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► **To cite this version:**

Edin Omerspahic, Kjell Mattiasson. Oriented Damage in Ductile Sheets: Constitutive Modeling and Numerical Integration. International Journal of Damage Mechanics, 2007, 16 (1), pp.35-56. 10.1177/1056789507060776 . hal-00571165

**HAL Id: hal-00571165**

**<https://hal.science/hal-00571165>**

Submitted on 1 Mar 2011

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# Oriented Damage in Ductile Sheets: Constitutive Modeling and Numerical Integration

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**ABSTRACT:** Thermodynamics with internal variables provides a framework for constitutive modeling of elasto-plastic deformations. Within the scope of the theory, constitutive and evolution equations for ductile, elasto-plastic materials with mixed (isotropic and kinematic) hardening and anisotropic damage have been developed. Postulates within continuum damage mechanics were used in order to incorporate damage as an internal variable. Owing to this, and to a simplified definition of the inverted damage effect tensor, a general expression for degradation of the elastic properties in materials has been obtained. The corresponding numerical algorithm for integration of the constitutive equations is based on an elastic predictor – plastic/damage corrector procedure. The plastic/damage corrector is uncoupled, which further simplifies and expedites the corrector procedure.

**KEY WORDS:** constitutive modeling, anisotropic damage, small strain elasticity, large strain plasticity, mixed hardening, numerical integration.

## INTRODUCTION

**T**HERE HAS BEEN much discussion about modeling the progressive material stiffness degradation (which the material damage essentially is)

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that leads to the initiation of a macroscopic crack. From the pioneering work of Kachanov (1958), through the early works on creep damage by Rabotnov (1968) and Odqvist and Hult (1961), until today, damage modeling has been extended to different application fields.

In order to describe ductile failure in metals, a number of damage models have been developed; most of them being based on an isotropic damage variable, specifying the same damage value in all directions. As the isotropic damage considers microcracks or microdefects with a uniform orientation, which is not always the case, the concept of anisotropic damage was developed. The work of Cardebois and Sidoroff (1982) was pioneering in modeling a ductile anisotropic damage. Their concept is based on elastic energy equivalence.

Anisotropic damage is presented in the theory of damage mechanics through the damage effect tensor,  $\tilde{\mathbf{M}}$ , introduced by means of the concept of the effective stress, relating the so-called effective stress to the area that efficiently resists the load. Chow and Wang (1987a, b) introduced a generalized description of this tensor and the corresponding material model in the principal damage directions. Zhu and Cescotto (1995) used this anisotropic concept and the energy equivalence stated by Cardebois and Sidoroff (1982) to develop a damage model at finite strains for ductile fracture. Their model takes into account three major anisotropies (elasticity, plasticity, and damage) and proposes a specific damage characteristic tensor,  $\mathbf{J}$  (due to damage anisotropy). A micro-crack opening and closing mechanism is considered as well, taking into consideration different effects of tensile and compressive states. Further concepts of the damage effect tensor have been developed and discussed in more detail by Betten (1983, 2001a, b, 2005), Chow and Lu (1992), Skrzypek and Ganczarski (1999, 2003), Zheng and Betten (1996), among others.

The anisotropic aspect of the material damage was exploited in the irreversible thermodynamic theory of Hayakawa et al. (1998). They derived the elastic law from an isotropic scalar function, expressed as the combination of ten basic invariants of two symmetric tensors of rank two. In this article, in comparison to the work of Zhu and Cescotto, the formulation of plasticity and damage i.e, the way the damage tensor was included in definition (reduction) of the plastic and damage yield surface, is somewhat simpler. Hayakawa et al. incorporated Chaboche's model (1993) of the crack closure effects under compressive stresses.

The model considered in the present study is partly based on the above-mentioned works, and is an extension of the previous work on rate-dependent viscoplastic 'orthotropic' damage (Omerspahic and Mattiasson, 2003), where only normal damage components influenced the damage

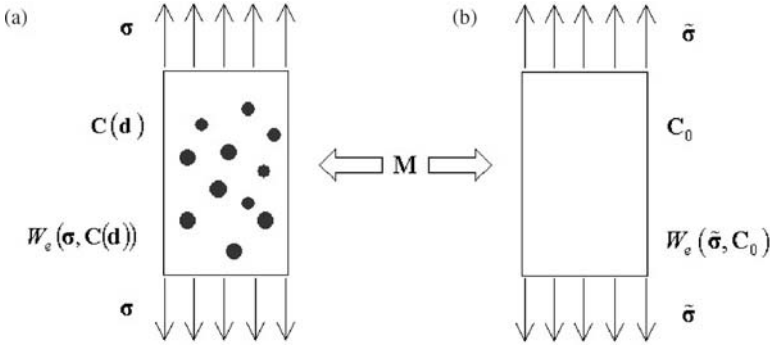
effect tensor (therefore the name orthotropic). In this concept, strain-rate dependency of damage was observed through the viscoplastic response, as damage components depend on the elastic strain components.

Chow and Wang's (1987a, b) model formulation in the principal damage directions has been used with the intention to construct a simple anisotropic damage model. Because the application field for the material model is cause for the damage development in sheets made of moderate ductile materials, simplifications concerning small elastic deformations and corotational formulations (the coordinate system is attached to and rotates together with shell elements), are possible. No micro-crack opening and closing mechanism is considered. The proposed model possesses a generalized derivation of the damage effect tensor. Owing to this, a simplified expression for the elastic stiffness degradation is obtained.

To sum up, the object of the study has been to construct an 'oriented' damage model, which can predict crack initiation in ductile sheets. In order to highlight this purpose, the methodology has been divided into five sections. The section on 'Integration of anisotropic damage' presents the incorporation of the damage variable into the expression for material stiffness. In the section 'Thermodynamic framework of elasto-plasticity', damage is discussed, followed by the 'Derivation of the constitutive and evolution equations' section. Finite deformation elasto-plastic laws proposed by Haupt (2000), Tsakmakis (1996), and Diegele et al. (2000), have been simplified by assuming small elastic deformations. These constitutive laws exhibit nonlinear kinematic and isotropic hardening. Numerical features of the resulting material model are presented in the section 'Numerical features of the constitutive model', and finally, an example involving a thin metal sheet with a central circular hole in the 'Computational results' section, which concludes the methodology part.

## INTEGRATION OF ANISOTROPIC DAMAGE

In order to incorporate damage as an internal variable into constitutive models, the concept of effective stress, stated by Rabotnov (1968), and the energy equivalence hypothesis, proposed by Cordebois and Sidoroff (1982), have been used. Essentially, the basic idea is the introduction of a fictitious material state (in a manner of small deformations), where damage effects are compensated through the effective Cauchy stress tensor  $\tilde{\boldsymbol{\sigma}}$  (Figure 1). This material state is equivalent to a (real) material state, where damage is seen through the stiffness degradation. Figure 1 shows two equivalent material states that are mapped by means of the two mentioned concepts.



