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Transmission Problem Between Two Herschel-Bulkley Fluids

Farid Messelmi*

Abstract

The paper is devoted to the study of transmission problem between two Herschel-Bulkley fluids with different viscosities, yield limits and power law index.

Keywords. Herschel-Bulkley fluid, interface, transmission.

2000 Mathematics Subject Classification. 76A05, 49J40, 76B15.

1 Introduction

The rigid viscoplastic and incompressible fluid of Herschel-Bulkley has been scrutinized and studied by mathematicians, physicists and engineers as intensively as the Navier-Stokes. While this model describes adequately a large class of flows. It has been used to model the flow of metals, plastic solids and a variety of polymers like paints. Due to existence of yield limit, the model can capture phenomena connected with the development of discontinuous stresses. The literature concerning this topic is extensive; see e.g. [1], [6], [7], [8] and references therein.

The purpose of this paper is to formulate and prove the existence of weak solutions for a class of boundary transmission problems for the Herschel-Bulkley fluid. To this aim we consider a transmission problem between two Herschel-Bulkley fluids with different viscosities, yield limits and power law index. We suppose that there is no-slip at the contact interface. Such problem can model the superposition of two paints in drawing process.

The paper is organized as follows. In Section 2 we present the steady-state mechanical problem of transmission between two Herschel-Bulkley fluids. We introduce some notations and preliminaries. Moreover, we derive the variational formulation of the problem. In Section 3, we are interested in the existence of weak solutions.

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2 Problem Statement

We consider a mathematical problem modelling the steady-state transmission problem between two rigid, viscoplastic and incompressible Herschel-Bulkley fluids flow. The first fluid occupies a bounded domain $\Omega_1 \subset \mathbb{R}^n$ ($n = 2, 3$) with the boundary $\partial\Omega_1$ of class C^1 . The second one occupies a bounded domain $\Omega_2 \subset \mathbb{R}^n$ ($n = 2, 3$) with the boundary $\partial\Omega_2$ of class C^1 . We denote by Ω the domain $\Omega_1 \cup \Omega_2$ and we suppose that

$$\partial\Omega_1 = \Gamma_0 \cup \Gamma_1 \text{ and } \partial\Omega_2 = \Gamma_0 \cup \Gamma_2,$$

where $\Gamma_0, \Gamma_1, \Gamma_2$ are measurable domains and $meas(\Gamma_1), meas(\Gamma_2) > 0$. The fluids are acted upon by given volume forces of densities $\mathbf{f}_1, \mathbf{f}_2$ respectively.

We denote by \mathbb{S}_n the space of symmetric tensors on \mathbb{R}^n . We define the inner product and the Euclidean norm on \mathbb{R}^n and \mathbb{S}_n , respectively, by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \quad \text{and} \quad \sigma \cdot \tau = \sigma_{lm} \tau_{lm} \quad \forall \sigma, \tau \in \mathbb{S}_n.$$

$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{\frac{1}{2}} \quad \forall \mathbf{u} \in \mathbb{R}^n \quad \text{and} \quad |\sigma| = (\sigma \cdot \sigma)^{\frac{1}{2}} \quad \forall \sigma \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and Einstein's convention is used. We denote by $\tilde{\sigma}$ the deviator of $\sigma = (\sigma_{lm})$ given by

$$\tilde{\sigma} = (\tilde{\sigma}_{lm}), \quad \tilde{\sigma}_{lm} = \sigma_{lm} - \frac{\sigma_{ll}}{n} \delta_{lm},$$

where $\delta = (\delta_{lm})$ denotes the identity tensor.

Let $1 < p \leq 2$. We consider the rate of deformation operator defined for every $\mathbf{u} \in W^{1,p}(\Omega)^n$ by

$$D(\mathbf{u}) = (D_{lm}(\mathbf{u})), \quad D_{lm}(\mathbf{u}) = \frac{1}{2}(u_{l,m} + u_{l,m}).$$

We denote by \mathbf{n} the unit outward normal vector on the boundary Γ_0 oriented to the exterior of Ω_1 and to the interior of Ω_2 , see the figure below. For every vector field $\mathbf{v} \in W^{1,p}(\Omega_i)^n$ we also write \mathbf{v} for its trace on $\partial\Omega_i$, $i = 1, 2$.

The steady-state transmission problem for the Herschel-Bulkley fluids is given by the following mechanical problem.

