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# SETS OF UNIQUENESS FOR DIRICHLET-TYPE SPACES

KARIM KELLAY

ABSTRACT. We study the uniqueness sets on the unit circle for weighted Dirichlet spaces.

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane and let  $\mathbb{T} = \partial\mathbb{D}$  be the unit circle. For  $\mu$  a positive Borel measure on the unit circle  $\mathbb{T}$ , the Dirichlet-type space  $\mathcal{D}(\mu)$  consists of all analytic functions  $f$  defined on  $\mathbb{D}$  such that

$$\mathcal{D}_\mu(f) := \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty.$$

where  $\mathcal{D}_\xi(f)$  is the local Dirichlet integral of  $f$  at  $\xi \in \mathbb{T}$  given by

$$\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \frac{|f(e^{it}) - f(\xi)|^2}{|e^{it} - \xi|^2} \frac{dt}{2\pi}.$$

The space  $\mathcal{D}(\mu)$  is endowed with the norm

$$\|f\|_\mu^2 := \|f\|_{H^2}^2 + \mathcal{D}_\mu(f).$$

Note that if  $d\mu(e^{it}) = dt/2\pi$ , the normalized arc measure on  $\mathbb{T}$ , then the space  $\mathcal{D}(\mu)$  coincides with the classical space of functions with finite Dirichlet integral. These spaces were introduced by Richter [11] and by Aleman [1]. The space  $\mathcal{D}(\mu)$  were studied in [1, 11, 12, 13, 14, 15, 16].

Let  $\mathcal{D}^h(\mu)$  the harmonic version of  $\mathcal{D}(\mu)$  given by

$$\mathcal{D}^h(\mu) := \{f \in L^2(\mathbb{T}) : \mathcal{D}_\mu(f) < \infty\}.$$

We define the capacity  $C_\mu$  of a set  $E \subset \mathbb{T}$  by

$$C_\mu(E) := \inf \{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ a.e. on a neighborhood of } E \},$$

see [4, 5]. We say that the property hold  $C_\mu$ -quasi-everywhere ( $C_\mu$ -q.e) if it holds everywhere outside a set of zero  $C_\mu$  capacity. We have

$$C_\mu(E) := \inf \{ \|f\|_\mu^2 : f \in \mathcal{D}^h(\mu) \text{ and } |f| \geq 1 \text{ } C_\mu\text{-q.e on } E \}.$$

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see [7, Theorem 4.2]. Since  $\mathcal{D}(\mu) \subset H^2$ , every function  $f \in \mathcal{D}(\mu)$  has non-tangential limits  $f^*$  almost everywhere on  $\mathbb{T}$ . We have more, the radial limit of  $f \in \mathcal{D}^h(\mu)$  given by  $f^*(\zeta) := \lim_{r \rightarrow 1^-} f(r\zeta)$  exists  $C_\mu$ -quasi-everywhere [4, Theorem 2.1.9].

Let  $E$  be a subset of  $\mathbb{T}$ . The set  $E$  is said to be a uniqueness set for  $\mathcal{D}(\mu)$  if, for each  $f \in \mathcal{D}(\mu)$  such that  $f^* = 0$  on  $E$ , we have  $f = 0$ .

In order to state our main result, we introduce some notion. Given  $E \subset \mathbb{T}$ , we write  $|E|$  for the Lebesgue measure of  $E$ . Let  $0 \leq \alpha < 1$ , we call the measure  $\mu$  is  $\alpha$ -regular if for all open set  $O \subset \mathbb{T}$ , all open arc  $J$  and  $\gamma \in (0, 1)$ , we have

$$\left\{ \begin{array}{l} \mu(O) \geq c(\alpha)|O|^{1-\alpha}, \\ \int_{\mathbb{T} \setminus J} \frac{d\mu(\zeta)}{d(\zeta, \gamma J)^2} \leq \frac{c(\alpha, \gamma)}{|J|^{1+\alpha}}. \end{array} \right.$$

The Lebesgue measure  $d\mu(\zeta) = |d\zeta|/2\pi$  is 0-regular, if  $d\mu(\zeta) = |1 - \zeta|^{-\alpha}|d\zeta|$ , the measure  $\mu$  is  $\alpha$  regular for  $\alpha \in (0, 1)$  and  $d\mu(\zeta) = d(\zeta, E)^{-\alpha}|d\zeta|$  is also  $\alpha$ -regular where  $E$  is a Cantor set see [6].

