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Karim Kellay

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SETS OF UNIQUENESS FOR DIRICHLET-TYPE SPACES

KARIM KELLAY

ABSTRACT. We study the uniqueness sets on the unit circle for weighted Dirichlet spaces.

1. INTRODUCTION

Let \mathbb{D} be the open unit disc in the complex plane, and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle. Let H^2 denote the Hardy space of analytic functions on \mathbb{D} . If μ is a positive Borel measure on the unit circle \mathbb{T} , the Dirichlet-type space $\mathcal{D}(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty,$$

where $dA(z) = dx dy / \pi$ stands for the normalized area measure in \mathbb{D} and P_μ is the Poisson integral of μ

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

The space $\mathcal{D}(\mu)$ is endowed with the norm

$$\|f\|_\mu^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

Since $\mathcal{D}(\mu) \subset H^2$, every function $f \in \mathcal{D}(\mu)$ has non-tangential limits almost everywhere on \mathbb{T} . We denote by $f(\zeta)$ the non-tangential limit of f at $\zeta \in \mathbb{T}$ if it exists. It turns out that there is a useful formula for expressing the norm of the Dirichlet-type space in terms of the local Dirichlet integral

$$\mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty.$$

where $\mathcal{D}_\xi(f)$ is the local Dirichlet integral of f at $\xi \in \mathbb{T}$ given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \frac{|f(e^{it}) - f(\xi)|^2 dt}{|e^{it} - \xi|^2} \frac{dt}{2\pi}.$$

For a proof of this see [13, Proposition 2.2]. Note that if $d\mu(e^{it}) = dt/2\pi$, the normalized arc measure on \mathbb{T} , then the space $\mathcal{D}(\mu)$ coincides with the classical space of functions with finite Dirichlet integral. These spaces were introduced by Richter [11] and generalized by Aleman [1] for nonnegative finite Borel measure on $\overline{\mathbb{D}}$. The spaces $\mathcal{D}(\mu)$ were studied in [1, 11, 12, 13, 14, 15, 17].

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Let $\mathcal{D}^h(\mu)$ be the harmonic version of $\mathcal{D}(\mu)$ given by

$$\mathcal{D}^h(\mu) := \{f \in L^2(\mathbb{T}) : D_\mu(f) < \infty\}.$$

We define the capacity C_μ of a set $E \subset \mathbb{T}$ by

$$C_\mu(E) := \inf \{\|f\|_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ a.e. on a neighborhood of } E\},$$

see [4, 5]. If $C_\mu(E) = 0$, then E has Lebesgue measure zero. Indeed, if $C_\mu(E) = 0$, then there exists a sequence $(f_i) \in \mathcal{D}^h(\mu)$ such that $\|f_i\|_\mu \leq 2^{-i}$ and $|f_i| \geq 1$ a.e. on a neighborhood of E . Then $f = \sum_i f_i \in \mathcal{D}^h(\mu)$ and $|f| = \infty$ on E . We have $\infty > \mathcal{D}_\mu(f) \geq \int_E D_\xi(f) d\mu(\xi)$ and this forces E to have measure zero. We say that a property holds C_μ -quasi-everywhere (C_μ -q.e.) if it holds everywhere outside a set of zero C_μ capacity. Note that C_μ -q.e. implies a.e. We have

$$C_\mu(E) := \inf \{\|f\|_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ } C_\mu\text{-q.e. on } E\}.$$

see [6, Theorem 4.2]. Every function $f \in \mathcal{D}(\mu)$ has non-tangential limits C_μ -quasi-everywhere on \mathbb{T} [4, Theorem 2.1.9]. Let E be a subset of \mathbb{T} . The set E is said to be a uniqueness set for $\mathcal{D}(\mu)$ if, for each $f \in \mathcal{D}(\mu)$ such that its non-tangential limit $f = 0$ on E , we have $f = 0$.

In order to state our main result, we define some notions. Given $E \subset \mathbb{T}$, we write $|E|$ for the Lebesgue measure of E . For $w \in L^1(\mathbb{T})$, we denote by $I(w)$ the mean of w over I

$$I[w] = \frac{1}{|I|} \int_I w(\zeta) |d\zeta|.$$

A nonnegative function w is a Muckenhoupt A_2 -weight if for all arc $I \subset \mathbb{T}$

$$\sup_{I \subset \mathbb{T}} I[w] I[w^{-1}] < +\infty.$$

Theorem 1.1. *Let μ be an absolutely continuous measure with respect to the Lebesgue measure on \mathbb{T} , $d\mu(\zeta) = w(\zeta) |d\zeta|$ and w is a Muckenhoupt A_2 -weight. Let E be a Borel subset of \mathbb{T} of Lebesgue measure zero. We assume that there exists a family of pairwise disjoint open arcs (I_n) of \mathbb{T} such that $E \subset \bigcup_n I_n$. Suppose*

$$\sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} = -\infty;$$

then E is a uniqueness set for $\mathcal{D}(\mu)$.