**Figure 1.** Mapping between a real damaged state (a), and a fictitious state with the effective stresses (b).

The concept of effective stress relates an effective stress state to the area that effectively resists the load. This is achieved by using the damage effect tensor,  $\tilde{M}$ . In general, the effective stress tensor is a function of the damage effect tensor,

$$\tilde{\sigma} = \tilde{M} : \sigma. \quad (1)$$

The simplest way of introducing the damage effect tensor is in the stress tensor principal directions with a second-order tensor. Such a description is, however, unrealistic. For that reason, Chow and Wang (1987a, b) stated development of a generalized form of the fourth-order damage effect tensor in principal damage directions. In their formulation the damage effect tensor is reduced to a scalar for a case when the principal damage components are equal (isotropic damage). The inverse matrix form of the denoted tensor is given through a diagonal matrix,

$$\begin{aligned} [\tilde{M}]^{-1} = \text{diag} & \left[ 1 - d_{11} \quad 1 - d_{22} \quad 1 - d_{33} \quad \sqrt{1 - d_{22}} \sqrt{1 - d_{33}} \right. \\ & \left. \sqrt{1 - d_{11}} \sqrt{1 - d_{33}} \quad \sqrt{1 - d_{11}} \sqrt{1 - d_{22}} \right]. \end{aligned} \quad (2)$$

In order to define the anisotropic damage in an arbitrary coordinate system, one has to propose the corresponding tensor expression. In finite element implementations, second-order matrices are converted from fourth-order tensors by means of the Voigt rule. To facilitate differentiation of the constitutive and evolution equations, an approximation to the fourth-order damage effect tensor, which converts to the second-order matrix

Equation (2), can be derived<sup>1</sup>, Omerspahic (2002) defined the inverted damage effect tensor,  $\mathbf{M} = \tilde{\mathbf{M}}^{-1}$ , as

$$\mathbf{M} = \frac{1}{2}(\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{\delta}}) - \frac{1}{2}(\underline{\underline{\mathbf{d}}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{\mathbf{d}}})^1. \quad (3)$$

In addition, to obtain the equivalence between the two material states in Figure 1, the energy equivalence hypothesis is used. It states that the elastic energy for a damaged material is equal to the elastic energy for a fictitious undamaged material, loaded by the effective stress. More details about this hypothesis (and the strain equivalence) have been elaborated by Betten (2001a), Chow and Lu (1992), Skrzypek and Ganczarski (1999, 2003), Zheng and Betten (1996).

Assuming Hooke's elasticity, this hypothesis leads to a relationship between a damaged ( $\mathbf{C}(\mathbf{d})$ ) and an undamaged ( $\mathbf{C}_0$ ) elastic stiffness (fourth-order) tensor:

$$\mathbf{C}(\mathbf{d}) = \mathbf{M} : \mathbf{C}_0 : \mathbf{M}^T. \quad (4)$$

Making use of the index notation, the multiplication in the equation above is easy to perform. Furthermore, if second-order damage terms are neglected in the performed multiplication, the following expression for the stiffness tensor is obtained, Omerspahic (2002):

$$\begin{aligned} \mathbf{C}(\mathbf{d}) &= \mathbf{C}_0 - \mathbf{C}_d, \\ \mathbf{C}_0 &= \mu(\underline{\underline{\delta}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{\delta}}) + \lambda \underline{\underline{\delta}} \otimes \underline{\underline{\delta}}, \\ \mathbf{C}_d &= \mu(\underline{\underline{\mathbf{d}}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{\mathbf{d}}} + \underline{\underline{\mathbf{d}}} \otimes \underline{\underline{\mathbf{d}}} + \underline{\underline{\delta}} \otimes \underline{\underline{\mathbf{d}}}) + \lambda(\underline{\underline{\mathbf{d}}} \otimes \underline{\underline{\delta}} + \underline{\underline{\delta}} \otimes \underline{\underline{\mathbf{d}}}), \end{aligned} \quad (5)$$

where  $\mu$  and  $\lambda$  are Lamé constants. As a result, the damaged stiffness tensor is symmetric. To sum up, the stiffness degradation in this model is obtained through damage propagation (Figure 1(a)).

### THERMODYNAMIC FRAMEWORK OF ELASTO-PLASTICITY

Once damage is incorporated in the expression for elastic stiffness, Equation (5), constitutive modeling can be performed. As intended in the introductory section, the model has been derived with the object of incorporating damage propagation in elasto-plastic materials, exhibiting

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<sup>1</sup>The tensor notations and products are highlighted in the Appendix.

kinematic and isotropic hardening. Finite deformation plasticity laws in Haupt (2000), Tsakmakis (1996), and Diegele et al. (2000), are derived in the fictitious, stress-free, intermediate configuration using the so-called multiplicative decomposition of the deformation gradient. This means that the elasticity law is given as a relation between the Lagrangian elastic strain tensor, and the second Piola-Kirchhoff stress tensor, both defined in the intermediate configuration by the push-forward of the corresponding tensors in the reference configuration,

$$\bar{\mathbf{S}}_2 = f(\bar{\mathbf{E}}^e). \quad (6)$$

In this case, the total strain tensor has been additively decomposed in its elastic part and its plastic part,  $\bar{\mathbf{E}}^p$ , which creates the kinematic basis for a mathematical description of the plasticity.

In order to define a plastic deformation process, thermodynamics with internal variables have been successfully used by Lemaitre and Chaboche (1978), Lemaitre (1992) and Maugin (1999), among others. This theory supplies us with a framework for the global derivation of constitutive and evolution equations. In order to define the thermodynamic state of a system, this theory introduces internal variables of state, in addition to observable ones.

The internal variables in this model describe micro-structural changes generated by a plastic deformation process and a damage process. These changes are reflected through material hardening and damage propagation. In the mathematical description of the plasticity, isotropic hardening denotes an expansion of the yield surface, while kinematic hardening implies a translation of the yield surface in the stress space. Additionally, as mentioned in the ‘Integration of anisotropic damage’ section, the damage propagation generates the stiffness degradation.