**Theorem 1.** *Let  $\mu$  be a  $\alpha$ -regular measure. Let  $E$  be a Borel subset of  $\mathbb{T}$  of Lebesgue measure zero. We assume that there exists a family of pairwise disjoint open arcs  $(I_n)$  of  $\mathbb{T}$  such that  $E \subset \bigcup_n I_n$ . Suppose*

$$\sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} = -\infty,$$

*then  $E$  is a uniqueness set for  $\mathcal{D}(\mu)$ .*

The case of Dirichlet space,  $d\mu = dm$ , was obtained by Khavin and Mazya [9], see also [4, 3, 8]. In [10], we give the generalization of their result in the Dirichlet spaces  $\mathcal{D}_s$ ,  $0 < s \leq 1$ , which consists of all analytic functions  $f$  on the open disc  $\mathbb{D}$  such that

$$\mathcal{D}_s(f) = \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(z) - f(w)|^2 |dz| |dw|}{|z - w|^{1+s} 2\pi 2\pi}.$$

## 2. PROOF

Let  $I$  be an open arcs of  $\mathbb{T}$  and  $f$  be a function. We set

$$\mathcal{D}_{I,\mu}(f) := \int_I \int_I \frac{|f(z) - f(w)|^2 |dz|}{|z - w|^2} \frac{d\mu(w)}{2\pi} \quad \text{and} \quad m_I(f) := \frac{1}{|I|} \int_I |f(\xi)| |d\xi|$$

We begin with the following key lemma

**Lemma 2.** *Let  $\mu$  be a  $\alpha$ -regular measure. Suppose that  $0 < \gamma < 1$ . Let  $E \subset \mathbb{T}$  and  $f \in \mathcal{D}(\mu)$  be such that  $f^*|_E = 0$ . Then, for any open arc  $I \subset \mathbb{T}$  with  $|I| \leq \gamma\pi$*

$$m_I(f)^2 \lesssim \frac{\mathcal{D}_{I,\mu}(f)}{C_\mu(E \cap I)},$$

*where the implied constants depend only on  $\alpha$  and  $\gamma$ .*

*Proof.* We assume that  $I = (e^{-i\theta}, e^{i\theta})$  with  $\theta < \gamma\pi/2$ . Let  $J = (e^{-2i\theta/(1+\gamma)}, e^{2i\theta/(1+\gamma)})$  and  $I_\gamma = (e^{-i\theta_\gamma}, e^{i\theta_\gamma})$  with  $\theta_\gamma = \frac{3+\gamma}{2(1+\gamma)}\theta$ . Note that  $I \subset I_\gamma \subset J$ . Let  $\phi$  be a positive function on  $\mathbb{T}$  such that  $\text{supp } \phi = I_\gamma$ ,  $\phi = 1$  on  $I$  and  $|\phi(z) - \phi(w)| \lesssim |z - w|/|J|$  for  $z, w \in \mathbb{T}$ .

Let  $\tilde{f}$  be such that

$$\tilde{f}(e^{it}) = \begin{cases} f(e^{it}), & e^{it} \in I, \\ f(e^{i\frac{3\theta-|t|}{2}}), & e^{it} \in J \setminus I. \end{cases}$$

Then by a change of variable, we get

$$\mathcal{D}_{I,\mu}(f) \asymp \mathcal{D}_{J,\mu}(\tilde{f}) \quad (1)$$

see [10]. Now, we consider the function

$$F(z) = \phi(z) \left| 1 - \frac{|\tilde{f}(z)|}{m_J(f)} \right|, \quad z \in \mathbb{T}.$$

