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Nonlinear Behavior of Ductile Quasi-homogeneous Solids

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ABSTRACT: First M.T. Huber in 1904, and later Mises and Hencky suggested equivalent yield criteria for elastic-perfectly plastic solids in three-dimensional stress states. The H–M–H criterion is commonly used in structural design. But, the Huber–Hencky distortion energy formula and the Huber–Mises reduced stress formula do not give unique yielding measures for elastic-nonlinearly plastic solids. The yielding probability κ , which has been introduced by the author in 1954, serves the purpose for ductile elastic-nonlinearly plastic solids. This idea has been a part of a more general probability-based theory such that the yielding ratio κ and a cracking tensor λ are the damage measures for quasi-homogeneous continuous media. Structural concrete has been analyzed in earlier studies. In this study, nominally ductile materials are taken into consideration such as structural steel and aluminum alloys in normal temperatures. The log-normal probability distributions of plastic micro-strength and microstress are accepted. Constitutive equations are derived with the yielding ratio κ as the coordinate of state. The Ramberg–Osgood σ – ε curve is taken as the empirical basis for evaluation of the probability distribution parameters. Two points of the curve are taken into account: the conventional yield strength f_y and the ultimate strength f_u . A numerical example indicates that both elastic and plastic compressible phases of the quasi-homogeneous solid is a likely model of behavior. A shear stress–strain curve is analytically derived. The conventional 0.2% permanent strain for the characteristic plastic strength f_y in a simple tension test applies approximately also to shear cases for the same yielding ratio κ_y at the characteristic strength level. The ultimate strength f_u will occur when the effective stress $\sigma_{\text{eff}}(\kappa)$ attains its maximum level for a critical yielding ratio κ_{cr} ; however, it is not the maximum point $\sigma_{\text{eff}}(\varepsilon)$ of the monotone Ramberg–Osgood curve. The characteristic κ_y and critical κ_{cr} values are verified in the case of shear.

KEY WORDS: elastic-plastic solids, brittle and ductile solids, material damage, stress–strain relations.

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INTRODUCTION

IN 1904, MAXIMILIAN Titus Huber, Professor of the Austrian and later Polish Technical University in Lwów, suggested that specific work of distortion W_f may be accepted as the yielding criterion of elastic-plastic solids unless excessive tensile stresses are applied (Huber, 1904).

$$W_f = \frac{E}{6(1 + \nu)} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] < W_{pl} \quad (1)$$

The material parameters of Equation (1) are: E is the Young's modulus and ν is the Poisson's ratio. The principal stresses $\sigma_1, \sigma_2, \sigma_3$ are invariant with reference to rotations of the coordinate system x, y, z in the physical space.

Huber was inspired by the ideas of E. Beltrami. The same criterion (1) was given in 1924 by H. Hencky. Collected works of Huber have been edited by the Polish Academy of Sciences in 1957. The centenary of the Huber criterion was celebrated in 2004 during an International Symposium in Kraków, Poland.

Huber (1930) concluded that the reduced stress σ_{red} , comparable with the uniaxial stress, might be the yielding criterion equivalent to the distortion energy, Equation (1)

$$\sigma_{red} = \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} < R_{pl} \quad (2)$$

Equivalent stress, formally identical with Equation (2), had been defined by Mises, in 1911, but he treated it as an approximation of the Tresca slip-plasticity condition. Both W_f and σ_{red} reach their specific values $W_{pl} = \text{constant}$ and $R_{pl} = \text{constant}$ at the yield point, therefore Conditions (1) and (2) are equivalent as the yield point criteria and they are called Huber–Mises–Hencky (H–M–H) yield criterion. The H–M–H criterion together with the Hooke's law is sufficient to assess the behavior of elastic-perfectly plastic solids. The uniaxial stress–strain law $\sigma_P(\varepsilon_x)$ has been represented by the Prandtl diagram (solid line in Figure 2). It may be written using the symbolic notation of the Mathcad as:

$$\sigma_P(\varepsilon) = \text{if}(|\sigma_P| < R_{pl}, E\varepsilon_x, R_{pl}) \quad (3)$$

The last term in brackets shall be taken as $\sigma_P = R_{pl}$, if the inequality $|\sigma_P| < R_{pl}$ is not actual. The limit value R_{pl} is constant for elastic-perfectly plastic solids. No such point can be seen for elastic-nonlinearly plastic solids (dotted line in Figure 1).

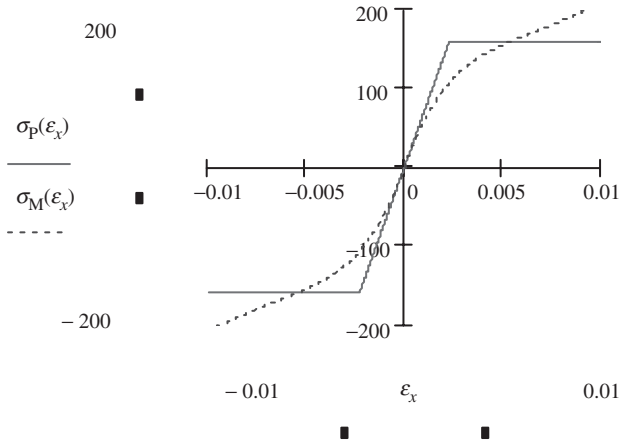


Figure 1. Elastic-perfectly plastic $\sigma_P(\epsilon)$ and elastic-nonlinearly plastic $\sigma_M(\epsilon)$ stress-strain diagrams.