The Helmholtz free energy per unit mass is a thermodynamic potential from which variables that characterize every thermodynamic property of the system are derived. Consequently, in this case, it is a function of observable variables, internal variables, and temperature. The Helmholtz free energy per mass unit depends on the following set of internal state variables: accumulated plastic strain,  $\kappa$ , describing the isotropic hardening; the backstrain tensor,  $\bar{\mathbf{z}}$ , representing the kinematic hardening; the second-order damage tensor,  $\mathbf{d}$ , and the internal variable describing damage hardening,  $\beta$ . Finally, by assuming an isothermal and adiabatic elastoplastic process with mixed hardening and damage (with damage hardening), the energy potential function depends on five variables,

$$\psi = \bar{\psi}(\bar{\mathbf{E}}^e, \bar{\mathbf{z}}, \kappa, \mathbf{d}, \beta). \quad (7)$$

Irreversibility of the isothermal and adiabatic thermodynamic process is provided by the Clausius-Duhem inequality. Haupt (2000) defined this inequality in the intermediate configuration as

$$-\rho_0 \dot{\psi} + \bar{\mathbf{S}}_2 : \dot{\bar{\mathbf{E}}}^e + \bar{\mathbf{P}} : \dot{\bar{\mathbf{E}}}^p \geq 0, \quad (8)$$

where  $\bar{\mathbf{P}}$  is the Mandel stress tensor in the intermediate configuration, and  $\dot{\bar{\mathbf{E}}}^p$  is the Oldroyd strain rate of the plastic strain tensor. The Mandel stress is equivalent to the second Piola-Kirchhoff stress for the case of small elastic deformations. This assumption implies as well that the intermediate and deformed configurations coincide, i.e.,  $\bar{\boldsymbol{\epsilon}}^e \approx \boldsymbol{\epsilon}^e$ . Consequently, the Helmholtz energy potential is  $\psi = \bar{\psi}(\boldsymbol{\epsilon}^e, \bar{\mathbf{z}}, \kappa, \mathbf{d}, \beta)$ . Besides the assumption of small elastic deformations the fact that the change of mass density can be neglected, leads to  $\bar{\mathbf{S}}_2 \approx \boldsymbol{\sigma}$ . Hence, the inequality above can be simplified as

$$-\rho \dot{\psi} + \boldsymbol{\sigma} : \left( \dot{\boldsymbol{\epsilon}}^e + \dot{\bar{\mathbf{E}}}^p \right) \geq 0. \quad (9)$$

Rewriting this inequality in order to further simplify the differentiation of the equations, gives the following:

$$\begin{aligned} & -\rho \left( \frac{\partial \psi}{\partial \boldsymbol{\epsilon}^e} \dot{\boldsymbol{\epsilon}}^e + \frac{\partial \psi}{\partial \mathbf{d}} \dot{\mathbf{d}} + \frac{\partial \psi}{\partial \bar{\mathbf{z}}} \dot{\bar{\mathbf{z}}} + \frac{\partial \psi}{\partial \kappa} \dot{\kappa} + \frac{\partial \psi}{\partial \beta} \dot{\beta} \right) + \boldsymbol{\sigma} : \left( \dot{\boldsymbol{\epsilon}}^e + \dot{\bar{\mathbf{E}}}^p \right) \\ & = \left( \boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\epsilon}^e} \right) : \dot{\boldsymbol{\epsilon}}^e + \boldsymbol{\sigma} : \dot{\bar{\mathbf{E}}}^p - \rho \frac{\partial \psi}{\partial \mathbf{d}} \dot{\mathbf{d}} - \rho \frac{\partial \psi}{\partial \bar{\mathbf{z}}} \dot{\bar{\mathbf{z}}} - \rho \frac{\partial \psi}{\partial \kappa} \dot{\kappa} - \rho \frac{\partial \psi}{\partial \beta} \dot{\beta} \\ & = \boldsymbol{\sigma} : \dot{\bar{\mathbf{E}}}^p - \bar{\mathbf{Y}} : \dot{\mathbf{d}} - \bar{\mathbf{X}} : \dot{\bar{\mathbf{z}}} - R \dot{\kappa} - B \dot{\beta} \geq 0. \end{aligned} \quad (10)$$

However, the dissipation inequality is the sum of products of rate variables and conjugated forces (dual variables to the set of internal and observable variables). Obviously, the set of thermodynamic forces is derived from the Helmholtz free energy: the Cauchy stress tensor ; the damage energy release rate,  $\bar{\mathbf{Y}}$ ; the backstress tensor,  $\bar{\mathbf{X}}$ ; the isotropic strain hardening scalar stress,  $R$ ; and  $B$ , the conjugated force of  $\beta$ :

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\epsilon}^e}, \quad (11)$$

$$\bar{\mathbf{Y}} = \rho \frac{\partial \psi}{\partial \mathbf{d}}, \quad (12)$$



$$\bar{\mathbf{X}} = \rho \frac{\partial \psi}{\partial \bar{\mathbf{z}}}, \quad (13)$$

$$R = \rho \frac{\partial \psi}{\partial \kappa}, \quad (14)$$

$$B = \rho \frac{\partial \psi}{\partial \beta}. \quad (15)$$

## DERIVATION OF THE CONSTITUTIVE AND EVOLUTION EQUATIONS

### Constitutive Equations

In order to establish the Helmholtz free energy, the resumé in Chaboche (1993) and Hayakawa et al. (1998) has been followed. Energies involved in the elastic process, plastic flow, and damage process are uncoupled. Consequently, the total free energy can be written as the sum of three contributions:

$$\psi = \psi_e + \psi_p + \psi_d. \quad (16)$$

Lemaitre (1992) indicated that the elastic strain energy (the free energy involved in an elastic process) is affected by damage development. Assuming Hooke's elasticity, and having in mind Equation (5), this indication becomes obvious. Hayakawa et al. (1998) elaborated that, as  $\psi_p$  is usually insignificant in comparison with  $\psi_e$ , the damage effects on it would not have to be taken into consideration. In other words, the damage propagates only during the elastic state.

Hence, the free energy is decomposed in the following way,

$$\psi(\boldsymbol{\epsilon}^e, \mathbf{d}, \bar{\mathbf{X}}, \kappa, \beta) = w_e(\boldsymbol{\epsilon}^e, \mathbf{d}) + \psi_p(\bar{\mathbf{z}}, \kappa) + \psi_d(\beta). \quad (17)$$

As mentioned above,  $w_e$  is the elastic strain energy. By using Equation (5), it becomes

$$w_e(\boldsymbol{\epsilon}^e, \mathbf{d}) = \frac{1}{2\rho} \boldsymbol{\epsilon}^e : \mathbf{C}(\mathbf{d}) : \boldsymbol{\epsilon}^e. \quad (18)$$

The potential  $\psi_p$ , consists of two quantities: the free energy due to the isotropic hardening, and the free energy due to the kinematic hardening,

$$\psi_p(\bar{\mathbf{z}}, \kappa) = \frac{R_\infty}{\rho} \left( \kappa + \frac{1}{\delta} e^{-\delta\kappa} \right) + \frac{c}{2\rho} \bar{\mathbf{z}} : \bar{\mathbf{z}}. \quad (19)$$

where  $R_\infty$ ,  $\delta$ , and  $c$  are material parameters, usually determined by conventional tension–compression uniaxial experiments.

