible. Then $H^3(F_{i,m}, \mathbb{Z}_p(i))_p = 0$ (see [14], (2.3.10)). In particular the image of the connecting homomorphism $H^2(F_{i,m}, \mu_{p^n}^{\otimes i}) \to H^3(F_{i,m}, \mathbb{Z}_p(i))$ relative to the exact sequence (9) is trivial which gives the second isomorphism of the lemma. □

We recall the definition and some properties of certain cohomological kernels which will be useful in the proof of Theorem 1.

Notation 5. Let j be a natural number. Let E/F an algebraic Galois extension with Galois group G and M a G-module. For any set T of prime of F we set

$$Ш_T^j(G, M) = \ker\left(H^j(G, M) \longrightarrow \prod_{v \in T} H^j(G_v, \mu_{p^n}^{\otimes i})\right)$$

where G_v is a decomposition group of v in E/F and the map is induced by the inclusion $G_v \hookrightarrow G$. It is standard to use the notation $Ш^j(F, M)$ when E is an algebraic closure of F and T is the set of all primes and $Ш_T^j(F, M)$ when $E = F_T$ for a set of primes T of F.

Notation 6. If M is a G_F-module, we denote by $M^\vee$ its Pontryagin dual, i.e. $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$.

Lemma 3.
- *There is an isomorphism* $Ш^1_{S_{i,n}}(F_{i,n}, (\mu_{p^n}^{\otimes i})^\vee) \cong \left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{p^n}^{\otimes i}\right)^\vee$.
- *For any set of primes T of F, there is an isomorphism* $Ш^2_T(F, \mu_{p^n}^{\otimes i+1}) \cong \left(Ш^1_T(F, (\mu_{p^n}^{\otimes i})^\vee)\right)^\vee$.

Proof. For the first assertion see [14], Lemma 8.6.3, for the second see [14], Theorem 8.6.8. □

Lemma 4. *Let G be a finite cyclic group and let M be a faithful G-module which is cyclic of order a power of p as an abelian group. Then*

- *if $p \neq 2$, then $H_j(G, M) = H^j(G, M) = 0$ for any $j \geq 1$ (or equivalently the norm map $M_G \to M^G$ is surjective);*
- *if $p = 2$ and $G \neq \{\pm 1\}$ (i.e. G is not a group of order 2 whose generator g satisfies $gm = -m$ for any $m \in M$), then $H_j(G, M) = H^j(G, M) = 0$ for any $j \geq 1$ (or equivalently the norm map $M_G \to M^G$ is surjective);*

Proof. See [24], Lemma 3.2 and Remark 3.2.1. □

Definition 3. We say that F satisfies $C(n, i, p)$ if at least one of the following holds

(i) p is odd;

(ii) i is even;

(iii) $n = 1$;

(iv) $\sqrt{-1} \in F$;

(v) $n \geq b^-(F) > 1$ where $b^-(F) = \max\{b \in \mathbb{N} \,|\, \zeta_{2^b} - \zeta_{2^b}^{-1} \in F\}$ where ζ_{2^b} is a generator of μ_{2^b}.

Remark 4. Note that F is nonexceptional if and only if $b^-(F) > 1$ (see [4], Lemma 2.1).

The next lemma shows the relation between Lemma 4 and the condition $C(n, i, p)$ for F.

Lemma 5. *The number field F satisfies $C(n, i, p)$ exactly when $\Gamma_{i,n}$ is cyclic and different from $\{\pm 1\}$ if $p = 2$. Moreover F is nonexceptional if and only if, for sufficiently large n, $\Gamma_{i,n}$ is cyclic and different from $\{\pm 1\}$ if $p = 2$.*

Proof. It is clear that if one of (i) or (iii) or (iv) holds, then $\Gamma_{i,n}$ is cyclic and different from $\{\pm 1\}$ if $p = 2$. For (ii), see [24], Application 3.3. For (v) see [5], Lemma 2.2 (3) (note that i can be taken to be odd and then $\Gamma_{i,n} = \mathrm{Gal}(F(\mu_{2^n})/F)$). On the other hand, suppose $\Gamma_{i,n}$ is cyclic and different from $\{\pm 1\}$ if $p = 2$ and none of (i), (ii), (iii) and (iv) holds. Then F is the nonexceptional (see [5], Lemma 2.2 (1)) and then $b^-(F) > 1$ by Lemma 2.1 of [5] and $n \geq b^-(F)$ again by Lemma 2.2 (3) of [5]. □

The next lemma will only be used for the 2-part of the proof of Theorem 1.