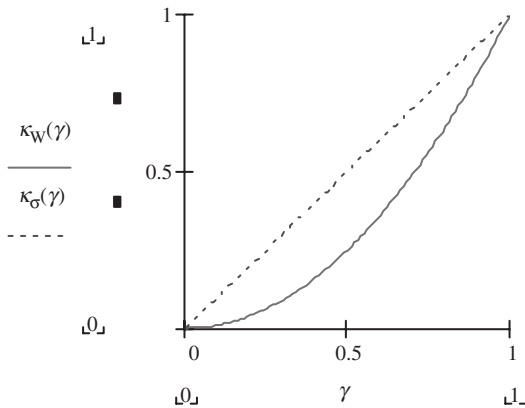


Figure 2. The Huber-Hencky (κ_W) and Huber-Mises (κ_σ) yielding measures.

The H-M-H yield criterion is not sufficient to assess the behavior of elastic-nonlinearly plastic solids. A yielding measure is necessary. The Huber-Hencky equation (1) and the Huber-Mises equation (2) do not give unique yielding measure.

For example the Huber-Hencky equation gives deterministic yielding measure $\kappa_W = W_i/W_{pl} = 0.25$ while the Huber-Mises equation gives $\kappa_\sigma = \sigma_{red}/R_{pl} = 0.5$ in the nondimensional coordinate system (Figure 2).

Fifty years after the Huber criterion, new ideas originating continuum damage mechanics were presented (Murzewski, 1954). Nonlinear plastic

deformation has been treated as an effect of simultaneous ductile and brittle damage of a quasi-homogeneous solid. More than 30 contributions have been presented by the author and his associates. Some publications are listed in the author's article in the first volume of '*Damage Mechanics*' (Murzewski, 1992). The developments have concerned mostly structural concrete.

A simpler presentation of the quasi-homogeneous theory is presented here. It will be confined to nominally ductile materials like low-carbon steel and aluminum alloys in conditions when the H–M–H criterion is applicable. The time-independent yielding measure κ will be applied to the quasi-homogeneous solids. The measure κ is a normed scalar variable ($0 < \kappa < 1$), which reflects cumulative damage of the material. Primarily, it was called plastification ratio (Murzewski, 1954). Some special aspects of the κ measure will be discussed in the next section.

In practical applications, the conventional plastic strength f_y relative to the permanent strain $\varepsilon_{pl} = 0.2\%$ is taken for quality control of elastic-nonlinearly plastic materials. It is determined in uniaxial laboratory tests. The conventional yield strength f_y is treated as the characteristic strength in structural design.

The equivalent stress σ_{red} according to Equation (2) is applied to safety verifications of structural elements at complex stress states. The design value of stress σ_{red} is determined for design values of applied loads; they are enhanced by load factors at the ultimate limit states. The design value σ_{red} shall not exceed the design strength f_d , which is reduced by a material factor γ_M . The symbols f_y , f_d , and γ_M are used in the Eurocodes for structural design. The question is whether the conventional plastic strain, $\varepsilon_{pl} = 0.002$, basic for the definition of the characteristic value of strength f_y , should correspond to a constant value of cumulative damage $\kappa_y = \text{constant}$ at any state of stress. Another question is how to explain the behavior at the ultimate stress limit $\sigma_{ult} = R_{ult}$ if the stress would still increase.

Kachanov (1958), defined damage parameter as a time-dependent variable of ductile materials subject to creep or relaxation of stresses. Kachanov has given impetus for the development of continuum mechanics of time-dependent elastic-plastic media. Microdamage problems and physical microstructural aspects have been analyzed in many subsequent works which are discussed in review articles (Krajcinovic, 1984; Ostoja-Starzewski, 2002). Thermomechanical framework has been also taken into consideration (Reckwerth and Tsamakis, 2003). Such considerations help to elucidate the real nature of material damage. Their scope goes often over the continuum perception of the solid with virtual microscopic structure. Structural materials like mild steel aluminum alloys in normal temperatures do not need any time-dependent analysis.

DAMAGE MEASURES OF A QUASI-HOMOGENEOUS SOLID

The scalar variable κ may be defined in a deterministic way in terms of geometric measures

$$\kappa = \frac{A_{\text{pl}}}{A_o} \quad \text{or} \quad \kappa = \frac{V_{\text{pl}}}{V_o} \quad (4)$$

where A_{pl} is the plastic part of a cross section, A_o is the total (elastic + plastic) area of the cross section, and V_{pl} , V_o are the plastic part and total volume of a three-dimensional microelement.

It may be defined also as a probabilistic measure

$$\kappa = 1 - \text{Prob}(\sigma_{\text{eff}} < R_D) \quad (5)$$

where

$$\sigma_{\text{eff}} = \sqrt{\frac{1}{3}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{\frac{2}{3}} \sigma_{\text{red}}$$

σ_i is the random principal microstresses, $i = 1, 2, 3$ and $R_D = \text{constant}$ is the random strength limiting the effective microstress σ_{eff} .

The deterministic definition of damage, Equation (4), is asymptotically equivalent to the probability-based formula, Equation (5), provided that probability is understood in the sense of ‘geometric definition’ given by Euler. Such meaning of probability is different from probability of failure in structural reliability, which is associated with the ‘statistical definition’ of probability given elsewhere by Mises. The advantage of the probabilistic approach to continuum mechanics is that strict definitions and well-founded theorems of probability may be applied to the theory of quasi-homogeneous solids without any special geometrical proofs.

Equation (5) is related to the Huber–Mises yielding measure. The yielding ratio $\kappa(\sigma_{\text{eff}})$ is understood as the probability that a particle belongs to the plastic phase of the quasi-homogeneous continuum. Any particle of the quasi-homogeneous solid will be elastic-perfectly plastic (piecewise linear line $\sigma_{\text{p}}(\varepsilon)$ in Figure 1). It is not so in the case of deformation of the quasi-homogeneous aggregate. The whole quasi-homogeneous aggregate may exhibit nonlinear macroscopic stress–strain relation $\sigma_{\text{M}}(\varepsilon)$ (the curve $\sigma_{\text{M}}(\varepsilon)$ in Figure 1).