Since the Armstrong-Frederick kinematic hardening is used in this study, the equation that describes how the backstress tensor is updated is not given as a constitutive law, but as a nonassociated evolution law.

The potential  $\psi_d$  represents the free energy increase during damage hardening. It is given as a function of the internal variable  $\beta$  and the material parameter  $k_d$ :

$$\psi_d = \frac{1}{2\rho} k_d \beta^2. \quad (20)$$

With the defined free energy potential, the constitutive equations become:

$$\boldsymbol{\sigma} = \mathbf{C}(\mathbf{d}) : \boldsymbol{\varepsilon}^e, \quad (21)$$

$$\bar{\mathbf{Y}} = \frac{1}{2} \boldsymbol{\varepsilon}^e : \frac{\partial \mathbf{C}}{\partial \mathbf{d}} : \boldsymbol{\varepsilon}^e = -[2\mu(\boldsymbol{\varepsilon}^e \cdot \boldsymbol{\varepsilon}^e) + \lambda \boldsymbol{\varepsilon}^e \text{tr}(\boldsymbol{\varepsilon}^e)], \quad (22)$$

$$R = R_\infty (1 - e^{-\delta \kappa}), \quad (23)$$

$$B = k_d \beta. \quad (24)$$

## Evolution Equations

As discussed in the previous subsection ‘Constitutive equations’, the evolution of the backstress tensor  $\bar{\mathbf{X}}$  (Oldroyd rate) is given through a non-associated Armstrong-Frederick law,

$$\overset{\nabla}{\bar{\mathbf{X}}} = c \overset{\Delta}{\bar{\mathbf{E}}}^p - b \dot{\kappa} \bar{\mathbf{X}}, \quad (25)$$

$$\dot{\kappa} = \left[ \frac{2}{3} \overset{\Delta}{\bar{\mathbf{E}}}^p : \overset{\Delta}{\bar{\mathbf{E}}}^p \right]^{1/2}, \quad (26)$$

where  $b$  in Equation (25) is another material parameter, determined by the conventional tension–compression uniaxial experiments.

Furthermore, the remaining evolution equations for the material model are derived from the potential of dissipation. This potential is a convex function of the Lemaitre–Chaboche type (1978):

$$F = \bar{F}(\boldsymbol{\sigma}, \mathbf{Y}, \bar{\mathbf{X}}, R, B). \quad (27)$$

In accordance with the statement about the uncoupled free energy potential, the dissipation potential is uncoupled as well. It consists of a plastic-dissipation potential and a damage-dissipation potential,

$$F = F_p + F_d. \quad (28)$$

Hence, from the plastic and damage dissipation potentials, the evolution equations are separately derived:

$$\dot{\bar{\mathbf{E}}}^p = \dot{\lambda}_p \frac{\partial F_p}{\partial \boldsymbol{\sigma}}, \quad (29)$$

$$\dot{\kappa} = -\dot{\lambda}_p \frac{\partial F_p}{\partial R}, \quad (30)$$

$$\dot{\mathbf{d}} = -\dot{\lambda}_d \frac{\partial F_d}{\partial \mathbf{Y}}, \quad (31)$$

$$\dot{\beta} = -\dot{\lambda}_d \frac{\partial F_d}{\partial B}. \quad (32)$$

The plastic and damage Lagrange multipliers,  $\dot{\lambda}_p$  and  $\dot{\lambda}_d$ , are the plastic and the damage consistency parameters, respectively. For plastic and damage loading/unloading conditions, they define the Kuhn–Tucker relations

$$\dot{\lambda}_p \geq 0, \quad F_p \leq 0, \quad \dot{\lambda}_p F_p = 0, \quad (33)$$

$$\dot{\lambda}_d \geq 0, \quad F_d \leq 0, \quad \dot{\lambda}_d F_d = 0. \quad (34)$$

### PLASTIC EVOLUTION

The von Mises yield condition defines a surface in the Cauchy stress space that encloses an elastic region. Due to isotropic and kinematic hardening, the yield surface expands and translates in the stress space. This give rise to the following yield function:

$$F_p = \sigma_{\text{eff}} - R - \sigma_y = \left[ \frac{3}{2} (\boldsymbol{\sigma}^D - \bar{\mathbf{X}}^D) : (\boldsymbol{\sigma}^D - \bar{\mathbf{X}}^D) \right]^{1/2} - R_\infty (1 - e^{-\delta\kappa}) - \sigma_y. \quad (35)$$

where  $\sigma_y$  is the initial yield stress (material parameter),  $\sigma_{\text{eff}}$  is the effective stress, and the deviatoric stress tensors are given by

$$\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{3} \text{tr}(\boldsymbol{\sigma}) \boldsymbol{\delta}, \quad (36)$$

$$\bar{\mathbf{X}}^D = \bar{\mathbf{X}} - \frac{1}{3} \text{tr}(\bar{\mathbf{X}}) \bar{\boldsymbol{\delta}}. \quad (37)$$

Differentiation of the plastic potential with respect to the stress tensor and isotropic stress scalar gives the following evolution equations:

$$\dot{\bar{\mathbf{E}}}^p = \mathbf{D}^p = \dot{\lambda}_p \sqrt{\frac{3}{2}} \frac{\boldsymbol{\sigma}^D - \bar{\mathbf{X}}^D}{\|\boldsymbol{\sigma}^D - \bar{\mathbf{X}}^D\|}, \quad (38)$$

$$\dot{\kappa} = \dot{\lambda}_p. \quad (39)$$

Here,  $\|\boldsymbol{\sigma}^D - \bar{\mathbf{X}}^D\|$  denotes the Euclidean norm of the deviatoric part of the expression, and  $\mathbf{D}^p$  the plastic strain rate.

For strain-rate-independent plasticity,  $\dot{\lambda}_p$  is determined by the consistency condition, i.e., by requiring that  $\dot{F}_p = 0$ .

#### DAMAGE EVOLUTION

By following the argumentation for the plastic dissipation potential, it is assumed that there is a surface,  $F_d = 0$ , in the space of the thermodynamic damage-conjugated forces, which separates the damaged domain from the undamaged one. The simple form of the dissipation potential below, was proposed by Cordebois and Sidoroff (1982),

$$F_d = Y_{\text{ekv}} - (Y_0 + B), \quad (40)$$

where  $Y_0$  is an energy barrier, a form of energy release rate threshold, which must be overcome in order to accomplish damage propagation. Here,  $Y_{\text{ekv}}$  is the equivalent damage energy release rate, defined as

$$Y_{\text{ekv}} = \left[ \frac{1}{2} \bar{\mathbf{Y}}^T : \mathbf{J} : \bar{\mathbf{Y}} \right]^{1/2}, \quad (41)$$

where  $\mathbf{J}$  is the damage characteristic tensor. Hayakawa et al. (1998) have defined it as the fourth-order tensor, which describes damage-induced change of the damage surface. By noting that the damage dissipation potential is a symmetric function of each component of the damage-conjugated forces, it is given as:

$$\mathbf{J} = \frac{1}{2} (\bar{\boldsymbol{\delta}} \otimes \bar{\boldsymbol{\delta}} + \underline{\boldsymbol{\delta}} \otimes \underline{\boldsymbol{\delta}}). \quad (42)$$