Problem P1. Find the velocity fields $\mathbf{u}_i = (u_{il}) : \Omega_i \rightarrow \mathbb{R}^n$ and the stress field $\sigma_i = (\sigma_{ilm}) : \Omega_i \rightarrow \mathbb{S}_n$, $i = 1, 2$ such that

$$\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 = \text{div } \sigma_1 + \mathbf{f}_1 \text{ in } \Omega_1. \quad (2.1)$$

$$\mathbf{u}_2 \cdot \nabla \mathbf{u}_2 = \text{div } \sigma_2 + \mathbf{f}_2 \text{ in } \Omega_2. \quad (2.2)$$

$$\left. \begin{array}{l} \tilde{\sigma}_1 = \mu_1 |D(\mathbf{u}_1)|^{p_1-2} D(\mathbf{u}_1) + g_1 \frac{D(\mathbf{u}_1)}{|D(\mathbf{u}_1)|} \text{ if } |D(\mathbf{u}_1)| \neq 0 \\ |\tilde{\sigma}_1| \leq g_1 \text{ if } |D(\mathbf{u}_1)| = 0 \end{array} \right\} \text{ in } \Omega_1. \quad (2.3)$$

$$\left. \begin{array}{l} \tilde{\sigma}_2 = \mu_2 |D(\mathbf{u}_2)|^{p_2-2} D(\mathbf{u}_2) + g_2 \frac{D(\mathbf{u}_2)}{|D(\mathbf{u}_2)|} \text{ if } |D(\mathbf{u}_2)| \neq 0 \\ |\tilde{\sigma}_2| \leq g_2 \text{ if } |D(\mathbf{u}_2)| = 0 \end{array} \right\} \text{ in } \Omega_2. \quad (2.4)$$

$$\operatorname{div} \mathbf{u}_1 = 0 \text{ in } \Omega_1. \quad (2.5)$$

$$\operatorname{div} \mathbf{u}_2 = 0 \text{ in } \Omega_2. \quad (2.6)$$

$$\mathbf{u}_1 = 0 \text{ on } \Gamma_1. \quad (2.7)$$

$$\mathbf{u}_2 = 0 \text{ on } \Gamma_2. \quad (2.8)$$

$$\mathbf{u}_1 - \mathbf{u}_2 = 0 \text{ on } \Gamma_0. \quad (2.9)$$

$$\sigma_1 \cdot \mathbf{n} - \sigma_2 \cdot \mathbf{n} = 0 \text{ on } \Gamma_0. \quad (2.10)$$

Here, the flows in the domains Ω_1, Ω_2 are given, respectively, by equations (2.1) and (2.2) where the densities is assumed equal to one for the two fluids. Equations (2.3) and (2.4) represent, respectively, the constitutive laws of the two Herschel-Bulkley fluids where $\mu_1, \mu_2 > 0$ and $g_1, g_2 > 0$ are the consistencies and yield limits of the two fluids, respectively, $1 < p_1, p_2 \leq 2$ are the power law index of the two fluids, respectively. (2.5) and (2.6) represent the incompressibility conditions for the two fluids, respectively. (2.7) and (2.8) give the velocities on the boundaries Γ_1 and Γ_2 , respectively. Finally, on the boundary part Γ_0 , (2.9) and (2.10) represent the transmission condition for liquid-liquid interface.

Existence of weak solutions for the flow of Herschel-Bulkley fluid was shown in 1969 for $p \geq \frac{3n}{n+2}$ for which the energy equality holds and higher differentiability techniques can be applied, in 1997 for $p \geq \frac{2n}{n+1}$, and recently for $p > \frac{2n}{n+2}$ using the Lipschitz truncation method. Moreover, some existence results has been obtained for the thermal flow in 2010 concerning the case $p \geq \frac{3n}{n+2}$, see [2], [4], [5], [6], [8] and [9]. Up to now, there are only a few results concerning the regularity of weak solutions, especially in three-dimensional domains.

We us consider the following function spaces

$$V(\Omega_i) = \{ \mathbf{v} \in W^{1,p_i}(\Omega_i)^n : \operatorname{div} \mathbf{v} \text{ in } \Omega_i \text{ and } \mathbf{v} = 0 \text{ on } \Gamma_i \}, \quad i = 1, 2.$$

$$V = \{ (\mathbf{v}_1, \mathbf{v}_2) \in V(\Omega_1) \times V(\Omega_2) : \mathbf{v}_1 - \mathbf{v}_2 = 0 \text{ on } \Gamma_0 \}.$$

$V(\Omega_i), i = 1, 2$ is a Banach space equipped with the norm

$$\| \mathbf{v} \|_{V(\Omega_i)} = \| \mathbf{v} \|_{W^{1,p_i}(\Omega_i)^n},$$

and V becomes a Banach space for the following norm

$$\| (\mathbf{v}_1, \mathbf{v}_2) \|_V = \| \mathbf{v}_1 \|_{V(\Omega_1)} + \| \mathbf{v}_2 \|_{V(\Omega_2)}.$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem and denoting by p' the conjugate of p .

