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# Isolatedness of characteristic points at blow-up for a semilinear wave equation in one space dimension \*

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We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \quad (1)$$

where  $u(t) : x \in \mathbb{R} \rightarrow u(x, t) \in \mathbb{R}$ ,  $p > 1$ ,  $u_0 \in H_{loc,u}^1$  and  $u_1 \in L_{loc,u}^2$  with  $\|v\|_{L_{loc,u}^2}^2 = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$  and  $\|v\|_{H_{loc,u}^1}^2 = \|v\|_{L_{loc,u}^2}^2 + \|\nabla v\|_{L_{loc,u}^2}^2$ .

The Cauchy problem for equation (1) in the space  $H_{loc,u}^1 \times L_{loc,u}^2$  follows from the finite speed of propagation and the wellposedness in  $H^1 \times L^2$  (see Ginibre, Soffer and Velo [6]). If the solution is not global in time, then we call it a blow-up solution. The existence of blow-up solutions is guaranteed by ODE techniques together with the finite speed of propagation, or also by the following blow-up criterion from Levine [9]:

*If  $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$  satisfies*

$$\int_{\mathbb{R}} \left( \frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|\partial_x u_0(x)|^2 - \frac{1}{p+1}|u_0(x)|^{p+1} \right) dx < 0,$$

*then the solution of (1) cannot be global in time.*

More blow-up results can be found in Caffarelli and Friedman [5], [4], Alinhac [1] and Kichenassamy and Littman [7], [8].

If  $u$  is a blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve  $\Gamma = \{(x, T(x))\}$  such that  $u$  cannot be extended beyond the set called the maximal influence domain of  $u$ :

$$D = \{(x, t) \mid t < T(x)\}. \quad (2)$$

$\bar{T} = \inf_{x \in \mathbb{R}} T(x)$  and  $\Gamma$  are called the blow-up time and the blow-up graph of  $u$ .

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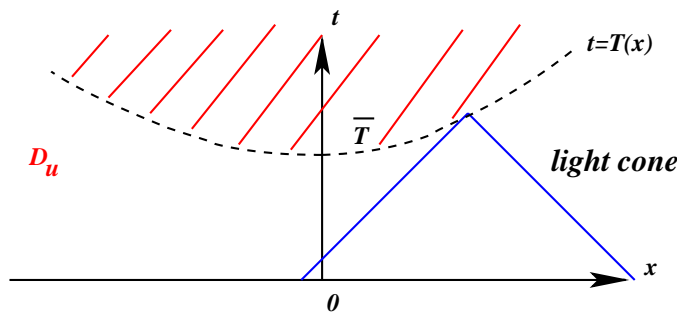


Fig. 1: The maximal influence domain of the solution

A point  $a$  is a non characteristic point (or a *regular* point) if there are  $\delta_0 \in (0,1)$  and  $t_0 < T(a)$  such that  $u$  is defined on  $\mathcal{C}_{a,T(a),\delta_0} \cap \{t \geq t_0\}$  where

$$\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x, t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}.$$

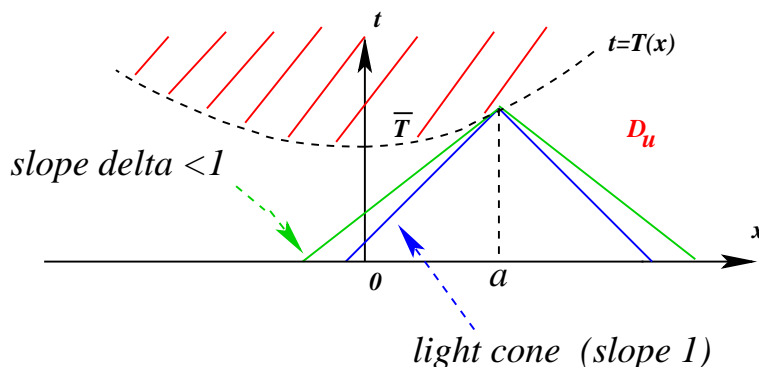


Fig. 2:  $a$  is a non characteristic point

If not, then we call  $a$  a characteristic point (or a *singular* point). Naturally, we denote by  $\mathcal{R}$  (resp.  $\mathcal{S}$ ) the set of non characteristic (resp. characteristic) points. Note then that

$$\mathcal{R} \cup \mathcal{S} = \mathbb{R}.$$

In our papers [14], [16], [15], [17], [18], [19] and [20], we made several contributions to the study of blow-up solutions of (1), namely the description of its blow-up graph and blow-up behavior in selfsimilar variables.

