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# Constitutive Relationships for Laminates with Ply Cracks in In-plane Loading

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**ABSTRACT:** A theoretical framework which allows determining the whole set of 2-D thermomechanical constants of a damaged laminate as a function of crack density in different layers is presented. In this approach, closed-form expressions, which contain thermoelastic ply properties, laminate layup, and crack density as the input information are obtained. It is shown that the crack opening displacement (COD) and crack face sliding displacement, normalized with respect to a load variable, are important parameters in these expressions influencing the level of the properties degradation. They are determined in this paper using generalized plain strain FEM analysis results for noninteractive cracks. The strong dependence of the COD on the relative stiffness and thickness of the surrounding layers, found in this study, is described by a power law. The methodology is validated and the possible error introduced by the noninteractive crack assumption is estimated by comparing with the 3-D FEM solution for a cross-ply laminate with two orthogonal systems of ply cracks. Experimental data and comparison with other models are used for further verification.

**KEY WORDS:** homogenization, intralaminar cracks, laminate stiffness.

## INTRODUCTION

COMPOSITE LAMINATES UNDER service loading undergo complex combinations of thermal and mechanical loading, leading to microdamage accumulation in the plies. The first mode of damage is usually intralaminar cracking with the crack plane transverse to the laminate middle-plane, spanning the whole width of the specimen. The density of cracks in a ply depends on layer orientation with respect to the load direction, temperature change, number of cycles in fatigue, laminate layup, ply thickness and,

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certainly, material fracture toughness. Relative displacements of crack surfaces during loading reduce the average strain and stress in the damaged layer, thus reducing the laminate stiffness. Many papers have been written on this subject, covering a broad range from micromechanics based to continuum damage mechanics (CDM)-based models (see review for example in Nairn and Hu (1994), Nairn (2000), and Talreja (1994)).

Most of the research has, however, been focused on cross-ply laminates which are excellent for academic studies of phenomena but are seldom used in practical applications. Laminates with a general layup containing cracks in several layers of different orientation are, therefore, a challenge for any constitutive model.

A two-dimensional shear-lag analysis is a simplest way to describe a doubly periodic matrix cracking in cross-ply laminates. It is used in Henaff-Gardin et al. (1996), where parabolic shape of the crack face is assumed to model the crack profile. It means that no distinction has been made between crack shape in the internal and the external layers. A model for general in-plane loading is derived for  $[0_m, 90_n]_s$  laminates averaging the equilibrium equations and obtaining second-order differential equations in a usual way. Unfortunately, there is no comparison with experimental data or with other models in this paper.

Hashin (1987) generalized his model (Hashin, 1985) to the case when cracks are in both 0- and 90-layers of a cross-ply laminate. Solution for an orthogonally cracked cross-ply laminate under tension was found constructing a simple admissible stress field in the context of the principle of minimum complementary energy. The chosen stress field satisfies equilibrium equations and all boundary and interface conditions in tractions. The assumed constant in-plane normal stress distribution over each layer thickness leads to linear and parabolic through-the-thickness distributions of out-of-plane shear and normal stresses, respectively. The principle of minimum complementary energy (which for approximate stress distributions is equivalent to satisfying the displacement continuity equations in average) is used to calculate the stress distributions. Expressions for damaged laminate E-modulus and Poisson's ratio were derived. This model does not involve any fitting parameters and is simple to use. Since Hashin's model renders a lower bound of the stiffness, its accuracy could be improved by more refined assumptions regarding the thickness coordinate dependence of stresses. The assumptions used are oversimplified and give too low stiffness of the damaged laminate.

McCartney (1992) applied his model, which is based on the same stress distribution assumptions as Hashin's model (Hashin, 1987) but the governing equations are obtained from Reissners principle, to doubly cracked cross-ply laminates assuming that the in-plane normal stress

dependence on the two in-plane coordinates is given by two independent functions.

A model of similar accuracy as Hashin's and McCartney's models was developed by Abdelrahman and Nayfeh (1999) to analyze stiffness of doubly cracked cross-ply laminates. In addition to the assumptions of a linear shear stress distribution across each layer, which is the same as in Hashin's model, authors assume linear distribution of out-of-plane displacements. These assumptions allow for exact satisfaction of all displacement and traction interface and boundary conditions. Since in derivations only the stress-strain relationships averaged over the layer thickness are used, the constitutive relationships are not satisfied pointwise. The governing equations are a system of two fourth-order partial differential equations with constant coefficients. Unfortunately, predictions and comparison with test data and other models are presented only for the case of one crack system.

The most accurate local stress state comparable with a very fine FE solution and, therefore, also accurate stiffness prediction can be obtained using semianalytical McCartney (1995), and Schoeppner and Pagano (1998) models. In the McCartney model, each layer in the laminate is divided into a certain number of thin sublayers and in each sublayer, the stress assumptions are as in Hashin's variational model (Hashin, 1985). All displacement and stress continuity conditions at sublayer interfaces are satisfied as are the stress-strain relationships, except one, which is satisfied in an average sense. It has been shown that this 'satisfying in average' is identical to minimization of the Reissner energy functional in the used approximation of the stress-strain state. The Schoeppner-Pagano model (1998), which is also based on Reissner's principle, considers a system of hollow concentric sub-cylinders with a large radius instead of laminate divided into sublayers. Each layer is divided into a number of cylinders. In order to simulate interface cracks, these cylinders may also be connected in parallel. Shape functions for each subcylinder in this model are different than in McCartney's model but the results converge with increasing number of sublayers (subcylinders) (McCartney et al., 2000). However, the calculation routines in these models are extremely complex which limits the application.

Neither of these models can be directly used for laminates containing several systems of cracks. However, considering these crack systems as noninteracting, one can first introduce crack system in 90-layer only and back calculate the effective stiffness of the damaged layer from the damaged laminate stiffness. Then the intralaminar cracks are introduced in the 0-layer only and similar problems as described above is solved in a system of coordinates rotated by 90°. Finally the effective properties of all damaged layers may be used in laminate theory to calculate the stiffness of laminate with cracks in both layers. The Schoeppner and Pagano model has been

used in this way to predict the reduction of thermal expansion coefficients of cross-ply laminates with cracks in both 0- and 90°-layers in (Kim et al., 2000).

Generally speaking, the CDM approaches (Allen et al., 1987; Ladeveze, 1990; Talreja, 1994) may be used to describe the stiffness of laminates with intralaminar cracks in off-axis plies of any orientation. The damage is represented by internal state variables (ISV) and the laminate constitutive equations are expressed in general forms containing ISV and a certain number of material constants. These constants must be determined for each considered laminate configuration either experimentally measuring stiffness for a laminate with a certain crack density or using FE analysis for the same reason. This limitation is partially removed in synergistic damage mechanics suggested by Talreja (1996), which incorporates micromechanics information to determine the material constants. For the same  $[\pm\theta, 90_2]_s$  class of laminates as in theoretical assessment (Talreja, 1996), Varna et al. (1999a) used experimentally measured crack opening displacement (COD) to identify the constraint parameter in CDM and to make stiffness predictions. For these measurements, a special device was designed and measurements were performed using optical microscopy on loaded specimens (Varna, 1993). The same technique was later applied to measure COD for cracks in off-axis plies of  $[0/\pm\theta_4/0_{1/2}]_s$  laminates and to perform CDM predictions (Varna et al., 1999b).

