Author-produced version of International Journal of Geographical Information Science, vol. 23 (7), pp. 877-921 Published: 2009
doi : 10.1080/13658810802022814
Original paper available at Taylor & Francis http://www.tandf.co.uk

Qualified Topological Relations between Spatial Objects with Possibly Vague Shape

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Broad boundaries are generally used to represent objects with fuzzy spatial extents. This concept is typically defined as a polygonal zone that should respect both connectedness and closeness conditions. Therefore, some real configurations, like regions with a *partially broad boundary* (e.g., *a lake with rocker and swamp banks*), are considered invalid. The main objective of this paper is to represent different levels of spatial fuzziness and consider these levels during the identification of topological relations. Then, we define a fuzzy spatial object as a *minimal extent* and a *maximal extent*. Topological relations are identified by a 4-Intersection matrix that describes four subrelations between the minimal and the maximal extents. For fuzzy regions, 242 relations are distinguished and classified into 40 clusters. This approach permits the representation of partially fuzzy objects as well as the expression of integrity constraints and spatial queries with different levels of fuzziness.

Keywords: Fuzzy spatial objects; topological relations; partial fuzziness; spatial query; integrity constraints; integration

1. Introduction

To satisfy the requirements of several categories of users, Geographic Information Systems (GIS) and spatial databases provide tools to store, retrieve, analyze, and display spatial data. Ensuring their usability requires controlling the spatial data quality, which can be degraded by several types of imperfections. Several approaches (Smithson 1989, Fisher 1999, Mowrer 1999, Duckham *et al.* 2001) proposed different categorizations of data imperfections that are generally caused by the complexity of reality and limitations of instruments and processes used in the measurements (Bédard 1987). Moreover, inappropriate spatial data representations can also be another source of data quality degradation (Yazici *et al.* 2001, Shu *et al.* 2003, Dilo 2006). Spatial reality is generally forced to be represented by *crisp spatial object types* (i.e., *points*, *lines,* and *regions*), whereas many spatial objects are inherently fuzzy (e.g., *forest stand*, *pollution zone*, *valley*, or *lake*). Fuzziness occurs when it is difficult to distinguish an object's boundaries geometry components from other spatial objects. Using crisp spatial object types to represent *fuzzy spatial objects* entails a clear gap between the spatial reality and its formal representation in databases and GIS (Cheng *et al.* 2001, Yazici *et al.* 2001).

Pertinent solutions were found to overcome the "*classical*" sources of spatial data quality degradation (Bédard 1987, Goodchild 1995, Guptill and Morisson 1995, Ubeda and Egenhofer 1997, Frank 2001, Van Oort 2006, Devillers *et al.* 2007, Duboisset *et al.* 2007). Several approaches (Burrough and Frank 1996, Cohn and Gotts 1996, Clementini and Di Felice 1997, Erwig and Schneider 1997, Schneider 2001, Xinming 2004, Pfoser *et al.* 2005, Dilo 2006) have studied specificities of fuzzy spatial objects to determine their appropriate representation. A review of the literature in this domain stresses that current GIS and spatial database systems do not offer the specific structure to formally represent this type of object (as pointed by Clementini and Di Felice 1997 ten years ago). With regard to this problem, researchers are increasingly more motivated to model spatial fuzziness in order to: (1) reduce the gap between the geographic reality and the spatial models (Cohn and Gotts 1996), (2) provide formal modeling tools to represent spatial fuzziness (Yazici *et al.* 2001), and (3) specify spatial queries involving *fuzzy spatial objects* (Erwig and Schneider 1997). In the same way, the spatial data integration requires the extraction of heterogeneous representations of the same objects from different data sources. The main difficulty lies in choosing one of them when no information exists about their quality (Rodriguez 2005). By using a spatial

model that supports spatial fuzziness, it becomes possible to merge different representations in such a way that the integration result looks like a fuzzy spatial object. For example, figure 1 shows a spatial object that has the representation *A* in a first source and the representation *B* in a second one. The integration result can correspond to one fuzzy geometry made up of *A* and *B* (figure 1). The intersection of *A* and *B* corresponds to the *certain part* (i.e., the part that exists in both representation *A* and representation *B*) or the *minimal extent* of the spatial object. However, the union is the *maximal extent* that the object can fill; it groups the certain and the uncertain parts (i.e., a geometry part is uncertain when it does not exist in all candidate representations for the integration) of the geometry. Indeed, there are strong and different motivations to present pertinent solutions in order to adequately model the spatial fuzziness.

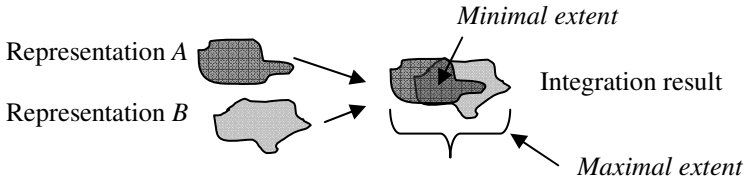


Figure 1. Integration of different spatial representations of a same object (e.g., lake)

To model fuzzy spatial objects, researchers were firstly inspired by **the** modeling of crisp spatial objects. In general point-set topology (Egenhofer and Herring 1990), crisp spatial objects are typically decomposed into three mutually disjoint topological invariants: an *interior*, a *boundary,* and an *exterior*. Several approaches (Clementini and Di Felice 1997, Xinming 2004, Reis *et al.* 2006) extend the crisp models by identifying other topological invariants for the fuzzy spatial objects. For example, Clementini and Di Felice (Clementini and Di Felice 1997) distinguish three topological invariants for fuzzy regions: an *interior*, a *broad boundary* (i.e., *a two –dimensional boundary*), and an *exterior*. In this approach, the fuzziness is correlated to the *broad boundary,* which should respect the closeness and the connectedness conditions (Clementini and Di Felice 1997, Xinming 2004). Thus, any representation that does not verify these conditions is considered invalid. Nonetheless, the fuzziness can also characterize only some parts of an object's geometry. For example, figure 2 shows a *lake* surrounded by rocker and swamp banks at the same time (figure 2). We denote this kind of feature as *partially fuzzy spatial objects* that cannot be represented by existing models. Then, the main questions are: How is it possible to define an exact model where different levels of fuzziness could be considered? How can we retain this expressivity during the specification of topological relations between such objects?

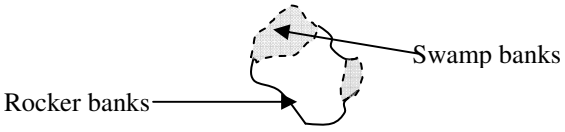


Figure 2. A lake with partially fuzzy boundaries

The first objective of this paper is to allow the representation of three levels of fuzziness: *crispness*, *partial fuzziness,* and *complete fuzziness*. Modeling fuzzy spatial objects requires a framework for identifying topological relations. The second objective is to consider the different levels of spatial fuzziness in the identification of topological relations between fuzzy spatial objects. In several studies (Clementini and Di Felice 1997, Xinming 2004, Reis *et al.* 2006), topological relations can be identified by enumerating the intersections between the topological invariants of the fuzzy objects involved. For each model, the number of relations depends upon the number of topological invariants. In this work, we look for an expressive model in which it is possible to specify the fuzziness level of the topological relation

instances. We think that it would be pertinent for the user to know whether objects are *weakly* or *strongly* disjoint. Accordingly, the third specific objective of this work is to classify the topological relations according to their fuzziness level.

The remainder of the paper is organized as follows. In section 2, we present previous work on the modeling of fuzzy spatial objects and their topological relations. In section 3, we present three fuzzy spatial object types: *fuzzy point*, *fuzzy line,* and *fuzzy region*. Then, section 4 gives a proposition based on the 9-Intersection model (Egenhofer and Herring 1990) in order to identify the topological relations among fuzzy spatial objects. The model is applied to fuzzy regions, and their topological relations are studied in detail in the appendix. As a result of this approach, 242 relations can be distinguished through a 4-Intersection matrix. Section 5 proposes a hierarchical clustering of topological relations between fuzzy regions, and section 6 explains how to use our approach to express spatial queries and integrity constraints. In section 7, our model is compared with existing approaches (Cohn and Gotts 1996, Clementini and Di Felice 1997, Xinming 2004). Finally, section 8 presents our conclusions and discusses future research.

2. Previous work

2.1 Spatial vagueness

According to (Erwig and Schneider 1997, Hazarika and Gotts 2001, Pfoser *et al.* 2005), spatial vagueness can characterize the position or shape of an object's spatial extent. From this perspective, the *shape fuzziness* refers to the difficulty of distinguishing one object's shape from other objects' shapes. Fuzziness is an intrinsic property of an object that certainly has an extent in a known position but cannot or does not have a well-defined shape (Erwig and Schneider 1997). For instance, a region can be fuzzy when it is surrounded by broad instead sharp boundaries. Spatial vagueness can also characterize well-defined (or crisp) objects when there is uncertainty about object positions despite their sharp shapes; we refer to this scenario as positional vagueness. Figure 3 shows this categorization of spatial vagueness into "*shape fuzziness*" and "*positional vagueness*". In this paper, we only deal with the formal representation of fuzzy spatial objects (i.e., objects that have fuzzy shapes) and the topological relations between them.

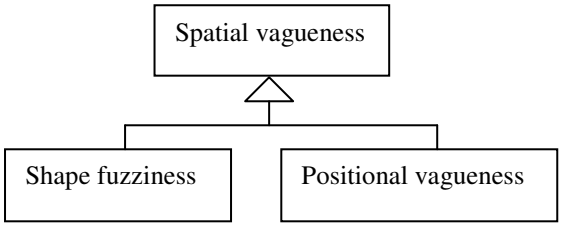


Figure 3. Categorization of spatial vagueness

In general, we distinguish **between** at least two categories of models used to represent spatial vagueness. In the first category, crisp spatial concepts are transferred and extended to formally express spatial vagueness; we speak about *exact models* as explained in the next section (Cohn and Gotts 1996, Clementini and Di Felice 1997, Erwig and Schneider 1997). In the second category, three principal mathematical theories are generally used: (1) models based on the fuzzy logic (Zadeh 1965) (e.g., Altman 1987, Burrough 1989, Brown 1998, Schneider 2001, Xinming 2004, Hwang and Thill 2005, Dilo 2006), which can be used to represent continuous phenomena such as temperature, (2) models based on rough sets (e.g., Ahlqvist *et al.* 1998, Worboys 1998), which represent the fuzzy spatial objects as a pair of

approximations (*upper and lower approximations*), and (3) models based on probability theory (e.g., Burrough and Frank 1996, Pfoser *et al.* 2005), which is principally used to model errors of position and attributes. In the next section, we present works that formally define fuzzy spatial objects.

2.2 Formal definitions of fuzzy spatial objects

2.2.1 Definitions based on exact models. The Egg-Yolk theory (Cohn and Gotts 1996) is an extension of the RCC (*Region Connection Calculus*) model (Randell and Cohn 1989, Cohn *et al.* 1997), it introduces the concept of *regions with large boundaries*. In fact, a fuzzy region is composed of two crisp regions. The inner region **called** «*yolk* » (i.e., the certain part of the geometry), and it is surrounded by an outer region **called** "*white*" (i.e., the large boundary or the uncertain part of the geometry). The union of the "*yolk*" and the "*white*" corresponds to the "*egg*". Because points and lines are ignored in RCC model, their *fuzziness* is not studied in this approach. In addition, fuzzy regions with empty "*yolk*" or empty "*egg*" are not admitted. Thus, the crisp regions cannot be represented through the Egg-Yolk theory.

In the same way, Clementini and Di Felice (Clementini and Di Felice 1997) define the *regions with broad boundaries* based on the general point-set topology (Egenhofer and Herring 1990). Thus, a region with a broad boundary *A* is made up of two simple crisp regions A_1 and A_2, where $A_1 \subseteq A_2$. The *broad boundary* represents the fuzziness, and it is equal to the closure of their difference $\Delta A = A_1 - A_2$. In this approach, A_1 and A_2 should be topologically consistent; this means that they should be bounded, regular, and closed sets in $\Re^2$ (Clementini and Di Felice 1997). Moreover, the authors distinguish two kinds of fuzzy lines: *broad lines* (i.e., all of the line is large) and *lines with broad boundaries* (i.e., the line's endpoints are ill-defined). Xinming (Xinming 2004) extends this approach and proposes a more detailed formal definition of fuzzy regions, where he distinguishes four mutually disjoint components: the *interior*, the *boundary's interior*, the *boundary's boundary,* and the *exterior* (figure 4).

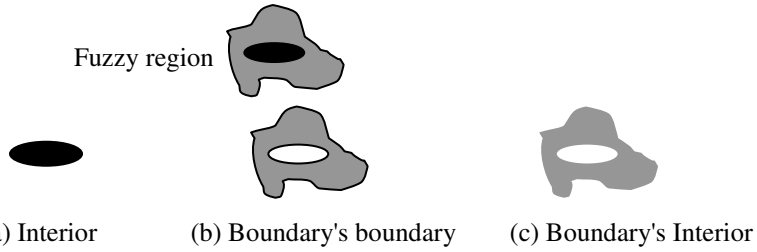


Figure 4. Topological invariants of a simple fuzzy region in (Xinming 2004)

The condition $A_1 \subseteq A_2$ in (Clementini and Di Felice 1997) does not exist in (Erwig and Schneider 1997). In fact, Erwig and Schneider (Erwig and Schneider 1997) are interested in another kind of vagueness, where a *vague region* is a composed geometry. The geometry's components belong to a pair of subsets. First, the *kernel* subset contains the subregions *definitely* belonging to the vague region. Second, the *boundary* subset contains the subregions *possibly* belonging to the vague region. Likewise, the *vague points* and the *vague lines* are respectively defined as a pair of subsets of points and lines. Crisp spatial objects can be expressed through this model when the *boundary* subset is empty. Figure 5 gives an example of a vague region A, in which the white subregions compose to the *boundary* subset and the gray ones compose the *kernel*.

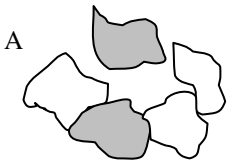


Figure 5. Vague region in (Erwig and Schneider 1997)

2.2.2 Models based on mathematical approaches. Roy and Stell (Roy and Stell 2001) deal with *indeterminacy,* which is defined as knowledge vagueness. They define an *indeterminate region* through rough sets (Pawlak 1994). An *indeterminate region* is composed of a lower and an upper approximation. The difference between these approximations represents the *fuzzy part of the region*. When this difference is empty, the region is crisp because the two approximations are equal (Roy and Stell 2001).

Fuzzy sets theory is also used to represent fuzzy spatial objects (Robinson and Thongs 1986, Altman 1987, Burrough 1989, Zhan 1997, Schneider 2001, Yongming and Sanjiang 2004, Dilo 2006). Zhan (Zhan 1997) and Dilo (Dilo 2006) interpret a fuzzy spatial object as a fuzzy subset. In (Zhan 1997), the membership function of the fuzzy spatial object is decomposed to n $\alpha-cuts$ (an $\alpha-cut$ is the set of points with a membership value α or higher (e.g. Godjjevac 1999)) in order to facilitate its interpretation. Xinming (Xinming 2004) defines the fuzzy spatial objects in two different ways. The first definition should respect the properties of the crisp topological space. The second definition should respect topological properties in a *fuzzy topological space*. In this context, the concept of *fuzzy topology* is a generalization of crisp topology, in which the sets belonging to the universe are fuzzy (i.e., the membership value is α, where $0 \le \alpha \le 1$). Schneider (Schneider 2001) presents formal definitions of fuzzy spatial object types through fuzzy subsets. For example, a fuzzy region is defined as a generalization of a crisp region in which an arbitrary point of space has a partial membership to the region (Schneider 2001).

In the next section, we highlight some approaches that study the specificities of the topological relations between fuzzy spatial objects.

2.3 Topological relations between fuzzy spatial objects

Topological relations are very important in GIS, especially in the specification of integrity constraints (Ubeda and Egenhofer 1997) and spatial queries. These relations are based only on the shape of objects and are totally independent of both the coordinates system and geometric transformations. The use of special shapes to express shape fuzziness should imply a different approach for identifying their topological relations. In the crisp context, several models (Egenhofer and Herring 1990, Egenhofer and Franzosa 1991, Mark and Egenhofer 1994, Cohn *et al.* 1997) studied the specification of topological relations in GIS and spatial databases. These models are based on two main approaches: general *point-set topology* (Egenhofer and Herring 1990) and *mereology*[1] (Varzi 2004). First, the 9-interesections model (Egenhofer and Herring 1990) is an extension of the 4-Intersection model (Egenhofer 1989). Based on point-set topology, the 9-Intersection model distinguishes 8 relations between two simple regions (*Disjoint*, *Meet*, *Overlap*, *Contains*, *Inside*, *Equal*, *Covers*, *Covered by*), 36 between two simple lines, 19 between a simple region and a simple line, 2 between two points, 3 between a point and a simple line, and 3 between a point and a simple region. In the context of mereology, a first model called RCC-5 (Randell and Cohn 1989) distinguished 5

[1] The region is defined as the elementary component of the space, i.e. points and lines are not considered

topological relations among two simple crisp regions. This model was then extended to distinguish 8 topological relations, and it became known as RCC-8 (Cohn *et al.* 1997).