## 1 The blow-up graph of equation (1)

It is clear from rather simple arguments that  $\mathcal{R} \neq \emptyset$  for *any* blow-up solution  $u(x, t)$  (if  $T(x)$  achieves its minimum at  $a$ , then  $a \in \mathcal{R}$ ; if the infimum of  $T(x)$  is at infinity, then there exists a large  $a$  such that  $a \in \mathcal{R}$  as we state in the remark following Theorem 1 in [18]). On the contrary, the situation was unclear for  $\mathcal{S}$ , and it was commonly conjectured

before our contributions that  $\mathcal{S}$  was empty. In particular, that was the case in the examples constructed by Caffarelli and Friedman in [5] and [4]. In [19], we prove that the conjecture was false. More precisely, we proved the following (see Proposition 1 in [19]):

**Proposition 1 (Existence of initial data with  $\mathcal{S} \neq \emptyset$ )** *If the initial data  $(u_0, u_1)$  is odd and  $u(x, t)$  blows up in finite time, then  $0 \in \mathcal{S}$ .*

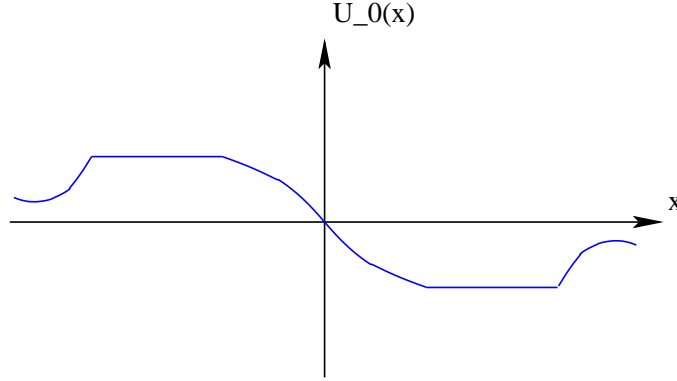


Fig 3: Odd initial data which makes the origin a characteristic point

As we explicitly show in Theorem 4 below, the existence of characteristic points is linked to sign changes in the solution near the singular point  $(a, T(a))$ . This enables us to give the following criterion for the non existence of characteristic points on some finite interval (see Theorem 4 in [19]):

**Proposition 2 (Non existence of characteristic points)** *Consider  $u(x, t)$  a blow-up solution of (1) such that  $u(x, t) \geq 0$  for all  $x \in (a_0, b_0)$  and  $t_0 \leq t < T(x)$  for some real  $a_0, b_0$  and  $t_0 \geq 0$ . Then,  $(a_0, b_0) \subset \mathcal{R}$ .*

**Remark:** This result can be seen as a generalization of the result of Caffarelli and Friedman [5] and [4], with no restriction on initial data. Indeed, from our result, taking non-negative initial data suffices to exclude the occurrence of characteristic points.