An extensive FE parametric analysis in plane stress formulation was performed by Joffe et al. (2001) to identify the main geometrical and stiffness parameters affecting the COD. It was found that average COD normalized with respect to the far field stress in the layer and the layer thickness is a very robust parameter: variation of shear moduli and Poisson's ratios has a negligible effect on the normalized COD. Only the stiffness and thickness ratios of the cracked to uncracked neighboring layers have a significant effect. Based on numerical results, the numerical COD values were fitted by power law. The main conclusion was that increasing stiffness and thickness of the constraint layer leads to significant reduction of the average normalized COD.

This power law for COD was used in the synergistic CDM predictions of stiffness reduction in  $[\pm\theta, 90_4]_s$  laminates (Varna et al., 2001) with cracks in 90°-layers only. Recently, it was demonstrated using micromechanics that all material parameters in CDM for this layup depend only on the material properties of the layer, not on the laminate layup (Varna et al., 2003). This finding was not proven for laminates with cracked layers other than 90° because analytical micromechanics solution for a general case does not exist.

Gudmundson and coworkers (1992, 1993) considered laminates with general layup and used the homogenization technique to derive expressions

for stiffness and thermal expansion coefficient of laminates with cracks in layers of 3-D laminates. These expressions in an exact form correlate damaged laminate thermoelastic properties with parameters characterizing crack behavior: the average COD and crack face sliding. These parameters follow from the solution of the local boundary value problem and their determination is a very complex task. Gudmundson and coworkers suggested to neglect the effect of neighboring layers on crack face displacements and to determine them using the known solution for a periodic system of cracks in an infinite homogeneous transversely isotropic medium ( $90^\circ$ -layer). The application of their methodology by other researchers has been rather limited due to the fairly complex form of the presented solutions.

In the present paper an attempt, similar to that performed by Gudmundson, is presented in the framework of the laminate theory. The biggest advantage is the transparency of derivations and the simplicity of application. Stiffness or compliance matrices and thermal expansion coefficients of an arbitrary symmetric laminate with damage in certain layers are presented in an explicit form. Derivation of constitutive relationships follows the same route as in the classical laminate theory. As an input from the homogenization theory, the relationships between volume-averaged and boundary surface-averaged quantities are used. The differences between undamaged and damaged laminate cases are indicated in each step of derivation. The damaged laminate stiffness and thermal expansion coefficients are calculated from the undamaged laminate stiffness and the crack face displacements normalized with respect to the far field stress in the layer.

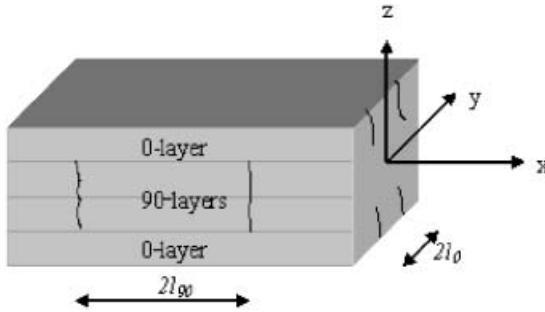
In contrast to Gudmundson's approach (Gudmundson and Östlund, 1992; Gudmundson and Zang, 1993), the normalized COD and crack face sliding are considered as dependent on the position of the cracked layer (outside or inside cracks) and on the constraint of the surrounding layers in terms of their stiffness and thickness. These dependences are analyzed using FEM calculated COD profiles in generalized plane strain formulation and presenting the results in the form of power laws.

In a special case of balanced laminates with cracks in  $90^\circ$ -layer only, expressions for thermoelastic properties are presented in an explicit and compact form.

## **STRESS-STRAIN RESPONSE OF DAMAGED LAMINATES**

### **Problem Formulation**

In this derivation, a symmetric laminate subjected to general in-plane loading is considered. To exclude bending effects, the laminate is assumed to be symmetric also in the damaged state (crack density is the same in



**Figure 1.** Schematic view over  $[0, 90]_s$  laminate with two orthogonal systems of cracks.

layers with symmetric location with respect to the midplane). Only in-plane loading is considered and the intralaminar cracks are assumed to run parallel to fibers with a crack plane transverse to the laminate midplane and to span the whole cross section of the layer. Laminate of thickness  $h$  contains  $N$  layers of which the  $k$ th layer is characterized by stiffness  $[Q]_k$ , thickness  $t_k$ , and fiber orientation angle, which determines the stress transformation matrix  $[T]_k$  between global and local coordinates. The overbar on the matrix and vectors denotes quantities in the global coordinate system. The crack density in a layer is  $\rho_{kn} = 1/2l_k$  and normalized crack density  $\rho_{kn}$  is defined as  $\rho_{kn} = t_k \rho_k$ . The geometry of the problem for the particular case of a doubly cracked cross-ply laminate can be seen in Figure 1.

The thermoelastic relation between applied stresses and strains experienced by the damaged laminate can be written in the following way:

$$\{\sigma\}^{\text{LAM}} = [Q]^{\text{LAM}}[\{\varepsilon\}^{\text{LAM}} - \{\alpha\}^{\text{LAM}}\Delta T] \quad (1)$$

where,

$$\Delta T = T - T_{\text{ref}} \quad (2)$$

In (1)  $\{\sigma\}^{\text{LAM}}$  and  $\{\varepsilon\}^{\text{LAM}}$  are macroscopic stress and strain vectors applied at the boundary of the representative volume element (RVE),  $[Q]^{\text{LAM}}$  and  $\{\alpha\}^{\text{LAM}}$  are the unknown stiffness matrix and thermal expansion coefficient vector of the damaged laminate to be determined.

### Homogenization Relationships

Introducing volume-averaged stresses and strains as in (Allen and Yoon, 1998) and using superscript  $a$  to denote average quantities, we have,

$$\sigma_{ij}^a = \frac{1}{V} \int_V \sigma_{ij} dV \quad \varepsilon_{ij}^a = \frac{1}{V} \int_V \varepsilon_{ij} dV \quad (3)$$

Here  $V$  is the volume of averaging, which may be one layer or the whole laminate volume, as needed. The average stress–strain relationships for a  $k$ th layer in the global coordinate system are:

$$\{\bar{\sigma}\}_k^a = [\bar{Q}]_k [\{\bar{\varepsilon}\}_k^a - \{\bar{\alpha}\}_k \Delta T] \tag{4}$$

Using the divergence theorem, it may be shown (Allen and Yoon, 1998; Varna, 2002) that stresses applied to the laminate boundary are equal to the stresses averaged over the volume of the whole laminate. Expressing the volume integral as a sum of integrals over volume of individual layers, we obtain

$$\{\sigma\}^{\text{LAM}} = \{\bar{\sigma}\}^a = \sum_{k=1}^N \{\bar{\sigma}\}_k^a \frac{t_k}{h} \tag{5}$$

Using the divergence theorem, it can also be shown (Allen and Yoon, 1998; Varna, 2002) that the volume average strains in each layer are equal to boundary-averaged strains defined as:

$$E_{ij} = \frac{1}{V} \int_S \frac{1}{2} (u_i n_j + u_j n_i) dS \tag{6}$$

Definition (6) is written for tensorial boundary-averaged strains. Using this definition, one can easily check that average strains at the external boundary of a layer are equal to the applied macroscopic strains, which are the same for all layers in the damaged laminate (isostrain condition in laminate theory).