Let us introduce the functional $B_i, i = 1, 2$ and the operator ϕ defined by

$$\begin{aligned} B_i &: V(\Omega_i) \times V(\Omega_i) \times V(\Omega_i) \longrightarrow \mathbb{R}, \\ B_i(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \int_{\Omega_i} \mathbf{v}_1 \cdot \nabla \mathbf{v}_2 \cdot \mathbf{v}_3 dx, \quad i = 1, 2. \end{aligned} \quad (2.11)$$

$$\begin{aligned}
\phi : V &\longrightarrow V', \quad (\mathbf{u}_1, \mathbf{u}_2) \longmapsto \phi(\mathbf{u}_1, \mathbf{u}_2) : \forall (\mathbf{v}_1, \mathbf{v}_2) \in V \\
\langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle_{V' \times V} &= \mu_1 \int_{\Omega_1} |D(\mathbf{u}_1)|^{p_1-2} D(\mathbf{u}_1) \cdot D(\mathbf{v}_1) dx + \\
&\quad \mu_2 \int_{\Omega_2} |D(\mathbf{u}_2)|^{p_2-2} D(\mathbf{u}_2) \cdot D(\mathbf{v}_2) dx. \tag{2.12}
\end{aligned}$$

We begin by recalling the following, which gives some properties of the convective operators B_i .

Lemma 2.1 1. *Suppose that*

$$\frac{3n}{n+2} \leq p_i \leq 2, \quad i = 1, 2. \tag{2.13}$$

Then,

$B_i, i = 1, 2$ is trilinear, continuous on $V(\Omega_i) \times V(\Omega_i) \times V(\Omega_i)$. Moreover, $\forall (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in V(\Omega_i) \times V(\Omega_i) \times V(\Omega_i)$ we have

$$\begin{aligned}
B_i(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) + B_i(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) &= \\
(-1)^{i+1} \int_{\Gamma_0} (\mathbf{v}_1 \cdot \mathbf{n})(\mathbf{v}_2 \cdot \mathbf{v}_3) d\gamma_0, \quad i &= 1, 2. \tag{2.14}
\end{aligned}$$

where $d\gamma_0$ represents the superficial measure on the boundary part Γ_0 .

2. The operator ϕ is hemi-continuous, strictly monotone, bounded and coercive on V .

Proof. 1. The proof of the continuity of $B_i, i = 1, 2$ on $V(\Omega_i) \times V(\Omega_i) \times V(\Omega_i)$ is an immediate consequence of Hölder's inequality and Sobolev embeddings, see [9].

Concerning the equality (2.13) it is enough to use an integration by parts and using the incompressibility condition (2.5), (2.6), the boundary conditions (2.7), (2.8) and the transmission condition (2.9).

2. We can easily prove that the operator ϕ can be written

$$\begin{aligned}
\langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle_{V' \times V} &= \langle dJ_1(D(\mathbf{u}_1)), D(\mathbf{v}_1) \rangle_{L^{p'_1}(\Omega_1)_s^{n \times n} \times L^{p_1}(\Omega_1)_s^{n \times n}} + \\
&\quad \langle dJ_2(D(\mathbf{u}_2)), D(\mathbf{v}_2) \rangle_{L^{p'_2}(\Omega_2)_s^{n \times n} \times L^{p_2}(\Omega_2)_s^{n \times n}},
\end{aligned}$$

where the functional $J_i, i = 1, 2$ is defined by

$$J_i : L^{p_i}(\Omega_i)_s^{n \times n} \subset \mathbb{S}_n \longrightarrow \mathbb{R}, \quad \sigma \longmapsto J_i(\sigma) = \frac{\mu_i}{p_i} \int_{\Omega_i} |\sigma|^{p_i} dx, \quad i = 1, 2,$$

and d represents the Gâteaux derivate.

Furthermore, it is easy to check that the functional J_i , $i = 1, 2$ is convex and Gâteaux differentiable on $L^{p_i}(\Omega_i)_s^{n \times n}$. Thus, dJ_i is hemi-continuous and monotone. The Gateaux derivate of J_i at any point $\sigma \in L^{p_i}(\Omega_i)_s^{n \times n}$ is given by

$$\langle dJ_i(\sigma), \tau \rangle_{L^{p_i}(\Omega_i)_s^{n \times n} \times L^{p_i}(\Omega_i)_s^{n \times n}} = \int_{\Omega_i} \mu_i |\sigma|^{p_i-2} \sigma \cdot \tau dx$$

$$\forall \tau \in L^{p_i}(\Omega_i)_s^{n \times n}, i = 1, 2.$$

This leads after some algebraic manipulations, that for $i = 1, 2$

$$\langle dJ_i(\sigma_1) - dJ_i(\sigma_2), \sigma_1 - \sigma_2 \rangle_{L^{p_i}(\Omega_i)_s^{n \times n} \times L^{p_i}(\Omega_i)_s^{n \times n}}$$

$$\geq \mu_i \int_{\Omega_i} (|dJ_i(\sigma_1)| - |dJ_i(\sigma_2)|) (|\sigma_1| - |\sigma_2|) dx.$$