Lemma 6. *Let $v \nmid p$ be a finite place of F. If w is a place of $F_{i,n}$ above v, then $(k_{i,n})_w = k_v(\mu_{p^n}^{\otimes i})$.*

Proof. Both $(k_{i,n})_w$ and $k_v(\mu_{p^n}^{\otimes i})$ are subextensions of $k_v(\mu_{p^n})/k_v$. Set D_v (resp. D'_v) for the decomposition group of v in $F_{i,n}/F$ (resp. $F(\mu_{p^n})/F$) and P for the Galois group of $F(\mu_{p^n})/F_{i,n}$. Identifying $\mathrm{Gal}(k_v(\mu_{p^n})/k_v)$ with D'_v we get

$$\mathrm{Gal}(k_v(\mu_{p^n})/k_v(\mu_{p^n}^{\otimes i})) = D'_v \cap P$$

$$\mathrm{Gal}((k_{i,m})_v/k_v) = D_v = D'_v/D'_v \cap P$$

which proves the result. □

We are now ready to state and prove the main result of the paper.

Theorem 1. *Let i be a positive integer and let F be a number field. If F satisfies $C(n, i, p)$, then there is an isomorphism*

$$\Omega_{i,n}^{(p)} \cong \left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}}$$

Moreover, if F is nonexceptional, then for any $n \in \mathbb{N}$ and any $i \geq 1$ we have

$$\Omega_{i,n}^{(2)} = 0 \Longleftrightarrow \left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{2^n}^{\otimes i}\right)_{\Gamma_{i,n}} = 0$$

In particular, if p is odd or F is nonexceptional or i is even, the localization sequence for $K_{2i}(F)_p$ (or equivalently for $H^2(F, \mathbb{Z}_p(i+1))$) splits if and only if for every $n \in \mathbb{N}$ we have $\left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}} = 0$.

Proof. Suppose first that F satisfies $C(n, i, p)$. Then note that

$$Ш_S^1(F, (\mu_{p^n}^{\otimes i})^\vee) \cong Ш_{S_{i,n}}^1(F_{i,n}, (\mu_{p^n}^{\otimes i})^\vee)^{\Gamma_{i,n}}$$

This follows from the exact sequence of terms of low degree associated to the exact sequence

$$1 \to G_{F_{i,n}, S_{i,n}} \to G_{F,S} \to \Gamma_{i,n} \to 1$$

using the fact that $H^j(\Gamma_{i,n}, (\mu_{p^n}^{\otimes i})^\vee) = 0$ for any $j \geq 1$ (use Lemma 5 and Lemma 4). Therefore, using Lemma 3, we get

$$Ш_S^2(F, \mu_{p^n}^{\otimes i+1}) \cong \left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}} \tag{10}$$

Now, again using Lemma 3, we get $Ш^2(F, \mu_{p^n}^{\otimes i+1}) \cong \left(Ш^1(F, (\mu_{p^n}^{\otimes i})^\vee)\right)^\vee$. Moreover $Ш^1(F, (\mu_{p^n}^{\otimes i})^\vee) \cong Ш^1(\Gamma_{i,n}, (\mu_{p^n}^{\otimes i})^\vee)$: this follows from the exact sequence

$$0 \to Ш^1(\Gamma_{i,n}, (\mu_{p^n}^{\otimes i})^\vee) \to Ш^1(F, (\mu_{p^n}^{\otimes i})^\vee) \to Ш^1(F_{i,n}, (\mu_{p^n}^{\otimes i})^\vee)^{\Gamma_{i,n}}$$

which comes from the exact sequence of terms of low degree associated to the exact sequence

$$1 \to G_{F_{i,n}} \to G_F \to \Gamma_{i,n} \to 1$$

together with the fact that $Ш^1(F_{i,n}, (\mu_{p^n}^{\otimes i})^\vee) = 0$ (see [14], Theorem 9.1.3). Finally we also have, $Ш^1(\Gamma_{i,n}, (\mu_{p^n}^{\otimes i})^\vee) = 0$ (use Lemma 5 and Lemma 4) and therefore $Ш^2(F, \mu_{p^n}^{\otimes i+1}) = 0$. Now considering the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ш_S^2(F, \mu_{p^n}^{\otimes i+1}) & \longrightarrow & H^2(G_{F,S}, \mu_{p^n}^{\otimes i+1}) & \longrightarrow & \prod_{v\in S} H^2(F_v, \mu_{p^n}^{\otimes i+1}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Ш^2(F, \mu_{p^n}^{\otimes i+1}) & \longrightarrow & H^2(F, \mu_{p^n}^{\otimes i+1}) & \longrightarrow & \prod_v H^2(F_v, \mu_{p^n}^{\otimes i+1}) \end{array}$$

we get

$$Ш_S^2(F, \mu_{p^n}^{\otimes i+1}) = \ker\left(H^2(G_{F,S}, \mu_{p^n}^{\otimes i+1}) \to H^2(F, \mu_{p^n}^{\otimes i+1})\right) \cong \Omega_{i,n}^{(p)}$$

thanks to Lemma 2. We conclude using (10).