The case of the Dirichlet space, $d\mu(\zeta) = |d\zeta|/2\pi$, was obtained by Khavin and Maz'ya [9]; see also [2, 3, 8]. In [10], we give the generalization of their result in the Dirichlet spaces \mathcal{D}_s , $0 < s \leq 1$, which consist of all analytic functions $f \in H^2$ such that

$$\mathcal{D}_s(f) := \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^{1+s}} \frac{|d\zeta|}{2\pi} \frac{|d\xi|}{2\pi} \asymp \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-s} dA(z).$$

The remaining of the note is devoted to proof of the theorem.

2. PROOF

To prove our theorem, we use the following lemmas,

Lemma 2.1. *Let w be a Muckenhoupt A_2 -weight and let $d\mu(\zeta) = w(\zeta)|d\zeta|$, then*

(a) *If I is an arc of \mathbb{T} and ξ_I its center, then*

$$|I| \int_{\mathbb{T} \setminus I} \frac{d\mu(\zeta)}{|\zeta - \xi_I|^2} \leq cI[w],$$

for some positive constant C independent of I .

(b) *for all nonnegative function g and all arcs I of \mathbb{T}*

$$\left(\frac{1}{|I|} \int_I g(\zeta) |d\zeta| \right)^2 \leq \frac{1}{\mu(I)} \int_I g(\zeta)^2 d\mu(\zeta).$$

(c) *for all open arcs I ,*

$$\mu(I) \geq \frac{\mu(\mathbb{T})}{\pi c} |I|^2 \left(\log \frac{2\pi}{|I|} \right)^2.$$

Proof. For a proof of (a), see [7, Lemma 1] and [16] p.200 for (b). Let now to prove (c). By (b), we have $|\mathbb{T} \setminus I|^2 / |\mathbb{T}|^2 \leq \mu(\mathbb{T} \setminus I) / \mu(\mathbb{T})$. By (a) and (b) we get

$$\frac{\mu(I)}{|I|^2} \geq \frac{1}{c} \int_{\mathbb{T} \setminus I} \frac{d\mu(\zeta)}{|\zeta - \xi_I|^2} \geq \frac{\mu(\mathbb{T} \setminus I)}{c} \left(\frac{1}{|\mathbb{T} \setminus I|} \int_{\mathbb{T} \setminus I} \frac{|d\xi|}{|\zeta - \xi_I|} \right)^2 \geq \frac{4\mu(\mathbb{T})}{c|\mathbb{T}|^2} \left(\log \frac{2\pi}{|I|} \right)^2.$$

□

Let I be an open arc of \mathbb{T} and f be a function. We set

$$\mathcal{D}_{I,\mu}(f) := \int_I \int_I \frac{|f(z) - f(w)|^2 |dz|}{|z - w|^2} \frac{d\mu(w)}{2\pi} \quad \text{and} \quad m_I(f) := \frac{1}{|I|} \int_I |f(\xi)| |d\xi|.$$

Lemma 2.2. *Let $d\mu = wdm$ be a measure such that $w \in (A_2)$. Suppose that $0 < \gamma < 1$. Let $E \subset \mathbb{T}$ and $f \in \mathcal{D}(\mu)$ be such that $f|_E = 0$. Then, for any open arc $I \subset \mathbb{T}$ with $|I| \leq \gamma\pi$*

$$m_I(f)^2 \leq \kappa \frac{\mathcal{D}_{I,\mu}(f)}{C_\mu(E \cap I)},$$

where κ depending only on γ .

Proof. Without loss generality, we assume that $I = (e^{-i\theta}, e^{i\theta})$ with $\theta < \gamma\pi/2$. Let $J = (e^{-2i\theta/(1+\gamma)}, e^{2i\theta/(1+\gamma)})$ and \tilde{f} be such that

$$\tilde{f}(e^{it}) = \begin{cases} f(e^{it}), & e^{it} \in I, \\ f(e^{i\frac{3\theta - |t|}{2}}), & e^{it} \in J \setminus I. \end{cases}$$

Then by a change of variable, we get

$$\mathcal{D}_{I,\mu}(f) \asymp \mathcal{D}_{J,\mu}(\tilde{f}) \quad \text{and} \quad m_I(f) \asymp m_J(\tilde{f}), \quad (1)$$

where the implied constants depend only on γ , see [10].

Let $I_\gamma = (e^{-i\theta_\gamma}, e^{i\theta_\gamma})$ with $\theta_\gamma = \frac{3+\gamma}{2(1+\gamma)}\theta$. Note that $I \subset I_\gamma \subset J$. Let ϕ be a positive function on \mathbb{T} , $0 \leq \phi \leq 1$, such that $\text{supp } \phi = I_\gamma$, $\phi = 1$ on I and

$$|\phi(z) - \phi(w)| \leq \frac{c_1}{|J|}|z - w|, \quad z, w \in \mathbb{T}.$$

where c_1 depending only on γ .