Hence, damage evolution equations are finally obtained:

$$\dot{\mathbf{d}} = -\dot{\lambda}_d \frac{1}{2} \frac{\mathbf{J} : \bar{\mathbf{Y}}}{Y_{ekv}} = \dot{\lambda}_d \frac{1}{2} \frac{\mathbf{J} : \mathbf{Y}}{Y_{ekv}}, \quad (43)$$

$$\dot{\beta} = \dot{\lambda}_d. \quad (44)$$

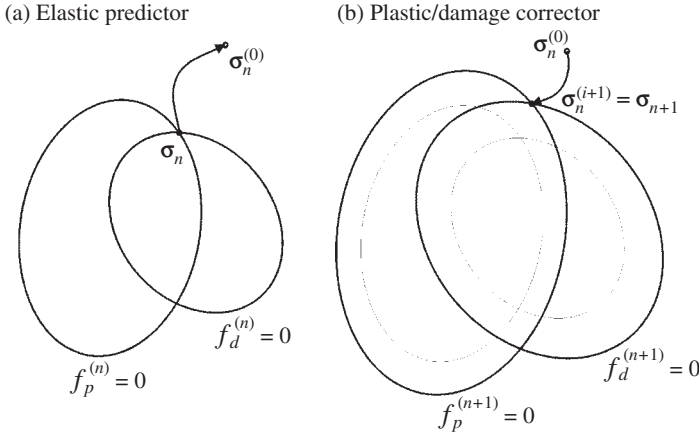
The differentiation of Equation (5) with respect to damage yields a negative sign of  $\bar{\mathbf{Y}}$  according to Equation (22). Further implementation of  $\bar{\mathbf{Y}}$  in Equation (43) yields positive damage propagation. For a strain rate-independent damage,  $\dot{\lambda}_d$  is determined by requiring that  $\dot{F}_d = 0$ .

### NUMERICAL FEATURES OF THE CONSTITUTIVE MODEL

The constitutive elasticity equation relates the Cauchy stress tensor,  $\boldsymbol{\epsilon}$ , to the infinitesimal elastic strain tensor,  $\boldsymbol{\epsilon}^e$ . Let us turn to the hypoelasticity law relating the rate of the Cauchy stress to the elastic part of the rate of deformation,  $(\mathbf{D} - \mathbf{D}^p)$ . In addition, the corotational formulation of shell elements avoids the difficulty with the objectivity of the Cauchy stress rate by embedding a coordinate system in the element. To sum up, the constitutive model is obtained as:

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= \mathbf{C}(\mathbf{d}) : (\mathbf{D} - \mathbf{D}^p), \\ \mathbf{Y} &= 2\mu(\boldsymbol{\epsilon}^e \cdot \boldsymbol{\epsilon}^e) + \lambda \boldsymbol{\epsilon}^e \text{tr}(\boldsymbol{\epsilon}^e), \\ R &= R_\infty(1 - e^{-\delta\kappa}), \\ B &= k_d \beta, \\ \dot{\mathbf{X}} &= c\mathbf{D}^p - b\dot{\kappa}\mathbf{X}, \\ \mathbf{D}^p &= \dot{\lambda}_p \sqrt{\frac{3}{2}} \frac{\boldsymbol{\sigma}^D - \mathbf{X}^D}{\|\boldsymbol{\sigma}^D - \mathbf{X}^D\|} = \dot{\lambda}_p \sqrt{\frac{3}{2}} \mathbf{N}, \\ \dot{\kappa} &= \dot{\lambda}_p, \\ \dot{\mathbf{d}} &= \dot{\lambda}_d \frac{\mathbf{J} : \mathbf{Y}}{2Y_{ekv}}, \\ \dot{\beta} &= \dot{\lambda}_d. \end{aligned} \quad (45)$$

The numerical algorithm for integration of the system of equations was developed with the objective of being implemented as a user material subroutine in the explicit Finite Element program LS-DYNA. The problem is a deformation controlled constitutive one. In the updating scheme, stresses and other variables at the end of the increment are updated using



**Figure 2.** Elastic predictor–plastic/damage corrector numerical procedure.

its values at the beginning of the increment. As proposed by Simo and Ortiz (1984), the operator splitting technology has been used, separating an elastic predictor and a plastic/damage corrector. In the first phase of every incremental step, an elastic material response is assumed ( $\Delta \boldsymbol{\epsilon}^e = \Delta \boldsymbol{\epsilon}$ ), giving the predictor or trial stress,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + (\mathbf{C}_0 - \mathbf{C}_{d_0}) : \Delta \boldsymbol{\epsilon}.^2 \quad (46)$$

Because damage propagates due to the elastic state (Equations (45)<sub>2,8</sub>), a damage corrector step with an update of the stiffness takes place if there is no plastic response. If the trial stress tensor is located outside both the plastic yield and the damage surfaces, the plastic/damage corrector is initiated in order to return the stress tensor to the updated surfaces (Figure 2). This return-mapping algorithm has been used for the time integration purpose. This method is based on a fully implicit backward Euler scheme.

Time discretization of the evolution equations gives the following set of equations:

$$\begin{aligned} \Delta \boldsymbol{\sigma} &= \mathbf{C} : \left( \Delta \boldsymbol{\epsilon} - \Delta \kappa \sqrt{\frac{3}{2}} \mathbf{N} \right), \\ \Delta \mathbf{X} &= c \Delta \kappa \sqrt{\frac{3}{2}} \mathbf{N} - b \Delta \kappa \mathbf{X}, \\ \Delta \mathbf{d} &= \Delta \beta \frac{\mathbf{J} : \mathbf{Y}}{2Y_{ekv}}. \end{aligned} \quad (47)$$

<sup>2</sup>Subscript '0' indicates the specific quantity at the beginning of the time increment. Otherwise, all quantities present values at the end of the time increment. Delta symbol 'Δ' represents increment in specific quantity.

The plastic/damage correction involves iterations until convergence to the updated yield surfaces is achieved. Although this procedure is coupled, the integration treatment can be uncoupled, because as soon as the plastic return has been carried out, all necessary unknowns for the damage update are given as well. In other words, as the damage increment and the update of the damage yield surface are related to the elastic strain state, they are performed following every iteration.

