The Gevrey hypoellipticity for a class of kinetic equations

Hua Chen, Weixi Li, Chao-Jiang Xu

THE GEVREY HYPOELLIPTICITY FOR A CLASS OF KINETIC EQUATIONS

HUA CHEN, WEI-XI LI, AND CHAO-JIANG XU

Abstract. In this paper, we study the Gevrey regularity of weak solutions for a class of linear and semi-linear kinetic equations, which are the linear model of spatially inhomogeneous Boltzmann equations without an angular cutoff.

1. Introduction

In this paper, we study the following kinetic operator:

$$\mathcal{P} = \partial_t + v\cdot\partial_x + a(t,x,v)(-\widetilde{\triangle}_v)^{\sigma}, \quad (t,x,v)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n, \tag{1.1}$$

where $0<\sigma<1$, $v\cdot\partial_x = \Sigma_{j=1}^n v_j\partial_{x_j}$, $a(t,x,v)\in C^\infty(\mathbb{R}^{2n+1})$ and $a(t,x,v)>0$ on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, the notation $(-\widetilde{\triangle}_v)^{\sigma}$ denotes the Fourier multiplier of symbol $p(\eta) = \left\{|\eta|^{\sigma}\omega(\eta) + |\eta|(1-\omega(\eta))\right\}^2$, with $\omega(\eta)\in C^\infty(\mathbb{R}^n)$, $0\le\omega\le 1$. Moreover, we have $\omega = 1$ if $|\eta|\ge 2$ and $\omega = 0$ if $|\eta|\le 1$. Throughout the paper, we denote by $\hat{u}(\tau,\xi,\eta)$ the Fourier transform of u with respect to the variables (t,x,v). $\mathcal{P}$ is not a classical pseudo-differential operator in $\mathbb{R}^{2n+1}$; for the coefficient in the kinetic part is not bounded in $\mathbb{R}^{2n+1}$. When $\sigma = 1$, the operator (1.1) is the so-called Vlasov-Fokker-Planck operator (see [12, 13]), it is then a Hörmander type operators, and we can apply the Gevrey hypoellipticity results of M. Derridj and C. Zuily [7] and M. Durand [10], see also [5] for the optimal G^3-hypoelliptic results.

As is well known, the operator (1.1) is a linear model of the spatially inhomogeneous Boltzmann equation without an angular cutoff (cf. [15]). This is the main motivation for the study of the regularizing properties of the operator (1.1) in this paper. In the past several years, a lot of progress has been made in the study of the spatially homogeneous Boltzmann equation without an angular cutoff, (see [2, 3, 8, 21] and references therein), in which the authors have proved that the singularity of the collision cross-section yields certain gain on the regularity for the weak solution of the Cauchy problem in the Sobolev space frame. That implies that there exists a C^∞ smoothness effect of the Cauchy problem for the spatially homogeneous Boltzmann equation without an angular cutoff. The Gevrey regularity of the local solutions has been constructed in [20] for the initial data having the same Gevrey regularity, and the propagation of Gevrey regularity is proved recently in [9]. In [17], the Gevrey smoothness effect of the Cauchy problem has been established for the spatially homogeneous linear Boltzmann equation. In [16], they obtain the ultra-analytical effect results for the non linear homogeneous Landau equations and inhomogeneous linear Landau equations.

Date: 2008-03-14 Research supported partially by the NSFC grant 10631020.
2000 *Mathematics Subject Classification.* 35 B, 35 H, 35 N.
Key words and phrases. Gevrey regularity, kinetic equation, microlocal analysis.

However, there is no general result for the smoothness effect of the spatially inhomogeneous problem, which is actually related with the regularity of the kinetic equation with its diffusion part a nonlinear operator in the velocity variable v. Under the singularity assumption on the collision cross section, the behavior of the Boltzmann collision operator is similar to a fractional power of the Laplacian $(-\triangle_v)^\sigma$. In [1], by using the uncertainty principle of the micro-local analysis, the authors obtained C^∞ regularity for the weak solution of the linear spatially inhomogeneous Boltzmann equation without an angular cutoff.

On the other hand, in [15], the existence and the C^∞ regularity have been proved for the solutions of the Cauchy problem for linear and semi-linear equations associated with the kinetic operators (1.1). In this paper, we shall consider the Gevrey regularity for such problems.

Let us first recall the definition for the functions in the Gevery class. Let U be an open subset of $\mathbb{R}^d$ and $1 \le s < +\infty$, we say that $f \in G^s(U)$ if $f \in C^\infty(U)$ and for any compact subset K of U, there exists a constant (say Gevrey constant of f) $C = C_K$, depending only on K and f, such that for all multi-indices $\alpha \in \mathbb{N}^d$,

$$\|\partial^\alpha f\|_{L^\infty(K)} \le C_K^{|\alpha|+1}(\alpha!)^s. \tag{1.2}$$

If W is a closed subset of $\mathbb{R}^d$, $G^s(W)$ denote the restriction of $G^s(\tilde{W})$ on W where $\tilde{W}$ is an open neighborhood of W. The condition (1.2) is equivalent to the following estimate (e.g. see [6] or [18]):

$$\|\partial^\alpha f\|_{L^2(K)} \le C_K^{|\alpha|+1}(|\alpha|!)^s.$$

We say that an operator P is G^s hypoelliptic in U if $u \in \mathcal{D}'(U)$ and $Pu \in G^s(U)$, then it follows that $u \in G^s(U)$. Likewise, we say that the operator P is C^∞ hypoelliptic in U if $u \in \mathcal{D}'(U)$ and $Pu \in C^\infty(U)$, then it follows that $u \in C^\infty(U)$.

In [15], Morimoto-Xu proved that the operator (1.1) is C^∞ hypoelliptic if $1/3 < \sigma \le 1$. Our first main result of this paper is the following:

Theorem 1.1. *Let $0 < \sigma < 1$ and $\delta = \max\{\frac{\sigma}{4},\ \frac{\sigma}{2} - \frac{1}{6}\}$. Then the operator $\mathcal{P}$ given by (1.1) is G^s hypoelliptic in $\mathbb{R}^{2n+1}$ for any $s \ge \frac{2}{\delta}$, provided the coefficient $a(t,x,v) \in G^s(\mathbb{R}^{2n+1})$ and $a(t,x,v) > 0$.*

Compared with what is obtained in [15], the result of Theorem 1.1 implies that the operator (1.1) is also C^∞ hypoelliptic in the case of $0 < \sigma \le 1/3$.

Next, we consider the following semi-linear equation:

$$\partial_t u + v \cdot \nabla_x u + a(-\widetilde{\triangle}_v)^\sigma u = F(t,x,v;u) \tag{1.3}$$

where F is a nonlinear function of the real variables (t,x,v,q). The following is the second main result of the paper, which implies that the weak solution of equation (1.3) has Gevrey regularity:

Theorem 1.2. *Let $0 < \sigma < 1$ and $\delta = \max\{\frac{\sigma}{4},\ \frac{\sigma}{2} - \frac{1}{6}\}$. Suppose that $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ is a weak solution of Equation (1.3). Then $u \in G^s(\mathbb{R}^{2n+1})$ for any $s \ge \frac{2}{\delta}$, provided that the coefficient $a \in G^s(\mathbb{R}^{2n+1})$, $a(t,x,v) > 0$ and the nonlinear function $F(t,x,v,q) \in G^s(\mathbb{R}^{2n+2})$.*

Remark 1.1. Our results here are local interior regularity results. This implies that if there exists a weak solution in $\mathcal{D}'$, then the solution is in Gevrey class in the interior of the domain. Thus, the interior regularity of a weak solution does

not depend much on the regularity of the initial Cauchy data. Also, without loss of generality, we can assume that $c_0^{-1} \le a(t,x,v) \le c_0$ for all $(t,x,v) \in \mathbb{R}^{2n+1}$ with c_0 a positive constant, and all derivatives of the coefficient a are bounded in $\mathbb{R}^{2n+1}$.

The paper is organized as follows: in section 2, we prove that $\mathcal{P}$ is subelliptic by using the method of subelliptic multiplier developed by J. Kohn [14]. Section 3 is devoted to the study of the commutator of $(-\widetilde{\triangle}_v)^\sigma$ with the cut-off function in the v variable. In section 4, we use the subelliptic estimates to prove the Gevrey hypoellipticity of the operator $\mathcal{P}$. Section 5 is devoted to the proof of the Gevrey regularity for the weak solution of the semilinear kinetic equation (1.3).

2. Subelliptic estimates

In this paper, the notation, $\|\cdot\|_\kappa, \kappa \in \mathbb{R}$, is used for the classical Sobolev norm in $H^\kappa(\mathbb{R}^{2n+1})$, and $(f,\ g)$ is the inner product of $f, g \in L^2(\mathbb{R}^{2n+1})$. Moreover if $f,\ g \in C_0^\infty(\mathbb{R}^{2n+1})$, it is easy to see that

$$|(f,\ g)| \le \|f\|_\kappa \|g\|_{-\kappa} \le \frac{\varepsilon \|f\|_\kappa^2}{2} + \frac{\|g\|_{-\kappa}^2}{2\varepsilon}. \tag{2.1}$$

We have also the interpolation inequality in Sobolev space: For any $\varepsilon > 0$ and $r_1 < r_2 < r_3$,

$$\|f\|_{r_2} \le \varepsilon \|f\|_{r_3} + \varepsilon^{-(r_2-r_1)/(r_3-r_2)} \|f\|_{r_1}. \tag{2.2}$$

Let Ω be an open subset of $\mathbb{R}^{2n+1}$ and $S^m(\Omega), m \in \mathbb{R}$, be the symbol space of the classical pseudo-differential operators (when there is no risk to cause the confusion, we will simply write S^m for $S^m(\Omega)$). We say $P = P(t,x,v,D_t,D_x,D_v) \in \mathrm{Op}(S^m)$ to be a pseudo-differential operator of order m, if its symbol $p(t,x,v;\tau,\xi,\eta) \in S^m$. If $P \in \mathrm{Op}(S^m)$, then P is a continuous operator from $H_c^\kappa(\Omega)$ to $H_{loc}^{\kappa-m}(\Omega)$, where $H_c^\kappa(\Omega)$ is the subspace of $H^\kappa(R^{2n+1})$ which consists of the distributions having their compact support in Ω. $H_{loc}^{\kappa-m}(\Omega)$ consists of the distributions h such that $\phi h \in H^{\kappa-m}(\mathbb{R}^{2n+1})$ for any $\phi \in C_0^\infty(\Omega)$. For more details on the pseudo-differential operators, we refer to Treves [19]. Observe that if $P_1 \in \mathrm{Op}(S^{m_1})$, $P_2 \in \mathrm{Op}(S^{m_2})$, then $[P_1,\ P_2] \in \mathrm{Op}(S^{m_1+m_2-1})$.

We study now the operator $\mathcal{P}$ given by (1.1). For simplicity, we introduce the following notations

$$\widetilde{\Lambda}_v^\sigma = (-\widetilde{\triangle}_v)^{\frac{\sigma}{2}}, \quad X_0 = \partial_t + v\cdot\partial_x, \quad X_j = \partial_{v_j}, j = 1, \cdots, n,$$
$$\Lambda^\kappa = (1 + |D_t|^2 + |D_x|^2 + |D_v|^2)^{\kappa/2}.$$

Then $\mathcal{P}$ can be written as $\mathcal{P} = X_0 + a(t,x,v)\widetilde{\Lambda}_v^{2\sigma}$, and $\partial_{x_j} = [X_j,\ X_0]$. The following simple fact is used frequently: For any compact $K \subset \mathbb{R}^{2n+1}$ and $r \ge 0$, there exists $C_{K,r} > 0$ such that for any $f \in C_0^\infty(K)$,

$$\|\widetilde{\Lambda}_v^\sigma f\|_r \le C_{K,r}\big\{\|\mathcal{P}f\|_r + \|f\|_r\big\}. \tag{2.3}$$

In fact, a simple computation gives that

$$\begin{aligned}
\|\widetilde{\Lambda}_v^\sigma f\|_r^2 &= \mathrm{Re}(\mathcal{P}f,\ a^{-1}\Lambda^{2r}f) - \mathrm{Re}(X_0 f,\ a^{-1}\Lambda^{2r}f) \\
&= \mathrm{Re}(\mathcal{P}f,\ a^{-1}\Lambda^{2r}f) - \frac{1}{2}(f,\ [a^{-1}\Lambda^{2r},\ \widetilde{X}_0]f) - \frac{1}{2}(f,\ [\Lambda^{2r},\ a^{-1}]\ \widetilde{X}_0 f) \\
&\le C_{K,r}\big\{\|\mathcal{P}f\|_r + \|f\|_r\big\},
\end{aligned}$$

where $\widetilde{X}_0 = \partial_t + \tilde{\psi}(v)v\cdot\partial_x$ and $\tilde{\psi}\in C_0^\infty(\mathbb{R}^n_v)$ is a cutoff function in the v variable such that $\tilde{\psi}=1$ in the projection of K on $\mathbb{R}^n_v$. Remark that, with the choice of such a cutoff function, we have that

$$\widetilde{X}_0 P(t,x,v,D_t,D_x,D_v)f = X_0 P(t,x,v,D_t,D_x,D_v)f$$

for any $f\in C_0^\infty(K)$ and any partial differential operator $P(t,x,v,D_t,D_x,D_v)$.