Then, if $\sigma_1 \neq \sigma_2$ we find

$$\langle dJ_i(\sigma_1) - dJ_i(\sigma_2), \sigma_1 - \sigma_2 \rangle_{L^{p_i}(\Omega_i)_s^{n \times n} \times L^{p_i}(\Omega_i)_s^{n \times n}} > 0.$$

Which means that the functional dJ_i , $i = 1, 2$ is strictly monotone. Consequently, operator ϕ is hemi-continuous and strictly monotone on V . Moreover, we obtain from the definition of ϕ

$$|\langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle_{V' \times V}| \leq \mu_1 \|\mathbf{u}_1\|_{V(\Omega_1)}^{\frac{p_1}{p_1}} \|\mathbf{v}_1\|_{V(\Omega_1)} + \mu_2 \|\mathbf{u}_2\|_{V(\Omega_2)}^{\frac{p_2}{p_2}} \|\mathbf{v}_2\|_{V(\Omega_2)}$$

$$\leq \left(\mu_1 \|\mathbf{u}_1\|_{V(\Omega_1)}^{\frac{p_1}{p_1}} + \mu_2 \|\mathbf{u}_2\|_{V(\Omega_2)}^{\frac{p_2}{p_2}} \right) \|(\mathbf{v}_1, \mathbf{v}_2)\|_V \quad \forall (\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in V.$$

Then,

$$\|\phi(\mathbf{u}_1, \mathbf{u}_2)\|_{V'} \leq \mu_1 \|\mathbf{u}_1\|_{V(\Omega_1)}^{\frac{p_1}{p_1}} + \mu_2 \|\mathbf{u}_2\|_{V(\Omega_2)}^{\frac{p_2}{p_2}} \quad \forall (\mathbf{u}_1, \mathbf{u}_2) \in V.$$

Hence, ϕ is bounded on V .

Now, we find using the generalized Korn inequality

$$\langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_1, \mathbf{u}_2) \rangle_{V' \times V} \geq c \left(\|\mathbf{u}_1\|_{V(\Omega_1)}^{p_1} + \|\mathbf{u}_2\|_{V(\Omega_2)}^{p_2} \right).$$

It follows that

$$\frac{\langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_1, \mathbf{u}_2) \rangle_{V' \times V}}{\|(\mathbf{u}_1, \mathbf{u}_2)\|_V} \geq c \frac{\|\mathbf{u}_1\|_{V(\Omega_1)}^{p_1} + \|\mathbf{u}_2\|_{V(\Omega_2)}^{p_2}}{\|\mathbf{u}_1\|_{V(\Omega_1)} + \|\mathbf{u}_2\|_{V(\Omega_2)}}.$$

By passage to the limit when $\|(\mathbf{u}_1, \mathbf{u}_2)\|_V \rightarrow +\infty$, we find

$$\frac{\langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_1, \mathbf{u}_2) \rangle_{V' \times V}}{\|(\mathbf{u}_1, \mathbf{u}_2)\|_V} \geq c \lim_{r \rightarrow +\infty, \theta \in [0, \frac{\pi}{2}]} \frac{r^{p_1} \cos^{p_1} \theta + r^{p_2} \cos^{p_2} \theta}{r \cos \theta + r \cos \theta} = +\infty.$$

This proves that the operator ϕ is coercive.

Which permits us to conclude the proof. ■

Remark 2.2 In (1), the right hand side has sense, since the injection

$$W^{1-\frac{1}{p_i}, p_i}(\Gamma_0)^n \longrightarrow L^{\frac{(n-1)p_i}{n-p_i}}(\Gamma_0)^n, \quad i = 1, 2$$

is continuous. In particular the trace application

$$\gamma_0 : W^{1, p_i}(\Omega_i)^n \longrightarrow L^3(\Gamma_0)^n, \quad i = 1, 2$$

is continuous.

From now on, we take $\frac{3n}{n+2} \leq p_i \leq 2$, $i = 1, 2$. The use of Green's formula under the conditions (2.5)-(2.10) permits us to derive the following variational formulation of the mechanical problem (P1).