Since the integration in (6) involves the total boundary including the crack surface, the above mentioned equality of volume-averaged and boundary-averaged strains for the  $k$ th layer may be written as:

$$\left\{ \begin{matrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\gamma}_{12} \end{matrix} \right\}_k^a = \left\{ \begin{matrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{matrix} \right\}^{\text{LAM}} + \left\{ \begin{matrix} \bar{\beta}_{11} \\ \bar{\beta}_{22} \\ 2\bar{\beta}_{12} \end{matrix} \right\}_k \tag{7}$$

Here  $\{\bar{\beta}\}_k$  is the Vakulenko–Kachanov tensor defined by

$$\bar{\beta}_{ij} = \frac{1}{V} \int_{S_c} \frac{1}{2} (u_i n_j + u_j n_i) dS \tag{8}$$

The engineering strains and engineering form of the Vakulenko–Kachanov tensor  $\{\bar{\beta}\}_k$  are used in this paper. In layers with no cracks,



$\beta_{ij}$  is zero.  $S_c$  is the total surface of cracks in the layer,  $u_i$  are displacements of the points on the crack surface,  $n_i$  is outer normal to the crack surface, and  $V$  is the volume of the layer.

### Crack Face Relative Displacements and Vakulenko–Kachanov Tensor

Considering (8) in the local coordinate system related to fiber orientation in the  $k$ th layer, it is seen that the only nonzero components are  $\beta_{12}$  and  $\beta_{22}$  (1 is the fiber direction and 2 is in-plane orientation transverse to the fiber direction), given by,

$$\beta_{12}^k = -\rho_k u_{1a}^k \quad \beta_{22}^k = -2\rho_k u_{2a}^k \quad (9)$$

Here  $u_{1a}^k$  and  $u_{2a}^k$  are the average crack face sliding displacement and average crack face opening displacement, respectively defined as:

$$u_{1a}^k = \frac{1}{2t_k} \int_{-t_k/2}^{t_k/2} \Delta u_1(x_3) dx_3 \quad u_{2a}^k = \frac{1}{2t_k} \int_{-t_k/2}^{t_k/2} \Delta u_2(x_3) dx_3 \quad (10)$$

Here  $\Delta u_i$  are the separation distances of the two crack faces. Normalizing the displacements with respect to thickness of the cracked layer (length of the crack) and the far field (CLT) stresses in the layer corresponding to the same load applied to undamaged laminate (indicated by subscript 0) gives:

$$u_{1an}^k = u_{1a}^k \frac{G_{12}}{t_k \sigma_{120}^k} \quad u_{2an}^k = u_{2a}^k \frac{E_2}{t_k \sigma_{20}^k} \quad (11)$$

Using Equation (11) in Equation (9) provides expressions for components of Vakulenko–Kachanov tensor through normalized displacements and far field stresses:

$$\beta_{12}^k = -\rho_{kn} u_{1an}^k \frac{\sigma_{120}^k}{G_{12}} \quad \beta_{22}^k = -2\rho_{kn} u_{2an}^k \frac{\sigma_{20}^k}{E_2} \quad (12)$$

Introducing the displacement matrix  $U$  makes it possible to express the Vakulenko–Kachanov tensor in the Voigt notation as a matrix product.

$$[U]_k = 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & u_{2an}^k & 0 \\ 0 & 0 & \frac{E_2}{G_{12}} u_{1an}^k \end{bmatrix} \quad (13)$$

$$\{\beta\}_k = -\frac{\rho_{kn}}{E_2}[U]_k\{\sigma_0\}_k \quad (14)$$

From here on, the vectorial representation of the Vakulenko–Kachanov tensor is used.

Transforming Equation (14) to global coordinates is the same as engineering strain transformation in CLT.

$$\{\bar{\beta}\}_k = [T]_k^T\{\beta\}_k \quad (15)$$

The far field stress components in the cracked layer required in Equation (14) can be expressed using CLT.

$$\{\sigma_0\}_k = [T]_k[\bar{Q}]_k[\{\varepsilon_0\}^{\text{LAM}} - \{\bar{\alpha}_0\}_k\Delta T] \quad (16)$$

Substituting Equations (16) and (14) in Equation (15) gives:

$$\{\bar{\beta}\}_k = \frac{-\rho_{kn}}{E_2}[T]_k^T[U]_k[T]_k[\bar{Q}]_k[\{\varepsilon_0\}^{\text{LAM}} - \{\bar{\alpha}_0\}_k\Delta T] \quad (17)$$

### Constitutive Relationships for Damaged Laminates

Substituting expression (7) in the averaged stress–strain relationships (4) and using Equation (5) gives the following expression for laminate stresses.

$$\{\sigma\}^{\text{LAM}} = [Q_0]^{\text{LAM}}\{\varepsilon\}^{\text{LAM}} - \frac{1}{h}\sum_{k=1}^N[\bar{Q}]_k\{\bar{\alpha}\}_k\Delta T t_k + \frac{1}{h}\sum_{k=1}^N[\bar{Q}]_k\{\bar{\beta}\}_k t_k \quad (18)$$

The second term on the right-hand side of (18) can be identified with the ‘thermal force’ per unit thickness,  $\{\sigma\}_{\text{th}}^{\text{LAM}}$  known in laminate theory. Since it can be related to the strain response of undamaged laminate as:

$$\{\sigma\}_{\text{th}}^{\text{LAM}} = [Q_0]^{\text{LAM}}\{\varepsilon_0\}_{\text{th}}^{\text{LAM}} \quad (19)$$

Equation (18) can be rewritten as

$$\{\sigma\}^{\text{LAM}} = [Q_0]^{\text{LAM}}[\{\varepsilon\}^{\text{LAM}} - \{\varepsilon_0\}_{\text{th}}^{\text{LAM}}] + \frac{1}{h}\sum_{k=1}^N[\bar{Q}]_k\{\bar{\beta}\}_k t_k \quad (20)$$

Here  $[Q_0]^{\text{LAM}}$  is the stiffness matrix of the undamaged laminate calculated as in the classical laminate theory (CLT).  $\{\varepsilon_0\}^{\text{LAM}}$  is the strain in the undamaged laminate corresponding to the same applied load.