The quasi-homogeneous medium is the aggregate of nonhomogeneous microscopic elements; however, it is treated as a continuous solid with elastic-brittle and plastic-ductile phases. The elastic and plastic phases of a quasi-homogeneous medium are understood similarly as the phases are in

theory of dispersive media. The notional elastic and plastic phases have been introduced to pass from analysis of the discrete aggregate of elastic and plastic particles to continuum mechanics (Murzewski, 1969). A quasi-homogeneous solid may be defined in the strict sense and broad sense as a continuum characterized by identical probability distributions of mechanical properties at each material point. It may be called a stochastically homogeneous solid if autocorrelation of local properties are defined (Murzewski, 1958). A quasi-homogeneous material in a broader sense is characterized only by constant central values (mean or median) and constant standard deviations or coefficients of variation of mechanical properties.

The effective stress σ_{eff} has been introduced in Equation (5) instead of the Huber's reduced stress σ_{red} because of some good properties.

- (1) The scalar product of effective stress σ_{eff} and effective strain ε_{eff} gives the specific energy of distortion

$$W_f = \frac{1}{2} \sigma_{\text{eff}} \cdot \varepsilon_{\text{eff}} \quad (6)$$

with the effective strain similarly defined as:

$$\varepsilon_{\text{eff}} = \sqrt{\frac{1}{3} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}}.$$

- (2) The effective stress σ_{eff} is equal to the second coordinate of the cylindrical system $\sigma_A, \sigma_D, \omega_\sigma$ in the stress space where the principal stresses $\sigma_1, \sigma_2, \sigma_3$ are the Cartesian coordinates.

$$\begin{aligned} \sigma_A &= \sqrt{\frac{1}{3}} (\sigma_1 + \sigma_2 + \sigma_3) \\ \sigma_D &= \sqrt{\frac{1}{3}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sigma_{\text{eff}} \\ \omega_\sigma &= \arcsin \frac{\sigma_2 - \sigma_3}{\sqrt{2} \sigma_D} \end{aligned} \quad (7)$$

The cylindrical coordinates $\sigma_A, \sigma_D, \omega_\sigma$ are invariant in the stress space with any Cartesian coordinates x, y, z being functions of the invariant principal stresses $\sigma_1, \sigma_2, \sigma_3$.

- (3) The first invariant σ_A of the cylindrical system is the axial coordinate related to the axis $\sigma_1 = \sigma_2 = \sigma_3$ of the cylindrical system. The second invariant σ_D is the radius of the polar coordinate system on the deviator

plane $\sigma_A=0$. The angle ω_σ is counted from the projection of the principal stress σ_1 on the deviator plane. The stress axiator characterizes a uniform part of the 3-D stress, like the hydrostatic pressure. The second invariant σ_D is the norm of the stress deviator and it characterizes for example, simple shear. The invariants σ_A , σ_D are components of the Euclidean norm of the stress tensor $|\sigma|$ in the vector space,

$$\sigma_A^2 + \sigma_D^2 = |\sigma|^2 \tag{8}$$

Retransformation equations of the cylindrical invariants σ_A , σ_D , ω_σ into the principal stresses σ_i , $i=1, 2, 3$ are as follows:

$$\sigma_i = \sqrt{\frac{1}{3}}\sigma_A + \sqrt{\frac{2}{3}}\sigma_D \cos\left(\omega_\sigma + (i-1)\frac{2\pi}{3}\right) \tag{9}$$

Similar equations can be formulated for principal strains ε_i , $i=1, 2, 3$. Equations (7)–(9) are of geometric nature and they are valid not only in the quasi-homogeneous continuum but also in its elastic and plastic phases.

The scalar ratio κ might be an adequate measure in conditions of melting; however, it is acceptable also as the damage measure in conditions of slip plasticity of an isotropic quasi-homogeneous medium. It is helpful in derivation of constitutive equations of ductile elastic-nonlinear plastic solids provided that the elastic phase is not subject to microcracking.

Nonlinear stress–strain relations would be of another nature in case of brittle elastic solids. Tensor of decohesion λ (Murzewski, 1957), later called tensor of cracking (Murzewski, 1976), has been defined for brittle damage evaluation of the elastic-brittle phase of the quasi-homogeneous solids. Principal probabilities of cracking λ_1 , λ_2 , λ_3 have been defined as conditional probabilities of cracks in the three principal directions $i=1, 2, 3$

$$\lambda_i = 1 - \text{Prob}(\sigma_i < R_t | i) \tag{10}$$

where R_t is the cleavage strength in simple tension, lower than the compression strength $|R_c|$.

Existence of orthogonal independent directions in any vector space has been proved. Transformation rules of cracking components λ_{ij} in the physical space, $i, j=x, y, z$, were derived (Murzewski, 1960) to clear up objections which rose in the 1950s, whether tensor is the right geometrical object to be used in mechanics of damage. Despite that, some authors

Table 1. Damage measures in the development of damage mechanics.

Symbol	Year of first appearance	Author
$\mu(\kappa, \lambda)$	1954	J. Murzewski
$1-\psi$	1958	L.M. Kachanov
ω	1959	Yu.N. Rabotnov
D	1961	F.G.K. Odqvist and J. Hult

associate scalar damage parameter not only with the ductile form of damage (Table 1, compiled from Chrzanowski, 1978). They use different symbols in place of κ and λ for the cumulative microdamage measures.

An overall microdamage tensor has been denoted μ (Murzewski, 1954) or ω (Murzewski, 1992). It takes into account both microyielding and microcracking in the unified theory of strength. The tensor μ has been defined using the theorem of independent random events and tensor multiplication

$$\mu = I - (1 - \kappa \cdot I) \cdot (1 - \lambda) \quad (11)$$

where I is the unit tensor.