Problem P_v . For prescribed data $(\mathbf{f}_1, \mathbf{f}_2) \in V'$. Find $(\mathbf{u}_1, \mathbf{u}_2) \in V$ satisfying the variational inequality

$$\begin{aligned} & B_1(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}_1 - \mathbf{u}_1) + B_2(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}_2 - \mathbf{u}_2) + \\ & \quad \langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \rangle_{V' \times V} \\ & + g_1 \int_{\Omega_1} |D(\mathbf{v}_1)| dx - g_1 \int_{\Omega_1} |D(\mathbf{u}_1)| dx + g_2 \int_{\Omega_2} |D(\mathbf{v}_2)| dx - g_2 \int_{\Omega_2} |D(\mathbf{u}_2)| dx \\ & \geq \int_{\Omega_1} \mathbf{f}_1 \cdot (\mathbf{v}_1 - \mathbf{u}_1) dx + \int_{\Omega_2} \mathbf{f}_2 \cdot (\mathbf{v}_2 - \mathbf{u}_2) dx \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in V. \end{aligned} \quad (2.15)$$

3 Main Result

In this section we establish an existence result to the problems (P_v) .

Theorem 3.1 *The problem (P_v) admits a solution $(\mathbf{u}_1, \mathbf{u}_2) \in V$.*

The proof will be done in two steps.

First step. Take an arbitrary element $(\mathbf{w}_1, \mathbf{w}_2) \in V$ and consider the auxiliary problem.

Problem $P_{(\mathbf{w}_1, \mathbf{w}_2)}$. Find $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{w}_1, \mathbf{w}_2), \mathbf{u}_2(\mathbf{w}_1, \mathbf{w}_2)) \in V$ solution of the variational inequality

$$\begin{aligned} & B_1(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}_1 - \mathbf{u}_1) + B_2(\mathbf{w}_2, \mathbf{u}_2, \mathbf{v}_2 - \mathbf{u}_2) + \\ & \quad \langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \rangle_{V' \times V} \\ & + g_1 \int_{\Omega_1} |D(\mathbf{v}_1)| dx - g_1 \int_{\Omega_1} |D(\mathbf{u}_1)| dx + g_2 \int_{\Omega_2} |D(\mathbf{v}_2)| dx - g_2 \int_{\Omega_2} |D(\mathbf{u}_2)| dx \\ & \geq \int_{\Omega_1} \mathbf{f}_1 \cdot (\mathbf{v}_1 - \mathbf{u}_1) dx + \int_{\Omega_2} \mathbf{f}_2 \cdot (\mathbf{v}_2 - \mathbf{u}_2) dx \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in V. \end{aligned} \quad (3.1)$$

and it satisfies the estimate

$$\|(\mathbf{u}_1, \mathbf{u}_2)\|_V \leq R. \quad (3.2)$$

Lemma 3.2 *The problem $(P_{(\mathbf{w}_1, \mathbf{w}_2)})$ has a unique solution*

$$(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{w}_1, \mathbf{w}_2), \mathbf{u}_2(\mathbf{w}_1, \mathbf{w}_2)) \in V.$$

Proof. Let us introduce the operator

$$\begin{aligned} \phi_{(\mathbf{w}_1, \mathbf{w}_2)} : V &\longrightarrow V', \quad (\mathbf{u}_1, \mathbf{u}_2) \longmapsto \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{u}_1, \mathbf{u}_2) : \forall (\mathbf{v}_1, \mathbf{v}_2) \in V \\ \left\langle \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \right\rangle_{V' \times V} &= B_1(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}_1) + B_2(\mathbf{w}_2, \mathbf{u}_2, \mathbf{v}_2) + \\ &\quad \langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle_{V' \times V}. \end{aligned} \quad (3.3)$$

First, we get using lemma 2.1

$$\begin{aligned} B_1(\mathbf{w}_1, \mathbf{u}_1, \mathbf{u}_1) + B_2(\mathbf{w}_2, \mathbf{u}_2, \mathbf{u}_2) &= \frac{1}{2} \int_{\Gamma_0} |\mathbf{u}_1|^2 (\mathbf{w}_1 \cdot \mathbf{n}) d\gamma_0 - \\ &\quad \frac{1}{2} \int_{\Gamma_0} |\mathbf{u}_2|^2 (\mathbf{w}_2 \cdot \mathbf{n}) d\gamma_0 \quad \forall (\mathbf{u}_1, \mathbf{u}_2) \in V. \end{aligned}$$

The fact that $(\mathbf{w}_1, \mathbf{w}_2), (\mathbf{u}_1, \mathbf{u}_2) \in V$ implies that $\mathbf{w}_1 - \mathbf{w}_2 = 0$ and $\mathbf{u}_1 - \mathbf{u}_2 = 0$ on Γ_0 . Thus,

$$\begin{aligned} \left\langle \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_1, \mathbf{u}_2) \right\rangle_{V' \times V} &= \langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_1, \mathbf{u}_2) \rangle_{V' \times V} \\ \forall (\mathbf{u}_1, \mathbf{u}_2) &\in V. \end{aligned} \quad (3.4)$$