Substituting Equation (17) in Equation (20) gives the final form for damaged laminate thermomechanical stress–strain response:

$$\begin{aligned} \{\sigma\}^{\text{LAM}} &= [Q_0]^{\text{LAM}}[\{\varepsilon\}^{\text{LAM}} - \{\varepsilon_0\}_{\text{th}}^{\text{LAM}}] \\ &\quad - \frac{1}{hE_2} \sum_{k=1}^N \rho_{kn} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k [\{\varepsilon_0\}^{\text{LAM}} - \{\bar{\alpha}_0\}_k \Delta T] t_k \end{aligned} \quad (21)$$

### Stiffness and Compliance Matrices of the Damaged Laminate

Assuming only mechanical loading ( $\Delta T = 0$ ) in Equation (21) and using

$$\{\varepsilon_0\}^{\text{LAM}} = [S_0]^{\text{LAM}} \{\sigma\}^{\text{LAM}} \quad (22)$$

gives

$$\begin{aligned} \{\sigma\}^{\text{LAM}} &= [Q_0]^{\text{LAM}} \{\varepsilon\}^{\text{LAM}} \\ &\quad - \frac{1}{hE_2} \sum_{k=1}^N \rho_{kn} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k [S_0]^{\text{LAM}} \{\sigma\}^{\text{LAM}} t_k \end{aligned} \quad (23)$$

Expressing laminate stress from Equation (23) gives:

$$\{\sigma\}^{\text{LAM}} = \left( [I] + \frac{1}{hE_2} \sum_{k=1}^N \rho_{kn} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k [S_0]^{\text{LAM}} t_k \right)^{-1} [Q_0]^{\text{LAM}} \{\varepsilon\}^{\text{LAM}} \quad (24)$$

where  $[I]$  is the identity matrix.

Comparing (24) with Equation (1), with  $\Delta T = 0$ , the stiffness matrix and corresponding compliance matrix for the damaged laminate are found to be:

$$[Q]^{\text{LAM}} = \left( [I] + \frac{1}{hE_2} \sum_{k=1}^N \rho_{kn} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k [S_0]^{\text{LAM}} t_k \right)^{-1} [Q_0]^{\text{LAM}} \quad (25)$$

$$[S]^{\text{LAM}} = [S_0]^{\text{LAM}} \left( [I] + \frac{1}{hE_2} \sum_{k=1}^N \rho_{kn} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k [S_0]^{\text{LAM}} t_k \right) \quad (26)$$

These expressions may be used to calculate the degradation of mechanical properties for the damaged laminate.

**Thermal Expansion Coefficients of the Damaged Laminate**

If the reductions in thermal properties are of interest, a derivation based on thermal loading only has to be done. Applying thermal loads only, the global laminate stresses are equal to zero, and Equation (20) allows determining the thermal expansion strains of the damaged laminate.

$$\{\varepsilon\}^{\text{LAM}} = \{\varepsilon_0\}_{\text{th}}^{\text{LAM}} - [S_0]^{\text{LAM}} \frac{1}{h} \sum_{k=1}^N [\bar{Q}]_k \{\bar{\beta}\}_k t_k \tag{27}$$

Applying Equation (17) in (27), the following relationship can be obtained:

$$\begin{aligned} \{\varepsilon\}^{\text{LAM}} = & \{\varepsilon_0\}_{\text{th}}^{\text{LAM}} + [S_0]^{\text{LAM}} \frac{1}{hE_2} \sum_{k=1}^N \rho_{kn} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k \{\varepsilon_0\}_{\text{th}}^{\text{LAM}} t_k \\ & - [S_0]^{\text{LAM}} \frac{1}{hE_2} \Delta T \sum_{k=1}^N \rho_{kn} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k \{\bar{\alpha}\}_k t_k \end{aligned} \tag{28}$$

By dividing Equation (28) by  $\Delta T$ , the final expression for the thermal expansion coefficient for the damaged laminate is obtained.

$$\{\alpha\}^{\text{LAM}} = \left( [I] + \sum_{k=1}^N \frac{t_k}{h} \rho_{kn} [D]_k \right) \{\alpha\}_0^{\text{LAM}} - \sum_{k=1}^N \frac{t_k}{h} \rho_{kn} [D]_k \{\bar{\alpha}\}_k \tag{29}$$

where,

$$[D]_k = [S]_0^{\text{LAM}} \frac{1}{E_2} [\bar{Q}]_k [T]_k^T [U]_k [T]_k [\bar{Q}]_k \tag{30}$$

**Thermoelastic Properties of Laminates with Cracks in 90°-layers**

In balanced and symmetric laminates with cracks in 90°-layers only, the matrix relationships for stiffness and thermal expansion coefficients may be simplified and expressed in the explicit form. We consider a particular case

often used in research when the  $90^\circ$ -layer with crack density  $\rho_n$  is in the middle of the laminate. The set of layers surrounding the  $90^\circ$ -layer on either side may be considered as a sublaminates with thermoelastic properties calculated using CLT and denoted by upper index  $s$ . Using  $90^\circ$ -layer properties in the local system and denoting thickness of the sublaminates and  $90^\circ$ -layer by  $t_s$  and  $t_{90}$ , respectively, we obtain, after tedious work, the following relationships for engineering constants of the damaged laminate.

$$\frac{E_x}{E_{x0}} = \frac{1}{1 - Q_{22}(1 - \nu_{12}\nu_{xy0})g_3k\rho_n u_{2an}} \quad (31)$$

$$\frac{E_y}{E_{y0}} = \frac{1}{1 - Q_{22}(((\nu_{12} - \nu_{yx0})^2/(1 - \nu_{12}\nu_{xy0}))(\nu_{xy0}/\nu_{yx0}))g_3k\rho_n u_{2an}} \quad (32)$$

$$\frac{\nu_{xy}}{\nu_{xy0}} = \frac{1 + Q_{22}((\nu_{12} - \nu_{yx0})/\nu_{yx0})g_3k\rho_n u_{2an}}{1 - Q_{22}(1 - \nu_{12}\nu_{xy0})g_3k\rho_n u_{2an}} \quad (33)$$

$$\frac{\nu_{yx}}{\nu_{yx0}} = \frac{1 + Q_{22}((\nu_{12} - \nu_{yx0})/\nu_{yx0})g_3k\rho_n u_{2an}}{1 - Q_{22}(((\nu_{12} - \nu_{yx0})^2/(1 - \nu_{12}\nu_{xy0}))(\nu_{xy0}/\nu_{yx0}))g_3k\rho_n u_{2an}} \quad (34)$$

$$\frac{\alpha_x}{\alpha_{x0}} = 1 - Q_{22} \frac{[\alpha_{x0} - \alpha_2 + \nu_{12}(\alpha_{y0} - \alpha_1)]}{\alpha_{x0}} g_3k\rho_n u_{2an} \quad (35)$$

$$\begin{aligned} \frac{\alpha_y}{\alpha_{y0}} &= 1 - Q_{22} \frac{[\alpha_{x0} - \alpha_2 + \nu_{12}(\alpha_{y0} - \alpha_1)]}{\alpha_{y0}} \\ &\times \left( \frac{\nu_{12} - \nu_{yx0}}{1 - \nu_{12}\nu_{xy0}} \right) \left( \frac{\nu_{xy0}}{\nu_{yx0}} \right) g_3k\rho_n u_{2an} \end{aligned} \quad (36)$$

where,

$$g_3 = \frac{t_{90}}{2t_s} \left( \frac{S_{12}^s t_s + S_{xy}^s t_{90}/2}{S_{xy}^s S_{11}^s t_s + S_{yy}^s t_{90}/2} - S_{xx}^s \right) \quad (37)$$

$$\frac{1}{k} = \frac{E_2 t_{90}}{4 E_x^s t_s} \left[ 1 + \frac{2S_{22}^s t_s}{S_{xx}^s t_{90}} - \frac{2(S_{xy}^s t_{90}/2 + S_{12}^s t_s)^2}{S_{xx}^s t_{90} (S_{yy}^s t_{90}/2 + S_{11}^s t_s)} \right] \quad (38)$$