First we show $\mathcal{P}$ is a subelliptic operator on $\mathbb{R}^{2n+1}$ with a gain of order $\delta = \max\left\{\frac{\sigma}{4},\ \frac{\sigma}{2}-\frac{1}{6}\right\}$.

Proposition 2.1. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. For any $r\geq 0$, there exists a constant $C_{K,r}$, depending only on K and r, such that for any $f\in C_0^\infty(K)$,*

$$\|f\|_{r+\delta} \leq C_{K,r}\{\ \|\mathcal{P}f\|_r + \|f\|_0\ \}, \tag{2.4}$$

where $\delta = \max\left\{\frac{\sigma}{4},\ \frac{\sigma}{2}-\frac{1}{6}\right\}$.

In order to prove Proposition 2.1, we need the following two lemmas.

Lemma 2.2. *Let K be any compact subset of $\mathbb{R}^{2n+1}$. Then for any $f\in C_0^\infty(K)$, we have*

$$\|\Lambda^{-1/3}X_0 f\|_0 \leq C_K(\ \|\mathcal{P}f\|_0 + \|f\|_0\ ), \tag{2.5}$$

and

$$\|\Lambda^{-1}X_j f\|_\sigma \leq C_K(\ \|\mathcal{P}f\|_0 + \|f\|_0\ ), \qquad j=1,\cdots,n. \tag{2.6}$$

This is the result of Proposition 3.1 in [15]. The following lemma is to estimate the commutators, which is different from the calculation in [15] for the second part of the lemma.

Lemma 2.3. *Let K be a compact subset of $\mathbb{R}^{2n+1}$. Then for any $f\in C_0^\infty(K)$, we have*

$$\|[X_j,\ \Lambda^{-1}\widetilde{X}_0]f\|_{\sigma/2-1/6} \leq C_K(\ \|\mathcal{P}f\|_0 + \|f\|_0\ ), \qquad j=1,\cdots,n, \tag{2.7}$$

and

$$\|[\Lambda^{-1}X_j,\ \widetilde{X}_0]f\|_{\sigma/4} \leq C_K(\ \|\mathcal{P}f\|_0 + \|f\|_0\ ), \qquad j=1,\cdots,n. \tag{2.8}$$

Proof. We denote $Q_j = \Lambda^{\sigma-1/3-1}[X_j,X_0] = \Lambda^{\sigma-1/3-1}\partial_{x_j} \in \mathrm{Op}(S^{\sigma-1/3})$. Note that $[X_k,\ Q_j]=0$ for any $1\leq k\leq n$. Therefore for any $f\in C_0^\infty(K)$,

$$\begin{aligned}
&\|[X_j,\ \Lambda^{-1}\widetilde{X}_0]f\|^2_{\sigma/2-1/6} = \|[X_j,\ \Lambda^{-1}X_0]f\|^2_{\sigma/2-1/6}\\
&\leq |(X_j\Lambda^{-1}X_0 f,\ Q_j f)| + |(\Lambda^{-1}\widetilde{X}_0 X_j f,\ Q_j f)|\\
&\leq |(\Lambda^{-1}X_0 f,\ Q_j X_j f)| + |(X_j f,\ \widetilde{X}_0\Lambda^{-1}Q_j f)|\\
&\leq\ \|\Lambda^{-1}X_0 f\|_{2/3}\|Q_j X_j f\|_{-2/3} + |(X_j f,\ [\widetilde{X}_0,\ \Lambda^{-1}Q_j]f)| + |(X_j f,\ \Lambda^{-1}Q_j X_0 f)|\\
&\leq C_K\{\ \|\Lambda^{-1/3}X_0 f\|_0^2 + \|\Lambda^{-1}X_j f\|_\sigma^2 + \|f\|_0^2\ \},
\end{aligned}$$

where we have used the simple fact that $[\widetilde{X}_0,\ \Lambda^{-1}Q_j]\in\mathrm{Op}(S^{\sigma-1/3-1})$. Then (2.5) and (2.6) give immediately (2.7).

We now study (2.8). First of all, we have

$$\begin{aligned}
\|[\Lambda^{-1}X_j,\ \widetilde{X}_0]f\|^2_{\sigma/4} \ &=\ (\Lambda^{-1}X_j\widetilde{X}_0 f,\ \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]f)\\
&\quad -(\widetilde{X}_0\Lambda^{-1}X_j f,\ \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]f).
\end{aligned}$$

By a straightforward calculation, it follows that

$$
\begin{aligned}
&|(\widetilde{X}_0\Lambda^{-1}X_jf,\ \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]f)| = |(\Lambda^{-1}X_jf,\ \widetilde{X}_0\Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]f)|\\
&\le |(\Lambda^{-1}X_jf,\ \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]\widetilde{X}_0f)| + |(\Lambda^{-1}X_jf,\ [\Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0],\ \widetilde{X}_0]f)|\\
&\le C_K\big\{\ |(\Lambda^{-1}X_jf,\ \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]X_0f)| + \|\Lambda^{-1}X_jf\|^2_{\sigma/2} + \|f\|^2_0\ \big\}\\
&\le C_K\big\{\ |(\Lambda^{-1}X_jf,\ \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]X_0f)| + \|\mathcal{P}f\|^2_0 + \|f\|^2_0\ \big\}.
\end{aligned}
$$

In the last inequality, we have used (2.6) in Lemma 2.2.

Denote $P_{\sigma/2} = \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0] \in \mathrm{Op}(S^{\sigma/2})$. Recall that $X_0 = \mathcal{P} - a\widetilde{\Lambda}_v^{2\sigma}$. We have

$$
\begin{aligned}
&|(\Lambda^{-1}X_jf,\ \Lambda^{\sigma/2}[\Lambda^{-1}X_j,\ \widetilde{X}_0]X_0f)| = |(\Lambda^{-1}X_jf,\ P_{\sigma/2}X_0f)|\\
&\le |(\Lambda^{-1}X_jf,\ P_{\sigma/2}\mathcal{P}f)| + |(\Lambda^{-1}X_jf,\ P_{\sigma/2}a\widetilde{\Lambda}_v^{2\sigma}f)|\\
&\le C_K\{\ \|\Lambda^{-1}X_jf\|^2_{\sigma/2} + \|\mathcal{P}f\|^2_0 + |(\widetilde{\Lambda}_v^{\sigma}\Lambda^{-1}X_jf,\ \widetilde{\Lambda}_v^{-\sigma}P_{\sigma/2}a\widetilde{\Lambda}_v^{2\sigma}f)|\ \}\\
&\le C_K\{\ \|\Lambda^{-1}X_jf\|^2_{\sigma/2} + \|\mathcal{P}f\|^2_0 + \|\widetilde{\Lambda}_v^{\sigma}\Lambda^{-1}X_jf\|^2_{\sigma/2} + \|\widetilde{\Lambda}_v^{\sigma}f\|^2_0\ \}\\
&\le C_K\{\ \|\widetilde{\Lambda}_v^{\sigma}\Lambda^{\sigma/2}\Lambda^{-1}X_jf\|^2_0 + \|\mathcal{P}f\|^2_0 + \|f\|^2_0\ \}.
\end{aligned}
$$

For the last inequality, we used results from (2.3) and (2.6). Clearly, $[\widetilde{\Lambda}_v^{\sigma},\ \Lambda^{-1}X_j] = [\widetilde{\Lambda}_v^{\sigma},\ \Lambda^{\sigma/2}] = [\Lambda^{-1}X_j,\ \Lambda^{\sigma/2}] = 0$. Then we get

$$
\begin{aligned}
&\|\widetilde{\Lambda}_v^{\sigma}\Lambda^{\sigma/2}\Lambda^{-1}X_jf\|^2_0 = -\mathrm{Re}(\mathcal{P}f,\ a^{-1}\Lambda^{\sigma}\Lambda^{-2}X_j^2f) + \mathrm{Re}(\widetilde{X}_0f,\ a^{-1}\Lambda^{\sigma}\Lambda^{-2}X_j^2f)\\
&\le C_K\{\ \|\mathcal{P}f\|^2_0 + \|\Lambda^{-1}X_jf\|^2_{\sigma} + \frac{1}{2}|(f,\ [\Lambda^{\sigma}\Lambda^{-2}X_j^2,\ a^{-1}\widetilde{X}_0]f)|\\
&\qquad + \frac{1}{2}|(f,\ [a^{-1},\ \widetilde{X}_0]\Lambda^{\sigma}\Lambda^{-2}X_j^2f)|\ \}\\
&\le C_K\{\ \|\mathcal{P}f\|^2_0 + \|f\|^2_0 + \|\Lambda^{-1}X_jf\|^2_{\sigma} + |(f,\ \Lambda^{-1}X_j[\Lambda^{\sigma}\Lambda^{-1}X_j,\ a^{-1}\widetilde{X}_0]f)|\\
&\qquad + |(f,\ [\Lambda^{-1}X_j,\ a^{-1}\widetilde{X}_0]\Lambda^{\sigma}\Lambda^{-1}X_jf)|\ \}\\
&\le C_K\{\ \|\mathcal{P}f\|^2_0 + \|\Lambda^{-1}X_jf\|^2_{\sigma} + \|f\|^2_0\ \}\\
&\le C_K\{\ \|\mathcal{P}f\|^2_0 + \|f\|^2_0\ \}.
\end{aligned}
$$

The above three estimates show immediately

$$
|(\widetilde{X}_0\Lambda^{-1}X_jf,\ P_{\sigma/2}f)| \le C_K\{\ \|\mathcal{P}f\|^2_0 + \|f\|^2_0\ \}.
$$

Similarly, we can prove

$$
|(\Lambda^{-1}X_j\widetilde{X}_0f,\ P_{\sigma/2}f)| \le C_K\{\ \|\mathcal{P}f\|^2_0 + \|f\|^2_0\ \}.
$$

This completes the proof of Lemma 2.3. □

The rest of this section is devoted to the proof of proposition 2.1:

Proof of Proposition 2.1. Notice that $\partial_{x_j} = [X_j,\ X_0]$ and $\partial_t = X_0 - \sum_{j=1}^{n} v_j \cdot [X_j,\ X_0]$. Hence, for any $f \in C_0^\infty(K)$, we have

$$\begin{aligned}\|f\|_\delta^2 &= \|\partial_t f\|_{\delta-1}^2 + \sum_{j=1}^{n} \|\partial_{x_j} f\|_{\delta-1}^2 + \sum_{j=1}^{n} \|\partial_{v_j} f\|_{\delta-1}^2 + \|f\|_0^2 \\ &\le C_K \{\ \|\Lambda^{-1} X_0 f\|_\delta^2 + \sum_{j=1}^{n} \big(\|\tilde{\psi}(v) v_j [X_j,\ \widetilde{X}_0] f\|_{\delta-1}^2 \\ &\qquad + \|[X_j,\ \widetilde{X}_0] f\|_{\delta-1}^2 + \|\Lambda^{-1} X_j f\|_\delta^2\big) + \|f\|_0^2\ \}.\end{aligned}$$

Since $\delta = \max\{\sigma/4,\ \sigma/2 - 1/6\} \le \min\{2/3,\ \sigma\}$, applying (2.5) and (2.6) to Lemma 2.2, we have that

$$\|\Lambda^{-1} X_0 f\|_\delta + \sum_{j=1}^{n} \|\Lambda^{-1} X_j f\|_\delta \le C_K \{\ \|\mathcal{P} f\|_0 + \|f\|_0\ \}$$

and

$$\|\tilde{\psi}(v) v_j [X_j,\ \widetilde{X}_0] f\|_{\delta-1} \le C_K \{\|[X_j,\ \widetilde{X}_0] f\|_{\delta-1} + \|f\|_0\}.$$

It remains to treat the term $\|[X_j,\ \widetilde{X}_0] f\|_{\delta-1}$. We consider the following two cases.