Principal damage components μ_i keep their directions $i=1, 2, 3$, if the stress process is proportional. Otherwise, incremental equations would be needed; induced anisotropy and nonsymmetric elasticity should be taken into consideration.

$$\mu_i = \kappa - \kappa\lambda_i + \lambda_i \quad (12)$$

The tensor of damage $\mu(\sigma_1, \sigma_2, \sigma_3)$ has been applied to concrete in earlier works (e.g., Murzewski, 1954). The analysis will be much simpler if the scalar ratio $|\mu| \rightarrow \kappa$ is treated as the coordinate of state of elastic-plastic solid. It is suitable to solve the problems of nominally ductile solids like structural steel and aluminum alloys unless fatigue loading is applied.

The notion of material damage seems to be synonymous with what used to be translated into English as material effort (wytężenie in Polish, Anstrengung in German – in Huber's works). The term 'failure' in some earlier papers (Murzewski, 1958) shall be better used in theory of reliability. The term 'material defects' should be rather applied to mezzo-defects. The material defects are essential in the Weibull's theory of strength and size effect considerations, they were reviewed recently by Ostoja-Starzewski (2002).

STRESS-STRAIN RELATIONS OF THE QUASI-HOMOGENEOUS SOLID

The total probability theorem has the fundamental importance in the theory of quasi-homogeneous media. Symbols $\bar{\sigma}$ and $\bar{\varepsilon}$ with subscripts A or D are relative to the mean values in the elastic phase and $\overline{\bar{\sigma}}$, $\overline{\bar{\varepsilon}}$ are relative to those in the plastic phase. Symbols σ_A , σ_D , ω_σ will denote now the overall mean values in the quasi-homogeneous medium

$$\begin{aligned}\sigma_D &= (1 - \kappa)\overline{\bar{\sigma}}_D + \kappa\overline{\bar{\sigma}}_D \\ \varepsilon_A &= (1 - \kappa)\overline{\bar{\varepsilon}}_A + \kappa\overline{\bar{\varepsilon}}_A\end{aligned}\tag{13}$$

In addition, compatibility equation of elastic and plastic distortions is assumed

$$\overline{\bar{\varepsilon}}_D = \overline{\bar{\varepsilon}}_D = \varepsilon_D\tag{14}$$

as well as action-reaction equation for 3-D uniform stresses in plastic and elastic phases

$$\overline{\bar{\sigma}}_A = \overline{\bar{\sigma}}_A = \sigma_A\tag{15}$$

Proportional stress processes will be taken into consideration

$$\omega_\sigma = \omega_\varepsilon = \text{constant}\tag{16}$$

The cylindrical invariants ω_σ , ω_ε remain equal not necessarily during a proportional stress process if the analysis is confined to ductile solids such that the yielding process can be defined by a scalar quantity.

The classical Hooke's law is accepted in the elastic phase of the quasi-homogeneous solid unless microcracks occur. The elasticity equations are expressed in the cylindrical coordinate system as follows:

$$\overline{\bar{\varepsilon}}_A = \frac{1 - 2\nu}{E}\overline{\bar{\sigma}}_A, \quad \overline{\bar{\varepsilon}}_D = \frac{1 + \nu}{E}\overline{\bar{\sigma}}_D, \quad \overline{\bar{\omega}}_\varepsilon = \overline{\bar{\omega}}_\sigma\tag{17}$$

The Young's modulus E and the Poisson's ratio ν will be treated here as nonrandom constants. Statistical elasticity problems with random elastic moduli have been taken into consideration by Volkov (1960), but without taking plastic phenomena into account.

Two models of behavior are taken for the plastic phase of a quasi-homogeneous solid

Model 1: The solid remains compressible equally in the plastic and elastic phases

$$\overline{\overline{\varepsilon_A}} = \overline{\varepsilon_A} = \varepsilon \quad (18)$$

Model 2: The solid is incompressible in the plastic phase

$$\overline{\overline{\varepsilon_A}} = 0 \quad (19)$$

If the strains are time-invariant, the mean deviatoric component of micro-stress in the plastic phase will be equal to a constant mean yield stress R_D

$$\overline{\overline{\sigma_D}} = R_D \quad (20)$$

If the behavior of a ductile material was time-dependent, the stress would relax $\sigma_D \rightarrow 0$ while $\varepsilon_D = \text{constant}$ or the material would creep $\varepsilon_D \rightarrow 0$ while $\sigma_D = \text{constant}$. Such media would require kinematical equations. They have been analyzed by Kachanov (1958) and other authors.

The 3-D mean stress-strain relations for time-invariant ductile elastic-plastic quasi-homogeneous solids depend on the coordinate of state κ . They are derived from Equations (13) and (17), and either (18) or (19).

$$\text{Model 1: } \varepsilon_A = \frac{1-2\nu}{E} \sigma_A \quad \varepsilon_D = \frac{1+\nu}{E(1-\kappa)} (\sigma_D - \kappa R_D) \quad \omega_\varepsilon = \omega_\sigma \quad (21)$$

$$\text{Model 2: } \varepsilon_A = \frac{(1-2\nu)(1-\kappa)}{E} \sigma_A \quad \varepsilon_D = \frac{1+\nu}{E(1-\kappa)} (\sigma_D - \kappa R_D) \quad \omega_\varepsilon = \omega_\sigma \quad (22)$$

The theoretical uniaxial $\sigma_1(\varepsilon_1)$ equation, if $\sigma_2 = \sigma_3 = 0$, has been derived from Equations (7), (9) and (21) or (22), as follows:

$$\text{Model 1: } \varepsilon_1(\sigma_1) = \frac{((1-2\nu)/3)\sigma_1(1-\kappa) + (2(1+\nu)/3)[\sigma_1 - R_{pl}\kappa]}{E(1-\kappa)} \quad (23)$$

$$\text{Model 2: } \varepsilon_1(\sigma_1) = \frac{((1-2\nu)/3)\sigma_1(1-\kappa)^2 + (2(1+\nu)/3)[\sigma_1 - R_{pl}\kappa]}{E(1-\kappa)} \quad (24)$$

where $R_{pl} = R_D \sqrt{3/2}$ is the mean value of the microstrength relative to uniaxial stress and κ is the yielding ratio identified with the cumulative probability, Equation (5).