Furthermore, we find for every $(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in V$.

$$\begin{aligned} &\left\langle \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{v}_1, \mathbf{v}_2) - \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \right\rangle_{V' \times V} \\ &= B_1(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}_1 - \mathbf{u}_1) + B_2(\mathbf{w}_2, \mathbf{u}_2, \mathbf{v}_2 - \mathbf{u}_2) - B_1(\mathbf{w}_1, \mathbf{v}_1, \mathbf{v}_1 - \mathbf{u}_1) - \\ &\quad B_2(\mathbf{w}_2, \mathbf{v}_2, \mathbf{v}_2 - \mathbf{u}_2) + \langle \phi(\mathbf{v}_1, \mathbf{v}_2) - \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) - (\mathbf{u}_1, \mathbf{u}_2) \rangle_{V' \times V} \\ &\quad \forall (\mathbf{v}_1, \mathbf{v}_2), (\mathbf{u}_1, \mathbf{u}_2) \in V. \end{aligned}$$

This gives, keeping in mind lemma 2.1

$$\begin{aligned} &\left\langle \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{v}_1, \mathbf{v}_2) - \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \right\rangle_{V' \times V} \\ &= \frac{1}{2} \int_{\Gamma_0} |\mathbf{u}_1 - \mathbf{v}_1|^2 (\mathbf{w}_2 \cdot \mathbf{n}) d\gamma_0 - \frac{1}{2} \int_{\Gamma_0} |\mathbf{u}_2 - \mathbf{v}_2|^2 (\mathbf{w}_1 \cdot \mathbf{n}) d\gamma_0 \\ &\quad + \langle \phi(\mathbf{v}_1, \mathbf{v}_2) - \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \rangle_{V' \times V} \\ &= \langle \phi(\mathbf{v}_1, \mathbf{v}_2) - \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \rangle_{V' \times V} \\ &\quad \forall (\mathbf{v}_1, \mathbf{v}_2), (\mathbf{u}_1, \mathbf{u}_2) \in V. \end{aligned} \quad (3.5)$$

Consequently, lemma 2.1 leads making use (1) and (3.5) that the operator is $\phi_{(\mathbf{w}_1, \mathbf{w}_2)}$ is hemi-continuous, strictly monotone, bounded and coercive on V for every $(\mathbf{w}_1, \mathbf{w}_2) \in V$.

Consider now the following functional

$$j : V \longrightarrow \mathbb{R},$$

$$j(\mathbf{v}_1, \mathbf{v}_2) = g_1 \int_{\Omega_1} |D(\mathbf{v}_1)| dx + g_2 \int_{\Omega_2} |D(\mathbf{v}_2)| dx \quad (3.6)$$

We can easily verify that the functional j is proper, convex and lower semi-continuous on V .

The inequality (3.1) can be rewritten using the operator $\phi_{(\mathbf{w}_1, \mathbf{w}_2)}$ and the functional j as follows

$$\begin{aligned} & \left\langle \phi_{(\mathbf{w}_1, \mathbf{w}_2)}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \right\rangle_{V' \times V} + j(\mathbf{v}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{u}_2) \\ & \geq \int_{\Omega_1} \mathbf{f}_1 \cdot (\mathbf{v}_1 - \mathbf{u}_1) dx + \int_{\Omega_2} \mathbf{f}_2 \cdot (\mathbf{v}_2 - \mathbf{u}_2) dx \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in V. \end{aligned} \quad (3.7)$$

Consequently, the existence and uniqueness results from classical theories for inequalities with monotone operators and convex functionals, see [1].

Furthermore the estimate (3.2) can be easily deduced by setting $(\mathbf{v}_1, \mathbf{v}_2) = (0, 0)$ as test function in inequality (3.1), using lemma 2.1, Korn's inequality and some algebraic manipulations. ■

Second step. In order to obtain the solution of problem (P_v) from that of problem $P_{(\mathbf{w}_1, \mathbf{w}_2)}$, we use the Schauder fixed point theorem, see [5]. To this aim we introduce the ball

$$K = \{(\mathbf{w}_1, \mathbf{w}_2) \in V : \|(\mathbf{w}_1, \mathbf{w}_2)\|_V \leq R\}, \quad (3.8)$$

where R is the constant given by the estimate (3.2). The ball K is convex and from the Rellich compactness theorem the ball is compact in $L^{\frac{3n}{n-1}}(\Omega_1)^n \times L^{\frac{3n}{n-1}}(\Omega_2)^n$. Let us built the mapping $\mathcal{L} : K \longrightarrow K$, as follows

$$(\mathbf{w}_1, \mathbf{w}_2) \longmapsto \mathcal{L}(\mathbf{w}_1, \mathbf{w}_2) = (\mathbf{u}_1, \mathbf{u}_2).$$