In (37) and (38),  $S_{ij}$  are elements of the compliance matrices. In the particular case of cross-ply laminate, the sublaminates are the  $0^\circ$ -layer, and,

$$S_{xx}^s = S_{11}, \quad S_{yy}^s = S_{22}, \quad S_{xy}^s = S_{12}, \quad E_x^s = E_1 \quad (39)$$

### FINITE ELEMENT CALCULATIONS

Finite element calculations were used (a) to perform parametric analysis of the main factors governing the value of the normalized crack opening; and (b) to render data for validation of the developed analytical model. For all FE calculations, the commercial code ANSYS 6.1 was used. In order to model the repeating volume element (see Figure 2), a 3-D model was created. The SOLID185 elements were used in all calculations. The main reason for choosing a 3-D model was to use the same elements for 3-D calculations (two orthogonal crack systems) and for generalized plain strain case (one system of cracks).

Two geometrical configurations were considered; see Figure 2 for geometry and boundary conditions modeled. In the first configuration, the cracked layer is in the middle of the laminate (inside crack) and in the second case, the crack is in the surface layer (outside crack). The upper boundary of the laminate was always traction free. Analyzing the COD, the crack density was always chosen small enough to get noninteracting cracks ( $2l_0/t_{90} = 5$  for inside cracks) and the number of elements was 6400. The stiffness ratio between the sublaminates and  $90^\circ$ -layer as well as the layer thickness ratio were varied. Figure 2 represents a quarter of the RVE defined in Figure 1. For a more detailed analysis and parameter study using plane stress formulation, see Joffe et al. (2001). For outside crack, the sublaminates and the  $90^\circ$ -layer have interchanged places.

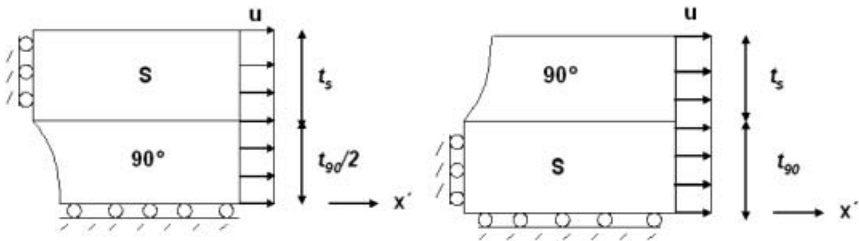


Figure 2. Load cases used for determination of average crack face opening displacement.

### Power Law for Crack Opening Displacement

Expressions for thermoelastic constants of the damaged laminate presented in the previous section can be applied only if the normalized crack opening and sliding displacements introduced in Equations (10) and (11) are known functions of laminate configuration and material properties. In this paper, we consider only laminates and properties with negligible sliding leaving the sliding effects for a separate paper. It has to be emphasized that Equations (10) and (11) are defined in the coordinate system where the cracked layer has a  $90^\circ$  orientation with respect to  $x'$ . Hence, an appropriate model to study the normalized COD is a cross-ply type symmetric laminate containing cracked 90-layer which is supported by a sublaminates.

A series of FEM calculations were performed and the displacement in  $x'$ -direction for the nodes at the crack surface was used to calculate the average value of the crack face displacement,  $u_{2a}$ . That value was then normalized with respect to thickness of the cracked layer and the far field stress in the layer transverse to the crack plane according to Equation (11).

Results were fitted by a power law as follows:

$$u_{2an} = A + B \left( \frac{E_2}{E_{x'}} \right)^n \quad (40)$$

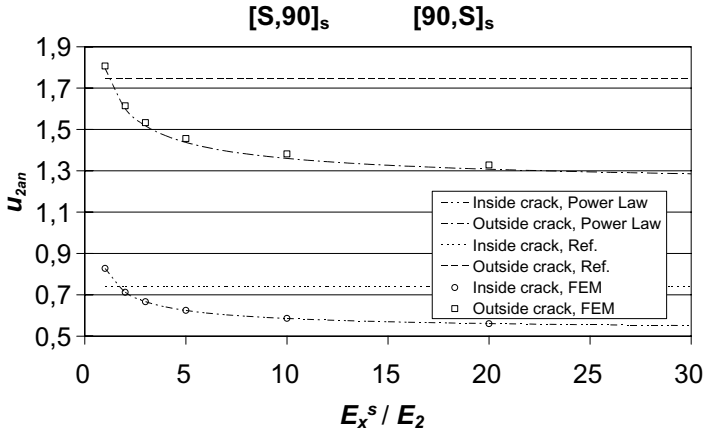
The obtained constants in the two power laws for inside and outside cracks, respectively are presented here.

For inside crack:

$$\begin{aligned} A &= 0.52 \\ B &= 0.3075 + 0.1652 \left( \frac{t_{90} - 2t_s}{2t_s} \right) \\ n &= 0.030667 \left( \frac{t_{90}}{2t_s} \right)^2 - 0.0626 \left( \frac{t_{90}}{2t_s} \right) + 0.7037 \end{aligned} \quad (41)$$

For outside crack:

$$\begin{aligned} A &= 1.2 \\ B &= 0.5942 + 0.1901 \left( \frac{t_{90} - 2t_s}{2t_s} \right) \\ n &= -0.13073 \left( \frac{t_{90}}{2t_s} \right)^2 + 0.4437 \left( \frac{t_{90}}{2t_s} \right) + 0.2576 \end{aligned} \quad (42)$$



**Figure 3.** The dependence of the normalized crack face opening displacement  $u_{2an}$  on the layer stiffness ratio for both inside ( $[S,90]_s$ ,  $2t_s = t_{90}$ ) and outside crack ( $[90,S]_s$ ,  $t_s = t_{90}$ ). Fitting by power law. "Ref" is the prediction according to Gudmundson et al. (1993).

The normalized CODs for inside and outside cracks versus layer stiffness ratio in case of  $[S,90]_s$  laminate is shown in Figure 3. Both, directly calculated by FEM and from the power law are in a very good agreement: the power law gives a very good description of the CODs for both crack systems. The normalized COD of the outside crack is significantly larger and both crack types show a strong dependence of COD on the surrounding layer thickness. The increasing constraint due to stiffer surrounding layers may lead to 30% reduction of the COD as compared to crack surrounded by an isotropic medium. The values corresponding to solution for a periodic system of cracks in an infinite transversally isotropic medium used in (Gudmundson and Zang, 1993) are also shown for comparison. Obviously, they do not depend either on the relative stiffness of layers nor on their thickness. The thickness of the constraint layer has a similar effect as its stiffness: increase leads to smaller normalized COD.