Hence  $F \geq 0$ ,  $F|_{E \cap I} = 1$   $C_\mu$ -q.e. Therefore,

$$C_\mu(E \cap I) \leq \|F\|_\mu^2. \quad (2)$$

We claim that

$$\|F\|_\mu^2 \asymp \frac{\mathcal{D}_{I,\mu}(f)}{m_I(f)^2}. \quad (3)$$

The Lemma follows from (2) and (3). Now, we proof the claim (3). We have

$$\begin{aligned} \|F\|_\mu^2 &= \int_{\mathbb{T}} |F(\zeta)|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|F(\zeta) - F(\xi)|^2 |d\zeta|}{|\zeta - \xi|^2} \frac{d\mu(\xi)}{2\pi} \\ &\leq \frac{1}{m_J(f)^2} \int_J |m_J(f) - |\tilde{f}(\zeta)||^2 \frac{|d\zeta|}{2\pi} + \int_J \int_J \frac{|F(\zeta) - F(\xi)|^2 |d\zeta|}{|\zeta - \xi|^2} \frac{d\mu(\xi)}{2\pi} \\ &\quad + \frac{1}{m_J(f)^2} \int_{\zeta \in \mathbb{T} \setminus J} \int_{\xi \in I_\gamma} \frac{|m_J(f) - |\tilde{f}(\xi)||^2 |d\zeta|}{|\zeta - \xi|^2} \frac{d\mu(\xi)}{2\pi} \\ &\quad + \frac{1}{m_J(f)^2} \int_{\xi \in \mathbb{T} \setminus J} \int_{\zeta \in I_\gamma} \frac{|m_J(f) - |\tilde{f}(\zeta)||^2 |d\zeta|}{|\zeta - \xi|^2} \frac{d\mu(\xi)}{2\pi} \\ &= \frac{A}{2\pi m_J(f)^2} + \frac{B}{4\pi^2} + \frac{C}{4\pi^2 m_J(f)^2} + \frac{D}{4\pi^2 m_J(f)^2}. \end{aligned} \quad (4)$$

Since a measure  $\mu$  is  $\alpha$ -regular, we have for l.s.c positive function

$$\int_J g(\xi) |d\xi| \lesssim |J|^\alpha \int_J g(\xi) d\mu(\xi). \quad (5)$$

Indeed

$$\begin{aligned}
\int_J g(\xi) d\mu(\xi) &= \int \mu(\{\xi \in J : g(\xi) > t\}) dt \\
&\gtrsim \int |\{\xi \in J : g(\xi) > t\}|^{1-\alpha} dt \\
&\gtrsim |J|^{-\alpha} \int |\{\xi \in J : g(\xi) > t\}| dt \asymp |J|^{-\alpha} \int_J g(\xi) |d\xi|.
\end{aligned}$$

Hence by (1) and (5)

$$\begin{aligned}
A &:= \int_J |m_J(f) - |\tilde{f}(\zeta)||^2 |d\zeta| \\
&\leq \frac{1}{|J|} \int_J \int_J |\tilde{f}(\xi) - \tilde{f}(\zeta)|^2 |d\xi| |d\zeta| \\
&\lesssim \frac{1}{|J|^{1-\alpha}} \int_J \int_J |\tilde{f}(\xi) - \tilde{f}(\zeta)|^2 |d\xi| d\mu(\zeta) \\
&\lesssim \int_J \int_J \frac{|\tilde{f}(\xi) - \tilde{f}(\zeta)|^2}{|\xi - \zeta|^2} |d\xi| d\mu(\zeta) \asymp \mathcal{D}_{I,\mu}(\tilde{f}).
\end{aligned} \tag{6}$$

If  $(z, w) \in J \times J$ , then we write

$$\begin{aligned}
|F(z) - F(w)| &= \left| \phi(z) \left( \left| 1 - \frac{|\tilde{f}(z)|}{m_J(f)} \right| - \left| 1 - \frac{|\tilde{f}(w)|}{m_J(f)} \right| \right) + (\phi(z) - \phi(w)) \left| 1 - \frac{|\tilde{f}(w)|}{m_J(f)} \right| \right| \\
&\lesssim \frac{1}{m_J(f)} |\tilde{f}(z) - \tilde{f}(w)| + \frac{1}{m_J(f)} \frac{|z - w|}{|J|} |m - |\tilde{f}(w)||.
\end{aligned}$$