For general blow-up solutions, we proved the following facts about  $\mathcal{R}$  and  $\mathcal{S}$  in [18] and [19] (see Theorem 1 (and the following remark) in [18], see Theorems 1 and 2 in [20]):

**Theorem 3 (Geometry of the blow-up graph)**

- (i)  $\mathcal{R}$  is a non empty open set, and  $x \mapsto T(x)$  is of class  $C^1$  on  $\mathcal{R}$ ;
- (ii)  $\mathcal{S}$  is made of isolated points, and given  $a \in \mathcal{S}$ , if  $0 < |x - a| \leq \delta_0$ , then

$$\frac{1}{C_0 |\log(x - a)|^{\frac{(k(a)-1)(p-1)}{2}}} \leq T'(x) + \frac{x - a}{|x - a|} \leq \frac{C_0}{|\log(x - a)|^{\frac{(k(a)-1)(p-1)}{2}}}$$

for some  $\delta_0 > 0$  and  $C_0 > 0$ , where  $k(a) \geq 2$  is an integer. In particular,  $T(x)$  is right and left differentiable at  $a$ , with  $T'_l(a) = 1$  and  $T'_r(a) = -1$ .

**Remark:** In [21], Nouaili improves the regularity of the restriction of  $x \mapsto T(x)$  to  $\mathcal{R}$  to  $C^{1,\alpha}$  for some  $\alpha > 0$ .

**Remark:** Integrating the estimate of (ii) in Theorem 3, we see that the blow-up set is corner-shaped near  $a$  in the sense that

$$\frac{|x - a|}{C_0 |\log(x - a)|^{\frac{(k(a)-1)(p-1)}{2}}} \leq T(x) - T(a) + |x - a| \leq \frac{C_0 |x - a|}{|\log(x - a)|^{\frac{(k(a)-1)(p-1)}{2}}}. \quad (3)$$

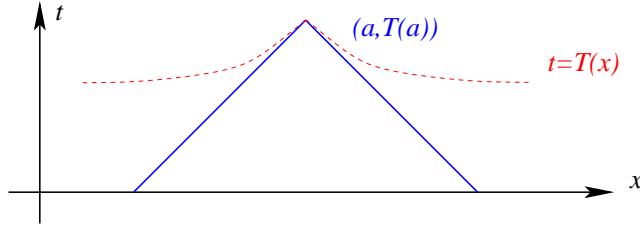


Fig. 4: The blow-up set is corner-shaped near characteristic points

In particular, there exists no solution of the semilinear wave equation (1) with a characteristic point  $a$  such that  $T(x)$  is differentiable at  $x = a$ .

Note from (3) that the blow-up set never touches the backward light cone with vertex  $(a, T(a))$  (except of course at  $a$ ), and that the distance between them is bounded from above and from below by the same rate, which is quantified in terms of the integer  $k(a) \geq 2$ . In particular, from the shape of the solution near  $(a, T(a))$ , we can recover the integer  $k(a) \geq 2$ , and  $k(a) - 1$  is the number of sign changes of the solution near  $(a, T(a))$  as we will see in (ii) of Theorem 4 below. In one word, the shape of the solution near  $(a, T(a))$  gives the topology of the solution and conversely.

**Remark:** The fact that the elements of  $\mathcal{S}$  are isolated points is not elementary. Direct arguments give no more than the fact that  $\mathcal{S} \neq \mathbb{R}$  (a point  $a$  such that  $T(a)$  is the blow-up time is non characteristic). The first step of the proof is done in [19] where we proved that  $\mathcal{S}$  has an empty interior and that in similarity variables, the solution splits in a non trivial decoupled sum of (at least 2) solitons with alternate signs (see (ii) of Theorem 4 below for a statement). The second step is done in [20]. It consists in using this decomposition and a good understanding of the dynamics of the equation in similarity variables (see equation (5) below) near a decoupled sum of “generalized” solitons. In fact, this is the first time where flows near an unstable sum of solitons are used and where such a result is obtained.

**Remark:** In higher dimensions  $N \geq 2$ , our result would be that the  $(N - 1)$ -dimensional Hausdorff measure of  $S$  is bounded. To prove that, we strongly need to characterize all selfsimilar solutions of equation (1) in the energy space. This is the main obstruction to extend this result to the higher dimension.