### Elastic Properties of Damaged Laminate

The calculation of elastic properties for cross-ply laminates with cracks in  $90^\circ$ -layer using FEM was performed for  $2l_{90}/t_{90} = 5$ . The number of elements was 6400. The stiffness of  $[0,90]_s$  cross-ply laminate with cracks in both  $90^\circ$ - and  $0^\circ$ -layers was calculated assuming the same crack density in both layers and the number of elements was same in  $x$ ,  $z$ - and  $y$ ,  $z$ -plane. The total number of elements used was 36,000 and for one particular case, it was 80,000. The number of elements in the mesh was varied to find the most



suitable mesh, taking both the calculation time and the accuracy into consideration.

## RESULTS AND DISCUSSION

The expressions in matrix form for damaged laminate stiffness (25) and thermal expansion coefficients (29) and (30) were used along with power law expressions (40)–(42) in predictions for damaged cross-ply laminates. Predictions for laminates with cracks in 90°-layer only are validated comparing with results from direct FEM calculations and further verified comparing with experimental data obtained in our laboratory (Varna and Krasnikovs, 1998; Joffe et al., 2001; Varna et al., 2001). Comparison is made for different types of glass fiber/epoxy and carbon fiber/epoxy systems and different laminate layups. The five materials used are defined in Table 1. Since no stiffness data for laminates with two orthogonal systems of cracks are available, comparison in this case is made with Hashin's model and with 3-D FEM calculations with a very fine mesh. Experimental data from (Kim et al., 2000) for reduction of thermal expansion coefficients of cross-ply laminates containing cracks in 90°-layer and for laminates with two crack systems are used.

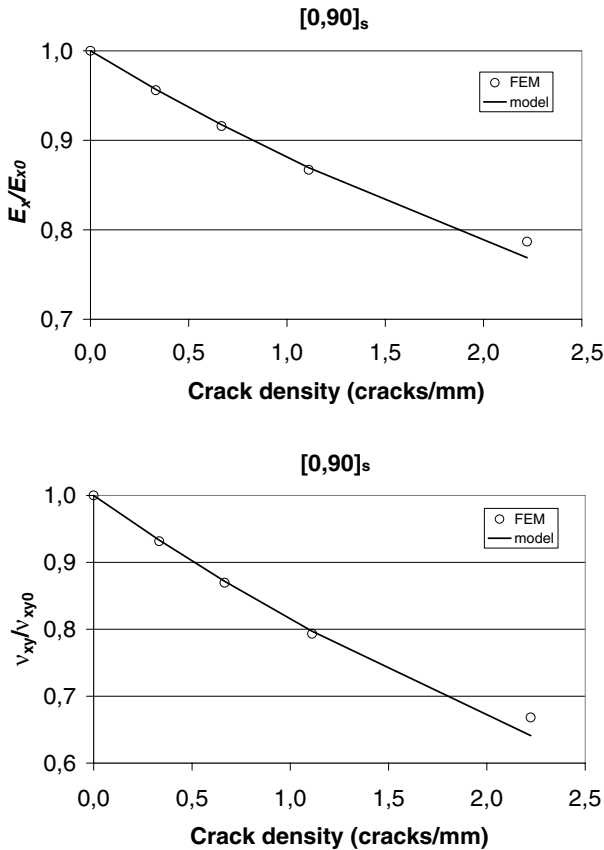
### Validation of the Analytical Approach using FEM

First, laminates with one crack system only were considered. The goal was (a) to validate the developed general expressions for calculation of all thermoelastic constants of the damaged laminate and (b) to study the crack interaction effects in order to establish the crack density region, where the concept of noninteractive cracks and the obtained power law can be used. Predictions of properties degradation were compared with direct FEM results. Considering cross-ply laminate with cracks in 90°-layer only, the axial modulus  $E_x$ , the Poisson's ratio  $\nu_{xy}$  and the thermal expansion

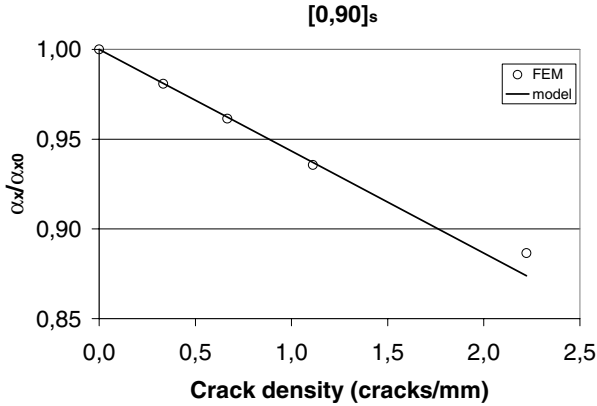
**Table 1. Material properties for materials used for calculations and validation.**

Materials	$E_1$ (GPa)	$E_2$ (GPa)	$G_{12}$ (GPa)	$\nu_{12}$	$\alpha_1$ ( $10^{-6} 1/^\circ\text{C}$ )	$\alpha_2$ ( $10^{-6} 1/^\circ\text{C}$ )	Lamina Thickness (mm)
GF/EP-1	46.50	22.82	8.60	0.30	10.00	20.00	0.150
GF/EP-2	41.70	13.00	3.40	0.30	–	–	0.150
GF/EP-3	44.73	12.76	3.50	0.30	–	–	0.138
GF/EP-4	44.73	12.76	3.50	0.30	–	–	0.148
CF/EP	138.00	10.30	5.50	0.30	0.43	25.87	0.125

coefficient  $\alpha_x$  were calculated using Equations (31), (33), and (35). The general expressions (25) and (29) were also used and the results were identical. Predictions presented in a normalized form for  $[0, 90]_s$  GF/EP-1 laminate are shown in Figure 4. Results of direct FE calculations in generalized plane strain formulations are also presented in the figure. Obviously, model predictions have very high accuracy for noninteractive cracks. Since the high accuracy of the power law for COD was already established, this proves the validity of the used relationships between global material response and local field parameters. With increasing crack density, deviations can be noticed: model, which uses CODs of noninteractive cracks, predicts too large change of thermoelastic properties. Noticeable deviations for the considered laminate start at crack density larger than



**Figure 4.** Thermoelastic properties degradation in GF/EP-1  $[0, 90]_s$  cross-ply laminate due to cracks in  $90^\circ$ -layer.



**Figure 4.** *Continued.*

1.5 cr/mm. The attempt to improve predictions using the crack interaction function for system of cracks in an infinite medium suggested by Gudmundson and Zang (1993) failed, the interaction using his model is significantly overestimated.

Second, cross-ply laminates with two orthogonal systems of cracks were investigated, see Figure 1. Here one of the goals was to investigate the interaction effects between cracks belonging to different crack systems. This is of importance because equations in Section “Stress–strain response of Damaged laminates” are valid also for interactive cracks but the power law for COD is established neglecting any interaction. From the analysis presented above, we know the interaction distance between cracks belonging to the same crack system. The same crack density  $2l_{90}/t_{90} = l_0/t_0 = 5$  corresponding to the noninteractive case was used in both layers. Considering the interaction between a crack in 90°-layer and a crack in 0°-layer, we can expect that crack in, for example, 0°-layer will slightly reduce the average stiffness of this layer. According to performed COD analysis, this will result in a slightly higher opening of the crack in the other layer, which implies that the stiffness using a very fine 3-D mesh should be slightly lower than that predicted by power law. Calculations were performed using very fine mesh with 36,000 elements to eliminate the effect of the artificially increased rigidity due to rough mesh.