So, by (1) again,

$$\begin{aligned}
B &:= \int_J \int_J \frac{|F(\zeta) - F(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\lesssim \frac{1}{m_J(f)^2} \int_J \int_J \frac{|\tilde{f}(\zeta) - \tilde{f}(\xi)|^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\quad + \frac{1}{m_J(f)^2 |J|^4} \int_J \int_J \left( \int_J |\tilde{f}(\eta) - \tilde{f}(w)| |d\eta| \right)^2 d\mu(\xi) |d\zeta| \\
&\lesssim \frac{1}{m_J(f)^2} \int_J \int_J \frac{|\tilde{f}(\eta) - \tilde{f}(\xi)|^2}{|\eta - \xi|^2} |d\eta| |d\mu(\xi)| \asymp \frac{\mathcal{D}_{I,\mu}(f)}{m_J(f)^2}.
\end{aligned} \tag{7}$$

Now,

$$\begin{aligned}
C &:= \int_{\zeta \in \mathbb{T} \setminus J} \int_{\xi \in I_\gamma} \frac{|m_J(f) - |\tilde{f}(\xi)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\zeta \in \mathbb{T} \setminus J} \frac{|d\zeta|}{d(\zeta, I_\gamma)^2} \int_{\xi \in I_\gamma} |m_J(f) - |\tilde{f}(\xi)||^2 d\mu(\xi) \\
&\lesssim \frac{1}{|J|^2} \int_{I_\gamma} \int_J |\tilde{f}(\eta) - \tilde{f}(\xi)|^2 |d\eta| d\mu(\xi) \\
&\lesssim \iint_{J \times J} \frac{|\tilde{f}(\eta) - \tilde{f}(\xi)|^2}{|\eta - \xi|^2} |d\zeta| d\mu(\xi) \asymp \mathcal{D}_{I, \mu}(f). \tag{8}
\end{aligned}$$

Finally, by (5)

$$\begin{aligned}
D &:= \int_{\xi \in \mathbb{T} \setminus J} \int_{\zeta \in I_\gamma} \frac{|m_J(f) - |\tilde{f}(\zeta)||^2}{|\zeta - \xi|^2} |d\zeta| d\mu(\xi) \\
&\leq \int_{\xi \in \mathbb{T} \setminus J} \frac{d\mu(\xi)}{d(\xi, I_\gamma)^2} \int_{\zeta \in I_\gamma} |m_J(f) - |\tilde{f}(\zeta)||^2 |d\zeta| \\
&\lesssim \frac{1}{|J|^{2+\alpha}} \int_{I_\gamma} \int_J |\tilde{f}(\eta) - \tilde{f}(\zeta)|^2 |d\eta| |d\zeta| \\
&\lesssim \frac{1}{|J|^2} \int_{I_\gamma} \int_J |\tilde{f}(\eta) - \tilde{f}(\zeta)|^2 |d\eta| d\mu(\zeta) \\
&\lesssim \iint_{J \times J} \frac{|\tilde{f}(\eta) - \tilde{f}(\zeta)|^2}{|\eta - \zeta|^2} |d\zeta| d\mu(\zeta) \asymp \mathcal{D}_{I, \mu}(f). \tag{9}
\end{aligned}$$

By (6), (7), (8), and (8) we get (3) and the proof is complete.  $\square$

**Proof of Theorem 1.** Since  $|E| = 0$ , we can assume that  $\sup_n |I_n| \leq \gamma\pi$  with  $\gamma \in (0, 1)$ . Let  $f \in \mathcal{D}(\mu)$  be such that  $f^*|E = 0$ . We set  $\ell = \sum_n |I_n|$ . By Lemma 2 and Jensen inequality

$$\begin{aligned}
\int_{\cup I_n} \log |f(\xi)| |d\xi| &\lesssim \sum_n |I_n| \log \left( \frac{c\mathcal{D}_{I_n, \mu}(f)}{C_\mu(E \cap I_n)} \right) \\
&= \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} + \ell \sum_n \frac{|I_n|}{\ell} \log \left( \frac{c\mathcal{D}_{I_n, \mu}(f)}{|I_n|} \right) \\
&\lesssim \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} + \ell \log \left( \frac{c}{\ell} \sum_n \mathcal{D}_{I_n, \mu}(f) \right) \\
&\lesssim \sum_n |I_n| \log \frac{|I_n|}{C_\mu(E \cap I_n)} + \ell \log \left( \frac{c}{\ell} D_\mu(f) \right) = -\infty.
\end{aligned}$$

By Fatou Theorem we obtain  $f = 0$  and the proof is complete.

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