To conclude the proof it is enough to verify the continuity of the mapping \mathcal{L} when the ball K is provided by the topology of space $L^{\frac{3n}{n-1}}(\Omega_1)^n \times L^{\frac{3n}{n-1}}(\Omega_2)^n$. To do this, we consider $(\mathbf{w}_1, \mathbf{w}_2), (\mathbf{w}'_1, \mathbf{w}'_2) \in K$ and denoting by $(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}'_1, \mathbf{u}'_2) \in K$ the elements $(\mathbf{u}_1, \mathbf{u}_2) = \mathcal{L}(\mathbf{w}_1, \mathbf{w}_2)$ and $(\mathbf{u}'_1, \mathbf{u}'_2) = \mathcal{L}(\mathbf{w}'_1, \mathbf{w}'_2)$.

Remembering that $(\mathbf{u}_1, \mathbf{u}_2)$ and $(\mathbf{u}'_1, \mathbf{u}'_2)$ are the solution of the problems below

$$\begin{aligned} & B_1(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}_1 - \mathbf{u}_1) + B_2(\mathbf{w}_2, \mathbf{u}_2, \mathbf{v}_2 - \mathbf{u}_2) + \\ & \quad \left\langle \phi(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1 - \mathbf{u}_1, \mathbf{v}_2 - \mathbf{u}_2) \right\rangle_{V' \times V} \\ & + g_1 \int_{\Omega_1} |D(\mathbf{v}_1)| dx - g_1 \int_{\Omega_1} |D(\mathbf{u}_1)| dx + g_2 \int_{\Omega_2} |D(\mathbf{v}_2)| dx - g_2 \int_{\Omega_2} |D(\mathbf{u}_2)| dx \\ & \geq \int_{\Omega_1} \mathbf{f}_1 \cdot (\mathbf{v}_1 - \mathbf{u}_1) dx + \int_{\Omega_2} \mathbf{f}_2 \cdot (\mathbf{v}_2 - \mathbf{u}_2) dx \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in V, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned}
& B_1(\mathbf{w}'_1, \mathbf{u}'_1, \mathbf{v}_1 - \mathbf{u}'_1) + B_2(\mathbf{w}'_2, \mathbf{u}'_2, \mathbf{v}_2 - \mathbf{u}'_2) + \\
& \quad \langle \phi(\mathbf{u}'_1, \mathbf{u}'_2), (\mathbf{v}_1 - \mathbf{u}'_1, \mathbf{v}_2 - \mathbf{u}'_2) \rangle_{V' \times V} \\
& + g_1 \int_{\Omega_1} |D(\mathbf{v}_1)| dx - g_1 \int_{\Omega_1} |D(\mathbf{u}'_1)| dx + g_2 \int_{\Omega_2} |D(\mathbf{v}_2)| dx - g_2 \int_{\Omega_2} |D(\mathbf{u}'_2)| dx \\
& \geq \int_{\Omega_1} \mathbf{f}_1 \cdot (\mathbf{v}_1 - \mathbf{u}'_1) dx + \int_{\Omega_2} \mathbf{f}_2 \cdot (\mathbf{v}_2 - \mathbf{u}'_2) dx \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in V, \quad (3.10)
\end{aligned}$$

Now, choosing $\mathbf{v}_1 = \mathbf{u}'_1$ and $\mathbf{v}_2 = \mathbf{u}'_2$ as test function in inequality (3.9) and $\mathbf{v}_1 = \mathbf{u}_1$ and $\mathbf{v}_2 = \mathbf{u}_2$ as test function in inequality (3.10). It follows by subtracting the two obtained inequalities and using lemma 3.1, the transmission conditions, the definition of the space V and some calculations

$$\begin{aligned}
& B_1(\mathbf{w}'_1 - \mathbf{w}_1, \mathbf{u}_1, \mathbf{u}'_1 - \mathbf{u}_1) + B_2(\mathbf{w}'_2 - \mathbf{w}_2, \mathbf{u}_2, \mathbf{u}'_2 - \mathbf{u}_2) + \\
& \quad \frac{1}{2} \int_{\Gamma_0} |\mathbf{u}'_1 - \mathbf{u}_1|^2 (\mathbf{w}'_1 \cdot \mathbf{n}) d\gamma_0 - \frac{1}{2} \int_{\Gamma_0} |\mathbf{u}'_2 - \mathbf{u}_2|^2 (\mathbf{w}'_2 \cdot \mathbf{n}) d\gamma_0 \\
& + \mu_1 \int_{\Omega_1} \left(|D(\mathbf{u}'_1)|^{p_1-2} D(\mathbf{u}'_1) - |D(\mathbf{u}_1)|^{p_1-2} D(\mathbf{u}_1) \right) \cdot D(\mathbf{u}'_1 - \mathbf{u}_1) dx \\
& + \mu_2 \int_{\Omega_2} \left(|D(\mathbf{u}'_2)|^{p_2-2} D(\mathbf{u}'_2) - |D(\mathbf{u}_2)|^{p_2-2} D(\mathbf{u}_2) \right) \cdot D(\mathbf{u}'_2 - \mathbf{u}_2) dx \leq 0. \quad (3.11)
\end{aligned}$$