In the fuzzy context, topological relations can be specified by extending the RCC and 9-Intersection models (Cohn and Gotts 1996, Clementini and Di Felice 1997, Erwig and Schneider 1997, Roy and Stell 2001, Xinming 2004). Erwig and Schneider (Erwig and Schneider 1997) use a three-valued logic to underline the spatial fuzziness. Then, an intersection between two topological invariants can be *true*, *false,* or *maybe* (i.e., when the boundary subset participates in the relation). In (Cohn and Gotts 1996), a topological relation between two Egg-Yolk regions *A* and *B* is specified through a 2*2-matrix that enumerates four subrelations: *Egg*(*A*) - *Egg*(*B*), *Egg*(*A*) - *Yolk*(*B*), *Yolk*(*B*) - *Egg*(*A*), and *Yolk*(*A*) - *Yolk*(*B*) (figure 4). These four subrelations are those defined in the RCC-5 model: *Partially Overlapping* (*PO*), *Proper Part* (*PP*), *Equal* (*E*), *Proper Part Inverse* (*PPI*), and *Distinct* (*D*). With this approach, only 46 combinations correspond to the 46 possible topological relations drawn in (Cohn and Gotts 1996). Figure 6 presents the relation number 15 in (Cohn and Gotts 1996). The main advantage of this approach is its simplicity in identifying the topological relations. However, point, lines, and crisp regions are not covered.

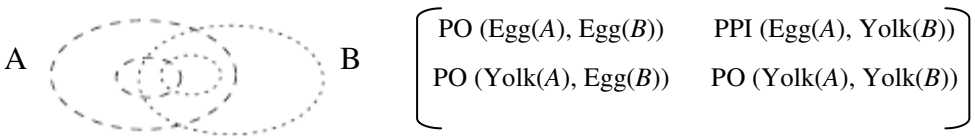


Figure 6. Identification of topological relations in (Cohn and Gotts 1996)

Clementini and Di Felice (Clementini and Di Felice 1997) studied *approximate topological relations,* which correspond to the topological relations between regions with broad boundaries. In fact, they use a 3*3-matrix, in which the sharp boundary is replaced by a broad boundary. After considering a set of consistency rules for the matrix (i.e., twelve rules to eliminate a matrix that cannot be drawn) (Clementini and Di Felice 1997), only 44 relations can be distinguished and drawn among two simple fuzzy regions. These relations can then be grouped into 17 clusters, for which a conceptual-neighborhood graph was drawn. This approach can be very interesting when it is necessary to coarsely identify the topological relations between fuzzy regions. However, the model is not sufficient when the needs are more specific and the user has a clear idea regarding the relation between fuzzy regions. For example, figure 7 shows an example of two different relations absorbed by the same cluster and identified through the same matrix.

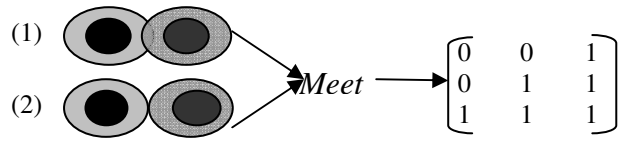
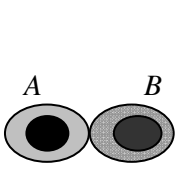


Figure 7. Identification of topological relations in (Clementini and Di Felice 1997)

In the same way, Reis *et al.* (Reis *et al.* 2006) use Clementini's model (Clementini and Di Felice 1997) in order to identify the topological relations between fuzzy lines. After considering the same 12 conditions defined in (Clementini and Di Felice 1997), 5 topological relations are identified between two *broad lines* and 77 between two *lines with broad boundaries*.

Xinming (Xinming 2004) presents an extension of the 9-Intersection model that identifies more topological relations than that of (Clementini and Di Felice 1997). He identifies

topological relations through a 4*4-matrix. Indeed, he distinguishes 152 topological relations and presents their correspondent matrices (see the example in the figure 8). The absence of relation clustering is the main limitation of the model. The model becomes useless in practice, however, since it is very difficult to easily and intuitively distinguish all of these relations. Furthermore, Xinming (Xinming 2004) does not distinguish between the inner and outer boundaries. Hence, many relations cannot be specified through a 4*4-matrix.



	B°	llB	lB°	B^{--}
A°	0	0	0	1
llA	0	1	0	1
lA°	0	0	0	1
A^{--}	1	1	1	1

with

A° : the interior of A

lA°: the boundary 's interior

llA : the boundary's boundary

A^{--} : the exterior

Figure 8. Identification of topological relations in (Xinming 2004)

Fuzzy sets' theory is also used to identify the topological relations between fuzzy spatial objects (Zhan 1997, Schneider 2001, Dilo 2006). A topological relation in (Zhan 1997) is called *R* (i.e., *a parameter used to replace the eight relations of the 4-Intersection model* (Egenhofer 1989)). For each pair of $\alpha-cuts$ of considered regions, a subrelation *r* is identified. Then, the *possibility* of the global relation *R* is deduced from the number of subrelations *r* that arise between the different $\alpha-cuts$. This approach is easy to use in practice, but it presents some complexity when the $\alpha-cuts$ are non-uniformly distributed between 0 and 1. In the same way, Dilo (Dilo 2006) identifies six possible topological relations (i.e. *Disjoint*, *Touches*, *Crosses*, *Overlaps*, *Within,* and *Equal*) between two vague spatial objects. A topological relation is defined based on fuzzy operations (e.g., *union*, *intersection*, *absolute difference,* and *bounded difference*) applied to fuzzy subsets that represent vague spatial objects. According to Dilo (Dilo 2006), many topological relations may exist at the same time with different *Truth degrees* (e.g., *Overlap*(*A*, *B*) with the *Truth degree* = 0,2; *Meet*(*A*,*B*) with the *Truth degree* = 0,3; *Disjoint*(*A*, *B*) with the *Truth degree* = 0,5).

2.4 Problem statement

The exact models presented earlier (Cohn and Gotts 1996, Clementini and Di Felice 1997, Erwig and Schneider 1997, Xinming 2004, Reis *et al.* 2006) have the advantage of explicitly distinguishing objects' topological invariants. Through this discrete viewpoint of space, the specification of topological relations can be improved (Clementini and Di Felice 1997). For these reasons, we propose an exact model in order to achieve objectives. Nevertheless, we think that the existing models do not distinguish between different levels of fuzziness and are not sufficiently expressive to represent partial fuzziness. In reality, a fuzzy spatial region is not always surrounded by a large boundary everywhere. For example, a lake boundary can be broad in some locations and sharp in some others. This situation cannot be represented by existing exact models, because the connectedness condition is violated. The same problem is present for lines. Only two cases of fuzziness are distinguished in lines (cf. section 2.2.1), but a line can easily have a partially fuzzy interior. Moreover, the studied models are not sufficiently expressive in terms of topological relations since there is no distinction between the inner and outer boundaries for a fuzzy region. Some work tries to offer more expressivity by increasing the number of topological invariants (Xinming 2004). Nevertheless, the absence of relation clustering limits their practical use. Indeed, the main research questions of our paper are the following:

1- How can we obtain more expressive definitions of the fuzzy spatial object types through an exact model? How can we represent partial fuzziness?
2- What are the topological relations between fuzzy spatial objects? How is it possible to identify topological relations between objects that have different levels of fuzziness? How can we formally identify these relations?
3- How can we classify the topological relations between fuzzy regions in order to facilitate their use in practice? How could resulting clusters reflect the fuzziness level of a topological relation?

3. Fuzzy spatial objects

In general, there is no agreement regarding the appropriate formal definition of fuzzy spatial objects, because fuzziness can be interpreted in different ways. It is not the objective of this work to unify these interpretations. We are interested in proposing an expressive and easy definition of fuzzy spatial objects through an exact model. In our approach, we transfer the Egg-Yolk's model into point-set topology context in order to both consider points and lines and permit the representation of *partially fuzzy objects*. From this perspective, we distinguish three *fuzzy spatial object types*: *fuzzy points*, *fuzzy lines,* and *fuzzy regions*. Each one of these is composed of *n* crisp object types (i.e., *point*, *line,* and *region*) distributed into a pair of sets called (1) the *minimal extent* and (2) the *maximal extent* (figure 9). Figure 9 presents an example of fuzzy points, fuzzy lines, and fuzzy regions. A fuzzy point is a zone that we approximate to a crisp region containing all of positions that the point can possibly fill. The minimal extent of point is equal to its maximal extent because the fuzziness concerns a unique topological invariant: *the interior* (cf. section 3.1 for more details). For a fuzzy line (cf. section 3.2), the minimal extent is the union of the linear parts. However, its maximal extent can contain some fuzzy parts (i.e., presented as fuzzy points in figure 9 (b)) at which the line can have any shape. For a fuzzy region (cf. section 3.3), the fuzziness concerns the boundary. The minimal extent refers to the geometry when the boundary is as close as possible (i.e. it is drawn around the area which certainly belongs to the region). The maximal extent is the geometry of the object when the boundary is as far away as possible (i.e. it is drawn around the area which contains all of points possibly belonging to the region).

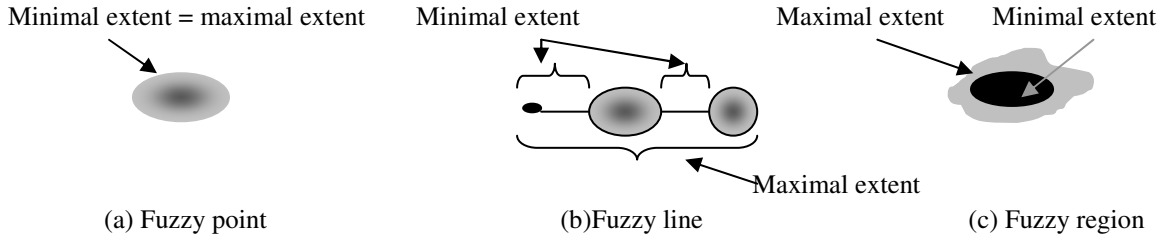


Figure 9. Minimal and maximal extents for (a) a fuzzy point, (b) a fuzzy line and (c) a fuzzy region

Generally, the minimal extent refers to the geometry's parts definitely belonging to the spatial object. The maximal extent corresponds to the object's geometry when fuzziness is taken into account and added to the minimal extent. Outside of the maximal extent, there are no spatial points that can possibly belong to the object. The number *n* of crisp object types composing the fuzzy spatial object type is 1 for *a fuzzy point* (i.e., a zone that we represent as a crisp region composed of the quasi-totality of possible positions of the point (cf. section 3.1)), 2 for a *fuzzy region* (i.e., two crisp regions (cf. section 3.3)), and *n* for *fuzzy lines* (i.e., 1 or *n* points of the line are fuzzy (cf. section 3.4)). For example, a fuzzy region corresponds to a pair of crisp regions that respectively represent the minimal and maximal extents. This general definition of fuzzy spatial objects is based on the following principles:

1- A fuzzy spatial object is a generalization of a crisp spatial object.
2- The minimal and the maximal extents are made up of crisp spatial object types. Only the combination of two extents corresponds to the fuzzy spatial object.
3- For the minimal and the maximal extents, the topological invariants should be mutually disjoint.

The first principle means that the spatial extent of a fuzzy object is crisp when its minimal extent is equal to its maximal one. The second principle requires that the minimal and maximal extents verify the topological consistency conditions of the crisp spatial object types (e.g., *a crisp region should be connected*). Finally, the third principle permits the identification of topological relations based on the intersections between the topological invariants of the minimal and maximal extents of two fuzzy spatial objects. In the next sections, we present our definitions of fuzzy point, fuzzy line, and fuzzy region.

3.1 Fuzzy point

In the crisp context, a point $p(x, y)$ is a 0-dimensional object type that represents one position (x, y) in space. The only topological invariant composing the shape of a crisp point is its interior. Because a point does not have a boundary (the boundary of an object with N dimensions has N-1 dimensions), the fuzziness can characterize only the interior and thus the point itself. Semantically, a fuzzy point occurs when an intrinsic property of the point or a lack of knowledge does not permit certainty regarding its position. For such a case, the spatial extent of the object is typically replaced by a zone that we represent as a crisp region composed of the family of positions that the point can fill (figure 10). The closure[2] of the crisp region represents an infinity of possible positions for the point. Consequently, a fuzzy point does not have a minimal extent; it only has a maximal extent.

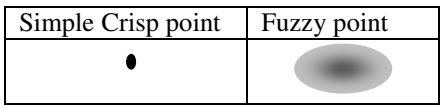


Figure 10. Fuzzy point

Since a simple fuzzy point corresponds, in fact, to a simple crisp region, it should verify the following conditions:
1- The closure is a non-empty connected regular closed set.
2- The interior is a non-empty connected regular open set.
3- The boundary and exterior are connected.

To provide an example of a fuzzy point, consider an application to help the fire brigades in their interventions. Generally, a fireman cannot precisely localize the fire source. However, he can draw an area in which the fire source should exist. This intervention area corresponds to a fuzzy point and can be represented through our model. It is clear that the size of the region representing the fuzzy point depends on the fuzziness level (i.e., *a larger region refers to a fuzzier point*).

3.2 Fuzzy line

A simple crisp line is a one-dimensional object type made up of an interior limited by two endpoints. The boundary of a crisp line corresponds to its endpoints, whereas the interior is

[2] The closure, in point set topology, is the union of the interior and the boundary.

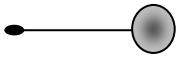
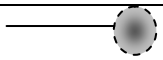
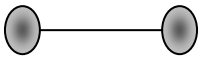
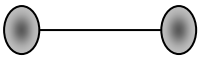
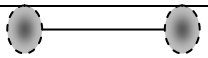
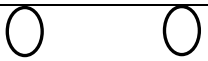
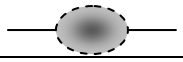
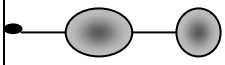
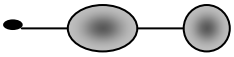
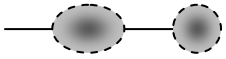
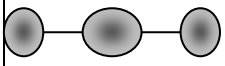
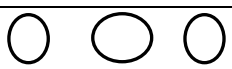
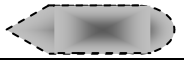
the set of points connecting them. The fuzziness can characterize any point of the line. Consequently, the line boundary can be partially or completely fuzzy while the interior remains well-defined; we then speak about lines with fuzzy boundaries. In the same way, the interior can be partially or completely fuzzy while the endpoints are well-defined; we then speak about lines with partially and completely fuzzy interiors, respectively (**figure 11**). The extreme case of line fuzziness arises when all topological invariants of the line (i.e. the interior and the boundary) are fuzzy (figure 11). Thus, a completely fuzzy line corresponds to a crisp region that represents the set of positions that the line can fill. However, a completely crisp line is a particular case of fuzzy lines, for which all of the interior and both endpoints are well defined. The different cases of line fuzziness can be combined as presented in figure 11. To underline the different levels of line fuzziness, we use four adverbs: (1) *weakly*, (2) *fairly*, (3) *strongly*, and (4) *completely*. The term "*weakly*" indicates that one of the topological invariants is partially fuzzy. The term "*fairly*" reflects either complete fuzziness of one of topological invariants or the case where the interior and endpoints are partially fuzzy at the same time. The term "*strongly*" specifies complete fuzziness for one of topological invariants and partial fuzziness for the second one. Finally, the term "*completely*" is used to express total fuzziness of the line's components. Figure 11 shows a symmetrical matrix, in which fuzziness increases from "*none*" in the upper-left cell to "*completely*" in the lower-right cell through a progression including "*weakly*", "*fairly,*" and "*strongly*".

	Crisp interior	Partially fuzzy interior	Completely fuzzy interior
Crisp boundary	none	weakly fuzzy	fairly fuzzy
Partially fuzzy boundary	weakly fuzzy	fairly fuzzy	strongly fuzzy
Completely fuzzy boundary	fairly fuzzy	strongly fuzzy	completely fuzzy

Figure 11. Fuzzy lines

In our model, a fuzzy line is typically composed of two-dimensional parts that correspond to the line's fuzzy points and one-dimensional parts that represent the minimal extent of the line. Thus, the maximal extent is the union of the one-dimensional and two-dimensional parts. The interior of maximal extent corresponds to the union of the interiors of the one-dimensional and two-dimensional parts. In the same way, the boundary of the maximal extent is the union of the boundary of the one-dimensional and two-dimensional parts. Figure 12 presents the different cases of fuzzy line decomposition. We should note that the different representations **of a line**, in figure 12, indicate a set of pictograms that we use to show different types of fuzziness for lines. In other terms, these representations are not based on a mathematical model that allows to consider error component of spatial data as in (Chrisman 1991).