Now, we consider the function

$$F(z) = \phi(z) \left| 1 - \frac{|\tilde{f}(z)|}{m_J(\tilde{f})} \right|, \quad z \in \mathbb{T}.$$

Hence $F \geq 0$ and $F = 1$ C_μ -q.e on $E \cap I$. Therefore,

$$C_\mu(E \cap I) \leq \|F\|_\mu^2. \quad (2)$$

We claim that

$$\|F\|_\mu^2 \leq \kappa \frac{\mathcal{D}_{I,\mu}(f)}{m_I(f)^2} \quad (3)$$

where κ depending only on γ . The Lemma 2.2 follows from (2) and (3).

Now, we prove the claim (3). We have

$$\begin{aligned} \|F\|_\mu^2 &= \int_{\mathbb{T}} |F(\zeta)|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} \frac{d\mu(\xi)}{2\pi} \\ &\leq \frac{1}{m_J(\tilde{f})^2} \int_J |m_J(\tilde{f}) - |\tilde{f}(\zeta)||^2 \frac{|d\zeta|}{2\pi} + \int_J \int_J \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} \frac{d\mu(\xi)}{2\pi} \\ &\quad + \frac{1}{m_J(\tilde{f})^2} \int_{\zeta \in \mathbb{T} \setminus J} \int_{\xi \in I_\gamma} \frac{|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} \frac{d\mu(\xi)}{2\pi} \\ &\quad + \frac{1}{m_J(\tilde{f})^2} \int_{\xi \in \mathbb{T} \setminus J} \int_{\zeta \in I_\gamma} \frac{|m_J(\tilde{f}) - |\tilde{f}(\zeta)||^2}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} \frac{d\mu(\xi)}{2\pi} \\ &= \frac{A}{2\pi m_J(\tilde{f})^2} + \frac{B}{4\pi^2} + \frac{C}{4\pi^2 m_J(\tilde{f})^2} + \frac{D}{4\pi^2 m_J(\tilde{f})^2}. \end{aligned} \quad (4)$$

Note that, by Lemma 2.1 (b)

$$|m_J(\tilde{f}) - |\tilde{f}(\zeta)||^2 \leq \left(\frac{1}{|J|} \int_J |\tilde{f}(\xi) - \tilde{f}(\zeta)| |d\xi| \right)^2 \leq \frac{1}{\mu(J)} \int_J |\tilde{f}(\xi) - \tilde{f}(\zeta)|^2 d\mu(\xi).$$

Hence by (1) and Lemma 2.1 (c)

$$\begin{aligned}
A &:= \int_J |m_J(\tilde{f}) - |\tilde{f}(\zeta)||^2 |d\zeta| \\
&\leq \frac{1}{\mu(J)} \int_J \int_J |\tilde{f}(\xi) - \tilde{f}(\zeta)|^2 d\mu(\xi) |d\zeta| \\
&\leq \frac{|J|^2}{\mu(J)} \int_J \int_J \frac{|\tilde{f}(\xi) - \tilde{f}(\zeta)|^2}{|\xi - \zeta|^2} d\mu(\xi) |d\zeta| \\
&\leq c_2 \mathcal{D}_{I,\mu}(f),
\end{aligned} \tag{5}$$

where c_2 depending only on γ .

Let us now estimate B . If $(\zeta, \xi) \in J \times J$, then we write

$$\begin{aligned}
|F(\zeta) - F(\xi)| &= \left| \phi(\zeta) \left(\left| 1 - \frac{|\tilde{f}(\zeta)|}{m_J(\tilde{f})} \right| - \left| 1 - \frac{|\tilde{f}(\xi)|}{m_J(\tilde{f})} \right| \right) + (\phi(\zeta) - \phi(\xi)) \left| 1 - \frac{|\tilde{f}(\xi)|}{m_J(\tilde{f})} \right| \right| \\
&\leq \frac{1}{m_J(\tilde{f})} |\tilde{f}(\zeta) - \tilde{f}(\xi)| + \frac{c_1}{m_J(\tilde{f})} \frac{|\zeta - \xi|}{|J|} |m_J(\tilde{f}) - |\tilde{f}(\xi)||.
\end{aligned} \tag{6}$$

Note that, by Cauchy-Schwarz,

$$|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2 \leq \frac{1}{|J|} \int_J |\tilde{f}(\eta) - \tilde{f}(\xi)|^2 |d\eta|. \tag{7}$$