Case (i). $\delta = \max\{\sigma/4,\ \sigma/2 - 1/6\} = \sigma/2 - 1/6$.
We apply (2.7) in Lemma 2.3 to get

$$\begin{aligned}\|[X_j,\ \widetilde{X}_0] f\|_{\delta-1} &\le \|[X_j,\ \Lambda^{-1}\widetilde{X}_0] f\|_\delta + \|[X_j,\ \Lambda^{-1}]\widetilde{X}_0 f\|_\delta \\ &\le C_K \{\ \|\mathcal{P} f\|_0 + \|\Lambda^{-1} X_0 f\|_\delta + \|f\|_0^2\ \}.\end{aligned}$$

Since $\delta < 2/3$, then applying (2.5) again, we get immediately

$$\|[X_j,\ \widetilde{X}_0] f\|_{\delta-1} \le C_K \{\ \|\mathcal{P} f\|_0 + \|f\|_0\ \}.$$

Case (ii). $\delta = \max(\sigma/4, \sigma/2 - 1/6) = \sigma/4$.
By (2.8) in Lemma 2.3, it follows that

$$\begin{aligned}\|[X_j,\ \widetilde{X}_0] f\|_{\delta-1} &\le \|[\Lambda^{-1} X_j,\ \widetilde{X}_0] f\|_\delta + \|[\Lambda^{-1},\ \widetilde{X}_0] X_j f\|_\delta \\ &\le C_K \{\ \|\mathcal{P} f\|_0 + \|\Lambda^{-1} X_j f\|_\delta + \|f\|_0\ \}.\end{aligned}$$

Note that $\delta < \sigma$, and hence from (2.6), we have

$$\|[X_j,\ \widetilde{X}_0] f\|_{\delta-1} \le C_K \{\ \|\mathcal{P} f\|_0 + \|f\|_0\ \}.$$

A combination of Case (i) and Case (ii) yields that for $\delta = \max\{\sigma/4,\ \sigma/2 - 1/6\}$,

$$\|[X_j,\ \widetilde{X}_0] f\|_{\delta-1} \le C_K \{\ \|\mathcal{P} f\|_0 + \|f\|_0\ \}.$$

Then we get

$$\|f\|_\delta \le C_K \{\ \|\mathcal{P} f\|_0 + \|f\|_0\ \}. \tag{2.9}$$

Choose now a cutoff function $\psi \in C_0^\infty(\mathbb{R}^{2n+1})$ such that $\psi|_K \equiv 1$ and Supp ψ is a neighborhood of K. Then for any $r \geq 0$, $\varepsilon > 0$ and $f \in C_0^\infty(K)$, by (2.9), we have

$$\|f\|_{r+\delta} \quad = \quad \|\Lambda^r \psi f\|_\delta \leq \|\psi \Lambda^r f\|_\delta + \|[\Lambda^r,\ \psi] f\|_\delta \leq C_K \{\ \|\mathcal{P}\psi\Lambda^r f\|_0 + \|f\|_r\ \}.$$

Furthermore, notice that

$$[a\widetilde{\Lambda}_v^{2\sigma}, \psi\Lambda^r] = 2a[\widetilde{\Lambda}_v^\sigma,\ \psi\Lambda^r]\widetilde{\Lambda}_v^\sigma + a[\widetilde{\Lambda}_v^\sigma,\ [\widetilde{\Lambda}_v^\sigma,\ \psi\Lambda^r]\ ] + [a,\ \psi\Lambda^r]\widetilde{\Lambda}_v^{2\sigma}.$$

Hence

$$\begin{aligned} \|\mathcal{P}\psi\Lambda^r f\|_0 \quad &\leq \quad \|\psi\Lambda^r \mathcal{P} f\|_0 + \|[\widetilde{X}_0,\ \psi\Lambda^r] f\|_0 + \|a[\widetilde{\Lambda}_v^\sigma,\ [\widetilde{\Lambda}_v^\sigma, \psi\Lambda^r]\ ] f\|_0 \\ &\quad +2\|a[\widetilde{\Lambda}_v^\sigma,\ \psi\Lambda^r]\widetilde{\Lambda}_v^\sigma f\|_0 + \|[a,\ \psi\Lambda^r]\widetilde{\Lambda}_v^{2\sigma} f\|_0 \\ &\leq \quad C_{K,r}\{\ \|\mathcal{P}f\|_r + \|f\|_r + \|\widetilde{\Lambda}_v^\sigma f\|_r\ \}, \end{aligned}$$

Combining with (2.3), we have

$$\|\mathcal{P}\psi\Lambda^r f\|_0 \leq C_{K,r}\{\ \|\mathcal{P}f\|_r + \|f\|_r\ \}.$$

The above three estimates show that

$$\|f\|_{r+\delta} \leq C_{K,r}\{\ \|\mathcal{P}f\|_r + \|f\|_r\ \}.$$

Applying the interpolation inequality (2.2), it follows that

$$\|f\|_{r+\delta} \leq C_{\varepsilon,r,K}\{\ \|\mathcal{P}f\|_r + \|f\|_0\ \} + \varepsilon\|f\|_{r+\delta}.$$

Taking ε small enough, we get the desired subelliptic estimate (2.4). This completes the proof of Proposition 2.1.

Since the subelliptic estimate in Proposition 2.1 is true for $0 < \sigma < 1$, we can now improve the C^∞-hypoellipticity result of [15](which is for $1/3 < \sigma < 1$) as in the following Theorem:

Theorem 2.4. *Let $0 < \sigma < 1$. Then the operator $\mathcal{P}$ given by (1.1) is C^∞ hypoelliptic in $\mathbb{R}^{2n+1}$, provided that the coefficient $a(t,x,v)$ is in the space $C^\infty(\mathbb{R}^{2n+1})$ and $a(t,x,v) > 0$.*

In fact, if we consider only the local regularity problem, as in Proposition 4.1 of [15], we can prove that if $f \in H^s_{loc}(\mathbb{R}^{2n+1}), u \in \mathcal{D}'(\mathbb{R}^{2n+1})$ and $\mathcal{P}u = f$ then $u \in H^{s+\delta}_{loc}(\mathbb{R}^{2n+1})$. By using the subelliptic estimate (2.4), the estimate for the commutators between the operator $\mathcal{P}$ and the mollifiers are exactly the same as in Section 4 of [15]. This gives the C^∞ hypoellipticity by the Sobolev embedding theorem. The same argument applies to the semi-linear equations.

Remark that the results of [15] are not only regularity results. The authors also proved a global estimate with weights (the moments). This is another important problem for the kinetic equation.

3. Cutoff functions and commutators

To prove the Gevrey regularity of a solution, we have to prove an uniformly iteration estimate (1.2). Our only tool is the subelliptic estimate (2.4). Since it is a local estimate, we have to control the commutators between the operator $\mathcal{P}$ and the cutoff functions. This is always the technical key step in the Gevrey regularity problem. Our additional difficulty comes from the complicated nature of the operator $\mathcal{P}$.

Since the Gevrey hypoellipticity is a local property, it suffices to show $\mathcal{P}$ is Gevrey hypoelliptic in the open domain $\Omega \subset \mathbb{R}^{2n+1}$ given by

$$\Omega = \Omega^1 \times \Omega^2 = \{(t,x) \in \mathbb{R}^{n+1};\, t^2 + |x|^2 < 1\} \times \left\{v \in \mathbb{R}^n;\, |v|^2 < 1\right\}.$$

Define W by setting

$$W = 2\Omega = \left\{(t,x,v)\,;\, |t|^2 + |x|^2 \le 2^2,\, |v| \le 2\right\}$$

For $0 \le \rho < 1$, set $\Omega_\rho = \Omega^1_\rho \times \Omega^2_\rho$ with Ω^1_ρ and Ω^2_ρ to be given by

$$\Omega^1_\rho = \left\{(t,x) \in \mathbb{R}^{n+1};\ \left(t^2 + |x|^2\right)^{1/2} < 1 - \rho\right\}, \qquad \Omega^2_\rho = \left\{v \in \mathbb{R}^n;\ |v|^2 < 1 - \rho\right\}.$$

Let χ_ρ be the characteristic function of the set Ω^2_ρ, and let $\phi \in C^\infty_0(\Omega^2)$ be a function satisfying $0 \le \phi \le 1$ and $\int_{\mathbb{R}^n} \phi(v)dv = 1$. For any $\varepsilon,\ \tilde\varepsilon > 0$, setting $\phi_\varepsilon(v) = \varepsilon^{-n}\phi\left(\frac{v}{\varepsilon}\right)$ and $\varphi_{\varepsilon,\tilde\varepsilon}(v) = \phi_{\varepsilon/2} * \chi_{\varepsilon/2+\tilde\varepsilon}(v)$. Then for a small $\varepsilon,\ \tilde\varepsilon > 0$,

$$\begin{aligned} &\varphi_{\varepsilon,\tilde\varepsilon} \in C^\infty_0(\Omega^2_{\tilde\varepsilon});\quad \varphi_{\varepsilon,\tilde\varepsilon} = 1 \ \text{ in } \ \Omega^2_{\varepsilon+\tilde\varepsilon};\\ &\sup_{v\in\mathbb{R}^n} |D^\alpha \varphi_{\varepsilon,\tilde\varepsilon}(v)| \le C_\alpha \varepsilon^{-|\alpha|} \quad \text{ for any } \quad \alpha \in \mathbb{N}^n. \end{aligned}$$

In the same way, we can find a function $\psi_{\varepsilon,\tilde\varepsilon}(t,x) \in C^\infty_0(\Omega^1_{\tilde\varepsilon})$ such that $\psi_{\varepsilon,\tilde\varepsilon} = 1$ in $\Omega^1_{\tilde\varepsilon+\tilde\varepsilon}$ and $\sup|D^\alpha\psi_{\varepsilon,\tilde\varepsilon}| \le C_\alpha\varepsilon^{-|\alpha|}$.

Now for any $N \in \mathbb{N}, N \ge 2$ and any $0 < \rho < 1$, we set

$$\Phi_{\rho,N}(t,x,v) = \psi_{\frac{\rho}{N},\frac{(N-1)\rho}{N}}(t,x)\varphi_{\frac{\rho}{N},\frac{(N-1)\rho}{N}}(v).$$

Then we have,

$$\begin{cases} \Phi_{\rho,N} \in C^\infty_0(\Omega_{\frac{N-1}{N}\rho})\\ \Phi_{\rho,N}(t,x,v) = 1, \quad (t,x,v) \in \Omega_\rho,\\ \sup|D^\alpha\Phi_{\rho,N}| \le C_\alpha(N/\rho)^{|\alpha|}. \end{cases} \tag{3.1}$$

For such cut-off functions, we have the following Lemma (see Corollary 0.2.2 of [10]).

Lemma 3.1. *There exists a constant C_n, depending only on n, such that for any $0 \le \mu \le n+2$, and $f \in \mathcal{S}(\mathbb{R}^{n+1})$, we have*

$$\left\|\left(D^\gamma\Phi_{\rho,N}\right)f\right\|_\mu \le C_n\left\{(N/\rho)^{|\gamma|}\left\|f\right\|_\mu + (N/\rho)^{|\gamma|+\mu}\left\|f\right\|_0\right\}, \quad |\gamma| \le 2. \tag{3.2}$$

We study now the commutator of above cutoff function with the operator $\mathcal{P}$. Since the operator is a differential operator with respect to the (t,x) variables, it is enough to consider the commutator of $\widetilde{\Lambda}^\sigma_v$ with a cut-off function in the v variable. We set $\varphi_{\rho,N}(v) = \varphi_{\frac{\rho}{N},\frac{(N-1)\rho}{N}}(v)$. The proof of the following Lemma is very similar to that of M. Durand [10]. Since our calculus is much easier and much more direct, we repeat it here.

Lemma 3.2. *There exists a constant $C_{\sigma,n}$, depending only on n and σ, such that for any κ with $1 \le \kappa \le n+3$, and $f \in \mathcal{S}(\mathbb{R}^{2n+1})$,*

$$\|[\widetilde{\Lambda}^\sigma_v,\ \varphi_{\rho,N}]f\|_\kappa \le C_{\sigma,n}\left\{(N/\rho)^\sigma\left\|f\right\|_\kappa + (N/\rho)^{\kappa+\sigma}\left\|f\right\|_0\right\} \tag{3.3}$$

and

$$\|[\widetilde{\Lambda}^\sigma_v,\ [\widetilde{\Lambda}^\sigma_v,\ \varphi_{\rho,N}]\ ]f\|_\kappa \le C_{\sigma,n}\left\{(N/\rho)^{2\sigma}\left\|f\right\|_\kappa + (N/\rho)^{\kappa+2\sigma}\left\|f\right\|_0\right\}. \tag{3.4}$$

Remark 3.1. Observe for $\tilde{\rho} = \frac{(N-1)\rho}{N}$, $\varphi_{\rho,N}\widetilde{\Lambda}_v^\sigma(1-\varphi_{\tilde{\rho},N})f = -\varphi_{\rho,N}\left[\widetilde{\Lambda}_v^\sigma, \varphi_{\tilde{\rho},N}\right] f$. Then as a consequence of (3.3), we have

$$\|\varphi_{\rho,N}\widetilde{\Lambda}_v^\sigma(1-\varphi_{\tilde{\rho},N})f\|_\kappa \le C_{\sigma,n}\left\{(N/\rho)^\sigma \|f\|_\kappa + (N/\rho)^{\kappa+\sigma}\|f\|_0\right\}.$$

Hence, in the following, we omit the detailed discussions for such terms.