There are three questions to discuss:

- (1) Should the permanent strain 0.2% characterize the Huber–Mises reduced stress $\sigma_{red}(\sigma_1, \sigma_2, \sigma_3)$ at any state of stress for the same value κ_y as it is in the uniaxial tension test?
- (2) Does the ultimate strength R_{ult} and the yielding ratio κ_{ult} occur at the maximum value of the effective stress $\sigma_{eff}(\varepsilon) = \max$?
- (3) Would the rupture occur when the yielding ratio tends to one in a continuous process, $\kappa \rightarrow 1$, or can it happen as a sudden event at a lower critical value $\kappa_{cr} < 1$?

MICROSTRESS AND MICROSTRENGTH DISTRIBUTIONS

The Huber–Mises yield criterion is expressed now in a multiplicative form equivalent to that which has been considered in Equation (5)

$$\frac{\overline{\sigma_D}}{R_D} < 1 \tag{25}$$

The quotient $\gamma = R_D / \overline{\sigma_D}$ of the yield microstrength $R_D = R_{pl} \sqrt{2/3}$ and the effective microstress $\overline{\sigma_D} = \overline{\sigma_{eff}}$ may be called the ‘plasticity factor’. It is analogous to the safety factor for materials in limit states design of structural members. The effective stress $\overline{\sigma_D}$ in the elastic phase is derived from Equations (13) and (20):

$$\overline{\sigma_D} = \frac{\sigma_D - R_D \kappa}{1 - \kappa} \tag{26}$$

The coordinate of state κ has been called yielding ratio or yielding probability. It is relative to the entire phase of the quasi-homogeneous solid, Equations (4) and (5). The elastic-perfectly plastic behavior is supposed for a small element of the quasi-homogeneous solid, that is $\sigma_p(\bar{\varepsilon})$ in Figure 1.

The actual stress σ_i at a point of the quasi-homogeneous solid is a superposition of initial microstress σ_i^0 and applied stresses $\sum \sigma_i$. The balanced initial microstresses, positive and negative, will be the Gauss-normal random variables. This theorem has been derived with very weak assumptions (Murzewski and Winiarska, 1970). The applied stresses $\Delta\sigma$ are of various origins and they may have various probability distributions. Notwithstanding, the Gauss-normal probability function is often assumed to be a fair approximation of the probabilistic composition $\sigma_i = \sigma_i^0 + \sum \sigma_i$.

This assumption is motivated by the central limit theorem of probability. But, the effective stress $\sigma_{\text{eff}} = \sigma_{\text{D}}$ is a nonlinear function of principal stresses σ_i and it cannot be the Gauss normal variable again. The Bessel function of imaginary argument $I_0(\sigma_{\text{D}}) = J_0(i\sigma_{\text{D}})$ will be the probability function of σ_{D} if the principal stresses σ_i are Gauss-normal (Murzewski, 1958). In addition, negative values σ_{D} are impossible and the Gauss-normal distribution does not respect this condition. That is why the log-normal probability function will be better accepted as the theoretical probability distribution of stress invariant σ_{D} .

Statistical tests have shown that the log-normal probability function is the most likely for distribution of plastic strength R_{pl} (Murzewski, 1976). The log-normal probability functions are 'stable' in reference to multiplication. Therefore, the random plasticity factor γ will be also log-normal and the Laplace function $\Phi(\cdot)$ will define its cumulative probability function

$$\kappa = 1 - \Phi\left(\frac{\ln(R_{\text{D}}/\overline{\sigma}_{\text{D}})}{\nu_{\text{D}}}\right) = \Phi\left(\frac{\ln(\overline{\sigma}_{\text{D}}/R_{\text{D}})}{\nu_{\text{D}}}\right) = \Phi(\beta) \quad (27)$$

Symbols $\overline{\sigma}_{\text{D}}$ and R_{D} denote central values of the log-normal distributions. Originally, they are medians, but the material factor $R_{\text{D}}/\overline{\sigma}_{\text{D}}$ may be equal to the quotient of mean values: $R_{\text{D}} \exp(\nu_{\text{R}}^2/2)$ and $\overline{\sigma}_{\text{D}} \exp(\nu_{\sigma}^2/2)$ also if the log-normal coefficients of variation ν_{R} and ν_{σ} are equal, $\nu_{\text{R}} \approx \nu_{\sigma}$.

The plasticity index $\beta = \ln(\overline{\sigma}_{\text{D}}/R_{\text{D}})/\nu_{\text{D}}$, so called, is like the reliability index in probability-based design. It depends on log-normal distribution parameters. The inverse Laplace function may be used to determine it as a function of the yield ratio κ .

$$\beta(\kappa) = \text{inv } \Phi(\kappa) \quad (28)$$

Therefore, the effective stress in the elastic phase

$$\overline{\sigma}_{\text{D}} = R_{\text{D}} \exp(\beta\nu_{\text{D}}) \quad (29)$$

If the random microstresses $\overline{\sigma}_{\text{D}}$ and microstrengths R_{D} are not correlated, the logarithmic coefficient of variation of the random plasticity factor (25) will be equal to the geometric sum of the coefficients of variation of the microstress and the microstrength,

$$\nu_{\text{D}} = \sqrt{\nu_{\sigma}^2 + \nu_{\text{R}}^2} \quad (30)$$

where ν_{σ} and ν_{R} are logarithmic coefficients of variation of the effective microstress σ_{D} and microstrength R_{D} .