We now focus on the case where $p = 2$ and F is nonexceptional. Suppose first that the localization sequence for $H^2(F, \mathbb{Z}_2(i+1))$ splits, in other words $\Omega_{i,n}^{(2)} = 0$ for any $n \in \mathbb{N}$. Observe that, since $\Gamma_{i,n}$ is a 2-group, Nakayama's lemma implies

$$(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{2^n}^{\otimes i})_{\Gamma_{i,n}} = 0 \iff (Cl_{F_{i,n}}^{S_{i,n}})_2 = 0 \tag{11}$$

Moreover, if we set $F_{i,\infty} = \cup_{n\in\mathbb{N}} F_{i,n}$, then $F_{i,\infty}/F$ is a $\mathbb{Z}_2$-extension, since F is nonexceptional. Now, since $C(1, i, 2)$ holds for F, we know that $(Cl_F^S)_2 = 0$ by the first part of the proof. Moreover, again by the first part of the proof, we also know that $(Cl_{F_{i,n}}^{S_{i,n}})_2 = 0$ for n large enough, since the fact that F is nonexceptional implies that $C(n, i, 2)$ is satisfied by F for n large enough. We have to show that $(Cl_{F_{i,n}}^{S_{i,n}})_2 = 0$ for any $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ be such that $F_{i,m} \neq F$: we can find a prime v above 2 in F which stays inert in $F_{i,m}/F$ because $(Cl_F^S)_2 = 0$ and only primes above 2 can ramify in $\mathbb{Z}_2$-extensions. In particular there is only one prime above v in $F_{i,\infty}/F$ (since $F_{i,\infty}/F$ is a $\mathbb{Z}_2$-extension). Now take $n \geq m$ such that $(Cl_{F_{i,n}}^{S_{i,n}})_2 = 0$: $F_{i,n}/F_{i,m}$ has to be disjointed from the 2-split Hilbert class field of F_m since there is a nonsplit prime above 2 in $F_{i,n}/F_{i,m}$. This means that the norm map

$$(Cl_{F_{i,n}}^{S_{i,n}})_2 \longrightarrow (Cl_{F_{i,m}}^{S_{i,m}})_2$$

has to be surjective, which implies $(Cl_{F_{i,m}}^{S_{i,m}})_2 = 0$.

Now suppose that $(Cl_{F_{i,n}}^{S_{i,n}})_2 = 0$ for every $n \in \mathbb{N}$: we prove by induction on n that $\Omega_{i,n}^{(2)} = 0$. Of course we trivially have $\Omega_{i,0}^{(2)} = 0$ (and indeed also $\Omega_{i,1}^{(2)} = 0$ by the first part of the proof). Next consider the following commutative diagram

$$\begin{array}{ccc} H^2(F_{i,n}, \mathbb{Z}_2(i+1))[2^n] & \xrightarrow{\partial_{F_{i,n}}^c} & \bigoplus_{w\nmid 2\infty} H^1((k_{i,n})_w, \mathbb{Z}_2(i))[2^n] \\ \downarrow cor_n^{(2)} & & \downarrow cor_n^{(1)} \\ H^2(F, \mathbb{Z}_2(i+1))[2^n] & \xrightarrow{\partial_F^c} & \bigoplus_{v\nmid 2\infty} H^1(k_v, \mathbb{Z}_2(i))[2^n] \end{array}$$

A generic element $x=(x_v)\in\bigoplus_{v\nmid 2\infty}H^1(k_v,\,\mathbb{Z}_2(i))[2^n]$ can be written as a sum $x=y_1+\ldots+y_n$ where, for any $j=1,\,\ldots,\,n$, $y_j\in\bigoplus_{v\nmid 2\infty}H^1(k_v,\,\mathbb{Z}_2(i))[2^n]$ and

$$(y_j)_v=\begin{cases} x_v & \text{if } x_v\in H^1(k_v,\,\mathbb{Z}_2(i))[2^j]\smallsetminus H^1(k_v,\,\mathbb{Z}_2(i))[2^{j-1}]\\ 0 & \text{otherwise}\end{cases}$$