Consequently, the set of nonlinear algebraic equations to be iterated, assures return to the plastic yield surface:

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\sigma}_0 + \mathbf{C} : \Delta \boldsymbol{\varepsilon}^e = \boldsymbol{\sigma}_0 + (\mathbf{C}_0 - \mathbf{C}_d) : \left( \Delta \boldsymbol{\varepsilon} - \Delta \kappa \sqrt{\frac{3}{2}} \mathbf{N} \right), \\ \mathbf{X} &= \frac{1}{1 + b \Delta s} \left( \mathbf{X}_0 + c \Delta \kappa \sqrt{\frac{3}{2}} \mathbf{N} \right), \\ F_p &= \boldsymbol{\sigma}_{\text{eff}} - \sigma_y - R_\infty (1 - e^{-\delta(\kappa_0 + \Delta \kappa)}),\end{aligned}\tag{48}$$

where  $\Delta \boldsymbol{\varepsilon}$  is the given total strain increment.

Using the expression for trial stress from the elastic predictor phase,  $\boldsymbol{\sigma}^{\text{trial}} = \boldsymbol{\sigma}_0 + (\mathbf{C}_0 - \mathbf{C}_{d_0}) : \Delta \boldsymbol{\varepsilon}$ , and the discretized damage evolution Equation, (47)<sub>3</sub>, the stress return is identified as

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\sigma}_0 + (\mathbf{C}_0 - \mathbf{C}_{d_0} - \mathbf{C}_{\Delta d}) : \left( \Delta \boldsymbol{\varepsilon} - \Delta \kappa \sqrt{\frac{3}{2}} \mathbf{N} \right) \\ &= \boldsymbol{\sigma}^{\text{trial}} - \mathbf{C}_{\Delta d} : \Delta \boldsymbol{\varepsilon} - \Delta \kappa \sqrt{\frac{3}{2}} \mathbf{C} : \mathbf{N}.\end{aligned}\tag{49}$$

Finally, residual equations referring to Equation (48) can be obtained:

$$\begin{aligned}\mathbf{R}_\sigma &= \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{trial}} + \mathbf{C}_{\Delta d} : \Delta \boldsymbol{\varepsilon} + \sqrt{\frac{3}{2}} \Delta \kappa \mathbf{C} : \mathbf{N} = 0, \\ \mathbf{R}_X &= (1 + b \Delta \kappa) \mathbf{X} - \mathbf{X}_0 - \sqrt{\frac{3}{2}} c \Delta \kappa \mathbf{N} = 0, \\ R_{F_p} &= F_p - \sigma_{\text{eff}} - \sigma_y - R_\infty (1 - e^{-\delta(\kappa_0 + \Delta \kappa)}) = 0.\end{aligned}\tag{50}$$

In compact matrix notation, the problem is the following:

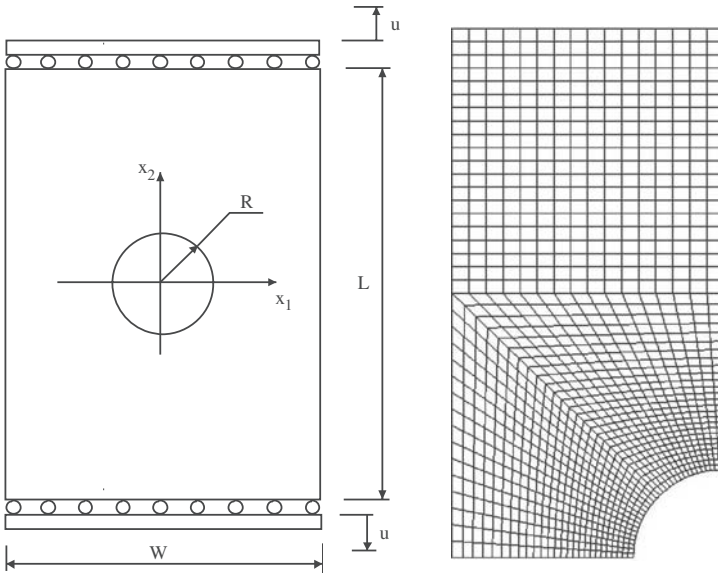
$$\{\mathbf{R}\{\xi\}\} = \{\mathbf{R}_\sigma \quad \mathbf{R}_X \quad R_{f_p}\}^T = \{0\}, \quad \{\xi\} = \{\sigma \quad X \quad \Delta\kappa\}^T. \quad (51)$$

Hence, increments in the unknowns,  $\{\delta\xi\}$ , are obtained using Newton–Raphson iterations,  $\{\mathbf{R}\} = -[\mathbf{J}] \cdot \{\delta\xi\}$ , followed by the subsequent update of the unknowns,  $\{\xi\}^{(i+1)} = \{\xi\}^{(i)} + \{\delta\xi\}$ . The Jacobian matrix,  $[\mathbf{J}] = \partial\{\mathbf{R}\}/\partial\{\xi\}$ , and the details about its derivatives are highlighted in the Appendix.

## COMPUTATIONAL RESULTS

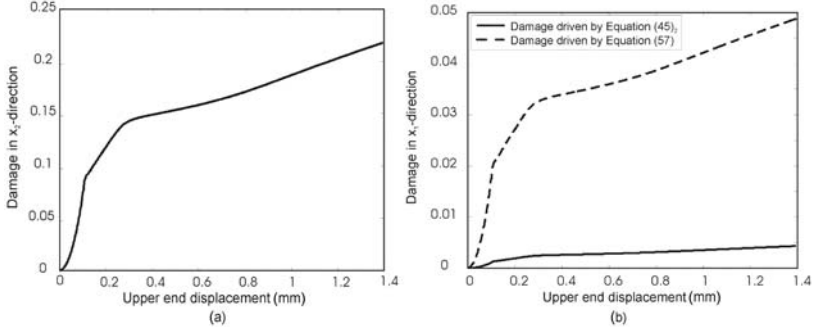
In order to apply the described formulation, LS-DYNA simulations of a problem involving a thin metal sheet with a central circular hole were performed, as seen in Figure 3. The following prototype set of geometrical and material parameters has been used (no damage threshold has been set, which means that damage propagates during the entire loading time).

$$\begin{aligned} \nu &= 0.3, & E &= 205 \text{ GPa}, & \sigma_y &= 265 \text{ MPa}, & k_d &= 2, & Y_0 &= 0, \\ R_\infty &= 200 \text{ MPa}, & \delta &= 2, & b &= 400, & c &= 2 \cdot 10^4 \text{ MPa}, & L &= 250 \text{ mm}, \\ W &= 125 \text{ mm}, & R &= 20.84 \text{ mm}, \end{aligned} \quad (52)$$



**Figure 3.** Geometry and loading for a metal sheet with a central circular hole and corresponding FE-mesh. Due to the double symmetry, only one quarter is analyzed.





**Figure 4.** Damage for the most damaged element vs upper end displacement of the metal sheet: (a) in the  $x_2$ -direction (loading direction) and (b) in the  $x_1$ -direction, driven by two constitutive equations.