Fuzzy line	Extents		Topological invariants	
	Minimal extent		Interior	
			Boundary	
	Maximal extent		Interior	
			Boundary	
	Minimal extent		Interior	
			Boundary	

	Maximal extent		Interior	
			Boundary	
	Minimal extent		Interior	
			Boundary	∅
	Maximal extent		Interior	
			Boundary	
	Minimal extent		Interior	
			Boundary	
	Maximal extent		Interior	
			Boundary	
	Minimal extent		Interior	
			Boundary	
	Maximal extent		Interior	
			Boundary	
	Minimal extent		Interior	
			Boundary	∅
	Maximal extent		Interior	
			Boundary	
	Minimal extent		Interior	∅
			Boundary	
	Maximal extent		Interior	
			Boundary	
	Minimal extent		Interior	∅
			Boundary	
	Maximal extent		Interior	
			Boundary	
	Minimal extent	∅	Interior	∅
			Boundary	∅

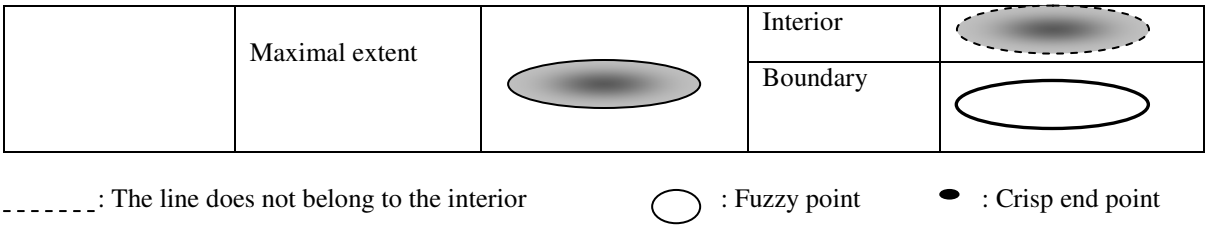


Figure 12. Topological invariants according to the line fuzziness

More formally, a fuzzy line $\tilde{L}$ is the union of a maximal extent $\tilde{L}_{max}$ and a minimal extent $\tilde{L}_{min}$. In our approach, we look for the most expressive model possible in terms of specification of the topological relations. Consequently, we focus on the definition of the topological invariants for the maximal $\tilde{L}_{max}$ and the minimal $\tilde{L}_{min}$ extents. For each one, we distinguish an interior and a boundary that can be empty according to the configuration of the line (figure 12). From a point-set topology view point, a simple fuzzy line should verify the following conditions:

1- Each one-dimensional part of the simple fuzzy line is connected.
2- Each one-dimensional part of the simple fuzzy line is not self-intersecting.
3- Each one-dimensional part of the simple fuzzy line does not form a loop.
4- Each two-dimensional part verifies the conditions defined for a fuzzy point (section 3.1).
5- If the endpoints are fuzzy, they do not overlap with each other.

The three first conditions are those defined for a crisp line in the general point-set topology. Then, we apply these conditions to each linear part of the fuzzy line. The fourth condition requires that every fuzzy point on the interior or the endpoints of a line respect the topological conditions of a crisp region applied to a fuzzy point (section 3.1). The last condition is defined to eliminate any risk of self-intersection or loop configurations. Figure 13 shows some cases of lines that are invalid according to our model.

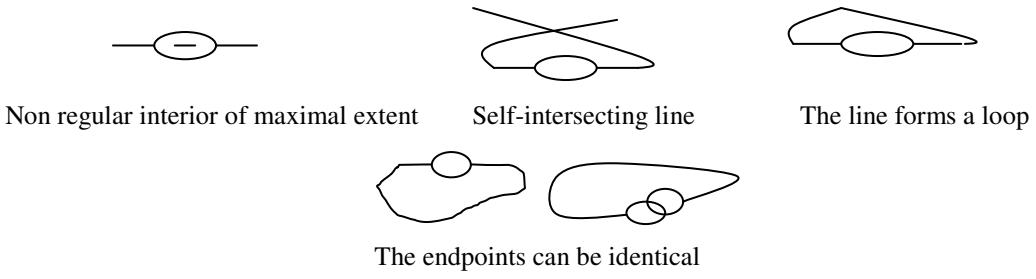
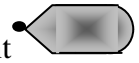
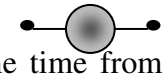


Figure 13. Examples of invalid lines

In the next cases, we give two examples of fuzzy lines:

- A *strongly fuzzy line* with only one crisp end point  :

The Bermuda triangle is a region in the Atlantic Ocean where some aircrafts and surface vessels have disappeared. We suppose that we have been appointed to redraw the trajectory of an engine that has disappeared. We will quickly discover that the only certain information is the last position of the engine before the communication interruption. In fact, this point refers to the minimal extent of the line. Far from this point, the trajectory can have any shape and position inside the fuzziness area. Indeed, our model is sufficiently expressive to represent this fuzziness case.

- A *fairly fuzzy line* with two crisp endpoints  :

We suppose that an aircraft disappeared for some time from radar screens. After that, the communication with the aircraft returns to normal and the engine arrives at its destination. In this case, the aircraft trajectory is composed of two crisp endpoints. However, the interior is *partially fuzzy,* because the trajectory can take any unpredictable shape inside the dangerous zone.

3.3 Fuzzy region

A crisp region is a two-dimensional spatial object type in which the shape is typically composed of an *interior*, a *boundary,* and an *exterior.* For a region, shape fuzziness occurs when there is difficulty in precisely distinguishing between the interior and the exterior via a sharp boundary. From this perspective, shape fuzziness is generally correlated with the boundary, which can itself be *crisp*, *partially fuzzy,* or *completely fuzzy*. It is possible to draw a *minimal spatial extent* by considering the boundary to be as close as possible (i.e. it is drawn around the area which *certainly* belongs to the region). In the same way, a *maximal spatial extent* can be drawn by considering the boundary to be as far as possible (i.e. it is drawn around the area which contains all of points *possibly* belonging to the region). Figure 14 represents an example of a fuzzy region in which the boundary is *partially fuzzy*. The spatial extent of a fuzzy region is composed of a portion called the *minimal extent* (i.e., *all of the points definitely belonging to the spatial object*) and covered by a *maximal extent* (i.e., *all of the points possibly belonging to the spatial object*).

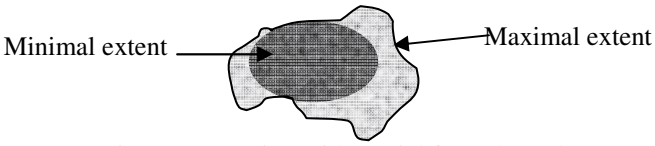


Figure 14. Region with partial fuzzy boundary

We consider that a simple fuzzy region is made up of two crisp regions: (1) the *maximal extent,* which can *be* "*Equal*", "*Contains*", or "*Covers*" (2) the *minimal extent* (figure 14). When the boundary is totally sharp (i.e. it does not contain any fuzzy point), the region is called *completely crisp*. This is a particular case of *fuzzy region* for which the *maximal extent* is *equal* to the *minimal extent*. In the second case, the region's boundary is fuzzy only in some locations. We speak about *partially fuzzy regions,* where the maximal extent *covers* the minimal extent. For example, a forest stand or a lake can have sharp boundaries (e.g., *rocker borders for a lake* and *a total cut for a forest stand*) and fuzzy boundaries (e.g., *swamp borders for a lake*) at the same time. The third case represents a typical fuzzy region for which the boundary is entirely fuzzy. For example, the boundary of a pollution zone is fuzzy everywhere since the pollution decreases from its kernel to the region's exterior. In the figure 15, we present an example of each of these three cases.

Fuzzy region	Representation	Maximal and minimal extents		Topological invariants of minimal and maximal extents	
				Interior	Boundary
Region with sharp boundary (i.e., *crisp region*)		Minimal extent			
		Maximal extent			

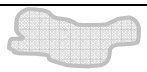
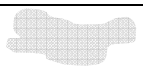
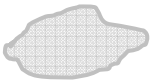
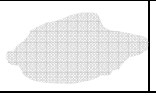
Region with partially fuzzy boundary (i.e., *partially fuzzy region*)		Minimal extent			
		Maximal extent			
Region with completely fuzzy boundary (i.e., *completely fuzzy region*)		Minimal extent			
		Maximal extent			

Figure 15. Fuzzy regions

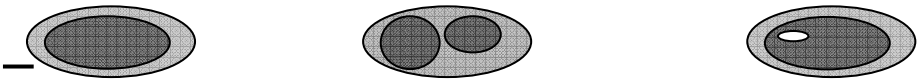
Since the minimal and maximal extents are crisp regions, we distinguish three mutually disjoint topological invariants for each of them: an *interior*, a *boundary,* and an *exterior*. Thus, a fuzzy region $\tilde{A}$ is made up of six topological invariants: *the interior of the minimal extent* $\tilde{A}^{\circ}_{\min}$, *the boundary of the minimal extent* $\partial\tilde{A}_{\min}$, *the exterior of the minimal extent* $\tilde{A}^{-}_{\min}$, *the interior of the maximal extent* $\tilde{A}^{\circ}_{\max}$, *the boundary of the maximal extent* $\partial\tilde{A}_{\max}$, and *the exterior of the maximal extent* $\tilde{A}^{-}_{\max}$ (figure 15).

- ***Definition 1***: A simple fuzzy region $\tilde{A}$ is composed of two simple crisp regions $\tilde{A}_{\max}$ and $\tilde{A}_{\min}$, *where Equal*[3]*(* $\tilde{A}_{\max}$, $\tilde{A}_{\min}$), *Contains(* $\tilde{A}_{\max}$, $\tilde{A}_{\min}$), *or Covers(* $\tilde{A}_{\max}$, $\tilde{A}_{\min}$). $\tilde{A}_{\min}$ is the *minimal extent* of $\tilde{A}$, $\partial\tilde{A}_{\min}$ is the inner boundary of $\tilde{A}$, $\tilde{A}_{\max}$ is the *maximal extent of* $\tilde{A}$, *and* $\partial\tilde{A}_{\max}$ is the outer boundary of $\tilde{A}$. $\tilde{A}_{\min}$ is the set of points *certainly* belonging to $\tilde{A}$. However, the maximal extent $\tilde{A}_{\max}$ is the union of the minimal extent and the set of points *possibly* belonging to the fuzzy region.

The following conditions should be respected for any fuzzy region:

1- The closures of the maximal and the minimal extents are non-empty regular connected closed subsets.
2- The interiors of the maximal and minimal extents are non-empty regular open sets.
3- The boundaries and exteriors of the maximal and minimal extents are connected.

Figure 16 presents some examples of invalid fuzzy regions. In case (a), the region is invalid because its closure is non-regular, i.e. there is an isolated line that belongs to the closure. In case (b), the interior of the region is non-connected because it is composed of two disjoint subregions. In the case (c), the exterior does not respect the connectedness condition since the interior contains a hole.



(a) Non-regular closed closure (b) Non-connected interior (c) Non-connected exterior

Figure 16. Examples of invalid fuzzy regions

This general definition covers the crisp regions occurring when *Equal(* $\tilde{A}_{\max}$, $\tilde{A}_{\min}$). Hereafter, we only focus on the typical fuzzy regions where *Contains(* $\tilde{A}_{\max}$, $\tilde{A}_{\min}$) or *Covers(* $\tilde{A}_{\max}$, $\tilde{A}_{\min}$).

[3] The spatial relations (i.e., *Equal*, *Contains*, *Covers*) used in this definition are those defined in (Egenhofer and Herring 1990).

4. Topological relations between fuzzy spatial objects

4.1 Principles

To identify the topological relations between two fuzzy objects, we interpret their maximal and minimal extents as independent crisp geometries. In fact, our methodology consists of identifying four specific topological relations between the minimal and maximal extents of the fuzzy spatial objects. For that, we define a 4-Intersection matrix containing the following four topological subrelations: $R_1(\tilde{A}_{\min}, \tilde{B}_{\min})$, $R_2(\tilde{A}_{\min}, \tilde{B}_{\max})$, $R_3(\tilde{A}_{\max}, \tilde{B}_{\min})$, and $R_4(\tilde{A}_{\max}, \tilde{B}_{\max})$(see example in figure 17). The topological subrelations assigned to the matrix's cells are those defined in the 9-Intersection model (Egenhofer and Herring 1990). For example, if we study the topological relations between two fuzzy regions, each cell receives one of the eight possible topological relations between two simple crisp regions (i.e., *Disjoint*, *Overlap*, *Meet*, *Equal*, *Contains*, *Inside*, *Covers*, *Covered by*). Then, the 4-Intersection matrix corresponds to the following representation:

$$\begin{array}{c} \\ \tilde{A}_{\min} \\ \tilde{A}_{\max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{\min} & \tilde{B}_{\max} \end{array} \\ \begin{bmatrix} R_1(\tilde{A}_{\min}, \tilde{B}_{\min}), & R_2(\tilde{A}_{\min}, \tilde{B}_{\max}) \\ R_3(\tilde{A}_{\max}, \tilde{B}_{\min}), & R_4(\tilde{A}_{\max}, \tilde{B}_{\max}) \end{bmatrix} \end{array}$$

Figure 17 shows the content of the matrix that describes the topological relation between a *partially fuzzy region* $\tilde{A}$ and a *completely fuzzy* region $\tilde{B}$. In the matrix (b), the letters *O* and *C* are used to denote the relations *Overlap* and *Contains,* respectively.

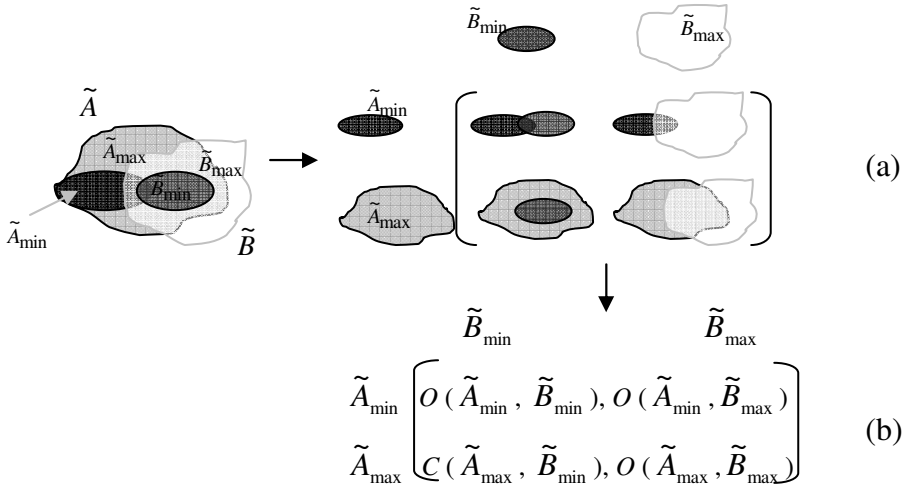


Figure 17. Description of the topological relation between two fuzzy regions: (a) visual content of the matrix, (b) formal identification of the relations between the minimal and maximal extents of the objects involved

The content of the matrix corresponds to the topological subrelations relating the minimal and maximal extents. We use the topological subrelation between the maximal extents $R_4(\tilde{A}_{\max}, \tilde{B}_{\max})$ to label the global topological relation. For example, if $R_4(\tilde{A}_{\max}, \tilde{B}_{\max})$ is *Overlap*, we consider that the fuzzy spatial objects globally *Overlap* each other. If $R_4(\tilde{A}_{\max}, \tilde{B}_{\max})$ is *Contains*, we consider that the global topological relation is *Contains*.

In figure 18, we present examples of the identification of topological relations between fuzzy spatial objects. The first example presents a description of the topological relation between two completely fuzzy regions $\tilde{A}$ and $\tilde{B}$. The second example concerns a fairly fuzzy line $\tilde{L}$

and a completely fuzzy region $\tilde{A}$. The third example shows the identification of the topological relation between two fairly fuzzy lines $\tilde{L}$ and $\tilde{K}$. Finally, the last example concerns a completely fuzzy region $\tilde{A}$ and a fuzzy point $\tilde{P}$.