The reduction in elastic properties for laminates with cracks in both layers is summarized in Table 2. At first, the difference between results is about 0.1% for E-moduli and thermal expansion coefficients, and about 2% for Poisson’s ratios. The second observation is that a very fine mesh leads to systematically slightly lower values than that obtained by using the analytical model. That may indicate an interaction effect between these two

**Table 2. Comparison between FEM and the present model for GF/EP-1  $[0, 90]_s$  cross-ply laminate with cracks in both 0- and 90°-layer.**

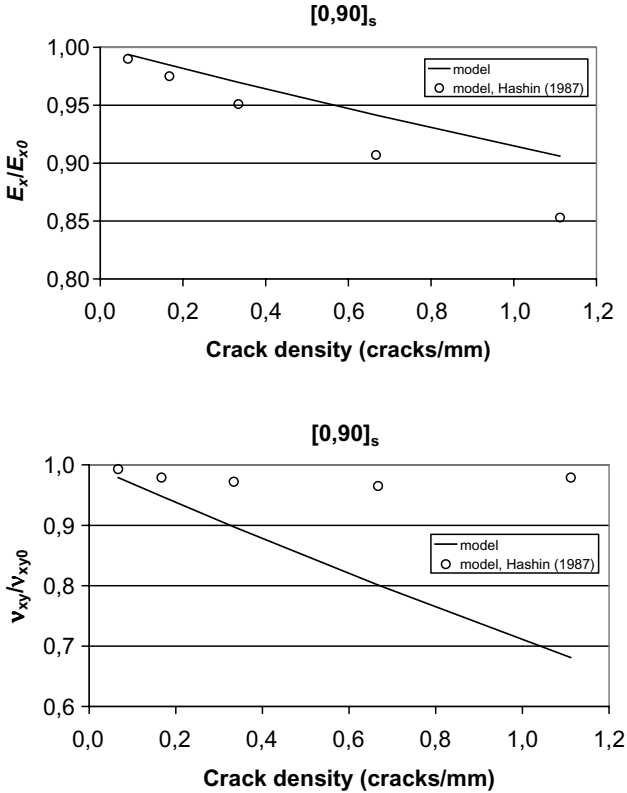
	FEM	Model
$E_x/E_{x0}$	0.9151	0.9162
$E_y/E_{y0}$	0.9076	0.9072
$\nu_{xy}/\nu_{xy0}$	0.7982	0.8195
$\nu_{yx}/\nu_{yx0}$	0.7935	0.8114
$\alpha_x/\alpha_{x0}$	0.9557	0.9576
$\alpha_y/\alpha_{y0}$	0.9521	0.9534

orthogonal cracks but may also be due to the mesh refinement or the approximate nature of the power law. Since the difference is small, we conclude that the interaction effects between cracks in 90°-layer and 0°-layer may be neglected. Using mesh with 80,000 elements led to further decrease of normalized E-modulus from 0.9151 to 0.9144.

The comparison between the developed model and Hashin's variational model (Hashin, 1987) is based on results for GF/EP-2  $[0,90]_s$  laminate with the same number of cracks in both layers is presented in Figure 5. The Young's modulus predicted by Hashin's model is significantly lower than that according to our model which we believe is more accurate. The too low modulus predicted by Hashin's model is a consequence of the very simple stress approximations used in combination with the principle of complementary energy which gives lower bound to the exact solution. Hashin's predicted Poisson's ratio is incorrect because of his wrong definition of the average transverse strain in the corresponding expression for Poisson's ratio: the average value over the 90°-layer should be taken instead of the whole laminate including the 0°-layer.

### Validation of the Model using Experimental Data

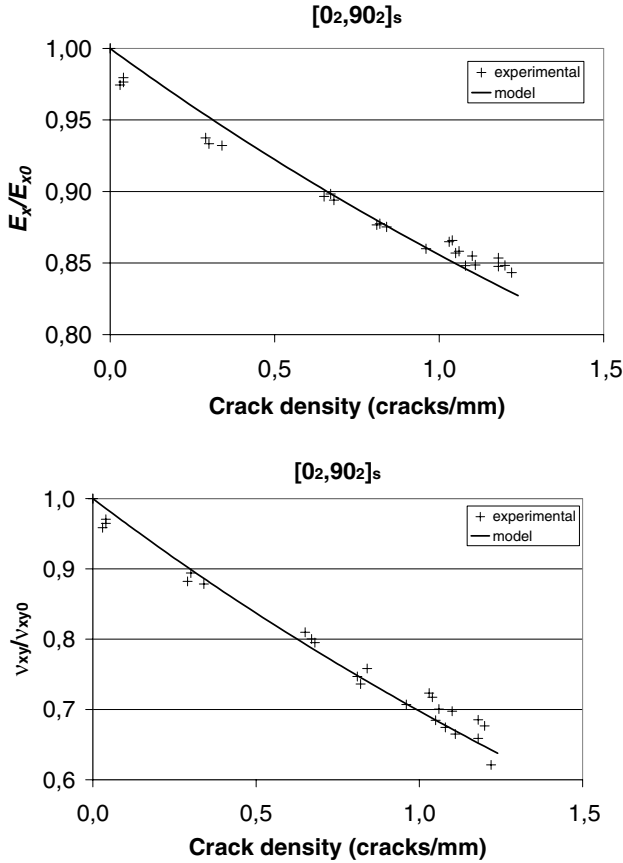
The model is also compared with experimental data for different layups and materials. Considering stiffness of laminates with cracks in 90°-layer only, shown in Figures 6–8, we see that the predictions are in good agreement with test data. Observed deviations may serve for more detailed analysis of the model and of the features of the phenomena. For example, in Figure 6 the reduction of the modulus and the Poisson's ratio of the damaged  $[0_2, 90_2]_s$  laminate at large crack densities is slowing down as compared with the model which is a clear indication of the interaction between cracks. Similar plots for  $[0, 90_2]_s$  laminate in Figure 7 shows the opposite trend: at high crack density the experimental values start to go



**Figure 5.** Elastic properties degradation in GF/EP-2  $[0,90]_s$  cross-ply laminate due to cracks in 0- and 90°-layers.

down faster. We explain this trend by local delaminations at the tip of transverse cracks which start at high loads and which is more pronounced in laminates with large ratio of damaged and supporting layer thickness. So the different layer thickness ratio makes the difference between laminates in Figures 6 and 7. The model also seems to predict the stiffness reduction for off-axis sublaminates in a good agreement with experimental data, see Figure 8.