By induction, for any $j=1,\,\ldots,\,n-1$, there is an element

$$z_j\in H^2(F,\,\mathbb{Z}_2(i+1))[2^j]\subseteq H^2(F,\,\mathbb{Z}_2(i+1))[2^n]$$

such that $\partial_F^c(z_j)=y_j$. Now $(y_n)_v$ is an element of order 2^n in $H^1(k_v,\,\mathbb{Z}_2(i))[2^n]\cong H^0(k_v,\,\mu_{2^n}^{\otimes i})$ (since $H^0(k_v,\,\mathbb{Z}_2(i))=0$). This means that $H^0(k_v,\,\mu_{2^n}^{\otimes i})=\mu_{2^n}^{\otimes i}$ which implies $(k_{i,n})_w=k_v(\mu_{2^n}^{\otimes i})=k_v$ for any $w|v$ in $F_{i,n}$ (by Lemma 6), i.e. v splits completely in $F_{i,n}/F$. In particular there is an element $w_n\in\bigoplus_{w\nmid 2\infty}H^1((k_{i,n})_w,\,\mathbb{Z}_2(i))[2^n]$ such that $cor_n^{(1)}(w_n)=y_n$. Now

$$H^2(F_{i,n},\,\mathbb{Z}_2(i+1))[2^n]\xrightarrow{\partial^c_{F_{i,n}}}\bigoplus_{w\nmid p}H^1((k_{i,n})_w,\,\mathbb{Z}_2(i))[2^n]$$

is surjective by the first part of the proof since $F_{i,n}$ satisfies $C(n,\,i,\,2)$ (if i is odd $F_{i,n}=F(\mu_{2^n})$ which contains $\sqrt{-1}$ since $n\geq 2$) and $(Cl^{S_{i,n}}_{F_{i,n}})_2=0$ and therefore there exists an element $t_n\in H^2(F_{i,n},\,\mathbb{Z}_2(i+1))[2^n]$ such that $\partial^c_{F_{i,n}}(t_n)=w_n$ and clearly $\partial_F^c(cor_n^{(2)}(t_n))=y_n$. This shows that

$$H^2(F,\,\mathbb{Z}_p(i+1))[2^n]\xrightarrow{\partial^c_F}\bigoplus_{v\nmid p}H^1(k_v,\,\mathbb{Z}_p(i))[2^n]$$

is surjective which is equivalent to $\Omega_{i,n}^{(2)}=0$. □

Remark 5. It is worth noting that, if $\mu_p\subseteq F$, then

$$\left(Cl^{S_{i,n}}_{F_{i,n}}\otimes\mu_{p^n}^{\otimes^i}\right)_{\Gamma_{i,n}}=0\Leftrightarrow Cl^{S_{i,n}}_{F_{i,n}}\otimes\mu_{p^n}^{\otimes^i}=0\Leftrightarrow Cl^{S_{i,n}}_{F_{i,n}}/p^n=0\Leftrightarrow(Cl^{S_{i,n}}_{F_{i,n}})_p=0$$

because in this situation $\Gamma_{i,n}$ is a (cyclic) p-group and hence Nakayama's lemma applies.

When p is odd or F is nonexceptional, the criterion of Theorem 1 is closely related with the triviality of the so-called i-th étale wild kernel of F, which is by defintion $Ш^2_S(F,\,\mathbb{Z}_p(i+1))$ and is isomorphic to $\mathrm{div}(K_{2i}(F)_p)$.

Theorem 2. *[Tate, Banaszak-Kolster, Keune, Schneider, Østvaer] Suppose that p is an odd prime or F is nonexceptional. For any $i\geq 1$, we have*

$$\mathrm{div}(K_{2i}(F)_p)\cong Ш^2_S(F,\,\mathbb{Z}_p(i+1))\cong\varprojlim\left(Cl^{S_{1,n}}_{F_{1,n}}\otimes\mu_{p^n}^{\otimes i}\right)_{\Gamma_{1,n}}$$