Observe that for every $x, y \in \mathbb{R}^n$,

$$\left(|x|^{p-2} x - |y|^{p-2} y \right) \cdot (x - y) \geq c \frac{|x - y|^2}{(|x| + |y|)^{2-p}}, \quad 1 < p \leq 2. \quad (3.12)$$

Then, inequality (3.11) becomes

$$\begin{aligned}
& \mu_1 \int_{\Omega_1} \frac{|D(\mathbf{u}'_1 - \mathbf{u}_1)|^2}{(|D(\mathbf{u}_1)| + |D(\mathbf{u}'_1)|)^{2-p_1}} dx + \mu_2 \int_{\Omega_2} \frac{|D(\mathbf{u}'_2 - \mathbf{u}_2)|^2}{(|D(\mathbf{u}_2)| + |D(\mathbf{u}'_2)|)^{2-p_2}} dx \\
& \leq c |B_1(\mathbf{w}'_1 - \mathbf{w}_1, \mathbf{u}_1, \mathbf{u}'_1 - \mathbf{u}_1)| + c |B_2(\mathbf{w}'_2 - \mathbf{w}_2, \mathbf{u}_2, \mathbf{u}'_2 - \mathbf{u}_2)| \quad (3.13)
\end{aligned}$$

On the other hand, the application of Korn's and Hölder's inequalities leads for $i = 1, 2$ to

$$\begin{aligned}
& \|\mathbf{u}'_i - \mathbf{u}_i\|_{V(\Omega_i)}^{p_i} \\
& \leq c \left(\int_{\Omega_i} \frac{|D(\mathbf{u}'_i - \mathbf{u}_i)|^2}{(|D(\mathbf{u}_i)| + |D(\mathbf{u}'_i)|)^{2-p_i}} dx \right)^{\frac{p_i}{2}} \left(\int_{\Omega_i} (|D(\mathbf{u}'_i)| + |D(\mathbf{u}_i)|)^{p_i} dx \right)^{\frac{2-p_i}{2}}. \quad (3.14)
\end{aligned}$$

This yields, taking into account (3.2), (3.13) and Hölder's inequality

$$\begin{aligned} & \|(\mathbf{u}'_1 - \mathbf{u}_1, \mathbf{u}'_2 - \mathbf{u}_2)\|_V^2 \leq \\ & c \|\mathbf{w}'_1 - \mathbf{w}_1\|_{L^{\frac{3n}{n-1}}(\Omega_1)^n} \|\mathbf{u}_1\|_{L^{p_1}(\Omega_1)^n} \|\mathbf{u}'_1 - \mathbf{u}_1\|_{L^{\frac{3n}{n-1}}(\Omega_1)^n} + \\ & c \|\mathbf{w}'_2 - \mathbf{w}_2\|_{L^{\frac{3n}{n-1}}(\Omega_2)^n} \|\mathbf{u}_2\|_{L^{p_2}(\Omega_2)^n} \|\mathbf{u}'_2 - \mathbf{u}_2\|_{L^{\frac{3n}{n-1}}(\Omega_2)^n}. \end{aligned}$$

Thus, Sobolev' embedding leads via the estimate (3.2) to

$$\begin{aligned} & \|(\mathbf{u}'_1 - \mathbf{u}_1, \mathbf{u}'_2 - \mathbf{u}_2)\|_{L^{\frac{3n}{n-1}}(\Omega_1)^n \times L^{\frac{3n}{n-1}}(\Omega_2)^n} \leq \\ & c \|(\mathbf{w}'_1 - \mathbf{w}_1, \mathbf{w}'_2 - \mathbf{w}_2)\|_{L^{\frac{3n}{n-1}}(\Omega_1)^n \times L^{\frac{3n}{n-1}}(\Omega_2)^n}. \end{aligned} \quad (3.15)$$

Hence, by virtue of Schauder's fixed point theorem, the mapping \mathcal{L} admits a fixed point $(\mathbf{u}_1, \mathbf{u}_2) = \mathcal{L}(\mathbf{u}_1, \mathbf{u}_2)$, which solves the problem (P_v) .

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