Finally, the thermal expansion coefficients were compared with data and predictions given in Kim et al. (2000). For CF/EP cross-ply laminates with one system of cracks, Figure 9, our predictions are in a very good agreement with experimental data and coincide for low crack densities with predictions based on model in Schoeppner and Pagano (1998). At large crack densities, the difference between predictions increases. However, even if conceptually

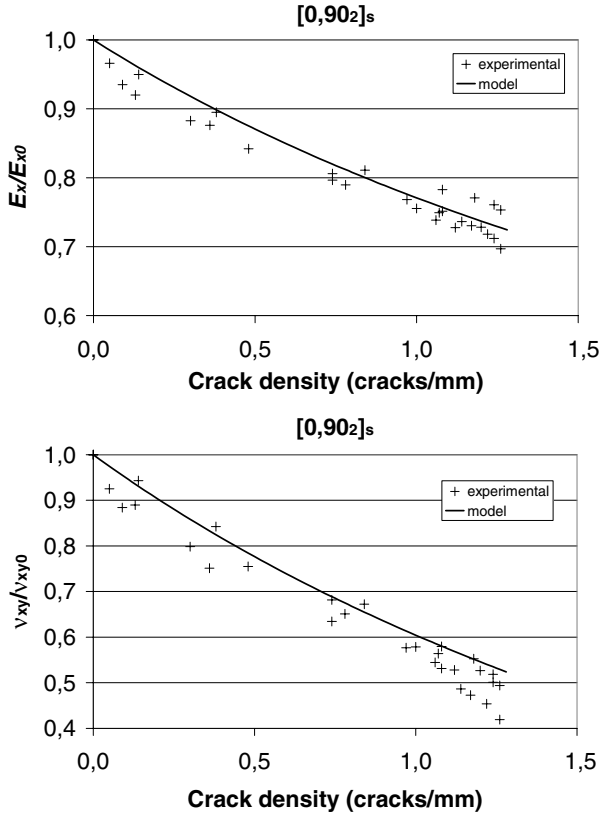


**Figure 6.** Reduction in elastic properties for GF/EP-3  $[0_2, 90_2]_s$  cross-ply laminate. Experimental data compared with model predictions.

incorrect for high crack density, our noninteractive COD-based predictions are approximately as good compared to test data as the results of the interactive model (Schoeppner and Pagano, 1998). Data and predictions according to both the models compared for double-cracked cross-ply laminate are presented in Table 3. Even in this case, our predictions are rather good and closer to the experimental data than the model (Schoeppner and Pagano, 1998).

### CONCLUSIONS

The stiffness matrix and the thermal expansion coefficients of a laminate with intralaminar cracks in layers may be predicted with confidence using

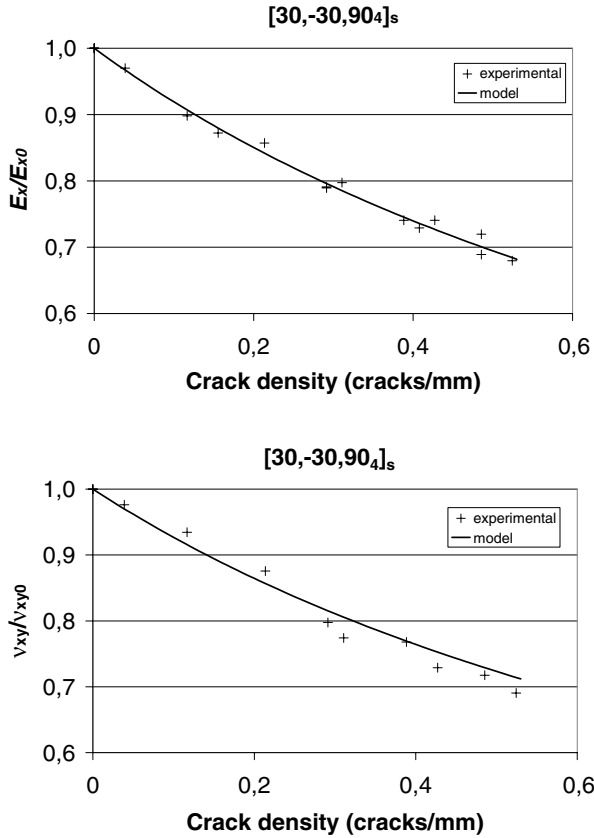


**Figure 7.** Reduction in elastic properties for GF/EP-3  $[0,90_2]_s$  cross-ply laminate. Experimental data compared with model predictions.

the exact expressions obtained in this paper. Relationships, expressing the laminate thermoelastic properties dependence on density of cracks in layers in a matrix form, depend on thermoelastic properties of layers, geometrical parameters characterizing laminate architecture and the normalized COD.

The normalized COD is load independent and depends only on the constraint of the surrounding layers. Analysis of noninteractive cracks by FEM showed that COD is a robust parameter which has a power law dependence on layer stiffness and thickness ratio, the effect of other stiffness constants being negligible.

The thermoelastic properties predictions based on the developed analytical method are in excellent agreement with direct 2-D FEM calculations for cross-ply laminates with one system of cracks.

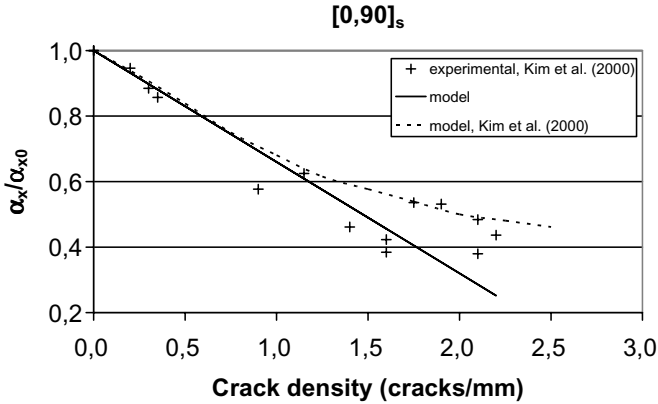


**Figure 8.** Reduction in elastic properties for GF/EP-4  $[30, -30, 90_4]_s$  laminate. Experimental data compared with model predictions.

The applicability of the power laws, obtained analyzing noninteractive cracks (only one system of cracks present and the distance between cracks is large), in problems with several crack systems and large crack density was inspected comparing predictions with direct 3-D FEM calculations. Three-dimensional FEM calculations for cross-ply laminates with two orthogonal crack systems showed that the interaction between cracks belonging to layers with different orientation is negligible, but the interaction between cracks of the same system is significant at large crack densities and must be included in the approximate expressions for normalized displacements.

The model is also in a good agreement with experimental data. However, agreement at large crack densities could be improved, introducing a function accounting for crack interaction. To be able to predict the reduced





**Figure 9.** Reduction in thermal expansion coefficient for CF/EP  $[0_2, 90_2]_s$  cross-ply laminate. Model compared with model and experiment from (Kim et al., 2000).

**Table 3.** Reduction in thermal expansion coefficients for CF/EP  $[0_2, 90_2]_s$  cross-ply laminate with cracks in both 0- and 90-layer.

Cracks/m		Model		Kim et al. (2000) Model		Kim et al. (2000) Exp	
$\rho_{90}$	$\rho_0$	$\alpha_x/\alpha_{x0}$	$\alpha_y/\alpha_{y0}$	$\alpha_x/\alpha_{x0}$	$\alpha_y/\alpha_{y0}$	$\alpha_x/\alpha_{x0}$	$\alpha_y/\alpha_{y0}$
670	390	0.73	0.79	0.64	0.69	0.72	0.78
940	670	0.61	0.65	0.56	0.60	0.58	0.74

elastic properties for a more complex laminate, the crack face sliding displacement has to be analyzed, which is left for a separate paper.

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