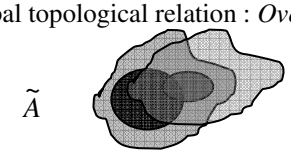
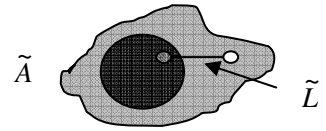
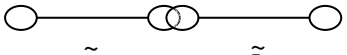
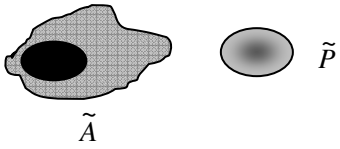
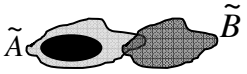
Spatial representation	Correspondent matrix
Global topological relation : *Overlap* 	$\begin{array}{c\|cc} & \tilde{B}_{min} & \tilde{B}_{max} \\ \tilde{A}_{min} & Overlap(\tilde{A}_{min}, \tilde{B}_{min}) & Overlap(\tilde{A}_{min}, \tilde{B}_{max}) \\ \tilde{A}_{max} & Contains(\tilde{A}_{max}, \tilde{B}_{min}) & Overlap(\tilde{A}_{max}, \tilde{B}_{max}) \end{array}$
Global topological relation : *Contains* 	$\begin{array}{c\|cc} & \tilde{L}_{min} & \tilde{L}_{max} \\ \tilde{A}_{min} & Overlap(\tilde{A}_{min}, \tilde{L}_{min}) & Overlap(\tilde{A}_{min}, \tilde{L}_{max}) \\ \tilde{A}_{max} & Contains(\tilde{A}_{max}, \tilde{L}_{min}) & Contains(\tilde{A}_{max}, \tilde{L}_{max}) \end{array}$
Global topological relation : *Overlap* 	$\begin{array}{c\|cc} & \tilde{L}_{min} & \tilde{L}_{max} \\ \tilde{K}_{min} & Disjoint(\tilde{K}_{min}, \tilde{L}_{min}) & Disjoint(\tilde{K}_{min}, \tilde{L}_{max}) \\ \tilde{K}_{max} & Disjoint(\tilde{K}_{max}, \tilde{L}_{min}) & Overlap(\tilde{K}_{max}, \tilde{L}_{max}) \end{array}$
Global topological relation : *Disjoint* 	$\begin{array}{c\|c} & \tilde{P}_{max} \\ \tilde{A}_{min} & Disjoint(\tilde{A}_{min}, \tilde{P}_{max}) \\ \tilde{A}_{max} & Disjoint(\tilde{A}_{max}, \tilde{P}_{max}) \end{array}$

Figure 18. Examples of identification of topological relations through a 4-Intersection matrix

4.2 Topological relations between a fuzzy region and a crisp one

In our approach, the 4-Intersection matrix highlights the subrelations that exist between the components of the fuzzy geometries. Indeed, this expressivity is highlighted when the maximal extent of the fuzzy spatial object is different from the minimal extent. In the other cases, some cells in the matrix will have the same values. For example, figure 19 shows a fuzzy region that overlaps a crisp region. The topological relation can be reduced to a 2-Intersections matrix, because the region $\tilde{B}$ is crisp and so its minimal extent equals its maximal one. Hereafter, we do not study topological relations that involve crisp regions.



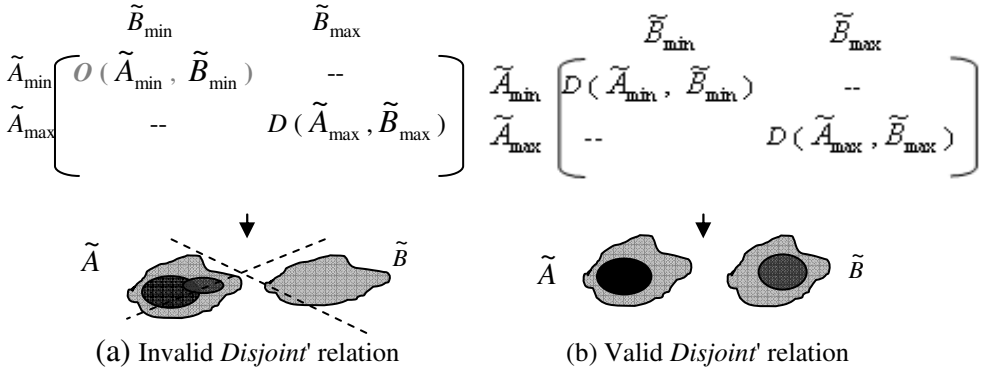
$$\begin{array}{c\|cc} & \tilde{B}_{min} & \tilde{B}_{max} \\ \tilde{A}_{min} & Overlap(\tilde{A}_{min}, \tilde{B}_{min}) & Overlap(\tilde{A}_{min}, \tilde{B}_{max}) \\ \tilde{A}_{max} & Overlap(\tilde{A}_{max}, \tilde{B}_{min}) & Overlap(\tilde{A}_{max}, \tilde{B}_{max}) \end{array} \Leftrightarrow \begin{array}{c\|c} & \tilde{B}_{min} \\ \tilde{A}_{min} & Overlap(\tilde{A}_{min}, \tilde{B}_{min}) \\ \tilde{A}_{max} & Overlap(\tilde{A}_{max}, \tilde{B}_{min}) \end{array}$$

Figure 19. Example of a topological relation between a fuzzy region and a crisp region

The values assigned to the different cells of the matrix should not be arbitrarily chosen. In general, the value of $R(\tilde{A}_{max}, \tilde{B}_{max})$ enforces the other values. In the next section, we study these aspects specifically for the topological relations between fuzzy regions.

4.3 Topological relations between fuzzy regions

Eight topological relations are possible between two simple crisp regions. By considering these as the possible values in the four cells of the matrix, there are $8^4 = 4096$ possible matrices. However, definition 1 imposes a condition mandating that the extents of a fuzzy region should be related by one of the following relations: *Equal*$(\tilde{A}_{max}, \tilde{A}_{min})$, *Contains*$(\tilde{A}_{max}, \tilde{A}_{min})$, *or Covers*$(\tilde{A}_{max}, \tilde{A}_{min})$. Indeed, a 4-Intersection matrix cannot identify a topological relation between two fuzzy regions when this condition is violated. Thus, the contents of the matrix's cells are not independent. For example, if the maximal extents are disjoint, **it is impossible for** an *Overlap* to exist between the minimal extents (**figure 20**). In figure 20, the subrelation $O(\tilde{A}_{min}, \tilde{B}_{min})$ is grey to denote that is not allowed whereas $D(\tilde{A}_{min}, \tilde{B}_{min})$ is black to show that is permitted. Consequently, several of the 4096 possible matrices are invalid because the dependency between the matrix's cells is not respected.



(a) Invalid *Disjoint*' relation (b) Valid *Disjoint*' relation

Figure 20. Controlling the validity of a *Disjoint* relation

In order to enumerate the valid 4-Intersection matrices, we firstly studied the possible values in the other three cells for each of the eight possible values of $R(\tilde{A}_{max}, \tilde{B}_{max})$. For example, if *Contains* $(\tilde{A}_{max}, \tilde{B}_{max})$, the only possible relation between $\tilde{A}_{max}$ and $\tilde{B}_{min}$ is *Contains;* otherwise, the expected relation cannot respect the general definition of a fuzzy region. The figure 20 shows an example of an inconsistent matrix in which *Disjoint* $(\tilde{A}_{max}, \tilde{B}_{max})$ and *Contains* $(\tilde{A}_{min}, \tilde{B}_{min})$. This matrix is inconsistent because $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$. In the second step, we also fix the relation between $\tilde{A}_{min}$ and $\tilde{B}_{min}$ to deduce the possible values of $R(\tilde{A}_{min}, \tilde{B}_{max})$. For example, when *Contains* $(\tilde{A}_{max}, \tilde{B}_{max})$ and *Contains* $(\tilde{A}_{min}, \tilde{B}_{min})$, $R(\tilde{A}_{min}, \tilde{B}_{max})$ should not be *Meet* or *Equal*. In this way, 31 rules (see table 4 in the appendix) are defined in order to ensure the consistency of matrices. In the rules' premises, we specify either $R(\tilde{A}_{max}, \tilde{B}_{max})$ or ($R(\tilde{A}_{max}, \tilde{B}_{max})$ and $R(\tilde{A}_{min}, \tilde{B}_{min})$). Then, we deduce the possible values in the remaining cells. In figure 20, the matrix on the left is not valid because it requires the minimal extent to be disjoint to the minimal extent (i.e., the definition of the fuzzy region is not respected, because $R(\tilde{A}_{max}, \tilde{B}_{max})$ should be *Contains*, *Covers,* or *Equal*).

This study proves that only 242 topological relations are possible between two simple fuzzy regions (see the appendix). More specifically, only one matrix is valid when *Disjoint*

($\tilde{A}_{\max}, \tilde{B}_{\max}$), 29 matrices are valid when *Contains* ($\tilde{A}_{\max}, \tilde{B}_{\max}$), 29 for *Inside* ($\tilde{A}_{\max}, \tilde{B}_{\max}$), 46 for *Covers* ($\tilde{A}_{\max}, \tilde{B}_{\max}$), 46 for *Covered by* ($\tilde{A}_{\max}, \tilde{B}_{\max}$), 65 for *Overlap* ($\tilde{A}_{\max}, \tilde{B}_{\max}$), 4 for *Meet* ($\tilde{A}_{\max}, \tilde{B}_{\max}$), and 22 when *Equal* ($\tilde{A}_{\max}, \tilde{B}_{\max}$). The topological relations are numbered from 1 to 242 according to the relation between $\tilde{A}_{\max}$ and $\tilde{B}_{\max}$. Table 1 shows this numbering (see the appendix to explore the relations).

5. Clustering of topological relations between fuzzy regions

5.1 Principles

In our work, the proposed model is expressive in terms of the topological relations between fuzzy regions. In this context, 242 topological relations are distinguished. Consequently, the clustering of relations into larger groups of relations is an important step, because it is very difficult to keep in the mind this great number of relations. It is additionally very difficult to find a name for each one of these regions, and so the user will have difficulty choosing the appropriate operator in order to express a query or integrity constraint. Mark and Egenhofer (Mark and Egenhofer 1994) studied the clustering of the topological relations between simple crisp regions and simple crisp lines both through a formal basis and by taking into account cognitive aspects. Clementini and Di Felice (Clementini and Di Felice 1997) defined a *topological distance* to classify the approximate topological relations between regions with broad boundaries. In this way, they deduced 17 clusters that they represent in a conceptual neighborhood graph.
In our approach, most of the distinguished topological relations are not completely different from each other. For example, two simple fuzzy regions can *weakly* or *completely overlap* each other depending on the content of the matrix. In the first case, only the maximal extents overlap. In the second case, however, *Overlap* is the unique value in the matrix's four cells. Thus, it is possible to deduce the relation's fuzziness level from the content of the 4-Intersection matrix. The objective of this section is to group the 242 topological relations into a limited number of clusters based on their fuzziness levels.

5.2 Clustering results

In section 4, we showed that the global topological relation is identified through a 4-Intersection matrix that enumerates four subrelations. Thus, a topological relation becomes possible if it appears at least once in the matrix. This possibility increases according to the number of similar subrelations. For example, a *Covers* topological relation in which *Covers* ($\tilde{A}_{\max}, \tilde{B}_{\max}$) and *Covers* ($\tilde{A}_{\min}, \tilde{B}_{\min}$) is stronger than another where only *Covers* ($\tilde{A}_{\max}, \tilde{B}_{\max}$). Because there are eight possible values for the matrix's cells, we distinguish eight basic clusters that we call: *DISJOINT*, *CONTAINS*, *INSIDE*, *COVERS*, *COVERED BY*, *EQUAL*, *MEET, and OVERLAP*. Each cluster contains all of the topological relations for which at least one of the four subrelations has the same name. For example, figure 21 shows a topological relation that belongs to the following clusters: *DISJOINT*, *CONTAINS,* and *COVERS*. Nevertheless, it belongs to the *DISJOINT cluster* more strongly than to the *CONTAINS* and *COVERS clusters.*

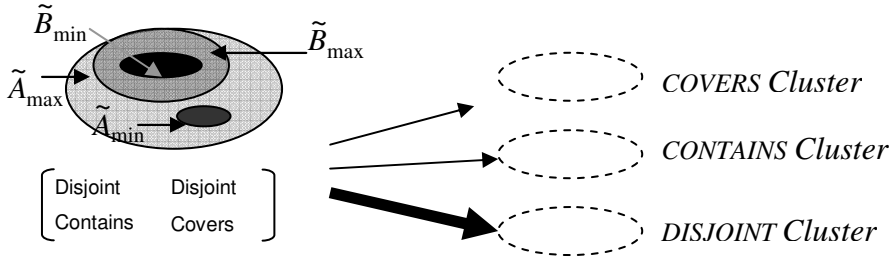


Figure 21. Example of clustering of a topological relation

For each one of the eight basic clusters, we identify four levels of relation membership: (1) *completely*, (2) *strongly*, (3) *fairly,* and (4) *weakly* (table 2). A topological relation belongs to the cluster *completely* when the four subrelations are similar. It belongs to the cluster *strongly* when only three subrelations have the same name as the cluster. The level labeled *fairly* contains all relations for which two subrelations have the same name as the cluster. Finally, the level called *weakly* contains the relations for which only one subrelation has the same name as the cluster. Figure 22 presents some relations that belong to different levels of *CONTAINS* and *DISJOINT* clusters, respectively, according to the contents of their correspondent matrices.

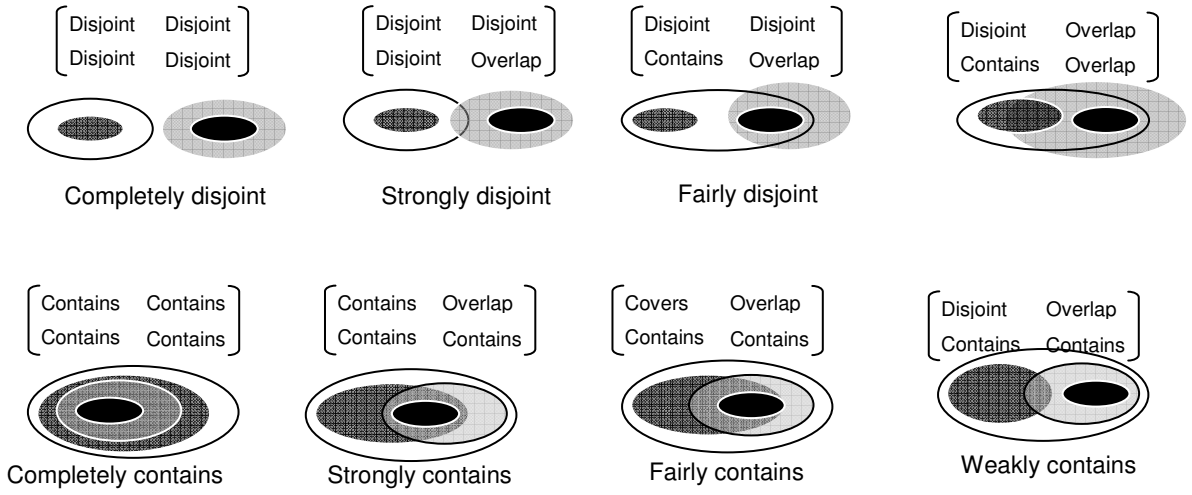


Figure 22. Evaluation of topological relation's fuzziness

5.3 Overlapping clusters

The main result of this clustering process is a hierarchical classification of the topological relations (figure 23). The top level is made up of eight basic clusters that each contains typically four levels: *completely*, *strongly*, *fairly,* and *weakly*. The resulting 32 subclusters overlap each other because a topological relation typically belongs to different levels of 1, 2, 3, or 4 clusters at the same time. For example, topological relation number 56 (see the appendix and the table 2) belongs *fairly* to the *CONTAINS* cluster and *weakly* to the *COVERS* and *INSIDE* clusters. The bottom level of the classification contains the 242 topological relations that appear in different subclusters.

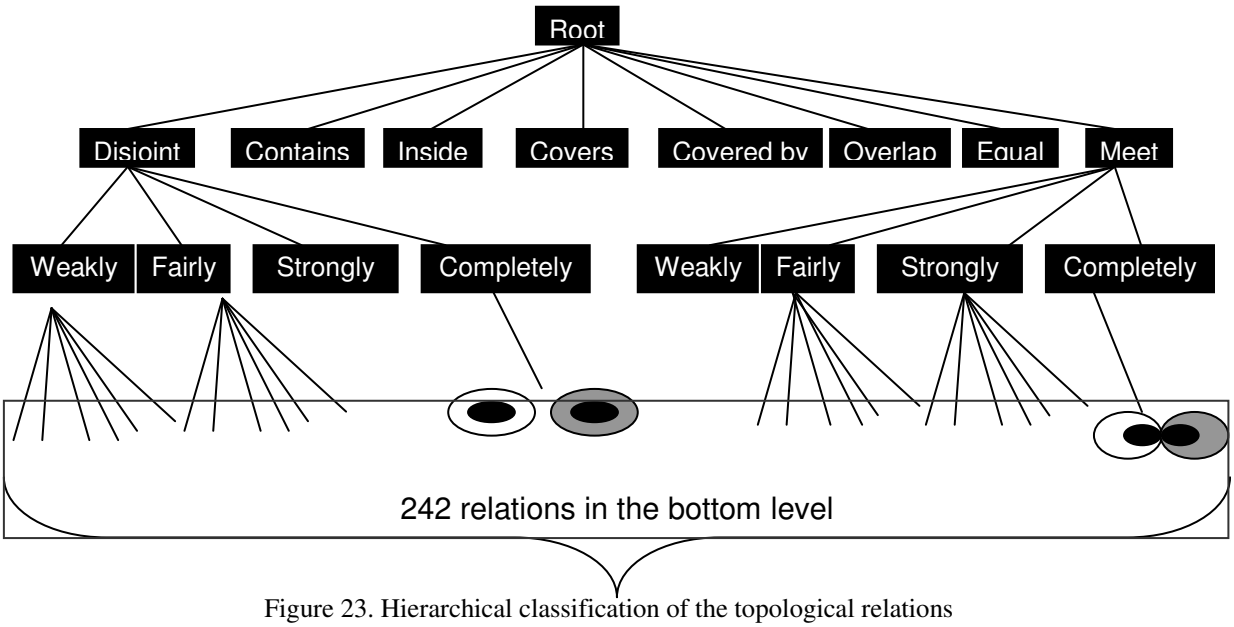


Figure 23. Hierarchical classification of the topological relations

6. Specification of spatial queries and integrity constraints

In the previous sections, we presented a framework for identifying topological relations between fuzzy regions. Because it uses the 9-Intersection model (Egenhofer and Herring 1990), our model can be easily integrated in a spatial database system. Indeed, the SQL language can be extended in order to retrieve fuzzy regions based on the qualitative information given by the user regarding their topological relation. In fact, a topological relation between two fuzzy regions can be recognized through the combination of four crisp spatial operators. For example, relation number 56 corresponds to (*Disjoint*, *Disjoint*, *Contains*, *Covers*). Hereafter, we suppose that we integrated our spatial model in a relational engine in order to give an example of its possible use in spatial queries involving fuzzy regions. We suppose that the spatial database stores pollution zones, which are represented as fuzzy regions. In the first query example, the user gives a coarse description of the topological relation when he introduces the specification *fairly DISJOINT*. The query's results should contain the *pollution zones* related to zone *A* by a topological relation belonging to this subcluster. In the second example, the query is more specific because the user identifies all topological subrelations that relate ($\tilde{A}_{\min}$, $\tilde{B}_{\min}$), ($\tilde{A}_{\min}$, $\tilde{B}_{\max}$), ($\tilde{A}_{\max}$, $\tilde{B}_{\min}$), and ($\tilde{A}_{\max}$, $\tilde{B}_{\max}$). The third example shows another use of our model, in which it is possible to display the different fuzziness levels of a relation (e.g., *Overlap*) that occurs between two fuzzy regions (cf. table 3). Table 3 shows a possible result for the query presented in example 3.