Proof. To simplify the notation, in the course of the proof, we shall use C to denote a constant which depend only on n and σ and may be different in different contexts. We denote by (τ,ξ,η) the Fourier transformation variable of (t,x,v). $\mathcal{F}_{t,x}(g)$, $\mathcal{F}_v(g)$ are the partial Fourier transforms, and $\hat{g}$ is the full Fourier transform with respect to (t,x,v). Set

$$h = [\widetilde{\Lambda}_v^\sigma,\ \varphi_{\rho,N}]f, \quad H(v) = H_{\tau,\xi}(v) = \mathcal{F}_{t,x}(f)(\tau,\xi,v)$$

In the following discussion, we always write $H(v)$ for $H_{\tau,\xi}(v)$, if there is no risk of causing the confusion. It is clear that

$$\mathcal{F}_{t,x}(h)(\tau,\xi,v) = [\widetilde{\Lambda}_v^\sigma,\ \varphi_{\rho,N}]\mathcal{F}_{t,x}(f)(\tau,\xi,v) = [\widetilde{\Lambda}_v^\sigma,\ \varphi_{\rho,N}]H(v). \tag{3.5}$$

Observe that the desired inequality (3.3) will follow if we show that, for each fixed pair (τ,ξ),

$$\left\|\left[\widetilde{\Lambda}_v^\sigma,\ \varphi_{\rho,N}(\cdot)\right]H(\cdot)\right\|_{H^\kappa(\mathbb{R}^n_v)} \le C\Big\{(N/\rho)^\sigma \|H(\cdot)\|_{H^\kappa(\mathbb{R}^n_v)} + (N/\rho)^{\kappa+\sigma}\|H(\cdot)\|_{L^2(\mathbb{R}^n_v)}\Big\}. \tag{3.6}$$

Indeed, a direct computation yields that

$$\begin{aligned}
\|h\|_\kappa^2 &= \int_{\mathbb{R}^{2n+1}} (1+\tau^2+|\xi|^2+|\eta|^2)^\kappa \left|\hat{h}(\tau,\xi,\eta)\right|^2 d\tau d\xi d\eta \\
&\le C\int_{\mathbb{R}^{2n+1}} \left\{(1+\tau^2+|\xi|^2)^\kappa + |\eta|^{2\kappa}\right\}\left|\hat{h}(\tau,\xi,\eta)\right|^2 d\tau d\xi d\eta \\
&\le C\int_{\mathbb{R}^{n+1}} (1+\tau^2+|\xi|^2)^\kappa \left(\int_{\mathbb{R}^n}\left(1+|\eta|^2\right)^\kappa \left|\hat{h}(\tau,\xi,\eta)\right|^2 d\eta\right) d\tau d\xi \\
&= C\int_{\mathbb{R}^{n+1}} (1+\tau^2+|\xi|^2)^\kappa \left(\left\|\left[\widetilde{\Lambda}_v^\sigma,\ \varphi_{\rho,N}(\cdot)\right]H_{\tau,\xi}(\cdot)\right\|_{H^\kappa(\mathbb{R}^n_v)}\right) d\tau d\xi.
\end{aligned}$$

This along with (3.6) yields the desired inequality (3.3).

Next, we shall prove (3.6). First, for any $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$|D_v|^\sigma\, g(v) = C_\sigma \int_{\mathbb{R}^n} \frac{g(v)-g(v-\tilde{v})}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} \tag{3.7}$$

with $C_\sigma \neq 0$ being a complex constant depending only on σ and the dimension n.

In fact,

$$\int_{\mathbb{R}^n} \frac{g(v)-g(v-\tilde{v})}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} = \int_{\mathbb{R}^n} \mathcal{F}_v(g)(\eta)\, e^{i\,v\cdot\eta}\left(\int_{\mathbb{R}^n} \frac{1-e^{-i\,\tilde{v}\cdot\eta}}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v}\right) d\eta$$

On the other hand, it is clear that

$$\int_{\mathbb{R}^n} \frac{1-e^{-i\,\tilde{v}\cdot\eta}}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} = |\eta|^\sigma \int_{\mathbb{R}^n} \frac{1-e^{-i\,u\cdot\frac{\eta}{|\eta|}}}{|u|^{n+\sigma}}\, du.$$

Observe that $\int_{\mathbb{R}^n} \frac{1-e^{i\,u\cdot\frac{\eta}{|\eta|}}}{|u|^{n+\sigma}}\, du \neq 0$ is a complex constant depending only on σ and the dimension n, but independent of η. Then the above two equalities give (3.7).

Next, we use (3.7) to get

$$|D_v|^\sigma \big(H(v)\varphi_{\rho,N}(v)\big) = C_\sigma \int_{\mathbb{R}^n} \frac{H(v)\varphi_{\rho,N}(v) - H(v-\tilde{v})\varphi_{\rho,N}(v-\tilde{v})}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v}$$
$$= \varphi_{\rho,N}(v)|D_v|^\sigma\, H(v) + C_\sigma \int_{\mathbb{R}^n} \frac{H(v-\tilde{v})\big(\varphi_{\rho,N}(v) - \varphi_{\rho,N}(v-\tilde{v})\big)}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v},$$

which gives that

$$\big[|D_v|^\sigma,\, \varphi_{\rho,N}(v)\big]H(v) = C_\sigma \int_{\mathbb{R}^n} \frac{H(v-\tilde{v})\big(\varphi_{\rho,N}(v) - \varphi_{\rho,N}(v-\tilde{v})\big)}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v}. \tag{3.8}$$

Let $\widetilde{\chi}_{\rho/N}$ be the characteristic function of the set $\{v;\, |v| \le \rho/N\}$. By the above expression, we compute

$$\|\big[|D_v|^\sigma,\, \varphi_{\rho,N}\big]H\|^2_{L^2(\mathbb{R}^n_v)} = |C_\sigma|^2 \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{H(v-\tilde{v})\big(\varphi_{\rho,N}(v) - \varphi_{\rho,N}(v-\tilde{v})\big)}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} \right|^2 dv$$
$$\le 2|C_\sigma|^2 \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{\widetilde{\chi}_{\rho/N}(\tilde{v}) H(v-\tilde{v})\big(\varphi_{\rho,N}(v) - \varphi_{\rho,N}(v-\tilde{v})\big)}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} \right|^2 dv$$
$$+ 2|C_\sigma|^2 \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{\big(1-\widetilde{\chi}_{\rho/N}(\tilde{v})\big)\, H(v-\tilde{v})\big(\varphi_{\rho,N}(v) - \varphi_{\rho,N}(v-\tilde{v})\big)}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} \right|^2 dv$$
$$\le C\big(\sup|\partial_v\, \varphi_{\rho,N}|\big)^2 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\widetilde{\chi}_{\rho/N}(\tilde{v})\, |H(v-\tilde{v})|}{|\tilde{v}|^{n+\sigma-1}}\, d\tilde{v} \right)^2 dv$$
$$+ C\big(\sup|\varphi_{\rho,N}|\big)^2 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\big(1-\widetilde{\chi}_{\rho/N}(\tilde{v})\big)\, |H(v-\tilde{v})|}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} \right)^2 dv$$
$$=: \mathcal{A}_1 + \mathcal{A}_2,$$

For the term $\mathcal{A}_1$, Young's inequality for convolutions gives

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\widetilde{\chi}_{\rho/N}(\tilde{v})\, |H(v-\tilde{v})|}{|\tilde{v}|^{n+\sigma-1}}\, d\tilde{v} \right)^2 dv \le \|H\|^2_{L^2(\mathbb{R}_v)} \left\| \frac{\widetilde{\chi}_{\rho/N}(v)}{|v|^{n+\sigma-1}} \right\|^2_{L^1(\mathbb{R}_v)}.$$

Then (3.1) with $|\alpha| = 1$ and the following inequality

$$\left\| \frac{\widetilde{\chi}_{\rho/N}(v)}{|v|^{n+\sigma-1}} \right\|^2_{L^1(\mathbb{R}_v)} \le C \left(\int_0^{\rho/N} \frac{dr}{r^\sigma} \right)^2 \le C\,(\rho/N)^{2(1-\sigma)}$$

deduce that

$$\mathcal{A}_1 \le C\,(N/\rho)^{2\sigma}\, \|H\|^2_{L^2(\mathbb{R}^n_v)}.$$

Similarly, we can use (3.1) with $|\alpha| = 0$ and the inequality

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\big(1-\widetilde{\chi}_{\rho/N}(\tilde{v})\big)\, |H(v-\tilde{v})|}{|\tilde{v}|^{n+\sigma}}\, d\tilde{v} \right)^2 dv \le \|H\|^2_{L^2(\mathbb{R}_v)} \left\| \frac{1-\widetilde{\chi}_{\rho/N}(v)}{|v|^{n+\sigma}} \right\|^2_{L^1(\mathbb{R}_v)}$$
$$\le C\,(\rho/N)^{-2\sigma}\, \|H\|^2_{L^2(\mathbb{R}_v)}$$

to get

$$\mathcal{A}_2 \le C\,(N/\rho)^{2\sigma}\, \|H\|^2_{L^2(\mathbb{R}^n_v)}.$$

On the other hand, it is trivial to see

$$\left\| \left[\left(|D_v|^\sigma - \widetilde{\Lambda}_v^\sigma \right),\, \varphi_{\rho,N} \right] H \right\|_{L^2(\mathbb{R}^n_v)} \le C \left\| H \right\|_{L^2(\mathbb{R}^n_v)}.$$

Now we combine these inequalities to conclude

$$\left\| \left[\widetilde{\Lambda}_v^\sigma,\, \varphi_{\rho,N} \right] H \right\|_{L^2(\mathbb{R}^n_v)} \le C \left(N/\rho \right)^\sigma \left\| H \right\|_{L^2(\mathbb{R}^n_v)}. \tag{3.9}$$

Next we treat $\left\| \left[\widetilde{\Lambda}_v^\sigma,\, \varphi_{\rho,N} \right] H \right\|_{H^\kappa(\mathbb{R}^n_v)}$. Similar to the above argument, we study only the commutator $\left\| \left[|D_v|^\sigma,\, \varphi_{\rho,N} \right] H \right\|_{H^\kappa(\mathbb{R}^n_v)}$. First, we consider the case when κ is a positive integer. Let α be an arbitrary multi-index with $|\alpha| \le \kappa$. Then taking derivatives in (3.8), and then using Leibnitz's formula; we get

$$\begin{aligned} &\partial_v^\alpha \left(\left[|D_v|^\sigma,\, \varphi_{\rho,N}(v) \right] H(v) \right) \\ &= C_\sigma \sum_{\beta \le \alpha} C_\alpha^\beta \int_{\mathbb{R}^n} \frac{\left(\partial_v^\beta H(v-\tilde{v}) \right) \cdot \left(\partial_v^{\alpha-\beta} \left(\varphi_{\rho,N}(v) - \varphi_{\rho,N}(v-\tilde{v}) \right) \right)}{|\tilde{v}|^{n+\sigma}} \, d\tilde{v}. \end{aligned}$$

Thus similar arguments as above show that

$$\left\| \partial_v^\alpha \left(\left[|D_v|^\sigma,\, \varphi_{\rho,N}(v) \right] H(v) \right) \right\|_{L^2(\mathbb{R}^n_v)} \le C \sum_{\beta \le \alpha} \left(N/\rho \right)^{|\alpha-\beta|+\sigma} \left\| \partial_v^\beta H \right\|_{L^2(\mathbb{R}^n_v)}.$$

Together with the interpolation inequality (2.2), we obtain

$$\begin{aligned} &\left\| \partial_v^\alpha \left(\left[|D_v|^\sigma,\, \varphi_{\rho,N}(v) \right] H(v) \right) \right\|_{L^2(\mathbb{R}^n_v)} \\ &\le C \left\{ \left(N/\rho \right)^\sigma \left\| H \right\|_{H^\kappa(\mathbb{R}^n_v)} + \left(N/\rho \right)^{|\alpha|+\sigma} \left\| H \right\|_{L^2(\mathbb{R}^n_v)} \right\}. \end{aligned}$$

Since $\alpha, |\alpha| \le \kappa$, is arbitrary, we conclude

$$\left\| \left[|D_v|^\sigma,\, \varphi_{\rho,N}(v) \right] H(v) \right\|_{H^\kappa(\mathbb{R}^n_v)} \le C \left\{ \left(N/\rho \right)^\sigma \left\| H \right\|_{H^\kappa(\mathbb{R}^n_v)} + \left(N/\rho \right)^{\kappa+\sigma} \left\| H \right\|_{L^2(\mathbb{R}^n_v)} \right\}.$$

This implies (3.6), when κ is a positive integer.