Proof. The first isomorphism is due to Tate (for $i=1$ and p odd, see [21]), Banaszak and Kolster (for any i and p odd, see [1], Theorem 3) and Østvaer (for any i, $p=2$ and F non exceptional, see [16], Theorem 9.5). The second isomorphism is due to Keune or Schneider (for p odd, see [10], Theorem 6.6, or [19], §6, Lemma 1) and Østvaer (for $p=2$ and F nonexceptional, see [16], Theorem 6.1). Note that the proof is now easy thanks to (10) since

$$Ш^2_S(F,\,\mathbb{Z}_p(i+1))\cong\varprojlim Ш^2_S(F,\,\mu_{p^n}^{\otimes i+1})\cong\varprojlim\left(Cl^{S_{i,n}}_{F_{i,n}}\otimes\mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}}\cong\varprojlim\left(Cl^{S_{1,n}}_{F_{1,n}}\otimes\mu_{p^n}^{\otimes i}\right)_{\Gamma_{1,n}}$$

(for the latter isomorphism see (12)). □

Remark 6. As we have already remarked in the introduction of this paper, the condition $\mathrm{div}(K_{2i}(F)_p) = 0$ is certainly necessary for the localization sequence for $K_{2i}(F)_p$ to split since

$$\mathrm{div}(K_{2i}(\mathcal{O}_F)_p) = \mathrm{div}\left(\bigoplus_{v\nmid p\infty} K_{2i-1}(k_v)_p\right) = 0$$

We can check that the criterion of Theorem 1 is indeed consistent with Theorem 2, namely we can give another proof of the fact that, if $\left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}} = 0$ for any $n \in \mathbb{N}$, then $\mathrm{div}(K_{2i}(F)_p) = 0$. This is not difficult thanks to the description given in Theorem 2. For any $i \geq 1$, set

$$F_{i,\infty} = \bigcup_{n\in\mathbb{N}} F_{i,n} \quad \text{and} \quad \Gamma_{i,\infty} = \mathrm{Gal}(F_{i,\infty}/F)$$

Next observe that $F_{i,\infty} \subseteq F_{1,\infty}$ and using the definition of $F_{i,n}$ one easily shows that the restriction gives an isomorphism

$$\Gamma_{1,\infty} = \Gamma_{i,\infty} \times \Delta_{i,\infty}$$

where $\Delta_{i,\infty} = \mathrm{Gal}(F_{1,\infty}/F_{i,\infty}) = \langle \gamma \in \Gamma_{1,\infty} \,|\, \gamma^i = 1\rangle$. In particular $\Delta_{i,\infty}$ is finite of order coprime with p (since $\Gamma_{1,n} \cong \mathbb{Z}_p \times \mathrm{Gal}(F(\mu_p)/F)$ and $\mathbb{Z}_p$ has no nontrivial finite subgroups) and acts trivially on $\mathbb{Z}_p(i)$. Now set

$$X'_{i,\infty} = \varprojlim Cl_{F_{i,n}}^{S_{i,n}}$$

and note that

$$\left(X'_{i,\infty} \otimes \mathbb{Z}_p(i)\right)_{\Gamma_{i,\infty}} = \varprojlim \left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}}$$

We have $(X'_{1,\infty})_{\Delta_{i,\infty}} = X'_{i,\infty}$ since $\Delta_{i,\infty}$ has order coprime with p and therefore

$$\mathrm{div}(K_{2i}(F)_p) \cong \left(X'_{1,\infty} \otimes \mathbb{Z}_p(i)\right)_{\Gamma_{1,\infty}} = \left(\left(X'_{1,\infty}\right)_{\Delta_{i,\infty}} \otimes \mathbb{Z}_p(i)\right)_{\Gamma_{i,\infty}} = \left(X'_{i,\infty} \otimes \mathbb{Z}_p(i)\right)_{\Gamma_{i,\infty}} \tag{12}$$

which shows that, if $\left(Cl_{F_{i,n}}^{S_{i,n}} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}} = 0$ for any $n \in \mathbb{N}$, then $\mathrm{div}(K_{2i}(F)_p) = 0$. We will see in the next section (Example 2) that the converse of this statement is not true in general: in other words, for any $i \geq 1$, there exists a prime number p and a number field F such that the localization sequence for $K_{2i}(F)_p$ does not split but $\mathrm{div}(K_{2i}(F)_p) = 0$.

4 Examples.

To begin with we analyze the simplest case, namely $F = \mathbb{Q}$. Let p be an odd prime and let $A_{i,n}$ denote the p-Sylow subgroup of $Cl_{F_{i,n}}^{S_{i,n}}$ where $F_{i,n} = \mathbb{Q}(\mu_{p^n}^{\otimes i})$. Let $K_{1,n}$ be the n-th level of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Set $\Delta_{1,n} = \mathrm{Gal}(F_{1,n}/K_n)$ and for every $j \in \mathbb{Z}$, let $A_{1,n}^{(j)}$ denote the ω^j-component of $A_{1,n}$ where $\omega : \Delta_n \to \mathbb{Z}_p^\times$ denotes the Teichmüller character. We need the following well known result.