Example 1: **Select** *Pollution_Zone.geometry* **From** *Pollution_Zone* **Where**
Fuzzy_Relate (*pollution_zone.geometry*, *A.geometry*, *fairly DISJOINT*);

Example 2: **Select** *Pollution_Zone.geometry* **From** *Pollution_Zone* **Where**
Fuzzy_Relate (*Pollution_Zone.geometry*, *A.geometry*, *Disjoint, Meet, Contains, Contains*);

Example 3: **Select** *P1.id, P2.id, determine (P1.geometry, P2.geometry, "Overlap")*
From *Pollution_Zone P1, P2* **Where** *P1.id<>P2.id*;

In the same way, it is possible to use the model to formally express spatial integrity constraints in a fuzzy context. For example, let the constraint saying that '*two different lakes*

*can be only **fairly meet** or **completely disjoint**'*. This constraint can be formally expressed by integrating new spatial operators (e.g., *completely Contains*, *weakly Covers*, etc.) in a formal constraint language like the Object Constraint Language (OCL) (Duboisset *et al.* 2007). The database storing the *lakes* is consistent only if the topological relations between the different entities belong to ***fairly MEET*** or ***completely DISJOINT*** subclusters (see example 4).

Example 4: **Context** Lake **inv**:
Lake.allInstances → **forAll** (a, b| a<>b implies ***fairly MEET***(a,b) **or** ***completely DISJOINT***(a,b));

7. Discussion

Clementini and Di Felice (Clementini and Di Felice 1997) propose an extension of the 9-Intersection model (Egenhofer and Herring 1990) that uses a broad boundary to replace the sharp boundary. In this approach, 44 topological relations are distinguished between two regions with broad boundaries. By considering topological distance, Clementini and Di Felice (Clementini and Di Felice 1997) draw a conceptual neighborhood graph that shows similarity degrees between relations classified into 17 clusters. The main advantage of this approach is the ability to support a coarser spatial reasoning involving regions with broad boundaries. When the needs are more specific, it becomes more difficult to use this model. Furthermore, the identification of a broad boundary as a two-dimensional topological invariant requires respecting consistency conditions related to closeness and connectedness. Xinming (Xinming 2004) presents a more expressive model than that of (Clementini and Di Felice 1997), because he decomposes the large boundary into the *boundary's interior* and the *boundary's boundary*. Based on this definition, Xinming (Xinming 2004) presents another extension of 9-Intersection model, in which topological relations are identified through a 4*4-Intersection matrix. He distinguishes 152 topological relations presented as variants of the 44 relations from (Clementini and Di Felice 1997). Nonetheless, this model does not distinguish between the boundaries of the *minimal* and *maximal extents*. Thus, many topological relations cannot be distinguished. Moreover, regions with *partially* fuzzy boundaries (figure 11) are considered invalid and cannot be presented through this model. In our approach, we resolve this problem by considering a simple fuzzy region as a maximal extent and a minimal extent, in which either *Equal* ($\tilde{A}_{max}, \tilde{A}_{min}$) (i.e., *a crisp region*) or *Contains* ($\tilde{A}_{max}, \tilde{A}_{min}$) or *Covers* ($\tilde{A}_{max}, \tilde{A}_{min}$). We do not define a large boundary as a topological invariant of fuzzy regions. Our main motivations for adopting this framework are (1) to consider partially fuzzy regions and (2) to present a more expressive model in terms of the identification of topological relations between simple fuzzy regions. In this work, topological relations are specified through a 4-Intersection matrix that enumerates the topological subrelations between ($\tilde{A}_{min}$, $\tilde{B}_{min}$), ($\tilde{A}_{min}, \tilde{B}_{max}$), ($\tilde{A}_{max}$, $\tilde{B}_{min}$), and ($\tilde{A}_{max}, \tilde{B}_{max}$). The topological relations used here are those defined by the 9-Intersection model (Egenhofer and Herring 1990). We distinguish 242 topological relations that we classify into 8 overlapping basic clusters. Each cluster has four membership levels (or subclusters): *completely*, *strongly*, *fairly,* and *weakly*. Nevertheless, our model is not able to quantify the gradual change inside the maximal extent in the same way as the fuzzy approaches do (Zhan 1997, Schneider 2001, Dilo 2006).

The Egg-Yolk model (Cohn and Gotts 1996) was our main inspiration to develop this framework for identifying topological relations. However, there are some fundamental differences between our model and that of (Cohn and Gotts 1996). For instance, the topological relations used in (Cohn and Gotts 1996) are those defined in the RCC-5 model (Randell and Cohn 1989, Cohn *et al.* 1997). In contrast, our model is based on point-set

topology, in which points and lines are considered as the basic crisp spatial object types. In (Cohn and Gotts 1996), a conceptual neighborhood graph was drawn with 44 topological relations are classified into 13 clusters. In our model, we define a hierarchical classification based on the content of the matrices we use to identify the topological relations.

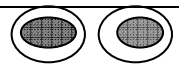
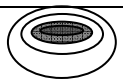
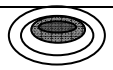
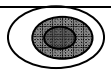
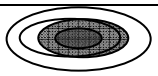
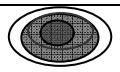
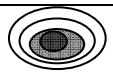
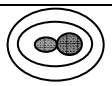
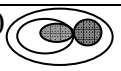
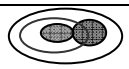
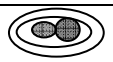
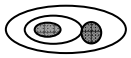
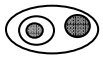
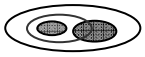
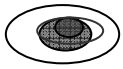
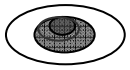
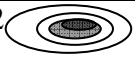
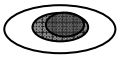
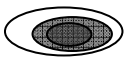
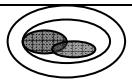
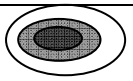
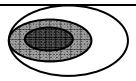
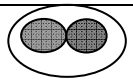
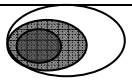
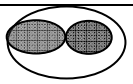
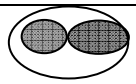
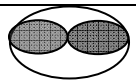
8. Conclusions and future work

Shape fuzziness is an inherent property of many spatial objects like lakes, valleys, and mountains. In GIS and spatial databases, it is a general practice to neglect shape fuzziness and formally represent fuzzy spatial objects as crisp geometries. Using such inappropriate representations can provide a source of spatial data quality degradation, because the reliability of spatial data is decreased. With emergence of prediction applications, data integration, and strategic decisional needs, researchers are increasingly more motivated to propose different methods for the formal representation of spatial fuzziness. A review of the literature regarding this topic proves that existing exact models do not permit the representation of *partially fuzzy objects*. For such objects, shape fuzziness partially characterizes one or several of its topological invariants. For example, a lake can have *rocker* and *swamp* banks at the same time; the boundary is fuzzy only for the *swamp* part. In this work, we have proposed an exact model in order to represent spatial objects that can be: *crisp*, *partially fuzzy*, or *completely fuzzy*. We have considered this categorization of fuzziness during the identification of topological relations.

More specifically, this paper contributes in three main ways. Based on point-set topology, we firstly define three basic *fuzzy spatial object types*: *fuzzy point*, *fuzzy line,* and *fuzzy region*. Each one of them is typically defined as a minimal extent $\tilde{A}_{\min}$ and a maximal extent $\tilde{A}_{\max}$, and these extents must verify some topological conditions in order to be valid. This model permits the representation of partially fuzzy spatial objects considered as invalid in the existing models of (Clementini and Di Felice 1997, Xinming 2004, Reis *et al.* 2006). Then, we identify a topological relation through use of a 4-Intersection matrix that permits the enumeration of four subrelations: R_1 ($\tilde{A}_{\min}, \tilde{B}_{\min}$), R_2 ($\tilde{A}_{\min}, \tilde{B}_{\max}$), R_3 ($\tilde{A}_{\max}, \tilde{B}_{\min}$), and R_4 ($\tilde{A}_{\max}, \tilde{B}_{\max}$). By using this formalism for simple fuzzy regions, 242 relations can be distinguished (cf. appendix). In order to retain our propositions useful in practice, we propose the clustering of these topological relations. A topological relation can belong to one or several clusters with various levels of membership: *completely*, *strongly*, *fairly,* and *weakly*. The objective of this qualitative clustering is to improve the specification of spatial queries and integrity constraints involving fuzzy spatial objects.

The main perspective of this work aims to improve the logical consistency of spatial databases involving fuzzy spatial objects. More specifically, we are interested in the specification of integrity constraints in spatial databases storing fuzzy spatial objects. We hope to identify both integrity constraint categories and the requirements for their formal expression. The framework presented earlier can provide a basis for the extension of a formal constraint language like OCL (Duboisset *et al.* 2007) to express *tolerant integrity constraints* for fuzzy spatial objects. In the same way, decisional applications require the integration of spatial data from heterogeneous sources before they are stored in a spatial data warehouse (Bédard *et al.* 2007). The main difficulty lies in choosing one of the available representations. We suggest merging the different representations in such way that the result looks like a fuzzy spatial object. The fuzzy integrity constraints can be used to increase the logical consistency of such data.

Appendix: 242 topological relations between fuzzy regions and required rules to deduce them