**Remark:** The fact that  $\mathcal{S}$  is made of isolated points certainly does not hold in general for quasilinear wave equations. Indeed, in [2], Alinhac gives an explicit solution  $u(x, t)$  for the following nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u + \partial_x u \partial_t u,$$

whose domain of definition is

$$D = \mathbb{R} \times [0, \infty) \setminus \{(x, t) \mid t \geq 1, |x| \leq t - 1\}$$

(when  $0 \leq t < 1$ ,  $u(x, t) = 4 \arctan\left(\frac{x}{1-t}\right)$ ). In this example, we clearly see that  $\mathcal{R} = \{0\}$ ,  $\mathcal{S} = \mathbb{R}^*$ , and the boundary of  $D$  is characteristic (i.e. has slope  $\pm 1$ ) on  $\mathcal{S}$ .

## 2 Asymptotic behavior near the blow-up graph

As one may guess from the above description, the asymptotic behavior will not be the same on  $\mathcal{R}$  and on  $\mathcal{S}$ . In both cases, we need to use the similarity variables which we recall in the following. Let us stress the fact that the keystone of our work is the existence of a Lyapunov functional in similarity variables.

Given some  $a \in \mathbb{R}$ , we introduce the following self-similar change of variables:

$$w_a(y, s) = (T(a) - t)^{\frac{2}{p-1}} u(x, t), \quad y = \frac{x - a}{T(a) - t}, \quad s = -\log(T(a) - t). \quad (4)$$

The function  $w = w_a$  satisfies the following equation for all  $y \in B = B(0, 1)$  and  $s \geq -\log T(a)$ :

$$\partial_{ss}^2 w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial_{y,s}^2 w \quad (5)$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho} \partial_y (\rho(1-y^2) \partial_y w) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \quad (6)$$

From Antonini and Merle [3], we know the existence of the following Lyapunov functional for equation (5):

$$E(w) = \int_{-1}^1 \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy, \quad (7)$$

defined for  $(\partial_s w, w) \in \mathcal{H}$  where

$$\mathcal{H} = \left\{ q \mid \|q\|_{\mathcal{H}}^2 \equiv \int_{-1}^1 \left( q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \quad (8)$$

Using this energy structure together with interpolation and the Gagliardo-Nirenberg estimate, we proved in [14], [16], [15] and [17] that  $(w_a(s), \partial_s w_a(s))$  is bounded in the energy space  $\mathcal{H}$ . Moreover, if  $a \in \mathcal{R}$ , then the bound holds in  $H^1 \times L^2(-1, 1)$  as well by a covering technique.

From Proposition 1 in [17], we know that the only stationary solutions of (5) in the space  $\mathcal{H}$  are  $q \equiv 0$  or  $w(y) \equiv \pm \kappa(d, y)$ , where  $d \in (-1, 1)$  and

$$\kappa(d, y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left( \frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \quad (9)$$

We will sometimes refer to  $\pm \kappa(d, y)$  as ‘‘solitons’’. Note that the set of stationary solutions is made of 3 connected components. One wonders whether these stationary solutions are

good candidates for the convergence of  $w_a(y, s)$ , at least when  $a \in \mathcal{R}$ . In the context of Hamiltonian systems, this question is delicate, and there is no natural reason for such a convergence, since the wave equation is time reversible. This kind of difficulties is also encountered for the critical generalized Korteweg de Vries equation (see Martel and Merle [10]) and for the Nonlinear Schrödinger equation (see Merle and Raphaël [11]).

As a matter of fact, there is convergence for  $w_a$  when  $a \in \mathcal{R}$  as we see from the following result (see Corollary 4 in [17] and Theorem 6 in [19]):

**Theorem 4 (Asymptotic behavior near the blow-up graph)**

(i) **Case where  $a \in \mathcal{R}$ : Existence of an asymptotic profile.** *There exist  $\delta_0(a) > 0$ ,  $|e(a)| = 1$ ,  $s_0(a) \geq -\log T(a)$  such that for all  $s \geq s_0$ :*

$$\left\| \begin{pmatrix} w_a(s) \\ \partial_s w_a(s) \end{pmatrix} - e(a) \begin{pmatrix} \kappa(T'(a), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_0)} \quad (10)$$

for some positive  $\mu_0$  and  $C_0$  independent from  $a$ . Moreover,  $E(w_a(s)) \rightarrow E(\kappa_0)$  as  $s \rightarrow \infty$ .