Theorem 3 (Kurihara)**.** *Let p be an odd prime. For any $i \geq 1$,*

$$H^2(G_{\mathbb{Q},S}, \mathbb{Z}_p(i+1)) = 0 \Longleftrightarrow A^{(-i)} = 0$$

where A is the p-Sylow of the class group of $\mathbb{Q}(\mu_p)$. In particular, the triviality of $H^2(G_{\mathbb{Q},S}, \mathbb{Z}_p(i+1))$ only depends on the class of i modulo $p-1$.

Proof. See [11], Corollary 1.5. □

Example 1. Let F be the field of rational numbers. Since $\mathrm{Gal}(F_{1,n}/F_{i,n})$ acts trivially on $\mu_{p^n}^{\otimes i}$, for every $n \geq 1$ we have

$$\left(A_{1,n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{1,n}} = \left((A_{1,n})_{\mathrm{Gal}(F_{1,n}/F_{i,n})} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}} = \left(A_{i,n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_{i,n}}$$

the last equality coming from the fact that there is only one ramified prime in $F_{1,n}/\mathbb{Q}$ and it is totally ramified (use [22], Proposition 13.22). By Nakayama's lemma

$$\left(\left(A_{1,n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Delta_{1,n}}\right)_{\mathrm{Gal}(K_{1,n}/\mathbb{Q})} = 0 \Longleftrightarrow \left(A_{1,n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Delta_{1,n}} = 0$$

Furthermore

$$\left(A_{1,n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Delta_{1,n}} \cong A_{1,n}^{(-i)}$$

Moreover, it is easy to see that for any $n \geq 1$

$$A_{1,n}^{(-i)} = 0 \Longleftrightarrow A_{1,1}^{(-i)} = 0$$

Hence the localization sequence for $K_{2i}(\mathbb{Q})_p$ splits if and only if $A_{1,1}^{(-i)}$ is trivial. Therefore, by Theorem 3, the localization sequence for $K_{2i}(\mathbb{Q})_p$ splits if and only if $H^2(G_{\mathbb{Q},S}, \mathbb{Z}_p(i+1)) = 0$ (the latter is equivalent to $K_{2i}(\mathbb{Z})_p = 0$ thanks to the Quillen-Lichtenbaum "conjecture"). Moreover, it can be easily proved that $Ш^2_S(\mathbb{Q}, \mathbb{Z}_p(i+1)) \cong H^2_{\acute{e}t}(G_{\mathbb{Q},S}, \mathbb{Z}_p(i+1))$: therefore in this case the triviality of $\mathrm{div}(K_{2i}(\mathbb{Q})_p)$ is a necessary and sufficient condition for the localization sequence for $K_{2i}(\mathbb{Q})_p$ to split.
It is worth noting that the order of $K_{4i-2}(\mathbb{Z})$ is known: if B_i denotes the i-th Bernoulli number and c_i denotes the numerator of $B_i/4i$, then $|K_{4i-2}(\mathbb{Z})|$ is c_i if i is even and $2c_i$ if i is odd. Furthermore $K_{4i-2}(\mathbb{Z})$ is known to be cyclic for $4i-2 < 20000$. As for $K_{4i}(\mathbb{Z})$, we know that $|K_{4i}(\mathbb{Z})|$ has no prime factor smaller than 10^7. Of course, Vandiver's conjecture predicts that $K_{4i-2}(\mathbb{Z})$ is cyclic and $K_{4i}(\mathbb{Z})$ is trivial for any $i \geq 1$ (for these final remarks, see for instance [25], Introduction).

Anyway in general the condition $Ш^2_S(F, \mathbb{Z}_p(i+1)) = 0$ is weaker than the condition of Theorem 1, as we will show in the next example. First we need the following criterion.

Proposition 3. *Let p be an odd prime and let $F/\mathbb{Q}$ be finite Galois extension such that*

- $\mu_p \subseteq F$;
- $(Cl_F^S)_p \cong \mathbb{Z}/p\mathbb{Z}$;
- *there is only one prime above p in F;*
- $F(\mu_{p^2})/F$ *is a nontrivial extension where every prime over p is totally split.*

Then, for any $i \geq 1$, $Ш^2_S(F, \mathbb{Z}_p(i+1))$ is trivial but the localization sequence for $K_{2i}(F)_p$ does not split.