1  Disjoint Disjoint / Disjoint Disjoint	2  Contains Inside / Contains Contains	3  Contains Covered by / Contains Contains	4  Contains Equal / Contains Contains
5  Contains Overlap / Contains Contains	6  Contains Contains / Contains Contains	7  Contains Covers / Contains Contains	8  Inside Inside / Contains Contains
9  Meet Inside / Contains Contains	10  Meet Meet / Contains Contains	11  Meet Overlap / Contains Contains	12  Meet Covered by / Contains Contains
13  Disjoint Inside / Contains Contains	14  Disjoint Covered by / Contains Contains	15  Disjoint Meet / Contains Contains	16  Disjoint Disjoint / Contains Contains
17  Disjoint Overlap / Contains Contains	18  Covers Inside / Contains Contains	19  Covers Overlap / Contains Contains	20  Covers Covers / Contains Contains
21  Covers Covered by / Contains Contains	22  Covers Inside / Contains Contains	23  Inside Covered by / Contains Contains	24  Equal Equal / Contains Contains
25  Covers Equal / Contains Contains	26  Equal Covered by / Contains Contains	27  Overlap Overlap / Contains Contains	28  Overlap Inside / Contains Contains
29  Equal Inside / Contains Contains	30  Equal Covered by / Covers Equal	31  Equal Inside / Contains Equal	32  Contains Inside / Contains Equal
33  Overlap Covered by / Contains Contains	34  Inside Inside / Contains Equal	35  Inside Inside / Covers Equal	36  Meet Inside / Contains Equal
37  Contains Covered by / Contains Equal	38  Meet Inside / Covers Equal	39  Meet Covered by / Contains Equal	40  Meet Covered by / Covers Equal
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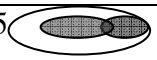
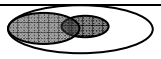
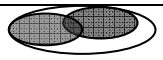
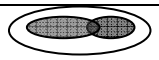
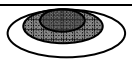
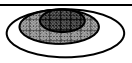
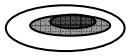
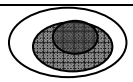
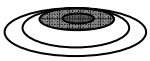
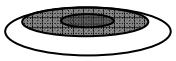
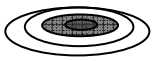
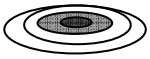
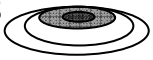
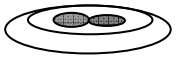
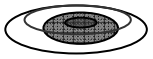
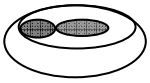
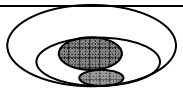
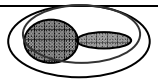
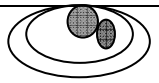
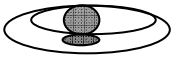
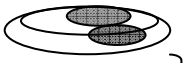
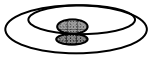
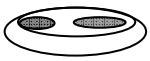
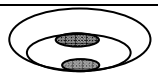
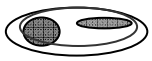
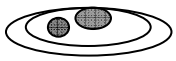
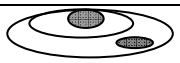
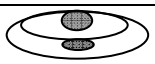
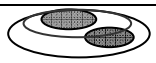
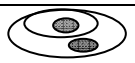
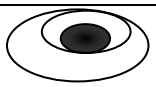
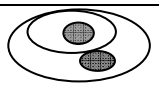
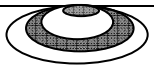
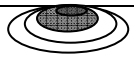
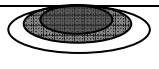
45  Overlap Covered by Contains Equal	46  Overlap Inside Covers Equal	47  Overlap Covered by Covers Equal	48  Overlap Inside Contains Equal
49  Covers Covered by Covers Equal	50  Covered by Covered by Covers Equal	51  Covered by Inside Contains Equal	52  Covers Inside Contains Equal
53  Contains Covered by Contains Covers	54  Contains Covers Contains Covers	55  Contains Equal Contains Covers	56  Contains Inside Contains Covers
57  Inside Inside Contains Covers	58  Inside Inside Covers Covers	59  Meet Inside Contains Covers	60  Contains Overlap Contains Covers
61  Meet Covered by Contains Covers	62  Meet Covered by Covers Covers	63  Meet Overlap Contains Covers	64  Meet Inside Covers Covers
65  Meet Meet Covers Covers	66  Meet Overlap Covers Covers	67  Disjoint Inside Contains Covers	68  Meet Meet Contains Covers
69  Disjoint Covered by Contains Covers	70  Disjoint Covered by Covers Covers	71  Disjoint Overlap Contains Covers	72  Disjoint Inside Covers Covers
73  Disjoint Disjoint Contains Covers	74  Disjoint Meet Covers Covers	75  Disjoint Overlap Covers Covers	76  Disjoint Disjoint Contains Covers
77  Equal Covered by Contains Covers	78  Equal Covered by Covers Covers	79  Equal Inside Contains Covers	80  Disjoint Meet Contains Covers
81  Covers Covers Covers Covers	82  Covers Inside Contains Covers	83  Covers Covered by Covers Covers	84  Covers Equal Covers Covers
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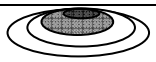
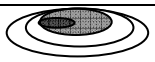
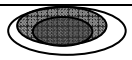
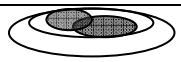
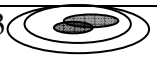
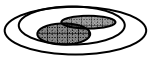
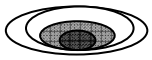
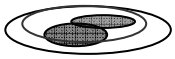
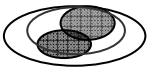
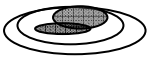
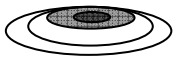
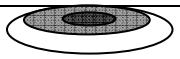
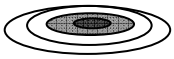
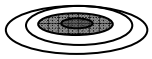
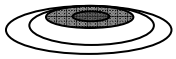
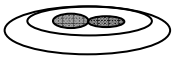
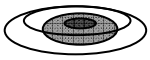
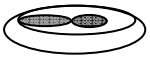
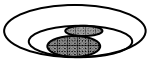
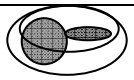
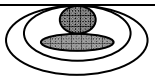
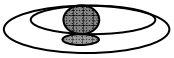
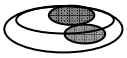
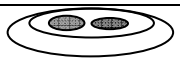
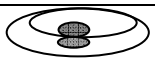
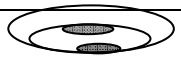
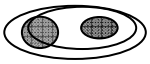
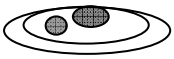
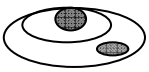
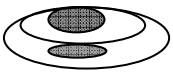
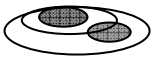
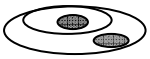
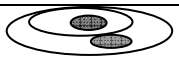
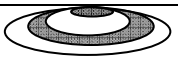
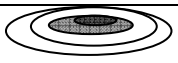
89  Covered by Covered by Covers Covers	90  Covered by Inside Covers Covers	91  Covers Equal Contains Covers	92  Overlap Covered by Covers Covers
93  Overlap Inside Contains Covers	94  Overlap Covered by Contains Covers	95  Covered by Covered by Contains Covers	96  Overlap Overlap Contains Covers
97  Overlap Overlap Covers Covers	98  Overlap Inside Covers Covers	99  Inside Inside Covers Covered by	100  Inside Inside Covered by Covered by
101  Inside Inside Equal Covered by	102  Inside Inside Contains Covered by	103  Contains Inside Contains Covered by	104  Contains Covered by Contains Covered by
105  Meet Inside Contains Covered by	106  Inside Inside Overlap Covered by	107  Meet Inside Covers Covered by	108  Meet Covered by Covers Covered by
109  Meet Inside Overlap Covered by	110  Meet Covered by Contains Covered by	111  Meet Covered by Meet Covered by	112  Meet Covered by Overlap Covered by
113  Disjoint Inside Contains Covered by	114  Meet Inside Meet Covered by	115  Disjoint Inside Covers Covered by	116  Disjoint Covered by Covers Covered by
117  Disjoint Inside Overlap Covered by	118  Disjoint Covered by Contains Covered by	119  Disjoint Covered by Disjoint Covered by	120  Disjoint Covered by Meet Covered by
121  Disjoint Covered by Overlap Covered by	122  Disjoint Inside Disjoint Covered by	123  Equal Inside Covers Covered by	124  Equal Covered by Covers Covered by
125  Equal Inside Contains Covered by	126  Disjoint Inside Meet Covered by	127  Covered by Covered by Covered by Covered by	128  Covered by Inside Contains Covered by
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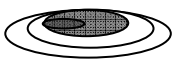
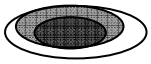
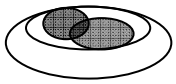
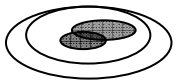
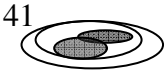
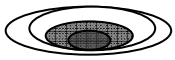
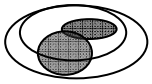
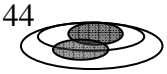
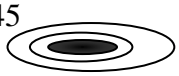
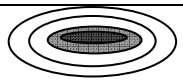
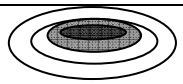
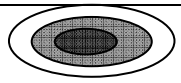
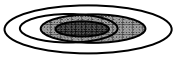
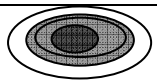
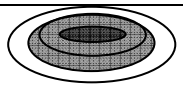
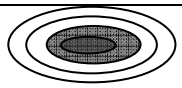
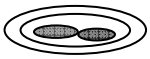
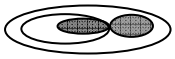
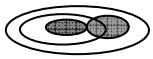
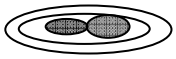
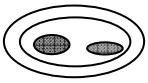
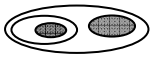
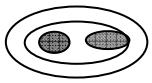
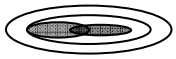
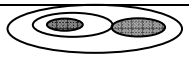
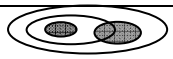
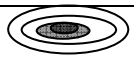
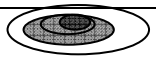
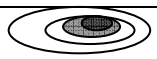
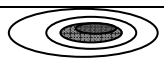
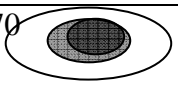
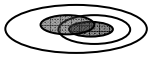
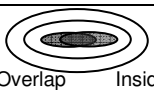
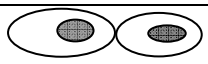
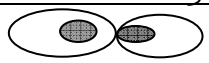
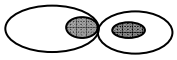
137  Covers Covered by Contains Covered by	138  Covered by Inside Equal Covered by	139  Overlap Covered by Covers Covered by	140  Overlap Inside Contains Covered by
141  Overlap Inside Covers Covered by	142  Covers Inside Covers Covered by	143  Overlap Inside Overlap Covered by	144  Overlap Covered by Overlap Covered by
145  Equal Inside Contains Inside	146  Inside Inside Contains Inside	147  Inside Inside Covers Inside	148  Inside Inside Equal Inside
149  Inside Inside Overlap Inside	150  Inside Inside Inside Inside	151  Inside Inside Covered by Inside	152  Contains Inside Contains Inside
153  Meet Inside Contains Inside	154  Meet Inside Meet Inside	155  Meet Inside Overlap Inside	156  Meet Inside Covers Inside
157  Disjoint Inside Contains Inside	158  Disjoint Inside Disjoint Inside	159  Disjoint Inside Covers Inside	160  Overlap Inside Covers Inside
161  Disjoint Inside Meet Inside	162  Disjoint Inside Overlap Inside	163  Covered by Inside Contains Inside	164  Covered by Inside Overlap Inside
165  Covered by Inside Covered by Inside	166  Covered by Inside Covers Inside	167  Covers Inside Contains Inside	168  Covers Inside Covers Inside
169  Equal Inside Equal Inside	170  Covered by Inside Equal Inside	171  Equal Inside Covers Inside	172  Overlap Inside Overlap Inside
173  Overlap Inside Contains Inside	174  Disjoint Disjoint Disjoint Meet	175  Disjoint Meet Disjoint Meet	176  Disjoint Disjoint Meet Meet
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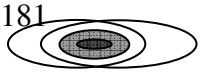
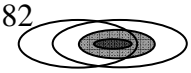
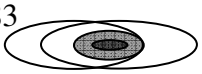
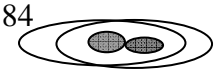
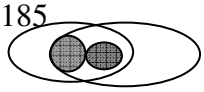
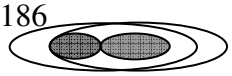
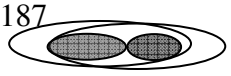
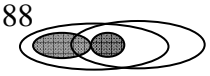
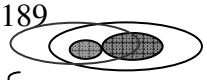
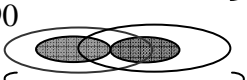
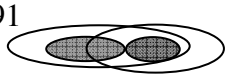
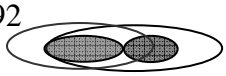
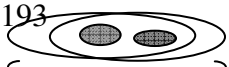
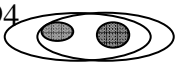
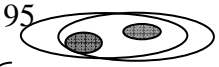
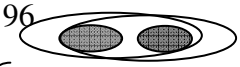
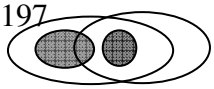
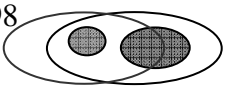
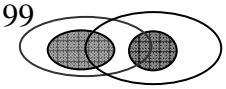
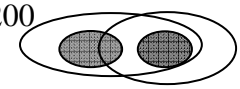
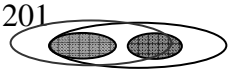
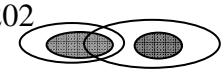
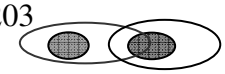
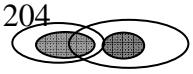
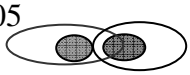
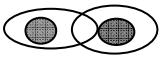
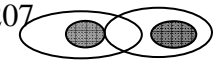
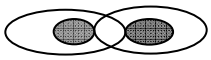
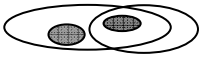
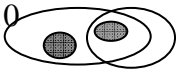
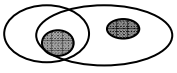
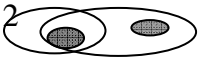
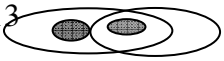
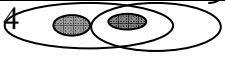
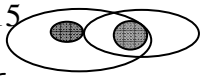
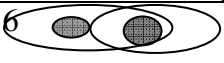
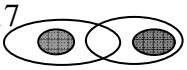
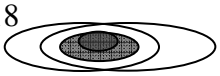
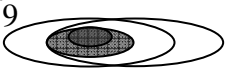
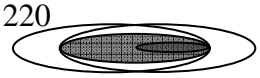
181  Inside Inside Contains Overlap	182  Inside Inside Overlap Overlap	183  Inside Inside Covers Overlap	184  Meet Inside Contains Overlap
185  Meet Inside Covers Overlap	186  Meet Covered by Contains Overlap	187  Meet Covered by Covers Overlap	188  Meet Inside Overlap Overlap
189  Meet Overlap Contains Overlap	190  Meet Overlap Overlap Overlap	191  Meet Covered by Overlap Overlap	192  Meet Overlap Covers Overlap
193  Disjoint Inside Contains Overlap	194  Disjoint Inside Covers Overlap	195  Disjoint Covered by Contains Overlap	196  Disjoint Covered by Covers Overlap
197  Disjoint Inside Overlap Overlap	198  Disjoint Overlap Contains Overlap	199  Disjoint Overlap Overlap Overlap	200  Disjoint Covered by Overlap Overlap
201  Disjoint Overlap Covers Overlap	202  Disjoint Disjoint Overlap Overlap	203  Disjoint Overlap Disjoint Overlap	204  Disjoint Meet Overlap Overlap
205  Disjoint Overlap Meet Overlap	206  Disjoint Meet Disjoint Overlap	207  Disjoint Disjoint Meet Overlap	208  Disjoint Meet Meet Overlap
209  Disjoint Inside Disjoint Overlap	210  Disjoint Disjoint Contains Overlap	211  Disjoint Disjoint Covers Overlap	212  Disjoint Covered by Disjoint Overlap
213  Disjoint Meet Contains Overlap	214  Disjoint Inside Meet Overlap	215  Disjoint Meet Covers Overlap	216  Disjoint Covered by Meet Overlap
217  Disjoint Disjoint Disjoint Overlap	218  Covers Inside Contains Overlap	219  Covers Covered by Contains Overlap	220  Covers Covered by Covers Overlap
221 Covers Overlap Contains Overlap	222 Covers Overlap Covers Overlap	223 Covers Covered by Contains Overlap	224 Covered by Covered by Overlap Overlap

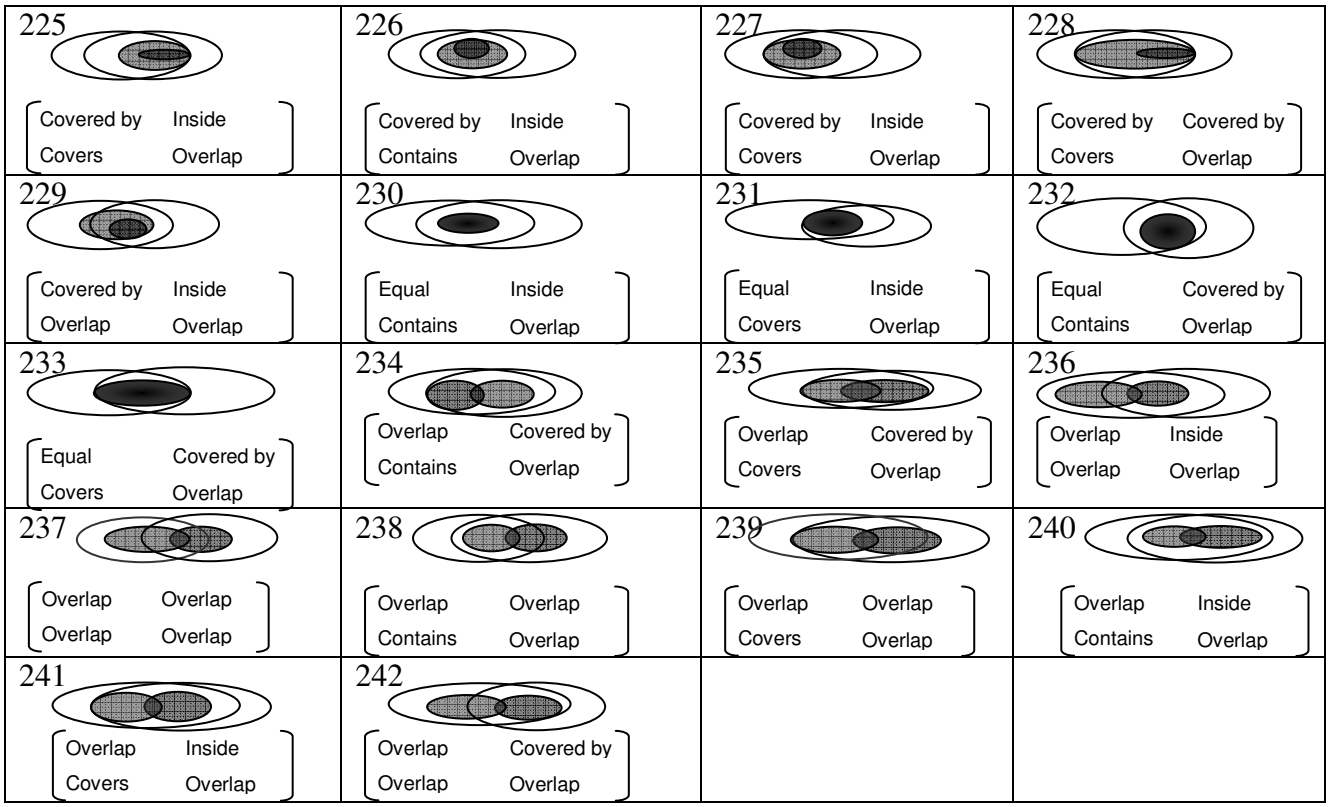


Figure 24. 242 topological relations between fuzzy regions

Rules

Table 4. Required rules for topological relations between fuzzy regions

Rule 1: Let $\tilde{A}$ and $\tilde{B}$ are two simple fuzzy regions, if ***Disjoint*** ($\tilde{A}_{max}, \tilde{B}_{max}$) then *Disjoint* ($\tilde{A}_{min}$, $\tilde{B}_{min}$).	$\begin{array}{c} \\ \tilde{A}_{min} \\ \tilde{A}_{max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{min} & \tilde{B}_{max} \end{array} \\ \begin{bmatrix} D(\tilde{A}_{min}, \tilde{B}_{min}) & -- \\ -- & D(\tilde{A}_{max}, \tilde{B}_{max}) \end{bmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Disjoint* ($\tilde{A}_{max}, \tilde{B}_{max}$). Now, we suppose that *Disjoint* ($\tilde{A}_{min}$, $\tilde{B}_{min}$) $\neq$ Disjoint. In this case, the relation between minimal extent $\tilde{A}_{min}$ and maximal extent $\tilde{A}_{max}$ of a fuzzy region $\tilde{A}$ or that between $\tilde{B}_{max}$ and $\tilde{B}_{min}$ does not correspond to *Contains*, *Covers, Equal*. Thus, there is a ***contradiction with definition 1***.	
Rule 2: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Meet*** ($\tilde{A}_{max}, \tilde{B}_{max}$) then $R(\tilde{A}_{min}, \tilde{B}_{min}) \in \{D, M\}$.	$\begin{array}{c} \\ \tilde{A}_{min} \\ \tilde{A}_{max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{min} & \tilde{B}_{max} \end{array} \\ \begin{bmatrix} R(\tilde{A}_{min}, \tilde{B}_{min}) \in \{D, M\} & -- \\ -- & M(\tilde{A}_{max}, \tilde{B}_{max}) \end{bmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Meet* ($\tilde{A}_{max}, \tilde{B}_{max}$). Now, we suppose that $R(\tilde{A}_{min}, \tilde{B}_{min}) \notin \{$*Disjoint*, *Meet*$\}$. In this case, relation between minimal extent $\tilde{A}_{min}$ and maximal extent $\tilde{A}_{max}$, *R'* ($\tilde{A}_{max}, \tilde{A}_{min}$) or that between $\tilde{B}_{max}$ and $\tilde{B}_{min}$, *R''*($\tilde{B}_{max}, \tilde{B}_{min}$) does not correspond to *Contains*, *Covers, Equal*. Thus, there is a ***contradiction with definition 1***.	

Rule 3: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Contains*** $(\tilde{A}_{max}, \tilde{B}_{max})$ then *Contains* $(\tilde{A}_{max}, \tilde{B}_{min})$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	--	--
$\tilde{A}_{max}$	$C(\tilde{A}_{max}, \tilde{B}_{min})$	$C(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains* $(\tilde{A}_{max}, \tilde{B}_{max})$. According to definition 1, any fuzzy region $\tilde{A}$ should respect the principal following condition: *Equal*$(\tilde{A}_{max}, \tilde{A}_{min})$, *Contains*$(\tilde{A}_{max}, \tilde{A}_{min})$ *or Covers*$(\tilde{A}_{max}, \tilde{A}_{min})$. Moreover, Contains is a transitive topological relation according to *Proposition 1*. Then, since *Contains* $(\tilde{A}_{max}, \tilde{B}_{max})$ and $R(\tilde{B}_{max}, \tilde{B}_{min}) = \{Contains, Covers, Equal\}$ then *Contains* $(\tilde{A}_{max}, \tilde{B}_{min})$ and vice versa.