Once the distribution parameters R_{D} , ν_{D} of the deviator component of microstress are specified, the strains can be evaluated for any stress state

from Equation (21) or (22). The conventional yield strength f_y and the ultimate strength f_u occur with values of yielding ratio κ_y and κ_u , respectively.

The log-normal probability may give a simpler and more realistic solution than the truncated Gauss distributions of absolute random variables $|\sigma| > 0$ and $|R| > 0$, which had been supposed for the non-negative stress invariant $\overline{\sigma_D}$ and yield strength R_D in earlier works (e.g., Murzewski, 1954).

The microstructure parameters R_D , v_D hardly can be verified in laboratory tests. There is a gap between ideal continuum theory and physical investigations of material aggregates like concrete and metal alloys. A semi-empirical inverse method will be applied to specify the general deformation law for elastic-nonlinearly plastic solids. Two parameters R_D and v_D of the microstructure of the solid will be determined so that two points of its empirical uniaxial $\sigma_1-\varepsilon_1$ curve will coincide with two points of the theoretical curve defined by Equation (23) or (24). The involved yielding ratio κ is defined by Equation (27).

The advantage of the quasi-homogeneous model is that strength and deformation of elastic-nonlinearly plastic solids can be predicted with the aid of constitutive equation (21) or (22) at any state of stress taking only a simple uniaxial stress-strain curve as the empirical basis. The microstrength characteristics R_D , v_D are auxiliary parameters for calculations.

The ultimate strength f_u and the rupture shall occur at a critical point κ_{cr} of yielding, where $\sigma(\kappa) = \max$; however, it cannot be the local maximum point $\sigma(\varepsilon) = \max$ of the Ramberg-Osgood curve because it is a monotone curve. The critical point, σ_{cr} is calculated from the necessary condition of a local extreme value

$$\frac{d\sigma_D(\kappa, \beta(\kappa))}{d\kappa} = 0 \tag{31}$$

Equations (26) and (29) define the same value $\overline{\varepsilon_D}$. Differentiation of the implicit function $\sigma_D(\kappa, \beta(\kappa))$ gives the equation

$$\frac{d\sigma_D}{d\kappa} = 1 - \exp(v_{pl}\beta) \cdot \left[1 - \frac{v_{pl}(1-\kappa)}{\varphi(\beta)} \right] R_D = 0 \tag{32}$$

where $\varphi(\beta) = (1/\sqrt{2\pi}) \exp(-\beta^2/2)$ is the Gauss function and $\beta = \text{inv } \Phi(\kappa)$.

Equation (32) has been derived using Equations (13), (29), and (28):

$$\sigma_D = (1-\kappa)\overline{\sigma_D} + \kappa R_D \quad \overline{\sigma_D} = R_D \exp(v_D\beta) \quad \text{and} \quad \frac{d\beta}{d\kappa} = \frac{1}{\varphi(\beta)}$$

The ultimate strength f_u should be reached for the critical yielding ratio κ_{cr} , such that $\sigma(\kappa_{cr}) = \max$. If the stress σ still increased, the internal equilibrium of the quasi-homogeneous material would not be possible.

The yielding ratio $\kappa(\sigma)$ will jump to the trivial solution $\kappa = 1$ and the plastic rupture will occur. Thus, the critical point is associated with an unstable state of microstresses.

NUMERICAL EXAMPLES

The Ramberg–Osgood σ – ε curve is applied to numerical examples. There are three parameters: E , R_{02} , and n

$$\varepsilon_{RO}(\sigma) = \frac{\sigma}{E} + 0.002 \cdot \left(\frac{\sigma}{f_y} \right)^n \quad (33)$$

Mechanical properties of an exemplary aluminum alloy AlMgSi are as follows: $E = 70$ GPa, $\nu = 1/3$ are the elastic constants, $n = 12$ is the parameter of the Ramberg–Osgood stress–strain relation, $f_y = 240$ MPa is the conventional plastic strength for 0.2% permanent strain, and $f_u = 270$ MPa is the ultimate strength in uniaxial tensile test. $(1 - 2\nu)/3 \approx 1/9$, $2(1 + \nu)/3 \approx 8/9$ are the coefficients used in the three-dimensional stress–strain relations.

Elastic-plastic strains relative to the strength limits f_y and f_u are derived from Equation (33):

$$\varepsilon_y = \varepsilon_{RO}(240) = 0.00543, \quad \varepsilon_u = \varepsilon_{RO}(270) = 0.0121.$$

The yielding probabilities κ_y , κ_u and relative microstrength parameters in tension R_{pl} , ν_{pl} will be determined from a set of nonlinear equations. They are derived in such a way that collocation of the empirical curve and theoretical curve are done at two points: f_y , ε_y and f_u , ε_u . The yielding ratio $\kappa_y = \text{constant}$ is supposed to define the characteristic strength at any stress state. The yielding ratio κ_u is presumably equal to the critical value κ_{cr} , where $\sigma(\kappa) = \text{max}$.

Example 1. This concerns the quasi-homogeneous solid in uniaxial stress state

$$|\sigma_1| > 0, \quad \sigma_2 = 0, \quad \sigma_3 = 0$$

First, two theoretical models defined earlier are taken into account.

Model 1: Equations (23) and (27) are taken at the points f_y and f_u , after rearrangements

$$\begin{aligned} 70000 \cdot 0.0054 \cdot (1 - \kappa_y) &= \frac{1}{9} \cdot 240 \cdot (1 - \kappa_y) + \frac{8}{9} \cdot (240 - \kappa_y \cdot R_{pl}) \\ 70000 \cdot 0.0121 \cdot (1 - \kappa_u) &= \frac{1}{9} \cdot 270 \cdot (1 - \kappa_u) + \frac{8}{9} \cdot (270 - \kappa_u \cdot R_{pl}) \end{aligned}$$

$$\begin{aligned} \nu_{pl}\beta(\kappa_y) &= \ln(240 - \kappa_y R_{pl}) - \ln[(1 - \kappa_y)R_{pl}] \\ \nu_{pl}\beta(\kappa_u) &= \ln(240 - \kappa_u R_{pl}) - \ln[(1 - \kappa_u)R_{pl}] \end{aligned}$$

where $R_{pl} = R_D\sqrt{3/2}$, $\nu_{pl} = \nu_D$, $\beta(\kappa) = \text{inv } \Phi(\kappa)$ – plasticity index.