Proof. We are going to use Jaulent's theory of logarithmic classes: for notation and basic results the reader is referred to [7]. For any $F/\mathbb{Q}$ finite and Galois (even not satisfying the hypotheses) we have an exact sequence (see [2], §3)

$$0 \to \widetilde{Cl}_F(p) \to \widetilde{Cl}_F \xrightarrow{\varphi} (Cl_F^S)_p \longrightarrow \deg_F \mathcal{D}\ell/(\deg_F \mathfrak{p})\mathbb{Z}_p \to 0 \tag{13}$$

where $\mathfrak{p}$ is any prime of F over p. Moreover

$$\deg_F \mathcal{D}\ell = p[F \cap \widehat{\mathbb{Q}}^c : \mathbb{Q}]\mathbb{Z}_p \quad \text{and} \quad \deg_F \mathfrak{p} = \tilde{f}_{\mathfrak{p}} \cdot \deg p = [F_{\mathfrak{p}} \cap \widehat{\mathbb{Q}}_p^c : \mathbb{Q}_p] \cdot p$$

where $\widehat{\mathbb{Q}}^c$ (resp. $\widehat{\mathbb{Q}}_p^c$) is the cyclotomic $\widehat{\mathbb{Z}}$-extension of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$). Now we want to compare $[F \cap \widehat{\mathbb{Q}}^c : \mathbb{Q}]$ and $[F_{\mathfrak{p}} \cap \widehat{\mathbb{Q}}_p^c : \mathbb{Q}_p]$. We have $v_p([F \cap \widehat{\mathbb{Q}}^c : \mathbb{Q}]) = 0$ thanks to the first and the fourth hypothesis. The fourth hypothesis also implies that $v_p([F_{\mathfrak{p}} \cap \widehat{\mathbb{Q}}_p^c : \mathbb{Q}_p]) = s \geq 1$. In other words

$$\deg_F \mathcal{D}\ell/(\deg_F \mathfrak{p})\mathbb{Z}_p \cong \mathbb{Z}/p^s\mathbb{Z}$$

with $s \geq 1$. Now the third hypothesis implies $\widetilde{Cl}(p) = 0$ (see [2], Lemma 4): therefore (13) implies that $s = 1$ and $\widetilde{Cl}_F = 0$ because of the second hypothesis. Since $\mu_p \subseteq F$, we can use the isomorphism (see [8], Théorème 3)

$$\mu_p^{\otimes i} \otimes \widetilde{Cl}_F \cong Ш_S^2(F, \mathbb{Z}_p(i+1))/pШ_S^2(F, \mathbb{Z}_p(i+1))$$

to deduce that $Ш_S^2(F, \mathbb{Z}_p(i+1)) = 0$. On the other hand $(Cl_F^S)_p$ is nontrivial, hence Theorem 1 tells us that the localization sequence for $K_{2i}(F)_p$ does not split. □

Example 2. We have to find a number field F and an odd prime p satisfying the hypotheses of Proposition 3. We proceed as follows: we choose an odd prime p and a prime ℓ such that $\ell \equiv 1 \ (mod\ p)$: this ensures that $\mathbb{Q}(\mu_\ell)$ has exactly one subextension of degree p which we call E. Let K be the subextension of degree p of $\mathbb{Q}(\mu_{p^2})$: then EK is an abelian number field whose Galois group is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Let F' be a subextension of degree p of EK which is different from E and K. Now, if the order of p modulo ℓ is not divisible by p, then E has to be totally split at 3. In particular, EK/F' is totally split at p and $F'/\mathbb{Q}$ has only one prime above p (which is totally ramified). We may then choose $F = F'(\mu_p)$: then the first, the third and the fourth hypotheses of Proposition 3 are satisfied. So we are left to find such a prime ℓ with the additional requirement that $(Cl_F^S)_p$ is cyclic of order p.
Choose $p = 3$ and $\ell = 61$: of course we have $61 \equiv 1 \ (mod\ 3)$ and 3 has order 10 modulo 61. We only have one choice for F' and computations with PARI ([17]) reveal that $F = F'(\mu_3) = \mathbb{Q}(\theta)$ where θ is a root of the polynomial $X^6 - 793X^3 + 226981$. We clearly have only one (totally ramified) prime above 3 in F and furthermore $(Cl_F^S)_3 \cong \mathbb{Z}/3\mathbb{Z}$. Then by Proposition 3, for any $i \geq 1$, we deduce that $Ш_S^2(F, \mathbb{Z}_3(i+1)) = 0$ but the localization sequence for $K_{2i}(F)_3$ does not split.