Rule 4: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Covers*** $(\tilde{A}_{max}, \tilde{B}_{max})$ then $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Contains, Covers\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	--	--
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{C, CV\}$	$C(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Covers* $(\tilde{A}_{max}, \tilde{B}_{max})$. According to definition 1, any fuzzy region $\tilde{A}$ should respect the principal following condition: *Equal*$(\tilde{A}_{max}, \tilde{A}_{min})$, *Contains*$(\tilde{A}_{max}, \tilde{A}_{min})$ *or Covers*$(\tilde{A}_{max}, \tilde{A}_{min})$. Moreover, *Covers* is a transitive topological relation according to Proposition 1. By considering Proposition 2, if *Contains* $(\tilde{B}_{max}, \tilde{B}_{min})$ then *Contains* $(\tilde{A}_{max}, \tilde{B}_{min})$ else if $R(\tilde{B}_{max}, \tilde{B}_{min}) \in \{Covers, Equal\}$ then *Covers* $(\tilde{A}_{max}, \tilde{B}_{min})$ and vice versa.

Rule 5: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Equal*** $(\tilde{A}_{max}, \tilde{B}_{max})$ then $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Contains, Covers, Equal\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	--	--
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{C, CV, E\}$	$E(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Equal* $(\tilde{A}_{max}, \tilde{B}_{max})$. According to definition 1, any fuzzy region $\tilde{A}$ should respect the principal following condition: *Equal*$(\tilde{A}_{max}, \tilde{A}_{min})$, *Contains*$(\tilde{A}_{max}, \tilde{A}_{min})$ *or Covers*$(\tilde{A}_{max}, \tilde{A}_{min})$. Moreover, Equal is a transitive topological relation according to Proposition 1. By considering Proposition 2, if *Contains* $(\tilde{B}_{max}, \tilde{B}_{min})$ then *Contains* $(\tilde{A}_{max}, \tilde{B}_{min})$ (1) else if $R(\tilde{B}_{max}, \tilde{B}_{min}) \in \{Covers, Equal\}$ then $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Covers, Equal\}$ (2). Then, $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Contains, Covers, Equal\}$.

Rule 6: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Contains*** $(\tilde{A}_{max}, \tilde{B}_{max})$ **and** ***Contains*** $(\tilde{A}_{min}, \tilde{B}_{min})$ then $R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{Meet, Equal\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$C(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{D, M\}$
$\tilde{A}_{max}$	--	$C(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains* $(\tilde{A}_{max}, \tilde{B}_{max})$ and *Contains* $(\tilde{A}_{min}, \tilde{B}_{min})$. According to definition 1, we have *Equal*$(\tilde{A}_{max}, \tilde{A}_{min})$, *Contains*$(\tilde{A}_{max}, \tilde{A}_{min})$ *or Covers*$(\tilde{A}_{max}, \tilde{A}_{min})$. We suppose now that *Disjoint* $(\tilde{A}_{min}, \tilde{B}_{max})$ or *Meet* $(\tilde{A}_{min}, \tilde{B}_{max})$ (1). By considering definition 1 and proposition 2, since $R(\tilde{B}_{max}, \tilde{B}_{min})\{Contains, Covers, Equal\}$ then *Contains* $(\tilde{A}_{max}, \tilde{B}_{min})$ (2). In addition, since *Contains* $(\tilde{A}_{min},$

$\tilde{B}_{min}$) and (1) then R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\neq$ {*Contains*, *Covers*, *Equal*}. Thus, there is a contradiction with definition 1.

Rule 7: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Contains*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Inside***($\tilde{A}_{min}$, $\tilde{B}_{min}$) then *Inside* ($\tilde{A}_{min}, \tilde{B}_{max}$) and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$I(\tilde{A}_{min}, \tilde{B}_{min})$	$I(\tilde{A}_{min}, \tilde{B}_{max})$
$\tilde{A}_{max}$	--	$C(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Inside* ($\tilde{A}_{min}$, $\tilde{B}_{min}$). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Inside*} (1). By considering definition 1 and proposition 2, since R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\in$ {*Contains*, *Covers*, *Equal*} and *Inside* ($\tilde{A}_{min}, \tilde{B}_{min}$) then *Inside* ($\tilde{A}_{min}, \tilde{B}_{max}$) (2). Thus, there is contradiction among (1) and (2).

Rule 8: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Contains*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Meet***($\tilde{A}_{min}$, $\tilde{B}_{min}$) then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Contains*, *Equal*, *Covers*, *Disjoint*} and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$M(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{C, E, CV, D\}$
$\tilde{A}_{max}$	--	$C(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Meet* ($\tilde{A}_{min}$, $\tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\in$ {*Contains*, *Equal*, *Covers*, *Disjoint*} (2). By considering definition 1 and proposition 2, if *Contains* ($\tilde{A}_{min}, \tilde{B}_{max}$) then there is a contradiction because R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\in$ {*Contains*, *Covers*, *Equal*} and (1). If *Equal* ($\tilde{A}_{min}, \tilde{B}_{max}$) then there is a contradiction because R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\in$ {*Contains*, *Covers*, *Equal*} and (1). If Covers($\tilde{A}_{min}, \tilde{B}_{max}$) then R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\notin$ {*Contains*, *Covers*, *Equal*}or (1) is *false*. Finally, if *Disjoint* ($\tilde{A}_{min}, \tilde{B}_{max}$) then there is a contradiction because R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\in$ {*Contains*, *Covers*, *Equal*} and (1). Thus, (2) cannot be true.

Rule 9: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Contains*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Covers***($\tilde{A}_{min}$, $\tilde{B}_{min}$) then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Meet*, *Disjoint*} and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$CV(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{M, D\}$
$\tilde{A}_{max}$	--	$C(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Covers* ($\tilde{A}_{min}$, $\tilde{B}_{min}$). We suppose now that *Disjoint* ($\tilde{A}_{min}, \tilde{B}_{max}$) or *Meet* ($\tilde{A}_{min}, \tilde{B}_{max}$) (1). By considering definition 1 and proposition 2, since R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\in$ {*Contains*, *Covers*, *Equal*} then *Contains* ($\tilde{A}_{max}, \tilde{B}_{min}$) (2). In addition, since *Covers* ($\tilde{A}_{min}, \tilde{B}_{min}$) and (1) then R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\notin$ {*Contains*, *Covers*, *Equal*}. Thus, there is a contradiction with definition 1.

Rule 10: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Contains*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Equal*** ($\tilde{A}_{min}, \tilde{B}_{min}$) then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Contains*, *Covers*, *Disjoint, Meet, Overlap*}and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$E(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{C, CV, D, M, O\}$
$\tilde{A}_{max}$	--	$C(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Equal* ($\tilde{A}_{min}$, $\tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\in$ {*Contains*, *Covers*, *Disjoint, Meet, Overlap*} (2). If (2) then R

($\tilde{B}_{max}, \tilde{B}_{min}$) $\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.	
Rule 11: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Contains*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Covered by***($\tilde{A}_{min}$, $\tilde{B}_{min}$) then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\in$ {*Covered by*, *Inside*} and vice versa.	$\begin{array}{c} \\ \tilde{A}_{min} \\ \tilde{A}_{max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{min} & \tilde{B}_{max} \end{array} \\ \begin{bmatrix} E(\tilde{A}_{min}, \tilde{B}_{min}) & R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{CVB, I\} \\ -- & C(\tilde{A}_{max}, \tilde{B}_{max}) \end{bmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Covered by*($\tilde{A}_{min}$, $\tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Covered by, Inside*} then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\in$ {*Contains*, *Covers*, *Disjoint, Meet, Overlap, Equal*} (2). If (2) then R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.	
Rule 12: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Contains*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Overlap***($\tilde{A}_{min}$, $\tilde{B}_{min}$) then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\in$ {*Covered by*, *Inside, Overlap*} and vice versa.	$\begin{array}{c} \\ \tilde{A}_{min} \\ \tilde{A}_{max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{min} & \tilde{B}_{max} \end{array} \\ \begin{bmatrix} O(\tilde{A}_{min}, \tilde{B}_{min}) & R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{CVB, I, O\} \\ -- & C(\tilde{A}_{max}, \tilde{B}_{max}) \end{bmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Contains*($\tilde{A}_{max}, \tilde{B}_{max}$) and *Overlap* ($\tilde{A}_{min}$, $\tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\in$ {*Contains*, *Covers*, *Disjoint, Meet, Equal*} (2). If (2) then R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.	
Rule 13: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Covers*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Contains***($\tilde{A}_{min}$, $\tilde{B}_{min}$) then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Disjoint*, *Meet*} and vice versa.	$\begin{array}{c} \\ \tilde{A}_{min} \\ \tilde{A}_{max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{min} & \tilde{B}_{max} \end{array} \\ \begin{bmatrix} C(\tilde{A}_{min}, \tilde{B}_{min}) & R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{D, M\} \\ -- & CV(\tilde{A}_{max}, \tilde{B}_{max}) \end{bmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Covers* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Contains* ($\tilde{A}_{min}, \tilde{B}_{min}$) (1). We suppose now that *Disjoint* ($\tilde{A}_{min}, \tilde{B}_{max}$) or *Meet* ($\tilde{A}_{min}, \tilde{B}_{max}$) (2). If (2) then R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.	
Rule 14: Let $\tilde{A}$ and $\tilde{B}$ two fuzzy regions, if ***Covers*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Inside***($\tilde{A}_{min}, \tilde{B}_{min}$) then *Inside*($\tilde{A}_{min}$, $\tilde{B}_{max}$) and vice versa.	$\begin{array}{c} \\ \tilde{A}_{min} \\ \tilde{A}_{max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{min} & \tilde{B}_{max} \end{array} \\ \begin{bmatrix} I(\tilde{A}_{min}, \tilde{B}_{min}) & I(\tilde{A}_{min}, \tilde{B}_{max}) \\ -- & CV(\tilde{A}_{max}, \tilde{B}_{max}) \end{bmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Covers* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Inside* ($\tilde{A}_{min}, \tilde{B}_{min}$). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Inside*} (1). By considering definition 1 and proposition 2, since R ($\tilde{B}_{max}, \tilde{B}_{min}$) $\in$ {*Contains*, *Covers*, *Equal*} and *Inside* ($\tilde{A}_{min}, \tilde{B}_{min}$) then *Inside* ($\tilde{A}_{min}, \tilde{B}_{max}$) (2). Thus, (1) cannot be true.	
Rule 15: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Covers*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***R***($\tilde{A}_{min}$, $\tilde{B}_{min}$) $\in$ **{*Disjoint*, *Meet*}** then R ($\tilde{A}_{min}, \tilde{B}_{max}$) $\notin$ {*Contains*, *Covers*, *Disjoint, Equal*} and vice versa.	$\begin{array}{c} \\ \tilde{A}_{min} \\ \tilde{A}_{max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{min} & \tilde{B}_{max} \end{array} \\ \begin{bmatrix} R(\tilde{A}_{min}, \tilde{B}_{min}) \in \{D, M\} & R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{C, CV, D, E\} \\ -- & CV(\tilde{A}_{max}, \tilde{B}_{max}) \end{bmatrix} \end{array}$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Covers* ($\tilde{A}_{max}, \tilde{B}_{max}$) and {*Meet, Disjoint*} ($\tilde{A}_{min}, \tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$)$\in$ {*Contains*, *Covers*, *Disjoint, Equal*} (2). If (2) then R ($\tilde{B}_{max}, \tilde{B}_{min}$)$\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.

Rule 16: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Covers*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***R***($\tilde{A}_{min}$, $\tilde{B}_{min}$)$\in$ **{*Equal*, Covered by}** then $R(\tilde{A}_{min}, \tilde{B}_{max})\in$ {*Covered by, Inside*} and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$R(\tilde{A}_{min}, \tilde{B}_{min})\in$ {*E, CVB*}	$R(\tilde{A}_{min}, \tilde{B}_{max})\in$ {*CVB, I*}
$\tilde{A}_{max}$	--	$CV(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Covers*($\tilde{A}_{max}, \tilde{B}_{max}$) and R ($\tilde{A}_{min}, \tilde{B}_{max}$)$\in$ {*Covered by, Inside*} (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$)$\notin$ {*Covered by*, *Inside*} (2). If (2) then R ($\tilde{B}_{max}, \tilde{B}_{min}$)$\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.

Rule 17: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Covers*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Overlap*** ($\tilde{A}_{min}$, $\tilde{B}_{min}$) then $R(\tilde{A}_{min}, \tilde{B}_{max})\in$ {*Covered by, Inside, Overlap*} and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$O(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max})\in$ {*CVB, I, O*}
$\tilde{A}_{max}$	--	$CV(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Covers* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Overlap* ($\tilde{A}_{min}, \tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$)$\notin$ {*Covered by*, *Inside, Overlap*} (2). If (2) then R ($\tilde{B}_{max}, \tilde{B}_{min}$)$\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (2) cannot be true.

Rule 18: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Meet*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Meet*** ($\tilde{A}_{min}$, $\tilde{B}_{min}$) then *Meet*($\tilde{A}_{min}, \tilde{B}_{max}$) and *Meet*($\tilde{A}_{max}, \tilde{B}_{min}$) and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$M(\tilde{A}_{min}, \tilde{B}_{min})$	$M(\tilde{A}_{min}, \tilde{B}_{max})$
$\tilde{A}_{max}$	$M(\tilde{A}_{max}, \tilde{B}_{min})$	$M(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Meet* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Meet* ($\tilde{A}_{min}$, $\tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$)$\neq$ *Meet* (2) and R ($\tilde{A}_{max}, \tilde{B}_{min}$)$\neq$ *Meet* (3). If (2) then R ($\tilde{B}_{max}, \tilde{B}_{min}$)$\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. Thus, (2) cannot be true. In the same way, if (3) then there is a contradiction because R ($\tilde{B}_{max}, \tilde{B}_{min}$)$\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a contradiction and (3) cannot be true.

Rule 19: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Meet*** ($\tilde{A}_{max}, \tilde{B}_{max}$) **and** ***Disjoint*** ($\tilde{A}_{min}$, $\tilde{B}_{min}$) then $R(\tilde{A}_{min}, \tilde{B}_{max})\in$ {*Meet, Disjoint*} and $R(\tilde{A}_{max}, \tilde{B}_{min})\in$ {*Meet, Disjoint*} and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$D(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max})\in$ {*M, D*}
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min})\in$ {*M, D*}	$M(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Meet* ($\tilde{A}_{max}, \tilde{B}_{max}$) and *Disjoint* ($\tilde{A}_{min}$, $\tilde{B}_{min}$) (1). We suppose now that R ($\tilde{A}_{min}, \tilde{B}_{max}$)$\notin$ {*Meet, Disjoint*} (2) and R ($\tilde{A}_{max}, \tilde{B}_{min}$)$\notin$ {*Meet, Disjoint*} (3). If (2) then there R ($\tilde{B}_{max}, \tilde{B}_{min}$)$\notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and (1), there is a

contradiction and (2) cannot be true. In the same way, if (3) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and (1), there is a contradiction and (3) cannot be true.

Rule 20 : Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** $(\tilde{A}_{max}, \tilde{B}_{max})$ then $R(\tilde{A}_{max}, \tilde{B}_{min}) \notin \{Equal, Inside, Covered\ by\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	--	--
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \notin \{E, I, CVB\}$	$O(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Overlap* $(\tilde{A}_{max}, \tilde{B}_{max})$. According to definition 1, any fuzzy region $\tilde{A}$ should respect the principal following condition: *Equal*$(\tilde{A}_{max}, \tilde{A}_{min})$, *Contains*$(\tilde{A}_{max}, \tilde{A}_{min})$ *or* *Covers*$(\tilde{A}_{max}, \tilde{A}_{min})$ (1). We suppose now that $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Equal, Inside, Covered\ by\}$ (2). By considering definition 1 and proposition 2, if (1) and (2) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$. Thus, there is a contradiction with ***definition 1***.

Rule 21: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** $(\tilde{A}_{max}, \tilde{B}_{max})$ **and** ***Contains*** $(\tilde{A}_{min}, \tilde{B}_{min})$ then $R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{Overlap, Inside, Covered\ by\}$ and $Contains(\tilde{A}_{max}, \tilde{B}_{min})$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$C(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{O, I, CVB\}$
$\tilde{A}_{max}$	$C(\tilde{A}_{max}, \tilde{B}_{min})$	$O(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Overlap* $(\tilde{A}_{max}, \tilde{B}_{max})$ and *Contains* $(\tilde{A}_{min}, \tilde{B}_{min})$ (1). We suppose now that $R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{Overlap, Inside, Covered\ by\}$ (2) and $R(\tilde{A}_{max}, \tilde{B}_{min}) \neq$ Contains (3). By considering definition 1 and proposition 2, if (2) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. Thus, (2) cannot be true because there is a contradiction. In the same way, if (3) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and proposition 2, (3) cannot be true because there is also a contradiction.