Now we consider the case when κ is not a integer. Without loss of generality, we may assume $0 < \kappa < 1$. Write $\kappa + \sigma = 1 + \mu$. Then $0 \le \mu < 1$, and

$$\begin{aligned} \left\| \left[|D_v|^{\kappa+\sigma},\, \varphi_{\rho,N}(v) \right] H(v) \right\|_{L^2(\mathbb{R}^n_v)} &\le \left\| \left[|D_v|^\mu,\, \varphi_{\rho,N}(v) \right] H(v) \right\|_{H^1(\mathbb{R}^n_v)} \\ &\quad + \left\| \left[|D_v|^1,\, \varphi_{\rho,N}(v) \right] |D_v|^\mu H(v) \right\|_{L^2(\mathbb{R}^n_v)}. \end{aligned}$$

We have treated the first term on the right, that is,

$$\left\| \left[|D_v|^\mu,\, \varphi_{\rho,N}(v) \right] H(v) \right\|_{H^1(\mathbb{R}^n_v)} \le C \left\{ \left(N/\rho \right)^\mu \left\| H \right\|_{H^1(\mathbb{R}^n_v)} + \left(N/\rho \right)^{1+\mu} \left\| H \right\|_{L^2(\mathbb{R}^n_v)} \right\}.$$

On the other hand, one has

$$\left\| \left[|D_v|^1,\, \varphi_{\rho,N}(v) \right] |D_v|^\mu H(v) \right\|_{L^2(\mathbb{R}^n_v)} \le C \left(N/\rho \right) \left\| H \right\|_{H^\mu(\mathbb{R}^n_v)}.$$

For the proof of this estimate, we refer to [10] for instance. Hence

$$\begin{aligned} \left\| \left[|D_v|^{\kappa+\sigma},\, \varphi_{\rho,N}(v) \right] H(v) \right\|_{L^2(\mathbb{R}^n_v)} \le C \Big\{ &\left(N/\rho \right)^\mu \left\| H \right\|_{H^1(\mathbb{R}^n_v)} + \left(N/\rho \right)^{1+\mu} \left\| H \right\|_{L^2(\mathbb{R}^n_v)} \\ &+ \left(N/\rho \right) \left\| H \right\|_{H^\mu(\mathbb{R}^n_v)} \Big\}. \end{aligned}$$

Notice that $\kappa \geq 1$,. The interpolation inequality (2.2) gives

$$\begin{aligned}
&\Big\| \big[|D_v|^{\kappa+\sigma} \,,\, \varphi_{\rho,N}(v) \big] H(v) \Big\|_{L^2(\mathbb{R}^n_v)} \\
&\leq C \Big\{ (N/\rho)^{\sigma} \|H\|_{H^{\kappa}(\mathbb{R}^n_v)} + (N/\rho)^{\kappa+\sigma} \|H\|_{L^2(\mathbb{R}^n_v)} \Big\}.
\end{aligned}$$

Since $0 < \kappa < 1$, then

$$\begin{aligned}
&\big\| \big[|D_v|^{\kappa} \,,\, \varphi_{\rho,N}(v) \big] \, |D_v|^{\sigma} H(v) \big\|_{L^2(\mathbb{R}^n_v)} \\
&\leq C \Big\{ (N/\rho)^{\kappa} \|H\|_{H^{\sigma}(\mathbb{R}^n_v)} + (N/\rho)^{\kappa+\sigma} \|H\|_{L^2(\mathbb{R}^n_v)} \Big\} \\
&\leq C \Big\{ (N/\rho)^{\sigma} \|H\|_{H^{\kappa}(\mathbb{R}^n_v)} + (N/\rho)^{\kappa+\sigma} \|H\|_{L^2(\mathbb{R}^n_v)} \Big\}.
\end{aligned}$$

In the last inequality, we have used the interpolation inequality (2.2). The above two inequalities yield that

$$\begin{aligned}
&\big\| |D_v|^{\kappa} \big[|D_v|^{\sigma} \,,\, \varphi_{\rho,N}(v) \big] H(v) \big\|_{L^2(\mathbb{R}^n_v)} \\
&\leq \Big\| \big[|D_v|^{\kappa+\sigma} \,,\, \varphi_{\rho,N}(v) \big] H(v) \Big\|_{L^2(\mathbb{R}^n_v)} \\
&\quad + \big\| \big[|D_v|^{\kappa} \,,\, \varphi_{\rho,N}(v) \big] \, |D_v|^{\sigma} H(v) \big\|_{L^2(\mathbb{R}^n_v)} \\
&\leq C \Big\{ (N/\rho)^{\sigma} \|H\|_{H^{\kappa}(\mathbb{R}^n_v)} + (N/\rho)^{\kappa+\sigma} \|H\|_{L^2(\mathbb{R}^n_v)} \Big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\big\| \big[|D_v|^{\sigma} \,,\, \varphi_{\rho,N}(v) \big] H(v) \big\|_{H^{\kappa}(\mathbb{R}^n_v)} \\
&\leq C \Big\{ \big\| |D_v|^{\kappa} \big[|D_v|^{\sigma} \,,\, \varphi_{\rho,N}(v) \big] H(v) \big\|_{L^2(\mathbb{R}^n_v)} \\
&\quad + \big\| \big[|D_v|^{\sigma} \,,\, \varphi_{\rho,N}(v) \big] H(v) \big\|_{L^2(\mathbb{R}^n_v)} \Big\} \\
&\leq C \Big\{ (N/\rho)^{\sigma} \|H\|_{H^{\kappa}(\mathbb{R}^n_v)} + (N/\rho)^{\kappa+\sigma} \|H\|_{L^2(\mathbb{R}^n_v)} \Big\}.
\end{aligned}$$

This implies (3.6) for general $\kappa, 1 \leq \kappa \leq n+2$, and thus (3.3) follows. The inequality (3.4) can be handled quite similarly. Thus the proof of Lemma 3.2 is complete. □

4. Gevrey regularity of linear operators

In this section, we prove the Gevrey hypoellipticity of $\mathcal{P}$. We will follow the idea of M.Durand [10]. We consider the following linear equation

$$\mathcal{P}u = \partial_t u + v \cdot \partial_x u + a(t,x,v)(-\widetilde{\triangle}_v)^{\sigma} u = f, \quad (t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \tag{4.1}$$

where $0 < \sigma < 1$. From Theorem 2.4, any weak solution of the above equation is in $C^{\infty}(\mathbb{R}^{2n+1})$ if $f \in C^{\infty}(\mathbb{R}^{2n+1})$. Hence, we start from a C^{∞} solution, and prove the Gevrey hypoellipticity in the following proposition, where Ω and $W = 2\Omega$ are open domains of $\mathbb{R}^{2n+1}$ defined in the section 3.

Proposition 4.1. *Set $\delta = \max\{\frac{\sigma}{4}, \ \frac{\sigma}{2} - \frac{1}{6}\}$ and let $s \geq \frac{2}{\delta}$. Suppose the coefficient $a(t,x,v) \in G^s(\bar{\Omega}), a > 0$, and $u \in C^{\infty}(\bar{W})$ be such that $\mathcal{P}u = f \in G^s(\bar{\Omega})$. Then*

there exits a constant L such that for any $r \in [0,1]$ and any $N \in \mathbb{N}$, $N \geq 4$,

$$(E)_{r,N} \qquad \begin{aligned} &\|\Phi_{\rho,N} D^\alpha u\|_{r+n+1} + \|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \\ &\leq \tfrac{L^{|\alpha|-1}}{\rho^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s \left(\tfrac{N}{\rho}\right)^{\frac{8}{3}r} \end{aligned}$$

holds for any $\alpha \in \mathbb{N}^{2n+1}$, $|\alpha| = N$ and any $0 < \rho < 1$. Here and in the sequel we denote $\widetilde{\Lambda}^\sigma = \widetilde{\Lambda}_v^\sigma = (-\widetilde{\triangle}_v)^{\frac{\sigma}{2}}$ for simplification.

Remark 4.1. Here the Gevrey constant L of u is determined by the Gevrey constants B_a and B_f of the functions $a, f \in G^s(\bar{\Omega})$, and depends only on $s, \sigma, n, \|u\|_{H^{n+6}(W)}$ and $\|a\|_{C^{2n+2}(\Omega)}$. This can be seen in the proof of Lemma 4.3, Lemma 4.4 and Lemma 4.5. .

As an immediate consequence of the above proposition, we have

Proposition 4.2. *Under the same assumption as in Proposition 4.1, we have $u \in G^s(\Omega)$.*

Indeed, for any compact subset K of Ω, we have $K \subset \Omega_{\rho_0}$ for some ρ_0, $0 < \rho_0 < 1$. Then for any $\alpha \in \mathbb{N}^{2n+1}$, $|\alpha| = N \geq 4$, $(E)_{0,N}$ gives

$$\|D^\alpha u\|_{L^2(K)} \leq \|\Phi_{\rho_0,N} D^\alpha u\|_{n+1} \leq \tfrac{L^{|\alpha|-1}}{{\rho_0}^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s \leq \big(\tfrac{L}{{\rho_0}^{s+n}}\big)^{|\alpha|} (|\alpha|!)^s.$$

Taking $C_K = \frac{L}{{\rho_0}^{s+n}} + \|u\|_{C^4(K)}$, then for all α,

$$\|D^\alpha u\|_{L^2(K)} \leq C_K^{|\alpha|+1} (|\alpha|!)^s.$$

The conclusion of Proposition 4.2 follows.

Proof of Proposition 4.1. We prove the esitimate $(E)_{r,N}$ by induction on N. In the proof, we use C_n to denote constants which depend only on n, which may be different in different contexts. Let Φ be an arbitrary fixed function compactly supported in W such that $\Phi = 1$ in Ω. First, we prove the first step of the induction for $N = 4$. For all $|\alpha| = 4$, we use (3.2) in Lemma 3.1 to compute

$$\begin{aligned} &\|\Phi_{\rho,3} D^\alpha u\|_{r+n+1} + \|\Phi_{\rho,3} \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \\ &\leq C_n \left(\frac{3}{\rho}\right)^{n+2} \left\{ \|\Phi D^\alpha u\|_{r+n+1} + \|\Phi \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \right\}, \end{aligned}$$

On the other hand , since $|\alpha| = 4$,

$$\|\Phi D^\alpha u\|_{r+n+1} + \|\Phi \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \leq C_n \|u\|_{H^{n+6}(W)}.$$

The term on the left side is bounded by the smoothness of u. Combing these, we obtain

$$(E)_{r,4} \qquad \|\Phi_{\rho,3} D^\alpha u\|_{r+n+1} + \|\Phi_{\rho,3} \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \leq \frac{C_n \|u\|_{H^{n+6}(W)}}{\rho^{(n+2)}} \leq \frac{L_0^3}{\rho^{s+n}}.$$

Thus $(E)_{r,4}$ is true if we take $L \geq C_n \|u\|_{H^{n+6}(W)} + 1$. Let now $N > 4$ and assume that $(E)_{r,N-1}$ holds for any $r \in [0,1]$. We need to show $(E)_{r,N}$ still holds with a constant L independents of N or $r \in [0,\ 1]$. We denote

$$\|D^j u\|_r = \sum_{|\gamma|=j} \|D^\gamma u\|_r.$$

In the following discussion, we fix N. For each $0<\rho<1$, define $\tilde{\rho}=\frac{N-1}{N}\rho$, $\tilde{\tilde{\rho}}=\frac{N-2}{N}\rho$. Let $\Phi_{\rho,N}$ be the cutoff function constructed in the previous section which satisfies the property (3.1). The following fact will be used frequently, for $k=1,2,\cdots,N$ with $N\geq 4$,

$$\frac{1}{\rho^{(s+n)k}}\leq\frac{1}{\tilde{\rho}^{(s+n)k}}\leq\frac{1}{\tilde{\tilde{\rho}}^{(s+n)k}}=\frac{1}{\rho^{(s+n)k}}\times\big(\frac{N}{N-2}\big)^{(s+n)k}\leq\frac{36^{s+n}}{\rho^{(s+n)k}}. \tag{4.2}$$

We shall proceed to prove the truth of $(E)_{r,N}$ by the following four lemmas. The first one is a technical lemma, and the second lemma is devoted to the proof of the truth of $(E)_{r,N}$ for $r=0$. In the third one, we prove that $(E)_{r,N}$ holds for $0\leq r\leq\frac{\delta}{2}$, and in the last one we prove that $(E)_{r,N}$ holds for all r with $0\leq r\leq 1$.

Lemma 4.3. *Let $s\geq 3$ be a given real number and $k\geq 5$ be any given integer. Assume the estimate $(E)_{0,m}$ holds, i.e.*

$$\|\Phi_{\rho,m}D^{\gamma}u\|_{n+1}+\Big\|\Phi_{\rho,m}\widetilde{\Lambda}^{\sigma}D^{\gamma}u\Big\|_{-\frac{\delta}{2}+n+1}\leq\frac{L^{m-1}}{\rho^{(s+n)(m-3)}}\big((m-3)!\big)^{s} \tag{4.3}$$

holds for all γ with $|\gamma|=m<k$, and all $0<\rho<1$. Then if $L\geq 4^{n+3}(\|u\|_{H^{n+6}(W)}+1)$, one has, for all β with $|\beta|=k$,

$$(k/\rho)^{n+3}\left\|\Phi_{\rho,k}D^{\beta}u\right\|_0+(k/\rho)^{n+3}\left\|\Phi_{\rho,k}\widetilde{\Lambda}^{\sigma}D^{\beta}u\right\|_0\leq\frac{L^{k-2}}{\rho^{(s+n)(k-3)}}\big((k-3)!\big)^{s}. \tag{4.4}$$

Proof. Without loss of generality, we may assume $k>n+4$, for, otherwise, in the case when $5\leq k\leq n+4$, it is obvious that for all β with $|\beta|=k\leq n+4$,

$$\begin{aligned}&(k/\rho)^{n+3}\left\|\Phi_{\rho,k}D^{\beta}u\right\|_0+(k/\rho)^{n+3}\left\|\Phi_{\rho,k}\widetilde{\Lambda}^{\sigma}D^{\beta}u\right\|_0\\&\leq(1/\rho)^{(s+n)(k-3)}2^{(n+3)k}\left\|u\right\|_{H^{n+6}(W)}.\end{aligned}$$

Then the desired inequality (4.4) follows if $L\geq 4^{n+3}\left(\|u\|_{H^{n+6}(W)}+1\right)$.