Remark 7. (i) It seems reasonable to conjecture that, for any $i \geq 1$ and any rational prime p, there exist infinitely many number fields F such that the localization sequence for $K_{2i}(F)_p$ does not split but $\mathrm{div}(K_{2i}(F)_p) = 0$.
(ii) As we have seen there exist a number field F and a prime p such that (for any $i \geq 1$) $\mathrm{div}(K_{2i}(F)_p) = 0$ but the localization sequence for $K_{2i}(F)_p$ does not split. However, for any number field F, any prime p and any $i \geq 1$, we have $\mathrm{div}(K_{2i}(F)_p) = 0$ if and only if $K_{2i}(F)_p$ is isomorphic to a direct sum of finite cyclic groups. This follows from a theorem of Prüfer (see [9], Theorem 11).

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References

[1] G. Banaszak, *Generalization of the Moore exact sequence and the wild kernel for higher K-groups*, Compositio Math. 86 no. 3 (1993), 281-305.

[2] F. Diaz y Diaz and F. Soriano-Gafiuk, *Approche algorithmique du groupe des classes logarithmiques*, J. Number Theory 76 (1999), 1-15.

[3] W.G. Dwyer and E.M. Friedlander, *Algebraic and étale K-theory*, Trans. Am. Math. Soc. 292 no. 1. (1985), 247-280.

[4] K. Hutchinson, *The 2-Sylow subgroup of the wild kernel of exceptional number fields*, J. Number Theory 87 (2001), 222-238.

[5] K. Hutchinson, *On tame and wild kernels of special number fields*, J. Number Theory 107 (2004), 368-391.

[6] K. Hutchinson, *Étale wild kernels of exceptional number fields*, J. Number Theory 120 (2006), 47-71.

[7] J.-F. Jaulent, *Classes logarithmiques des corps de nombres*, J. Théor. Nombres Bordeaux 6 (1994), 301-325.

[8] J.-F. Jaulent and A. Michel, *Approche logarithmique des noyaux étales sauvages des corps de nombres*, J. Number Theory 120 no. 1 (2006), 72-91.

[9] I. Kaplansky, *Infinite abelian groups*, University of Michigan Publications n. 2, 1954.

[10] F. Keune, *On the structure of the K_2 of the ring of integers in a number field*, K-Theory 2 no. 5 (1989), 625-645.

[11] M. Kurihara, *Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbb{Z}$*, Compositio Math. 81 no. 2 (1992), 223-236.

[12] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. 9 (1969/70), 318-344.

[13] J. Milnor, *Introduction to algebraic K-theory*, Priceton University Press and Tokyo University Press, 1971.

[14] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren der mathematischen Wissenschaften 323, Springer-Verlag, 2000.

[15] T. Nguyen Quang Do, *Théorie d'Iwasawa des noyaux sauvages étales d'un corps de nombres*, Publ. Math. UFR Sci. Tech. Besançon, 2002, 9 pp.

[16] P. A. Østvaer, *Wild kernels at the prime* 2, Acta Arith. 112 no. 2 (2004), 109-130.

[17] PARI/GP, version 2.3.2, Bordeaux, 2007, http://pari.math.u-bordeaux.fr/.

[18] J. Rognes and C. Weibel, *Étale descent for two-primary algebraic K-theory of totally imaginary number fields*, K-Theory 16 no. 2 (1999), 101-104.

[19] P. Schneider, *Über gewisse Galoiscohomologiegruppen*, Math. Z. 168 no. 2 (1979), 181-205.

[20] C. Soulé, *K-théorie des anneaux d'entiers des corps de nombres et cohomologie étale*, Invent. Math. 55 (1979), 251-295.

[21] J. Tate, *Relation between K_2 and Galois cohomology*, Invent. Math. 36 (1976), 257-274.

[22] L. C. Washington, *Introduction to cyclotomic fields*, GTM 83, Springer-Verlag 1997.

[23] C. Weibel, *Étale Chern classes at the prime* 2, in *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluwer Acad. Publ., 1993, 249-286.

[24] C. Weibel, *Higher wild kernels and divisibility in the K-theory of number fields*, J. Pure Appl. Algebra 206 no. 1-2 (2006), 222-244.

[25] C. Weibel, *Algebraic K-theory of rings of integers in local and global fields*, n E. M. Friedlander and D. R. Grayson (Editors), *Handbook of K-theory volume 1*, Springer, 2005.

[26] C. Weibel, *The norm residue isomorphism theorem*, J. Topol. no. 2 (2009), 346-372.

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