Solution of the Model 1 equations: $R_{pl} = 176.8$ MPa, $\nu_{pl} = 1.437$, $\kappa_y = 0.714$, and $\kappa_u = 0.874$.

Model 2: Equations (24) and (27) are taken at the same two points of the σ - ε curve:

$$70000 \cdot 0.0054 \cdot (1 - \kappa_y) = \frac{1}{9} \cdot 240 \cdot (1 - \kappa_y)^2 + \frac{8}{9} \cdot (240 - \kappa_y \cdot R_{pl})$$

$$70000 \cdot 0.0121 \cdot (1 - \kappa_u) = \frac{1}{9} \cdot 270 \cdot (1 - \kappa_u)^2 + \frac{8}{9} \cdot (270 - \kappa_u \cdot R_{pl})$$

$$\nu_{pl}\beta(\kappa_y) = \ln(240 - \kappa_y R_{pl}) - \ln[(1 - \kappa_y)R_{pl}]$$

$$\nu_{pl}\beta(\kappa_u) = \ln(240 - \kappa_u R_{pl}) - \ln[(1 - \kappa_u)R_{pl}]$$

Solution of the Model 2 equations: $R_{pl} = 171.9$ MPa, $\nu_{pl} = 1.493$, $\kappa_y = 0.724$, and $\kappa_u = 0.873$.

The solutions have been derived with the aid of Mathcad computer program.

Verification: The chi-square test is taken to select the more likely theoretical model

$$\text{Model 1: } \chi_{M1}^2 = \int_0^{0.9} \frac{(\varepsilon_{M1}(\kappa) - \varepsilon_{RO}(\kappa))^2}{\varepsilon_{RO}(\kappa)} d\kappa = 41.19 \times 10^{-6}$$

$$\text{Model 2: } \chi_{M2}^2 = \int_0^{0.9} \frac{(\varepsilon_{M2}(\kappa) - \varepsilon_{RO}(\kappa))^2}{\varepsilon_{RO}(\kappa)} d\kappa = 46.96 \times 10^{-6}$$

The Model 1 with the compressible plastic phase appears to be better than the incompressibility assumption of the uniform component of the plastic strain tensor, provided that the Ramberg–Osgood curve accurately renders the experimental data for the aluminum alloy. Stress σ_M and strain ε_M in Figure 3 will refer to Model 1.

Now, the critical value κ_{cr} where $\sigma_{\text{eff}}(\kappa) = \max$ is determined as the root of Equation (32):

$$\kappa_{cr} = \text{root} \left[1 - \exp(1.437\beta(\kappa_{cr})) \cdot \left(1 - \frac{1.437(1 - \kappa_{cr})}{\varphi(\beta(\kappa_{cr}))} \right), \kappa_{cr} \right] = 0.894$$

$$\text{with } \beta(\kappa_{cr}) = 1.245$$

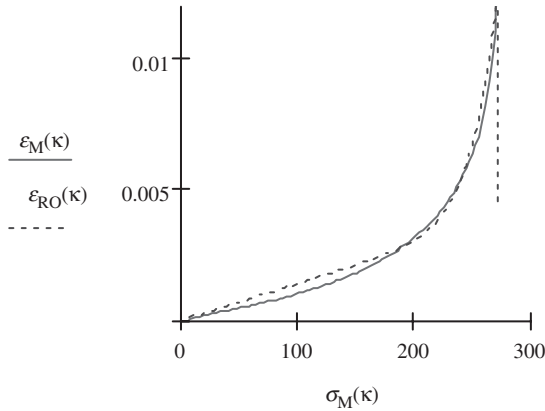


Figure 3. Theoretical curves $\sigma_M(\kappa) - \varepsilon_M(\kappa)$ and empirical one $\sigma_{R0}(\kappa)$ related to coordinate of state κ .

The maximum effective stress $\sigma_{cr} = \sigma(\kappa_{cr})$ is determined from Equations (13) and (29)

$$\begin{aligned}\sigma_{cr} &= [(1 - 0.894) \exp(0.894 \cdot 1.245) + 0.894] \cdot 176.8 \\ &= 270.6 \text{ MPa} \approx f_u = 270 \text{ MPa}\end{aligned}$$

The strain ε_{cr} relative to the critical stress σ_{cr} from Equation (24).

$$\varepsilon_{cr} = \frac{270.6(1 - 0.894) + 8(270.6 - 0.894 \cdot 176.8)}{9 \cdot 70000 \cdot (1 - 0.894)} = 0.0138 > \varepsilon_u = 0.0121$$

The $\sigma_M(\kappa)$ curve is presented in Figure 4. It confirms that the ultimate strength f_u will coincide with the critical stress σ_{cr} , which will cause rupture unless the loading σ_Q falls down; but the strain is 14% more at imminent rupture than the Ramberg–Osgood formula predicts.

Example 2. This concerns the quasi-homogeneous solid in simple shear,

$$\sigma_1 = \tau, \quad \sigma_2 = -\tau, \quad \sigma_3 = 0.$$

The cylindrical invariants in the case of the simple shear follow from Equations (7), (13), and (19)

$$\sigma_A = 0, \quad \sigma_D = \sqrt{2}\tau, \quad \omega_\sigma = \frac{-\pi}{6}$$

The shear stress $\tau(\kappa)$ is derived from Equation (32)

$$\tau(\kappa) = \frac{R_D}{\sqrt{2}} [(1 - \kappa) \exp(v_D \beta(\kappa)) + \kappa] \quad (34)$$

with $\beta(\kappa) = \text{inv } \Phi(\kappa)$ – plasticity index, Equation (27).