Now for all $\beta,|\beta|=k>n+4$, we can find a multi-index $\tilde{\beta}\leq\beta$ such that $|\tilde{\beta}|=n+1$. First we treat $(k/\rho)^{n+3}\left\|\Phi_{\rho,k}D^{\beta}u\right\|_0$. Since $\Phi_{\frac{(k-1)\rho}{k},k-n-1}=1$ in $\operatorname{Supp}\Phi_{\rho,k}$, then the following relation is clear:

$$\left\|\Phi_{\rho,k}D^{\beta}u\right\|_0=\left\|\Phi_{\rho,k}D^{\tilde{\beta}}\Phi_{\frac{(k-1)\rho}{k},k-n-1}D^{\beta-\tilde{\beta}}u\right\|_0\leq\left\|\Phi_{\frac{(k-1)\rho}{k},k-n-1}D^{\beta-\tilde{\beta}}u\right\|_{n+1}.$$

Observe $\left|\beta-\tilde{\beta}\right|=k-n-1$, then we use the above relation and the assumption (4.3) to compute, for $L\geq 4^{n+3}\left(\|u\|_{H^{n+6}(W)}+1\right)$,

$$\begin{aligned}(k/\rho)^{n+3}\left\|\Phi_{\rho,k}D^{\beta}u\right\|_0&\leq(k/\rho)^{n+3}\left\|\Phi_{\frac{(k-1)\rho}{k},k-n-1}D^{\beta-\tilde{\beta}}u\right\|_{n+1}\\&\leq(k/\rho)^{n+3}\frac{L^{k-n-2}}{\rho^{(s+n)(k-n-4)}}\big((k-n-4)!\big)^{s}\\&\leq\frac{5(n/L)^{n}L^{k-2}}{\rho^{(s+n)(k-3)}}\big((k-3)!\big)^{s}\\&\leq\frac{1}{2}\frac{L^{k-2}}{\rho^{(s+n)(k-3)}}\big((k-3)!\big)^{s}.\end{aligned}$$

In the same way, we can get the estimate on the term $(k/\rho)^{n+3}\left\|\Phi_{\rho,k}\widetilde{\Lambda}^{\sigma}D^{\beta}u\right\|_0$, that is,

$$(k/\rho)^{n+3}\left\|\Phi_{\rho,k}\widetilde{\Lambda}^{\sigma}D^{\beta}u\right\|_0 \leq \frac{1}{2}\frac{L^{k-2}}{\rho^{(s+n)(k-3)}}\big((k-3)!\big)^s.$$

Thus by the above two inequalities, we get the desired inequality (4.4). This completes the proof. □

Lemma 4.4. *Assume that $(E)_{r,N-1}$ is true for any $r \in [0,1]$. Then there exists a constant C_1, depending only on the Gevrey index s and the dimension n, such that, if $L \geq 4^{n+3}\left(\|u\|_{H^{n+6}(W)}+1\right)$,*

$$\|\Phi_{\rho,N}D^{\alpha}u\|_{n+1} + \|\Phi_{\rho,N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1} \leq \frac{C_1L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \tag{4.5}$$

for any $\alpha \in \mathbb{N}^{2n+1}$, $|\alpha| = N$, and any $0 < \rho < 1$.

Remark 4.2. In fact, this is $(E)_{r,N}$ for $r = 0$ if we choose L such that $L \geq C_1$ and $L \geq 4^{n+3}\left(\|u\|_{H^{n+6}(W)}+1\right)$.

Proof. We choose a multi-index β with $|\alpha| = |\beta| + 1$. Then $|\beta| = N-1$. Recall $\tilde{\rho} = \frac{N-1}{N}\rho$. By the construction, $\Phi_{\tilde{\rho},N-1} = 1$ in $\operatorname{Supp}\Phi_{\rho,N}$. Thus

$$\begin{aligned}
&\|\Phi_{\rho,N}D^{\alpha}u\|_{n+1} \leq \|\Phi_{\rho,N}D^{\beta}u\|_{1+n+1} + \|(D\Phi_{\rho,N})D^{\beta}u\|_{n+1}\\
&\leq \|\Phi_{\rho,N}\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_{1+n+1} + \|(D\Phi_{\rho,N})\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_{n+1}\\
&\leq C_n\left\{\|\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_{1+n+1} + (N/\rho)\|\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_{n+1} + (N/\rho)^{n+2}\|\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_0\right\},
\end{aligned}$$

In the last inequality, we have used Lemma 3.1. For the third term on the right-hand side, we use Lemma 4.3 with $k = N-1$ to obtain

$$\begin{aligned}
(N/\rho)^{n+2}\|\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_0 &= \frac{N-1}{\tilde{\rho}}\left\{\left(\frac{N-1}{\tilde{\rho}}\right)^{n+2}\|\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_0\right\}\\
&\leq \frac{N-1}{\tilde{\rho}}\left\{\left(\frac{N-1}{\tilde{\rho}}\right)^{n+3}\|\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_0\right\}\\
&\leq \frac{N-1}{\tilde{\rho}}\frac{L^{N-3}}{\tilde{\rho}^{(s+n)(N-4)}}\big((N-4)!\big)^s\\
&\leq \frac{2L^{|\alpha|-2}}{\tilde{\rho}^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.
\end{aligned}$$

Applying the relation (4.2), we get

$$(N/\rho)^{n+2}\|\Phi_{\tilde{\rho},N-1}D^{\beta}u\|_0 \leq \frac{20^{s+n}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.$$

On the other hand, by the induction assumption that $(E)_{r,N-1}$ holds for any r with $0 \le r \le 1$, we have immediately

$$\begin{aligned}
&\|\Phi_{\tilde{\rho},N-1} D^\beta u\|_{1+n+1} + (N/\rho)\|\Phi_{\tilde{\rho},N-1} D^\beta u\|_{n+1} \\
&\le \frac{L^{|\beta|-1}}{\tilde{\rho}^{(s+n)(|\beta|-3)}}\big((|\beta|-3)!\big)^s (N/\tilde{\rho})^s + (N/\rho)\frac{L^{|\beta|-1}}{\tilde{\rho}^{(s+n)(|\beta|-3)}}\big((|\beta|-3)!\big)^s \\
&\le \frac{2L^{|\alpha|-2}}{\tilde{\rho}^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \big(N/(N-3)\big)^s \\
&\le \frac{30^{s+n} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.
\end{aligned}$$

Thus

$$\|\Phi_{\rho,N} D^\alpha u\|_{n+1} \le \frac{30^{s+n} C_n L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.$$

By exactly the same calculation, we obtain

$$\|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{-\frac{\delta}{2}+n+1} \le \frac{30^{s+n} C_n L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.$$

Taking $C_1 = 60^{s+n} C_n$ with C_n being the constant appearing in Lemma 3.1, we obtain (4.5). This completes the proof of Lemma 4.4.

□

Lemma 4.5. *Assume that $(E)_{r,N-1}$ is true for any $r \in [0,1]$. Then there exists a constant C_2, depending only on σ, the Gevrey index s, the dimension n and $\|u\|_{H^{n+6}(W)}, \|a\|_{C^{n+2}(\bar{\Omega})}$, such that for any $0 \le r \le \frac{\delta}{2}$, if*

$$L \ge \max\left\{2^{s+1}B_a,\, B_f,\, 4^{n+3}\left(\|u\|_{H^{n+6}(W)} + 1\right)\right\}$$

with B_a, B_f being the Gevrey constants of $a, f \in G^s(\bar{\Omega})$, we have that

(4.6)
$$\|\Phi_{\rho,N} D^\alpha u\|_{r+n+1} + \|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \le \frac{C_2 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{rs},$$

for any $\alpha \in \mathbb{N}^{2n+1}$, $|\alpha| = N$.

Remark 4.3. The assumption that $L \ge 2^{s+1}B_a$ will be needed in Step 2 of the following proof of this lemma, while that $L \ge B_f$ will be required in Step 3. That $L \ge 4^{n+3}\left(\|u\|_{H^{n+6}(W)} + 1\right)$ is required because in the sequel we will use frequently the conclusion of Lemma 4.3 where such a assumption is presented.

Proof. In this proof, we shall use $\widetilde{C}_j, j \ge 0$, to denote different constants which are greater than 1 and depend only on $s, \sigma, n, \|u\|_{H^{n+6}(W)}$ and $\|a\|_{C^{2n+2}(\Omega)}$. The conclusion will follow if we prove that

$$\|\Phi_{\rho,N} D^\alpha u\|_{\frac{\delta}{2}+n+1} + \|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{n+1} \le \frac{\widetilde{C}_0 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s\delta}{2}}. \tag{4.7}$$

Indeed, from (4.7) we know that (4.6) is true for $r = \frac{\delta}{2}$. The truth of (4.6) for the general $r, 0 \le r \le \frac{\delta}{2}$, follows from the interpolation inequality (2.2) and Lemma 4.4.

To prove (4.7), we shall proceed in the following four steps.

Step 1. In this step we prove

$$\|a[\widetilde{\Lambda}^{2\sigma},\ \Phi_{\rho,N}D^{\alpha}]u\|_{-\frac{\delta}{2}+n+1} \leq \frac{\widetilde{C}_1 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s\delta}{2}}. \tag{4.8}$$

Recall $\Phi_{\rho,N}(t,x,v)=\psi_{\rho,N}(t,x)\varphi_{\rho,N}(v)$ with $\psi_{\rho,N},\varphi_{\rho,N}$ being the cut-off functions constructed in Section 3. First, notice that $\psi_{\tilde{\rho},N}=1$ in the support of $\psi_{\rho,N}$, and $\varphi_{\tilde{\rho},N}=1$ in the support of $\varphi_{\rho,N}$. It then follows that

$$\begin{aligned}
&\|a[\widetilde{\Lambda}^{2\sigma},\ \Phi_{\rho,N}D^{\alpha}]u\|_{-\frac{\delta}{2}+n+1} = \|a[\widetilde{\Lambda}^{2\sigma},\ \varphi_{\rho,N}]\psi_{\rho,N}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1}\\
&\leq C_a\Big\{\|[\widetilde{\Lambda}^{\sigma},\ \varphi_{\rho,N}]\psi_{\rho,N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1} + \|[\widetilde{\Lambda}^{\sigma},\ [\widetilde{\Lambda}^{\sigma},\ \varphi_{\rho,N}]\,]\psi_{\rho,N}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1}\Big\}\\
&\leq C_a\Big\{\|[\widetilde{\Lambda}^{\sigma},\ \varphi_{\rho,N}]\psi_{\rho,N}\,\psi_{\tilde{\rho},N}\,\varphi_{\tilde{\rho},N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1}\\
&\quad + \|[\widetilde{\Lambda}^{\sigma},\ [\widetilde{\Lambda}^{\sigma},\ \varphi_{\rho,N}]\,]\psi_{\rho,N}\,\psi_{\tilde{\rho},N}\,\varphi_{\tilde{\rho},N}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1}\Big\}\\
&= C_a\Big\{\|[\widetilde{\Lambda}^{\sigma},\ \varphi_{\rho,N}]\psi_{\rho,N}\Phi_{\tilde{\rho},N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1}\\
&\quad + \|[\widetilde{\Lambda}^{\sigma},\ [\widetilde{\Lambda}^{\sigma},\ \varphi_{\rho,N}]\,]\psi_{\rho,N}\Phi_{\tilde{\rho},N}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1}\Big\}\\
&=: (S_1)+(S_2),
\end{aligned}$$

where C_a is a constants depending only on the coefficient a through $\|a\|_{C^{n+2}(\bar{\Omega})}$. To estimate the term (S_1), we apply the inequality (3.3) in Lemma 3.2 and then (3.2) in Lemma 3.1. This gives

$$\begin{aligned}
(S_1) &\leq C_a C_{\sigma,n}\Big\{(N/\rho)^{\sigma}\Big\|\psi_{\rho,N}\Phi_{\tilde{\rho},N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\Big\|_{-\frac{\delta}{2}+n+1}\\
&\qquad + (N/\rho)^{n+1-\frac{\delta}{2}+\sigma}\Big\|\psi_{\rho,N}\Phi_{\tilde{\rho},N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\Big\|_{0}\Big\}\\
&\leq C_a C_{\sigma,n}\left\{(N/\rho)^{\sigma}\Big\|\Phi_{\tilde{\rho},N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\Big\|_{-\frac{\delta}{2}+n+1} + (N/\rho)^{n+1-\frac{\delta}{2}+\sigma}\Big\|\Phi_{\tilde{\rho},N}\widetilde{\Lambda}^{\sigma}D^{\alpha}u\Big\|_{0}\right\}\\
&=: (S_1)' + (S_1)''.
\end{aligned}$$