Rule 22 : Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** $(\tilde{A}_{max}, \tilde{B}_{max})$ **and** ***R*** $(\tilde{A}_{min}, \tilde{B}_{min}) \in$ ***{Overlap, Meet}*** then $R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{Overlap, Inside, Covered\ by\}$ and $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Overlap, Covers, Contains\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$R(\tilde{A}_{min}, \tilde{B}_{min}) \in \{O, M\}$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{O, I, CVB\}$
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{O, CV, C\}$	$O(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Overlap* $(\tilde{A}_{max}, \tilde{B}_{max})$ and $R(\tilde{A}_{min}, \tilde{B}_{min}) \in \{Overlap, Meet\}$ (1). We suppose now that $R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{Overlap, Inside, Covered\ by\}$ (2) $R(\tilde{A}_{max}, \tilde{B}_{min}) \notin \{Overlap, Covers, Contains\}$ (3). If (2) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and proposition 2, there is a contradiction and (2) cannot be true. In the same way, if (3) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and proposition 2, there is contradiction and (3) cannot be true.

Rule 23: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** $(\tilde{A}_{max}, \tilde{B}_{max})$ **and** ***Equal*** $(\tilde{A}_{min}, \tilde{B}_{min})$ **then** ***R*** $(\tilde{A}_{min}, \tilde{B}_{max}) \in$ ***{Overlap, Inside, Covered by}*** and $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Overlap, Covers, Contains\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$E(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{I, CVB\}$
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{CV, C\}$	$O(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Overlap* ($\tilde{A}_{\max}, \tilde{B}_{\max}$) and *Equal* ($\tilde{A}_{\min}, \tilde{B}_{\min}$) (1). We suppose now that $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \notin$ {*Inside, Covered by*} (2) $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin$ {*Covers, Contains*} (3). If (2) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, there is contradiction and (2) cannot be true. In the same way, if (3) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, (3) cannot be true because there is also a contradiction.

Rule 24: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** ($\tilde{A}_{\max}, \tilde{B}_{\max}$) **and** ***Inside*** ($\tilde{A}_{\min}, \tilde{B}_{\min}$) then $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in$ {*Inside*} and $(\tilde{A}_{\max}, \tilde{B}_{\min}) \in$ {*Overlap, Covers, Contains*} and vice versa.

	$\tilde{B}_{\min}$	$\tilde{B}_{\max}$
$\tilde{A}_{\min}$	$I(\tilde{A}_{\min}, \tilde{B}_{\min})$	$I(\tilde{A}_{\max}, \tilde{B}_{\max})$
$\tilde{A}_{\max}$	$R(\tilde{A}_{\max}, \tilde{B}_{\min}) \in$ {*C, CV, O*}	$O(\tilde{A}_{\max}, \tilde{B}_{\max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Overlap* ($\tilde{A}_{\max}, \tilde{B}_{\max}$) and *Inside* ($\tilde{A}_{\min}, \tilde{B}_{\min}$) (1). We suppose now that $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \notin$ {*Inside*}(2) $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin$ {*Covers, Contains, Overlap*} (3). If (2) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, there is a contradiction and (2) cannot be true. In the same way, if (3) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, (3) cannot be true because there is also a contradiction.

Rule 25: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** ($\tilde{A}_{\max}, \tilde{B}_{\max}$) **and** ***Covers*** ($\tilde{A}_{\min}, \tilde{B}_{\min}$) then $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in$ {*Inside, Covered by, Overlap*} and $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \in$ {*Covers, Contains*} and vice versa.

	$\tilde{B}_{\min}$	$\tilde{B}_{\max}$
$\tilde{A}_{\min}$	$CV(\tilde{A}_{\min}, \tilde{B}_{\min})$	$R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in$ {*I, CVB, O*}
$\tilde{A}_{\max}$	$R(\tilde{A}_{\max}, \tilde{B}_{\min}) \in$ {*CV, C*}	$O(\tilde{A}_{\max}, \tilde{B}_{\max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Overlap* ($\tilde{A}_{\max}, \tilde{B}_{\max}$) and *Covers* ($\tilde{A}_{\min}, \tilde{B}_{\min}$) (1). We suppose now that $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \notin$ {*Inside, Covered by, Overlap*} (2) $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin$ {*Covers, Contains*} (3). If (2) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, (2) cannot be true because there is a contradiction. In the same way, if (3) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, (3) cannot be true because there is also a contradiction.

Rule 26: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** ($\tilde{A}_{\max}, \tilde{B}_{\max}$) **and** ***Disjoint*** ($\tilde{A}_{\min}, \tilde{B}_{\min}$) then $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \notin$ {*Equal, Contains, Covers*} and $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin$ {*Equal, Covered by, Inside*} and vice versa.

	$\tilde{B}_{\min}$	$\tilde{B}_{\max}$
$\tilde{A}_{\min}$	$D(\tilde{A}_{\min}, \tilde{B}_{\min})$	$R(\tilde{A}_{\min}, \tilde{B}_{\max}) \notin$ {*E, C, CV*}
$\tilde{A}_{\max}$	$R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin$ {*E, CVB, I*}	$O(\tilde{A}_{\max}, \tilde{B}_{\max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Overlap* ($\tilde{A}_{\max}, \tilde{B}_{\max}$) and *Disjoint* ($\tilde{A}_{\min}, \tilde{B}_{\min}$) (1). We suppose now that $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \notin$ {*Equal, Contains, Covers*} (2) $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin$ {*Equal, Covered by, Inside*} (3). If (2) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, (2) cannot be true because there is a contradiction. In the same way, if (3) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin$ {*Contains*, *Covers*, *Equal*} or (1) is false. By considering definition 1 and proposition 2, (3)

cannot be true because there is also a contradiction.

Rule 27: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** $(\tilde{A}_{max}, \tilde{B}_{max})$ **and** ***Covered by*** $(\tilde{A}_{min}, \tilde{B}_{min})$ then $R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{Inside, Covered\ by\}$ and $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Covers, Contains, Overlap\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$CV(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{CVB, O\}$
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{CV, C, O\}$	$Overlap(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where $Overlap(\tilde{A}_{max}, \tilde{B}_{max})$ and $Covers(\tilde{A}_{min}, \tilde{B}_{min})$ (1). We suppose now that $R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{Inside, Covered\ by\}$ (2) $R(\tilde{A}_{max}, \tilde{B}_{min}) \notin \{Covers, Contains, Overlap\}$ (3). If (2) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and proposition 2, (2) cannot be true because there is a contradiction. In the same way, if (3) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and proposition 2, (3) cannot be true because there is also a contradiction.

Rule 28: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Overlap*** $(\tilde{A}_{max}, \tilde{B}_{max})$ **and** ***Covers*** $(\tilde{A}_{min}, \tilde{B}_{min})$ then $R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{Inside\}$ and $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Covers, Contains, Overlap\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$CV(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{CVB, O\}$
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{CV, C, O\}$	$O(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where $Overlap(\tilde{A}_{max}, \tilde{B}_{max})$ and $Covers(\tilde{A}_{min}, \tilde{B}_{min})$ (1). We suppose now that $R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{Inside, Covered\ by\}$ (2) $R(\tilde{A}_{max}, \tilde{B}_{min}) \notin \{Covers, Contains, Overlap\}$ (3). If (2) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and (1), (2) cannot be true because there is a contradiction. In the same way, if (3) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and (1), (3) cannot be true because there is also a contradiction.

Rule 29: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Covers*** $(\tilde{A}_{max}, \tilde{B}_{max})$ **and** ***Covers*** $(\tilde{A}_{min}, \tilde{B}_{min})$ then $R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{Inside, Covered\ by, Equal, Overlap, Covers\}$ and $R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{Covers, Contains\}$ and vice versa.

	$\tilde{B}_{min}$	$\tilde{B}_{max}$
$\tilde{A}_{min}$	$CV(\tilde{A}_{min}, \tilde{B}_{min})$	$R(\tilde{A}_{min}, \tilde{B}_{max}) \in \{CVB, O, E, CV, I\}$
$\tilde{A}_{max}$	$R(\tilde{A}_{max}, \tilde{B}_{min}) \in \{CV, C\}$	$O(\tilde{A}_{max}, \tilde{B}_{max})$

Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where $Covers(\tilde{A}_{max}, \tilde{B}_{max})$ and $Covers(\tilde{A}_{min}, \tilde{B}_{min})$ (1). We suppose now that $R(\tilde{A}_{min}, \tilde{B}_{max}) \notin \{Covered\ by, Overlap, Equal, Covers, Inside\}$ (2) $R(\tilde{A}_{max}, \tilde{B}_{min}) \notin \{Covers, Contains\}$ (3). If (2) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false.). By considering definition 1 and (1), (2) cannot be true because there is a contradiction. In the same way, if (3) then $R(\tilde{B}_{max}, \tilde{B}_{min}) \notin \{Contains, Covers, Equal\}$ or (1) is false.). By considering definition 1 and (1), (3) cannot be true because there is also a contradiction.

Rule 30: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Rule 29*** **and** $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in$ ***{Covered by}*** Then $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \in \{Covers\}$ and vice versa.	$\begin{array}{c} \\ \tilde{A}_{\min} \\ \tilde{A}_{\max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{\min} & \tilde{B}_{\max} \end{array} \\ \begin{pmatrix} CV(\tilde{A}_{\min}, \tilde{B}_{\min}) & R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in \{CVB\} \\ R(\tilde{A}_{\max}, \tilde{B}_{\min}) \in \{CV, C\} & CV(\tilde{A}_{\max}, \tilde{B}_{\max}) \end{pmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Rule 29* and $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in \{Covered\ by\}$ (1). We suppose now that $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin \{Covers\}$ (2). By considering definition 1 and proposition 2, if (2) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. Thus, (2) cannot be true because there is a contradiction.	
Rule 31: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions, if ***Rule 30*** **and** $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in$ ***{Inside}*** **t**hen $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \in \{Contains\}$ and vice versa.	$\begin{array}{c} \\ \tilde{A}_{\min} \\ \tilde{A}_{\max} \end{array} \begin{array}{c} \begin{array}{cc} \tilde{B}_{\min} & \tilde{B}_{\max} \end{array} \\ \begin{pmatrix} CV(\tilde{A}_{\min}, \tilde{B}_{\min}) & R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in \{I\} \\ R(\tilde{A}_{\max}, \tilde{B}_{\min}) \in \{C\} & CV(\tilde{A}_{\max}, \tilde{B}_{\max}) \end{pmatrix} \end{array}$
Proof: Let $\tilde{A}$ and $\tilde{B}$ two simple fuzzy regions where *Rule 30* and $R(\tilde{A}_{\min}, \tilde{B}_{\max}) \in \{Inside\}$ (1). We suppose now that $R(\tilde{A}_{\max}, \tilde{B}_{\min}) \notin \{Contains\}$ (2). If (2) then $R(\tilde{B}_{\max}, \tilde{B}_{\min}) \notin \{Contains, Covers, Equal\}$ or (1) is false. By considering definition 1 and (1), (2) cannot be true because there is a contradiction.	

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Table 1. Relations' numbers

The relation between ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	Correspondent matrices
Disjoint ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	1
Contains ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	2→30
Equal ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	31→52
Covers ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	53→98
Covered by ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	99→144
Inside ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	145→173
Meet ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	174→177
Overlap ($\tilde{A}_{max}$, $\tilde{B}_{max}$)	178→242

Table 2. Clustering results

Cluster's name	Fuzziness level	Topological relations' numbers (cf. appendix)
DISJOINT	*Weakly*	13, 14, 15, 17, 41, 42, 43, 44, 67, 69, 70, 71, 72, 74, 75, 80, 113, 115, 116, 117, 118, 120, 121, 126, 157, 159, 161, 162, 193, 194, 195, 196, 197, 198, 199, 200, 201, 204, 205, 208, 213, 214, 215, 216
	Fairly	16, 73, 76, 119, 122, 158, 175, 176, 202, 203, 206, 207, 209, 210, 211, 212
	Strongly	174, 217
	Completely	1
CONTAINS	*Weakly*	31, 34, 36, 39, 43, 44, 45, 48, 51, 52, 57, 59, 61, 63, 67, 68, 69, 71, 73, 76, 77, 79, 80, 82, 85, 86, 88, 91, 93, 94, 95, 96, 102, 105, 110, 113, 118, 125, 128, 130, 135, 137, 140, 145, 146, 153, 157, 163, 167, 173, 181, 184, 186, 189, 193, 195, 198, 210, 213, 218, 219, 221, 223, 226, 230, 232, 234, 238, 240
	Fairly	8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32, 33, 37, 53, 54, 55, 56, 60, 103, 104, 152, 178, 179, 180
	Strongly	2, 3, 4, 5, 7
	Completely	6
EQUAL	*Weakly*	4, 25, 26, 29, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 55, 77, 78, 79, 84, 91, 101, 123, 124, 125, 131, 138, 145, 148, 170, 171, 230, 231, 232, 233
	Fairly	24, 30, 31, 169
	Strongly	
	Completely	
COVERS	*Weakly*	7, 18, 19, 21, 25, 30, 35, 38, 40, 41, 42, 46, 47, 50, 52, 53, 55, 56, 57, 59, 60, 61, 63, 67, 68, 69, 71, 73, 76, 77, 79, 80, 88, 93, 94, 95, 96, 99, 107, 108, 115, 116, 123, 124, 129, 132, 135, 137, 139, 141, 147, 156, 159, 160, 166, 167, 171, 183, 185, 187, 192, 194, 196, 201, 211, 215, 218, 219, 221, 223, 225, 227, 228, 231, 233, 235, 239, 241
	Fairly	20, 49, 54, 58, 62, 64, 65, 66, 70, 72, 74, 75, 78, 82, 85, 86, 89, 90, 91, 92, 97, 98, 136, 142, 168, 220, 222
	Strongly	83, 84, 87
	Completely	81
COVERED BY	*Weakly*	3, 12, 14, 21, 22, 23, 26, 30, 33, 37, 39, 40, 42, 44, 45, 47, 49, 51, 53, 61, 62, 69, 70, 77, 78, 83, 85, 88, 90, 92, 94, 99, 101, 102, 103, 105, 106, 107, 109, 113, 114, 115, 117, 122, 123, 125, 126, 135, 140, 141, 142, 143, 151, 163, 164, 166, 170, 180, 186, 187, 191, 195, 196, 200, 212, 216, 219, 220, 223, 225, 226, 227, 229, 232, 233, 234, 235, 242
	Fairly	50, 89, 95, 100, 104, 108, 110, 111, 112, 116, 118, 119, 120, 121, 124, 128, 130, 132, 133, 136, 137, 138, 139, 144, 165, 224, 228
	Strongly	129, 131, 134
	Completely	127

INSIDE	*Weakly*	2, 9, 13, 18, 22, 23, 28, 29, 31, 32, 36, 38, 41, 43, 46, 48, 51, 52, 56, 59, 64, 67, 72, 79, 82, 88, 90, 93, 98, 103, 105, 107, 109, 113, 114, 115, 117, 122, 123, 125, 126, 128, 132, 133, 135, 138, 140, 141, 142, 143, 178, 184, 185, 188, 193, 194, 197, 209, 214, 218, 225, 226, 227, 229, 230, 231, 236, 240, 241
	Fairly	8, 34, 35, 57, 58, 99, 100, 101, 102, 106, 145, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 181, 182, 183
	Strongly	146, 147, 148, 149, 151
	Completely	150
MEET	*Weakly*	9, 11, 12, 15, 36, 38, 39, 40, 59, 61, 62, 63, 64, 66, 74, 80, 105, 107, 108, 109, 110, 112, 120, 126, 153, 155, 156, 161, 174, 184, 185, 186, 187, 188, 189, 190, 191, 192, 204, 205, 206, 207, 213, 214, 215, 216
	Fairly	10, 65, 68, 111, 114, 154, 175, 176, 208
	Strongly	
	Completely	177
OVERLAP	*Weakly*	5, 11, 17, 19, 28, 33, 45, 46, 47, 48, 60, 63, 66, 71, 75, 86, 87, 92, 93, 94, 98, 109, 112, 117, 121, 130, 133, 134, 139, 140, 141, 149, 155, 160, 162, 164, 173, 178, 180, 181, 183, 184, 185, 186, 187, 193, 194, 195, 196, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 223, 225, 226, 227, 228, 230, 231, 232, 233
	Fairly	27, 96, 97, 143, 144, 172, 179, 182, 188, 189, 191, 192, 197, 198, 200, 201, 202, 203, 204, 205, 221, 222, 224, 229, 234, 235, 240, 241
	Strongly	190, 199, 236, 238, 239, 242
	Completely	237

Table 3. Result of query 3

P1.id	*P2.id*	*determine*
11	23	*Weakly overlap*
45	14	---
18	26	*Strongly Overlap*

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