(ii) **Case where  $a \in \mathcal{S}$ : Decomposition into a sum of decoupled solitons.** *It holds that*

$$\left\| \begin{pmatrix} w_a(s) \\ \partial_s w_a(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(a)} e_i^*(a) \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \rightarrow 0 \text{ and } E(w_a(s)) \rightarrow k(a)E(\kappa_0) \quad (11)$$

as  $s \rightarrow \infty$ , for some

$$k(a) \geq 2, \quad e_i^*(a) = e_1^*(a)(-1)^{i+1} \quad (12)$$

and continuous  $d_i(s) = -\tanh \zeta_i(s) \in (-1, 1)$  for  $i = 1, \dots, k(a)$ . Moreover, for some  $C_0 > 0$ , for all  $i = 1, \dots, k(a)$  and  $s$  large enough,

$$\left| \zeta_i(s) - \left( i - \frac{(k(a)+1)}{2} \right) \frac{(p-1)}{2} \log s \right| \leq C_0. \quad (13)$$

**Remark:** It happens that (i) holds uniformly for all  $b$  in some neighborhood of  $a$ . Therefore, using the Sobolev injection in one dimension and a covering technique, we can show that the convergence of (10) holds in  $L^\infty(-1, 1)$  in the sense that

$$\|w_a(y, s) - e(a)\kappa(T'(a), y)\|_{L^\infty(-1,1)} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

**Remark:** To better see that the solitons in (11) are decoupled, we should use the change of variables

$$y = \tanh \xi \text{ where } \xi \in \mathbb{R}$$

and introduce

$$\bar{w}_a(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_a(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(s) = -\tanh^{-1} d_i(s).$$

In this case, estimate (11) yields the fact that

$$\|\bar{w}_a(\xi, s) - e_1^*(a) \sum_{i=1}^{k(a)} (-1)^i \cosh^{-\frac{2}{p-1}}(\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \rightarrow 0 \text{ as } s \rightarrow \infty,$$

It is worth noting that the  $\zeta_i(s)$  satisfy a Toda lattice system:

$$\frac{1}{c_1} \zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i(s) - \zeta_{i-1}(s))} - e^{-\frac{2}{p-1}(\zeta_{i+1}(s) - \zeta_i(s))} + R_i(s)$$

with

$$R_i(s) = o\left(\sum_{j=1}^{k-1} e^{-\frac{2}{p-1}(\zeta_{j+1}(s) - \zeta_j(s))}\right) \text{ as } s \rightarrow \infty.$$

An important step of the proof is to derive this system which gives two facts: the signs  $e_i^*$  in (12) are alternate and the solitons' centers  $\zeta_i(s)$  are equidistant (up to a constant) as written in (13).

**Remark:** Note that some elements in the description in similarity variables given above have a geometrical interpretation:

- when  $a \in \mathcal{R}$ , the solution in similarity variables converges to the profile  $e(a)\kappa(T'(a), \cdot)$  which has the slope  $T'(a)$  as a parameter;
- when  $a \in \mathcal{S}$ ,  $k(a) \geq 2$  is the number of solitons in the decomposition (11) appears also in the upper bound estimate on  $T(x)$  for  $x$  near  $a$  given in (3).

**Remark:** The proof of the convergence in (i) has two major difficulties:

- the linearized operator of equation (5) around the profile  $\kappa(d, y)$  is not selfadjoint, which makes the standard tools inefficient for the control of the negative part of the spectrum. Fortunately, the Lyapunov functional structure will be useful in this matter;
- all the non zero stationary solutions of equation (5) are non isolated, which generates a null eigenvalue difficult to control in the linearization of equation (5) around  $\kappa(d, y)$ . A modulation technique is then used to overcome this difficulty.