The metal sheet is subjected to a prescribed end displacement in the  $x_2$ -direction. Due to the double symmetry, only one quarter of the sheet is analyzed. The finite element discretization is shown in Figure 3. For the loaded sheet, damage–displacement (damage in the most damaged element) curves are plotted in Figure 4.

Figure 4(a) illustrates the damage in the loading ( $x_2$ ) direction of the mentioned element, while the perpendicular normal damage component for the same element (for the outlined set of constitutive and evolution equations), is represented by the solid line in Figure 4(b). It is obvious that the damage in the  $x_2$ -direction is considerably higher than in the  $x_1$ -direction. This is explained by the following reasoning, where we enlighten the constitutive Equation (45)<sub>2</sub> and the evolution Equation (45)<sub>8</sub>.

The evolution of damage in the sheet (plane stress condition) is driven by the damage-conjugated forces,  $\mathbf{Y}$ , obtained in the normal directions as

$$\begin{aligned} Y_{11} &= 2\mu \left[ (\varepsilon_{11}^e)^2 + (\varepsilon_{12}^e)^2 \right] + \lambda \varepsilon_{11}^e (\varepsilon_{11}^e + \varepsilon_{22}^e), \\ Y_{22} &= 2\mu \left[ (\varepsilon_{22}^e)^2 + (\varepsilon_{12}^e)^2 \right] + \lambda \varepsilon_{22}^e (\varepsilon_{11}^e + \varepsilon_{22}^e). \end{aligned} \quad (53)$$

Consequently, the damage growth is directly proportional to the damage energy release rate, since the damage characteristic tensor is the unity tensor, i.e.,

$$\Delta d_{11} = \frac{\Delta \beta}{2Y_{ekv}} Y_{11}, \quad \Delta d_{22} = \frac{\Delta \beta}{2Y_{ekv}} Y_{22}, \quad (54)$$

where  $\Delta \beta$  is obtained from the damage yield condition:

$$F_d = Y_{ekv} - (Y_0 + B_0 + \Delta \beta k_d) = 0. \quad (55)$$

As a particular damage increment is strongly dependent on the elastic strain component in its own direction, the second term in Equation (53)<sub>1</sub> is negative (because 11-direction is the compression direction). As a result, it levels out the first term so that the damage propagation in that direction is minor. In order to obtain a higher damage amount in the compression direction, the following form of constitutive law (45)<sub>2</sub> is proposed – a sum of absolute values of two terms:

$$\mathbf{Y} = 2\mu|\boldsymbol{\epsilon}^e \cdot \boldsymbol{\epsilon}^e| + \lambda|\boldsymbol{\epsilon}^e \text{tr}(\boldsymbol{\epsilon}^e)|. \quad (56)$$

This yields a considerable damage increase in the  $x_1$ -direction, which is illustrated by the dashed line in Figure 4(b). The above equation is in accordance with the requirement that damage must be positive.

## CONCLUSIONS

Constitutive and evolution equations for elasto-plastic damaged materials, exhibiting both isotropic and kinematic hardening, have been derived. The concept of small elastic and large plastic deformations applied on shell elements, with a coordinate system embedded in the element, is equivalent to the ‘whole’ small-deformation model. The second-order damage tensor was shown to give rise to elastic stiffness degradation.

Thermodynamics with internal variables were used to provide a framework for the derivation of the equations. Within the scope of this theory, the constitutive and evolution equations for the observable and internal variables, like damage, damage hardening, and variables that characterize isotropic and kinematic plastic hardening, and to them conjugated forces, could be derived from two thermodynamic potentials: the Helmholtz free energy and the uncoupled dissipation potential. The general expression for the inverted damage effect tensor, followed by the simplified expression for material stiffness degradation, facilitates the derivation of the equations. The numerical algorithm for calculation of the system of constitutive equations has been developed and implemented as a user material subroutine in the explicit finite element program LS-DYNA. On the whole, the model shows good convergence.

The elasticity law was derived assuming material isotropy, since no damage is present at the initial undamaged state. Furthermore, the development of damage, i.e., the damaged elastic response, introduces certain orthotropy. A general orthotropic law depends on nine constant

parameters, while in this case, three dynamic damage components (besides Young's modulus and Poisson's ratio) enforces certain limitations on the elastic response. Cordebois and Sidoroff (1982) remarked that the importance of these limitations could only be evaluated by experiments.

Zhu and Cescotto (1995) defined the plastic yield surface in the space of the effective stresses. In the definition of Hayakawa et al. (1998), increased effects of stress due to damage were specified for the plastic yield surface through a characteristic tensor containing the damage. On the contrary, the model described in this article proposes the yield surface in the space of the real stresses. However, damage is indirectly involved in the surface definition as a result of the elastic law, since the constitutive equation describing the real stress is a function of the degraded elastic stiffness caused by the propagating damage.

Cyclic loading causes a micro-crack opening and closing phenomenon. Since the main object of the present model is to predict cracks in ductile, strain-rate independent metal sheets during impact, this phenomenon is not considered. However, future work on identification of the material parameters, which will combine experiments and simulations, will show whether or not such a mechanism is of importance for the current application.

## APPENDIX

### General Notations

Let  $\Gamma$  and  $\Pi$  be two fourth-order tensors,  $\mathbf{A}$  and  $\mathbf{B}$  two second-order tensors (nonsymmetric in general) and  $\{\mathbf{e}_i\}$  the orthonormal basis in a Cartesian coordinate system:

$$\begin{aligned}\Gamma &= \Gamma_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \\ \Pi &= \Pi_{klmn} \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n, \\ \mathbf{A} &= A_{kl} \mathbf{e}_k \otimes \mathbf{e}_l, \\ \mathbf{B} &= B_{ij} \mathbf{e}_i \otimes \mathbf{e}_j.\end{aligned}\tag{57}$$

The following inner products are established:

$$\begin{aligned}\Gamma : \Pi &= \Gamma_{ijkl} \Pi_{klmn} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n, \\ \Gamma : \mathbf{A} &= \Gamma_{ijkl} A_{kl} \mathbf{e}_i \otimes \mathbf{e}_j, \\ \mathbf{A} \cdot \mathbf{B} &= A_{ik} B_{kj} \mathbf{e}_i \otimes \mathbf{e}_j.\end{aligned}\tag{58}$$

The trace and the Euclidean norm of a second tensor  $\mathbf{A}$  are:

$$\begin{aligned}\text{tr}(\mathbf{A}) &= A_{ii}, \\ \|\mathbf{A}\| &= \sqrt{\mathbf{A} : \mathbf{A}} = \sqrt{A_{ij}A_{ji}}.\end{aligned}\quad (59)$$

The identity tensors of second- and fourth-order's are given as:

$$\begin{aligned}\bar{\mathbf{\delta}} &= \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \\ \mathbf{I} &= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l,\end{aligned}\quad (60)$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \text{ is the Kronecker delta symbol.}$$