First, the estimate (4.5) in Lemma 4.4 yields

$$\begin{aligned}
(S_1)' &\leq C_a C_{\sigma,n} C_1\left(\frac{N}{\rho}\right)^{\sigma}\frac{L^{|\alpha|-2}}{\tilde{\rho}^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\\
&\leq \frac{\widetilde{C}_2 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}}.
\end{aligned}$$

In the last inequality, we used the fact $\frac{s\delta}{2}\geq 1>\sigma$. Next, we treat $(S_1)''$. By virtue of the induction assumption, the required condition (4.3) in Lemma 4.3 is satisfied with $k=N$. It thus follows from (4.4) that

$$\begin{aligned}
(S_1)'' &\leq C_a C_{\sigma,n}\left(\frac{N}{\rho}\right)^{\sigma}\frac{L^{|\alpha|-2}}{\tilde{\rho}^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\\
&\leq \frac{\widetilde{C}_2 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}}.
\end{aligned}$$

Thus

$$(S_1) \leq \frac{\widetilde{C}_3 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}}.$$

Now it remain to treat the term (S_2). By the similar arguments as above, the inequality (3.4) in Lemma 3.2 gives

$$(S_2) \le \widetilde{C}_4\,(N/\rho)^{2\sigma}\,\|\Phi_{\tilde\rho,N} D^\alpha u\|_{-\frac{\delta}{2}+n+1} + \widetilde{C}_4\,(N/\rho)^{n+1-\frac{\delta}{2}+2\sigma}\,\|\Phi_{\tilde\rho,N} D^\alpha u\|_0 := \mathcal{N}_1 + \mathcal{N}_2.$$

We first estimate $\mathcal{N}_1$. Choose a multi-index β with $|\alpha| = |\beta| + 1$. Then the similar arguments as the proof of Lemma 4.4 give

$$\begin{aligned}\|\Phi_{\tilde\rho,N} D^\alpha u\|_{-\frac{\delta}{2}+n+1} \le C_n\Big\{ &\|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_{(1-\frac{\delta}{2})+n+1} \\ &+ (N/\rho)\|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_{-\frac{\delta}{2}+n+1} + (N/\rho)^{n+2-\frac{\delta}{2}}\|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_0\Big\}.\end{aligned}$$

We recall $\tilde{\tilde\rho} = \frac{(N-2)\rho}{N}$. By the interpolation inequality (2.2),

$$(N/\rho)\|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_{-\frac{\delta}{2}+n+1} \le \|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_{(1-\frac{\delta}{2})+n+1} + \left(\frac{N}{\rho}\right)^{n+2-\frac{\delta}{2}} \|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_0\Big\}.$$

Therefore

$$\|\Phi_{\tilde\rho,N} D^\alpha u\|_{-\frac{\delta}{2}+n+1} \le C_n\Big\{ \|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_{(1-\frac{\delta}{2})+n+1} + (N/\rho)^{n+2-\frac{\delta}{2}}\|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_0\Big\}$$

Hence $\mathcal{N}_1 \le \mathcal{N}_{1,1} + \mathcal{N}_{1,2}$ with $\mathcal{N}_{1,1}$, $\mathcal{N}_{1,2}$ given by

$$\mathcal{N}_{1,1} = \widetilde{C}_5\left(\frac{N}{\rho}\right)^{2\sigma} \|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_{(1-\frac{\delta}{2})+n+1}, \quad \mathcal{N}_{1,2} = \widetilde{C}_5\left(\frac{N}{\rho}\right)^{n+2-\frac{\delta}{2}+2\sigma} \|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_0.$$

Since $(E)_{r,N-1}$ holds for all $r \in [0,1]$, then it follows that

$$\begin{aligned}\mathcal{N}_{1,1} &\le \widetilde{C}_5\left(\frac{N}{\rho}\right)^{2\sigma} \frac{L^{|\alpha|-2}}{\tilde{\tilde\rho}^{(s+n)(|\alpha|-4)}}\big((|\alpha|-4)!\big)^s \left(\frac{N-1}{\tilde{\tilde\rho}}\right)^{s(1-\frac{\delta}{2})} \\ &\le \widetilde{C}_6\left(\frac{N}{\rho}\right)^{2\sigma-\frac{s\delta}{2}} \frac{L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-4)}}\big((|\alpha|-4)!\big)^s \left(\frac{N-3}{\rho}\right)^s \\ &\le \widetilde{C}_6\left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}} \frac{L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.\end{aligned}$$

In the last inequality, we used again the fact $\frac{s\delta}{2} \ge \sigma$. For the term $\mathcal{N}_{1,2}$, we use Lemma 4.3 with $k = N-1$. This gives

$$\begin{aligned}\mathcal{N}_{1,2} &\le \widetilde{C}_5\left(\frac{N-2}{\tilde{\tilde\rho}}\right)^{n+2-\frac{\delta}{2}+2\sigma} \left(\frac{N-1}{\tilde{\tilde\rho}}\right)^{-(n+3)} \left\{\left(\frac{N-1}{\tilde{\tilde\rho}}\right)^{(n+3)} \|\Phi_{\tilde{\tilde\rho},N-1} D^\beta u\|_0\right\} \\ &\le \widetilde{C}_5\left(\frac{N-1}{\tilde{\tilde\rho}}\right)^{-1-\frac{\delta}{2}+2\sigma} \frac{L^{N-3}}{\tilde{\tilde\rho}^{(s+n)(N-4)}}\big((N-4)!\big)^s.\end{aligned}$$

Since $-1 - \frac{\delta}{2} + 2\sigma < s$, then it follows from the above inequality that

$$\mathcal{N}_{1,2} \le \frac{\widetilde{C}_7 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.$$

With the estimate on $\mathcal{N}_{1,1}$, one has

$$\mathcal{N}_1 = \mathcal{N}_{1,2} + \mathcal{N}_{1,2} \le \frac{\widetilde{C}_8 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s \left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}}.$$

In the following, we treat $\mathcal{N}_2 = \widetilde{C}_4 \left(N/\rho\right)^{n+1-\frac{\delta}{2}+2\sigma} \left\|\Phi_{\tilde{\rho},N} D^\alpha u\right\|_0$. Using Lemma 4.3 with $k = N$, we get

$$\begin{aligned}
\mathcal{N}_2 &\leq \widetilde{C}_4 \left(\frac{N}{\rho}\right)^{n+2-\frac{\delta}{2}+2\sigma} \left(\frac{N}{\tilde{\rho}}\right)^{-(n+3)} \left\{ \left(\frac{N}{\tilde{\rho}}\right)^{(n+3)} \left\|\Phi_{\tilde{\rho},N} D^\alpha u\right\|_0 \right\} \\
&\leq \widetilde{C}_4 \left(\frac{N}{\tilde{\rho}}\right)^{\sigma} \frac{L^{|\alpha|-2}}{\tilde{\rho}^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s \\
&\leq \frac{\widetilde{C}_9 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s \left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}}.
\end{aligned}$$

Thus,

$$(S_2) = \mathcal{N}_1 + \mathcal{N}_2 \leq \frac{\widetilde{C}_{10} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s \left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}}.$$

With the estimate on (S_1), we get the desired inequality (4.8). This completes the proof of Step 1.

Step 2. In this step, we prove

$$\left\|[\mathcal{P},\ \Phi_{\rho,N} D^\alpha] u\right\|_{-\frac{\delta}{2}+n+1} \leq \frac{\widetilde{C}_{11} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s\delta}{2}}. \tag{4.9}$$

Recall $\mathcal{P} = X_0 + a\widetilde{\Lambda}^{2\sigma}$ with $X_0 = \partial_t + v\cdot\partial_x$. Then a direct computation deduces that

$$\begin{aligned}
\left\|[\mathcal{P},\ \Phi_{\rho,N} D^\alpha] u\right\|_{-\frac{\delta}{2}+n+1} \quad &\leq \quad \left\|[X_0,\ \Phi_{\rho,N} D^\alpha] u\right\|_{-\frac{\delta}{2}+n+1} + \left\|a[\widetilde{\Lambda}^{2\sigma},\ \Phi_{\rho,N} D^\alpha] u\right\|_{-\frac{\delta}{2}+n+1} \\
&\qquad + \left\|\Phi_{\rho,N}[a,\ D^\alpha] \widetilde{\Lambda}^{2\sigma} u\right\|_{-\frac{\delta}{2}+n+1} \\
&=: \quad (I) + (II) + (III).
\end{aligned}$$

We have already handled the second term in Step 1. It remains to treat the first term (I) and the third term (III).

Observe that $[X_0,\ D^\alpha]$ equals to 0 or D^{α_0} for some α_0 with $|\alpha_0| \leq |\alpha|$. A direct verification yields

$$\begin{aligned}
(I) &\leq \left\|[X_0,\ \Phi_{\rho,N}] D^\alpha u\right\|_{n+1} + \left\|\Phi_{\rho,N} D^{\alpha_0} u\right\|_{n+1} \\
&\leq \left\|(D\Phi_{\rho,N}) \Phi_{\tilde{\rho},N} D^\alpha u\right\|_{n+1} + \left\|\Phi_{\rho,N} D^{\alpha_0} u\right\|_{n+1} \\
&\leq C_n \Big\{ (N/\rho) \left\|\Phi_{\tilde{\rho},N} D^\alpha u\right\|_{n+1} + (N/\rho)^{n+2} \left\|\Phi_{\tilde{\rho},N} D^\alpha u\right\|_0 + \left\|\Phi_{\rho,N} D^{\alpha_0} u\right\|_{n+1} \Big\}.
\end{aligned}$$

For the first term and the third term on the right-hand side, using (4.5) in Lemma 4.4, and noting that $\frac{s\delta}{2} \geq 1$, we obtain

$$\begin{aligned}
&C_n \Big\{ (N/\rho) \left\|\Phi_{\tilde{\rho},N} D^\alpha u\right\|_{n+1} + \left\|\Phi_{\rho,N} D^{\alpha_0} u\right\|_{n+1} \Big\} \\
&\leq C_n \big(N/\rho + 1\big) \frac{C_1 L^{|\alpha|-2}}{\tilde{\rho}^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s \\
&\leq \frac{\widetilde{C}_{12} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}} \big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s\delta}{2}}.
\end{aligned}$$

On the other hand, we use Lemma 4.3 with $k = N$ to get

$$C_n(N/\rho)^{n+2}\|\Phi_{\tilde{\rho},N}D^{\alpha}u\|_0 \le \frac{\widetilde{C}_{13}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.$$

Thus

$$(I) \le \frac{\widetilde{C}_{14}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.$$

Now it remains to eatimate (III). The Leibniz' formula yields

$$\begin{aligned}(III) &\le \sum_{0<|\gamma|\le|\alpha|} C_\alpha^\gamma \big\|\Phi_{\rho,N}(D^\gamma a)\widetilde{\Lambda}^{2\sigma}D^{\alpha-\gamma}u\big\|_{-\frac{\delta}{2}+n+1} \\ &\le C_n \sum_{0<|\gamma|\le|\alpha|} C_\alpha^\gamma \|D^\gamma a\|_{C^{n+2}(\bar{\Omega})}\cdot\|\Phi_{\rho,N}\widetilde{\Lambda}^{2\sigma}D^{\alpha-\gamma}u\big\|_{-\frac{\delta}{2}+n+1},\end{aligned} \tag{4.10}$$

where $C_\alpha^\gamma = \frac{\alpha!}{\gamma!(\alpha-\gamma)!}$ are the binomial coefficients. Since $a \in G^s(\bar{\Omega})$, letting B_a be the Gevrey constant of Gevrey function a on $\bar{\Omega}$, we have

(4.11)

$$\|D^\gamma a\|_{C^{n+2}(\bar{\Omega})} \le B_a^{|\gamma|-1}\big((|\gamma|-2)!\big)^s \text{ if } |\gamma|\ge 2, \quad \|D^\gamma a\|_{C^{n+2}(\bar{\Omega})} \le B_a \text{ if } |\gamma| = 0,\ 1.$$

On the other hand, observe that

$$\begin{aligned}\|\Phi_{\rho,N}\widetilde{\Lambda}^{2\sigma}D^{\alpha-\gamma}u\|_{-\frac{\delta}{2}+n+1} \le &\|[\widetilde{\Lambda}^\sigma,\ \Phi_{\rho,N}]\widetilde{\Lambda}^\sigma D^{\alpha-\gamma}u\|_{-\frac{\delta}{2}+n+1} \\ &+ \|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^{\alpha-\gamma}u\|_{(\sigma-\frac{\delta}{2})+n+1}.\end{aligned}$$