Extending the definition of  $k(a)$  defined for  $a \in \mathcal{S}$  after (11) by setting

$$k(a) = 1 \text{ for all } a \in \mathcal{R},$$

we proved the following energy criterion in [19] and using the monotonicity of the Lyapunov functional  $E(w)$ , we have the following consequence from the blow-up behavior in Theorem 4:

**Proposition 2.1 (An energy criterion for non characteristic points; see Corollary 7 in [19])**

(i) For all  $a \in \mathbb{R}$  and  $s_0 \geq -\log T(a)$ , we have

$$E(w_a(s_0)) \geq k(a)E(\kappa_0).$$

(ii) If for some  $a$  and  $s_0 \geq -\log T(a)$ , we have

$$E(w_a(s_0)) < 2E(\kappa_0),$$

then  $a \in \mathcal{R}$ .



### 3 A Liouville theorem and a trapping result near the set of stationary solutions

The following Liouville Theorem is crucial in our analysis:

**Theorem 5 (A Liouville Theorem for equation (5))** *Consider  $w(y, s)$  a solution to equation (5) defined for all  $(y, s) \in (-\frac{1}{\delta_*}, \frac{1}{\delta_*}) \times \mathbb{R}$  such that for all  $s \in \mathbb{R}$ ,*

$$\|w(s)\|_{H^1(-\frac{1}{\delta_*}, \frac{1}{\delta_*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta_*}, \frac{1}{\delta_*})} \leq C^* \quad (14)$$

for some  $\delta_* \in (0, 1)$  and  $C^* > 0$ . Then, either  $w \equiv 0$  or  $w$  can be extended to a function (still denoted by  $w$ ) defined in

$$\{(y, s) \mid -1 - T_0 e^s < d_0 y\} \supset \left(-\frac{1}{\delta_*}, \frac{1}{\delta_*}\right) \times \mathbb{R} \text{ by } w(y, s) = \theta_0 \kappa_0 \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(1 + T_0 e^s + d_0 y)^{\frac{2}{p-1}}}, \quad (15)$$

for some  $T_0 \geq T^*$ ,  $d_0 \in [-\delta_*, \delta_*]$  and  $\theta_0 = \pm 1$ , where  $\kappa_0$  defined in (9).

**Remark:** Note that deriving blow-up estimates through the proof of Liouville Theorems has been successful for different problems. For the case of the heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \quad (16)$$

where  $u : (x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}$ ,  $p > 1$  and  $(N - 2)p < N + 2$ , the blow-up time  $T$  is unique for equation (16). The blow-up set is the subset of  $\mathbb{R}^N$  such that  $u(x, t)$  does not remain bounded as  $(x, t)$  approaches  $(a, T)$ . In [22], the second author proved the  $C^2$  regularity of the blow-up set under a non degeneracy condition. A Liouville Theorem proved in [13] and [12] was crucially needed for the proof of the regularity result in the heat equation. The above Liouville Theorem is crucial for the regularity of the blow-up set for the wave equation (Theorem 4).

The second fundamental crucial result for our contributions is given by the following trapping result from [17] (See Theorem 3 in [17] and its proof):

**Proposition 3.1 (A trapping result near the sheet  $d \mapsto \kappa(d, y)$  of stationary solutions)** *There exists  $\epsilon_0 > 0$  such that if  $w \in C([s^*, \infty), \mathcal{H})$  for some  $s^* \in \mathbb{R}$  is a solution of equation (5) such that*

$$\forall s \geq s^*, \quad E(w(s)) \geq E(\kappa_0) \text{ and } \left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d^*, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \epsilon^* \quad (17)$$

for some  $d^* = -\tanh \xi^*$ ,  $\omega^* = \pm 1$  and  $\epsilon^* \in (0, \epsilon_0]$ , then there exists  $d_\infty = -\tanh \xi_\infty$  such that

$$|\xi_\infty - \xi^*| \leq C_0 \epsilon^* \text{ and } \left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d_\infty, \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \rightarrow 0. \quad (18)$$

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