Dyadic products  $\otimes$ ,  $\bar{\otimes}$ , and  $\underline{\otimes}$  are read as:

$$\begin{aligned}(A \otimes B)_{ijkl} &= (A)_{ij}(B)_{kl}, \\ (A \bar{\otimes} B)_{ijkl} &= (A)_{ik}(B)_{jl}, \\ (A \underline{\otimes} B)_{ijkl} &= (A)_{ii}(B)_{jk}.\end{aligned}\quad (61)$$

### Useful Notations and Derivatives

The stiffness degradation in Equations (46) and (49):

$$\begin{aligned}\mathbf{C}_{d_0} &= \mu(\mathbf{d}_0 \bar{\otimes} \bar{\mathbf{\delta}} + \bar{\mathbf{\delta}} \bar{\otimes} \mathbf{d}_0 + \mathbf{d}_0 \underline{\otimes} \bar{\mathbf{\delta}} + \bar{\mathbf{\delta}} \underline{\otimes} \mathbf{d}_0) + \lambda(\mathbf{d}_0 \otimes \bar{\mathbf{\delta}} + \bar{\mathbf{\delta}} \otimes \mathbf{d}_0), \\ \mathbf{C}_{\Delta d} &= \frac{\Delta\beta}{2Y_{\text{ekv}}} [\mu(\mathbf{Y} \bar{\otimes} \bar{\mathbf{\delta}} + \bar{\mathbf{\delta}} \bar{\otimes} \mathbf{Y} + \mathbf{Y} \underline{\otimes} \bar{\mathbf{\delta}} + \bar{\mathbf{\delta}} \underline{\otimes} \mathbf{Y}) + \lambda(\mathbf{Y} \otimes \bar{\mathbf{\delta}} + \bar{\mathbf{\delta}} \otimes \mathbf{Y})].\end{aligned}\quad (62)$$

The Jacobian matrix:

$$[\mathbf{J}] = \begin{bmatrix} \mathbf{J}_{\sigma\sigma} & \mathbf{J}_{\sigma X} & \mathbf{J}_{\sigma\kappa} \\ \mathbf{J}_{X\sigma} & \mathbf{J}_{XX} & \mathbf{J}_{X\kappa} \\ \mathbf{J}_{\kappa\sigma} & \mathbf{J}_{\kappa X} & \mathbf{J}_{\kappa\kappa} \end{bmatrix}.\quad (63)$$

Specific statements are given as:

$$\begin{aligned}
\mathbf{J}_{\sigma\sigma} &= \frac{\partial \mathbf{R}_\sigma}{\partial \boldsymbol{\sigma}} = \mathbf{I} + \sqrt{\frac{3}{2}} \Delta \kappa \mathbf{C} : \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}}, \\
\mathbf{J}_{\sigma X} &= \frac{\partial \mathbf{R}_\sigma}{\partial \mathbf{X}} = \sqrt{\frac{3}{2}} \Delta \kappa \mathbf{C} : \frac{\partial \mathbf{N}}{\partial \mathbf{X}}, \\
\mathbf{J}_{\sigma\kappa} &= \frac{\partial \mathbf{R}_\sigma}{\partial \Delta \kappa} = \sqrt{\frac{3}{2}} \mathbf{C} : \mathbf{N}, \\
\mathbf{J}_{X\sigma} &= \frac{\partial \mathbf{R}_X}{\partial \boldsymbol{\sigma}} = -\sqrt{\frac{3}{2}} c \Delta \kappa \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}}, \\
\mathbf{J}_{X\sigma} &= \frac{\partial \mathbf{R}_X}{\partial \mathbf{X}} = (1 + b \Delta \kappa) \mathbf{I} - \sqrt{\frac{3}{2}} c \Delta \kappa \frac{\partial \mathbf{N}}{\partial \mathbf{X}}, \\
\mathbf{J}_{X\kappa} &= \frac{\partial \mathbf{R}_X}{\partial \Delta \kappa} = b \mathbf{X} - \sqrt{\frac{3}{2}} c \mathbf{N}, \\
\mathbf{J}_{\kappa\sigma} &= \frac{\partial R_{F_p}}{\partial \boldsymbol{\sigma}} = \frac{\partial}{\partial \boldsymbol{\sigma}} \left[ \frac{3}{2} (\boldsymbol{\sigma}^D - \mathbf{X}^D) : (\boldsymbol{\sigma}^D - \mathbf{X}^D) \right]^{1/2} = \sqrt{\frac{3}{2}} \mathbf{N}^{\text{dev}}, \\
\mathbf{J}_{\kappa\sigma} &= \frac{\partial R_{F_p}}{\partial \mathbf{X}} = -\sqrt{\frac{3}{2}} \mathbf{N}^{\text{dev}}, \\
\mathbf{J}_{\kappa\kappa} &= \frac{\partial R_{F_p}}{\partial \Delta \kappa} = -R_\infty \delta e^{-\delta(\kappa_0 + \Delta \kappa)},
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
\frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} &= -\frac{\partial \mathbf{N}}{\partial \mathbf{X}} = \frac{1}{\|\boldsymbol{\sigma}^D - \mathbf{X}^D\|} (\mathbf{I}^{\text{dev}} - \mathbf{N} \otimes \mathbf{N}^{\text{dev}}), \\
\mathbf{I}^{\text{dev}} &= \frac{\partial (\boldsymbol{\sigma}^D - \mathbf{X}^D)}{\partial \boldsymbol{\sigma}} = -\frac{\partial (\boldsymbol{\sigma}^D - \mathbf{X}^D)}{\partial \mathbf{X}}, \\
\mathbf{N}^{\text{dev}} &= \frac{\partial \|\boldsymbol{\sigma}^D - \mathbf{X}^D\|}{\partial \boldsymbol{\sigma}} = -\frac{\partial \|\boldsymbol{\sigma}^D - \mathbf{X}^D\|}{\partial \mathbf{X}}, \\
\frac{\partial}{\partial \boldsymbol{\sigma}} \left[ \frac{1}{\|\boldsymbol{\sigma}^D - \mathbf{X}^D\|} \right] &= -\frac{\partial}{\partial \mathbf{X}} \left[ \frac{1}{\|\boldsymbol{\sigma}^D - \mathbf{X}^D\|} \right] = -\frac{\mathbf{N}^{\text{dev}}}{[(\boldsymbol{\sigma}^D - \mathbf{X}^D) : (\boldsymbol{\sigma}^D - \mathbf{X}^D)]^{3/2}} \\
&= -\frac{\mathbf{N}^{\text{dev}}}{\|\boldsymbol{\sigma}^D - \mathbf{X}^D\|^3}.
\end{aligned} \tag{65}$$

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