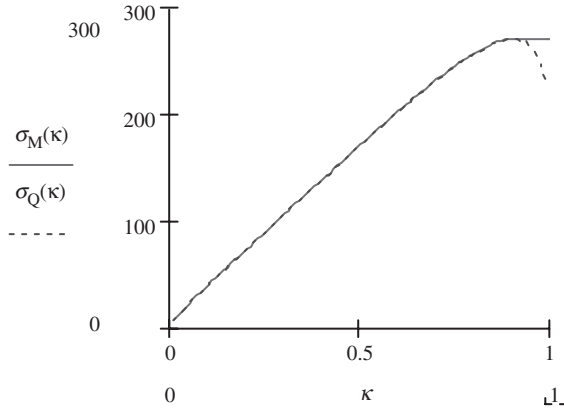


Figure 4. The tensile stress $\sigma(\kappa)$ and increasing probability of yielding κ .

The value $R_D = 174.5\sqrt{2/3} = 140.4$ MPa is the strength limit for the effective microstress and $\nu_D = \nu_{pl} = 1.493$ are equal coefficients of variation for the proportional variables R_D/R_{pl} . The parameters R_D, ν_D have been specified in Example 1 as constants of the elastic phase. The yielding probabilities $\kappa_y = 0.725$ and $\kappa_u = 0.873$ are taken for the characteristic shear strength τ_y and the ultimate shear strength τ_u , respectively. These values have been also determined in Example 1.

The permanent part of the shear strain is derived from Equations (21) and (17)

$$\eta_{pl}(\kappa) = \frac{1 + \nu}{E} \frac{\tau(\kappa)\sqrt{2} - \kappa R_D}{1 - \kappa} - \frac{1 + \nu}{E} \tau = \frac{1 + \nu}{E} \frac{\kappa \cdot (\tau(\kappa) - R_{pl}/\sqrt{3})}{1 - \kappa} \quad (35)$$

The characteristic shear strength τ_y is determined from Equation (34) for $\kappa_y = 0.725$

$$\tau_y = \frac{140.4}{\sqrt{2}} [(1 - 0.725) \cdot \exp(1.435\beta) + 0.725] = 138.6 \text{ MPa}$$

with $\beta_y = \text{inv } \Phi(0.725) = 0.597$.

The shear strength τ_y satisfies the H–M–H criterion exactly $f_y/\sqrt{3} = 240/\sqrt{3} = 138.6$ MPa. The plastic shear strain η_{pl} relative to the shear strength τ_y is determined from Equation (34)

$$\eta_{pl}(\kappa_y) = \frac{4}{3 \cdot 70000 \cdot \sqrt{2}} \frac{0.725 \cdot (138.6\sqrt{2} - 140.4)}{1 - 0.725} = 0.00197 \approx 0.002$$

The plastic strain η_y is insignificantly lower than the conventional value 0.2%.

The ultimate shear stress τ_u is determined from Equation (34) for the yielding ratio $\kappa_u=0.873$ and the strain η_u is from Equation (17) with $\sigma_D = \sqrt{2}\tau$, $\varepsilon_D = \sqrt{2}\eta$:

$$\tau_u = \frac{140.4}{\sqrt{2}} [(1 - 0.873) \cdot \exp(1.435\beta_u) + 0.873] = 155.9 \text{ MPa}$$

$$\text{with } \beta_u = \text{inv } \Phi(0.873) = 1.143$$

$$\eta_u = \frac{4}{3 \cdot 70000 \cdot \sqrt{2}} \frac{155.9\sqrt{2} - 0.873 \cdot 140.4}{1 - 0.873} = 0.0104$$

The critical yielding ratio $\kappa_{cr}=0.894$ in shear is the same as it has been in tension since the parameters in $\nu_D = \nu_{pl}$ in Equation (30) are equal. Maximum shear stress τ_{cr} and relative values β_{cr} and η_{cr} are determined for the critical yielding ratio $\kappa_{cr}=0.894$ from equations

$$\tau_{cr} = 156.5 \text{ MPa}, \quad \beta_{cr} = 1.245, \quad \eta_{cr} = 0.0121$$

The maximum shear stress τ_{cr} is approximately equal to the ultimate shear strength τ_u , which corresponds to the constant yield ratio κ_u like the ultimate strength f_u in the tension, but the shear strain η_{cr} at the maximum shear stress τ_{cr} is 16% more at imminent rupture than the ultimate strain η_u coherent with the Ramberg–Osgood curve.

CONCLUSIONS

- The Huber–Mises–Hencky yield criterion is proper to check the limit states of elastic-perfectly plastic solids; but the Huber–Hencky and the Huber–Mises yielding measures may be different for elastic-nonlinearly plastic solids. A scalar probability-based yielding ratio $\kappa(\sigma_{\text{eff}})$ may give a unique measure of damage at any state of stress unless microcracks would occur.
- A simple uniaxial tension test is sufficient to specify probability distribution parameters R_D , ν_D of random microstrengths of a quasi-homogeneous solid. Log-normal probability functions are acceptable for the plastic microstrengths and effective microstresses σ_{eff} . The probability of yielding κ is a function of the parameters of random plasticity factor R_{pl}/σ in the elastic phase of the solid.
- A yielding ratio κ_{pl} which is attributed to the characteristic plastic strength f_y , is constant at any state of stress for elastic-nonlinearly plastic solids. The characteristic plastic strain in shear is close to the

conventional value 0.2% in tension test. The rupture will occur at a critical value κ_{cr} when $\sigma(\kappa) = \max$, but not necessarily $\sigma(\varepsilon) = \max$. The critical strain ε_{cr} appears more than the value ε_u relative to ultimate strength f_u .

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