So, by (6), (7) and (1)

$$\begin{aligned}
B &:= \int_J \int_J \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \frac{2}{m_J(\tilde{f})^2} \int_J \int_J \frac{|\tilde{f}(\zeta) - \tilde{f}(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\quad + \frac{2c_1^2}{m_J(\tilde{f})^2 |J|^3} \int_J \int_J \int_J |\tilde{f}(\eta) - \tilde{f}(\xi)|^2 |d\eta| |d\zeta| d\mu(\xi) \\
&\leq \frac{2 + 2c_1^2}{m_J(\tilde{f})^2} \int_J \int_J \frac{|\tilde{f}(\eta) - \tilde{f}(\xi)|^2}{|\eta - \xi|^2} |d\eta| d\mu(\xi) \\
&\leq c_3 \frac{\mathcal{D}_{I,\mu}(f)}{m_I(f)^2}.
\end{aligned} \tag{8}$$

where c_3 depending only on γ .

Next, using again (7) and (1)

$$\begin{aligned}
C &:= \int_{\zeta \in \mathbb{T} \setminus J} \int_{\xi \in I_\gamma} \frac{|m_J(\tilde{f}) - |\tilde{f}(\xi)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\zeta \in \mathbb{T} \setminus J} \frac{|d\zeta|}{d(\zeta, I_\gamma)^2} \int_{\xi \in I_\gamma} |m_J(\tilde{f}) - |\tilde{f}(\xi)||^2 d\mu(\xi) \\
&\leq \frac{c_4}{|J|^2} \int_J \int_J |\tilde{f}(\eta) - \tilde{f}(\xi)|^2 |d\eta| d\mu(\xi) \\
&\leq c_4 \int_J \int_J \frac{|\tilde{f}(\eta) - \tilde{f}(\xi)|^2}{|\eta - \xi|^2} |d\eta| d\mu(\xi) \\
&\leq c_5 \mathcal{D}_{I, \mu}(f),
\end{aligned} \tag{9}$$

where c_4, c_5 depend only on γ .

Finally, by Lemma (2.1) (a) and (b) and (1)

$$\begin{aligned}
D &:= \int_{\xi \in \mathbb{T} \setminus J} \int_{\zeta \in I_\gamma} \frac{|m_J(\tilde{f}) - |\tilde{f}(\zeta)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\xi \in \mathbb{T} \setminus J} \frac{d\mu(\xi)}{d(\xi, I_\gamma)^2} \int_{\zeta \in I_\gamma} |m_J(\tilde{f}) - |\tilde{f}(\zeta)||^2 |d\zeta| \\
&\leq c_6 \frac{\mu(J)}{|J|^2} \int_{\zeta \in I_\gamma} \frac{1}{\mu(J)} \int_J |\tilde{f}(\eta) - \tilde{f}(\zeta)|^2 d\mu(\eta) |d\zeta| \\
&\leq \frac{c_6}{|J|^2} \int_{I_\gamma} \int_J |\tilde{f}(\eta) - \tilde{f}(\zeta)|^2 |d\eta| d\mu(\zeta) \\
&\leq c_6 \int_J \int_J \frac{|\tilde{f}(\eta) - \tilde{f}(\zeta)|^2}{|\eta - \zeta|^2} |d\eta| d\mu(\zeta) \\
&\leq c_7 \mathcal{D}_{I, \mu}(f),
\end{aligned} \tag{10}$$

where c_6, c_7 depend only on γ .

By (5), (8), (9) and (10) we get (3) and the proof is complete. \square

Proof of Theorem 1.1. Since $|E| = 0$, we can assume that $\sup_n |I_n| \leq \gamma\pi$ with $\gamma \in (0, 1)$. Let $f \in \mathcal{D}(\mu)$ be such that $f|_E = 0$. We set $\ell = \sum_n |I_n|$. By Lemma 2.2 and Jensen's

inequality

$$\begin{aligned}
\int_{\bigcup I_n} \log |f(\xi)| |d\xi| &= \sum_n |I_n| \frac{1}{|I_n|} \int_{I_n} \log |f(\xi)| |d\xi| \\
&\leq \sum_n |I_n| \log \frac{1}{|I_n|} \int_{I_n} |f(\xi)| |d\xi| \\
&\leq \sum_n |I_n| \log \left(\frac{\kappa \mathcal{D}_{I_n, \mu}(f)}{C_\mu(E \cap I_n)} \right) \\
&= \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} + \ell \sum_n \frac{|I_n|}{\ell} \log \left(\frac{\kappa \mathcal{D}_{I_n, \mu}(f)}{|I_n|} \right) \\
&\leq \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} + \ell \log \left(\frac{\kappa}{\ell} \sum_n \mathcal{D}_{I_n, \mu}(f) \right) \\
&\leq \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} + \ell \log \left(\frac{\kappa}{\ell} D_\mu(f) \right) = -\infty.
\end{aligned}$$

By Fatou's Theorem we obtain $f = 0$ and the proof is complete.

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