We have handled in Step 1 the first term on the right hand. This gives

$$\|[\widetilde{\Lambda}^\sigma,\ \Phi_{\rho,N}]\widetilde{\Lambda}^\sigma D^{\alpha-\gamma}u\|_{-\frac{\delta}{2}+n+1} \le \frac{\widetilde{C}_{15}L^{|\alpha|-|\gamma|-2}}{\rho^{(s+n)(|\alpha|-|\gamma|-3)}}\big((|\alpha|-|\gamma|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.$$

For the second term, note that $|\alpha|-|\gamma| \le N-1$ for $\gamma \ne 0$. We use the induction hypothesis that $(E)_{r,N-1}$ holds for all $r \in [0,1]$, to get, for $\gamma, 0<|\gamma|\le|\alpha|-3$, that

$$\|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^{\alpha-\gamma}u\|_{(\sigma-\frac{\delta}{2})+n+1} \le \frac{L^{|\alpha|-|\gamma|-1}}{\rho^{(s+n)(|\alpha|-|\gamma|-3)}}\big((|\alpha|-|\gamma|-3)!\big)^s(N/\rho)^{s\left(\sigma-\frac{\delta}{2}\right)}.$$

Observe that

$$\begin{aligned}(N/\rho)^{s\left(\sigma-\frac{\delta}{2}\right)} \le (N/\rho)^s &\le \frac{2^s(N-|\gamma|-2)^s+2^s(|\gamma|+2)^s}{\rho^s} \\ &\le 16^s(2^s)^{|\gamma|-1}(N-|\gamma|-2)^s\rho^{-s},\end{aligned}$$

Thus for γ with $0<|\gamma|\le|\alpha|-3 = N-3$, we have

$$\|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^{\alpha-\gamma}u\|_{(\sigma-\frac{\delta}{2})+n+1} \le \frac{16^s(2^s)^{|\gamma|-1}L^{|\alpha|-|\gamma|-1}}{\rho^{(s+n)(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s.$$

Note that the above inequality still holds for γ with $|\gamma| = |\alpha|-2$ if we take $L \ge 4^{n+1}\left(\|u\|_{H^{n+6}(W)}+1\right)$. Consequently, we combine these inequalities to obtain, for $0<|\gamma|\le|\alpha|-2$,

$$\|\Phi_{\rho,N}\widetilde{\Lambda}^{2\sigma}D^{\alpha-\gamma}u\|_{-\frac{\delta}{2}+n+1} \le \frac{\widetilde{C}_{16}(2^s)^{|\gamma|-1}L^{|\alpha|-|\gamma|-1}}{\rho^{(s+n)(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.$$

This together with (4.11) yields

$$\begin{aligned}
&\sum_{2\le|\gamma|\le|\alpha|-2} C_\alpha^\gamma \left\|D^\gamma a\right\|_{C^{n+2}(\bar\Omega)} \cdot \left\|\Phi_{\rho,N}\widetilde{\Lambda}^{2\sigma} D^{\alpha-\gamma}u\right\|_{-\frac{\delta}{2}+n+1}\\
&\le \left(\frac{N}{\rho}\right)^{\frac{s\delta}{2}} \sum_{2\le|\gamma|\le|\alpha|-2} \frac{|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!}(2^sB_a)^{|\gamma|-1}\big((|\gamma|-2)!\big)^s\\
&\qquad\times \frac{\widetilde{C}_{16}L^{|\alpha|-|\gamma|}}{\rho^{(s+n)(|\alpha|-|\gamma|-2)}}\big((|\alpha|-|\gamma|-2)!\big)^s\\
&\le \frac{\widetilde{C}_{16}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\,(N/\rho)^{\frac{s\delta}{2}} \sum_{2\le|\gamma|\le|\alpha|-2}\left(\frac{2^sB_a}{L}\right)^{|\gamma|-1}|\alpha|!\big((|\alpha|-4)!\big)^{s-1}\\
&\le \frac{\widetilde{C}_{16}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s\,(N/\rho)^{\frac{s\delta}{2}} \sum_{2\le|\gamma|\le|\alpha|-2}\left(\frac{2^sB_a}{L}\right)^{|\gamma|-1}\frac{|\alpha|^3}{(|\alpha|-3)^{s-1}}.
\end{aligned}$$

Observe that $s-1\ge 3$ and thus the series in the last inequality is bounded from above by a constant depending only on n if we take $L>2^{s+1}B_a$. Then we get

$$\begin{aligned}
&\sum_{2\le|\gamma|\le|\alpha|-2} C_\alpha^\gamma \left\|D^\gamma a\right\|_{C^{n+2}(\bar\Omega)} \cdot \left\|\Phi_{\rho,N}\widetilde{\Lambda}^{2\sigma} D^{\alpha-\gamma}u\right\|_{-\frac{\delta}{2}+n+1}\\
&\qquad\le \frac{\widetilde{C}_{17}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.
\end{aligned}$$

For $|\gamma|=1,\ |\alpha|-1$ or $|\alpha|$, we can compute directly

$$\begin{aligned}
&\sum_{|\gamma|=1,|\alpha|-1,|\alpha|} C_\alpha^\gamma \left\|D^\gamma a\right\|_{C^{n+2}(\bar\Omega)} \cdot \left\|\Phi_{\rho,N}\widetilde{\Lambda}^{2\sigma} D^{\alpha-\gamma}u\right\|_{-\frac{\delta}{2}+n+1}\\
&\qquad\le \frac{\widetilde{C}_{18}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.
\end{aligned}$$

Combination of the above two inequalities and (4.10) gives that

$$(III)\le \frac{\widetilde{C}_{19}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.$$

Consequently, the desired inequality (4.9) follows. This completes the proof of Step 2.

Step 3. In this step, we prove that if $\mathcal{P}u=f\in G^s(\bar\Omega)$ and if $L\ge\tilde B$ with $\tilde B$ the Gevrey constant of f,

$$\|\mathcal{P}\Phi_{\rho,N}D^\alpha u\|_{-\frac{\delta}{2}+n+1}\le \frac{\widetilde{C}_{20}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}. \tag{4.12}$$

Indeed, observe that

$$\|\mathcal{P}\Phi_{\rho,N}D^\alpha u\|_{-\frac{\delta}{2}+n+1}\le \|[\mathcal{P},\ \Phi_{\rho,N}D^\alpha]u\|_{-\frac{\delta}{2}+n+1}+\|\Phi_{\rho,N}D^\alpha\mathcal{P}u\|_{-\frac{\delta}{2}+n+1}.$$

Since $\mathcal{P}u=f\in G^s(\bar\Omega)$, then $\|D^\gamma\mathcal{P}f\|_{H^{n+2}(\Omega)}\le\tilde B$ if $|\gamma|<n+5$, and

$$\|D^\gamma\mathcal{P}f\|_{H^{n+2}(\Omega)}\le\tilde B^{|\gamma|-2}\left((|\gamma|-n-5)!\right)^s,\ \text{if}\ \ |\gamma|\ge n+5.$$

Hence,

$$\|\Phi_{\rho,N}D^\alpha\mathcal{P}u\|_{-\frac{\delta}{2}+n+1}\le C_n(N/\rho)^{n+2}\,\|D^\alpha\mathcal{P}f\|_{H^{n+2}(\Omega)}\le\frac{\widetilde{C}_{21}\tilde{B}^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s.$$

We take L such that $L>\tilde{B}$. Then the above inequality together with (4.9) in Step 2 yields immediately the inequality (4.12).

Step 4. In the last step we show (4.7). And hence the proof of Lemma 4.5 will be complete.

First we apply the subelliptic estimate (2.4), which is needed only here, to get

$$\|\Phi_{\rho,N}D^\alpha u\|_{\frac{\delta}{2}+n+1}\le C(\Omega)\big\{\|\mathcal{P}\Phi_{\rho,N}D^\alpha u\|_{-\frac{\delta}{2}+n+1}+\|\Phi_{\rho,N}D^\alpha u\|_{n+1}\big\}$$

with $C(\Omega)$ a constant depending only on the set Ω. Combining Lemma 4.4 with (4.12) in Step 3, we have

$$\|\Phi_{\rho,N}D^\alpha u\|_{\frac{\delta}{2}+n+1}\le\frac{\widetilde{C}_{22}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.\tag{4.13}$$

Next, we prove

$$\|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^\alpha u\|_{\frac{\delta}{2}-\frac{\delta}{2}+n+1}\le\frac{\widetilde{C}_{23}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.\tag{4.14}$$

Observe that

$$\|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^\alpha u\|_{\frac{\delta}{2}-\frac{\delta}{2}+n+1}\le\|[\widetilde{\Lambda}^\sigma,\ \Phi_{\rho,N}]D^\alpha u\|_{n+1}+\|\widetilde{\Lambda}^\sigma\Phi_{\rho,N}D^\alpha u\|_{n+1}.$$

By the same method as that in Step 1, we get the estimate on the first term of the right side, that is,

$$\|[\widetilde{\Lambda}^\sigma,\ \Phi_{\rho,N}]D^\alpha u\|_{n+1}\le\frac{\widetilde{C}_{24}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.$$

Then it remains to estimate the second term. A direct calculation gives that

$$\begin{aligned}
&\|\widetilde{\Lambda}^\sigma\Phi_{\rho,N}D^\alpha u\|_{n+1}^2\\
&=\mathrm{Re}\,\big(\mathcal{P}\Phi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\Phi_{\rho,N}D^\alpha u\big)-\mathrm{Re}\,\big(\widetilde{X}_0\Phi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\Phi_{\rho,N}D^\alpha u\big)\\
&=\mathrm{Re}\big(\mathcal{P}\Phi_{\rho,N}D^\alpha u,\ a^{-1}\Lambda^{2n+2}\Phi_{\rho,N}D^\alpha u\big)-\frac{1}{2}\big(\Phi_{\rho,N}D^\alpha u,\ [\Lambda^{2n+2},\ a^{-1}]\widetilde{X}_0\Phi_{\rho,N}D^\alpha u\big)\\
&\quad-\frac{1}{2}\big(\Phi_{\rho,N}D^\alpha u,\ [a^{-1}\Lambda^{2n+2},\ \widetilde{X}_0]\Phi_{\rho,N}D^\alpha u\big)\\
&\le\widetilde{C}_{25}\big\{\ \|\mathcal{P}\Phi_{\rho,N}D^\alpha u\|_{-\frac{\delta}{2}+n+1}^2+\|\Phi_{\rho,N}D^\alpha u\|_{\frac{\delta}{2}+n+1}^2\ \big\}.
\end{aligned}$$

This along with (4.12) and (4.13) shows at once

$$\Big\|\widetilde{\Lambda}^\sigma\Phi_{\rho,N}D^\alpha u\Big\|_{n+1}\le\frac{\widetilde{C}_{26}L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s(N/\rho)^{\frac{s\delta}{2}}.$$

and hence (4.14) follows if we choose $\widetilde{C}_{23}=\widetilde{C}_{24}+\widetilde{C}_{26}$. Now by (4.13) and (4.14), we obtain the desired inequality (4.7) if we choose $\widetilde{C}_0=\widetilde{C}_{22}+\widetilde{C}_{23}$. This completes the proof of Step 4.

□

In quite the similar way as that in the proof of Lemma 4.5, we can prove by induction the following

Lemma 4.6. *Assume that $(E)_{r,N-1}$ is true for any $r \in [0,1]$, then there exists a constant C_3, depending only on $\sigma, s, n, \|u\|_{H^{n+6}(W)}$ and $\|a\|_{C^{2n+2}(\Omega)}$, such that for any $r \in [\frac{\delta}{2}, \delta]$, if $L \geq \max\Big\{2^{s+1}B_a,\, B_f,\, 4^{n+3}\Big(\|u\|_{H^{n+6}(W)}+1\Big)\Big\}$, we have, for all α, $|\alpha| = N$,*

$$\|\Phi_{\rho,N}D^\alpha u\|_{r+n+1} + \|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \leq \frac{C_3 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{sr}.$$

Inductively, For any $m \in \mathbb{N}$ such that $\frac{m\delta}{2} < 1+\frac{\delta}{2}$, the above inequality still holds for any r with $\frac{(m-1)\delta}{2} \leq r \leq \frac{m\delta}{2}$, and hence for all r with $0 \leq r \leq 1$.

Proof. Since the arguments are quite similar as that in the previous lemma, we only present here a sketch of the proof. Assuming $(E)_{\frac{m\delta}{2},N}$ with $m \geq 0$ is valid, that is, for any α, $|\alpha| = N$,

$$\|\Phi_{\rho,N}D^\alpha u\|_{\frac{m\delta}{2}+n+1} + \|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^\alpha u\|_{\frac{(m-1)\delta}{2}+n+1} \leq \frac{C_2 L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{sm\delta}{2}},$$

we need to show the validity of $(E)_{\frac{(m+1)\delta}{2},N}$, and the validity of $(E)_{r,N}$ for $r \in [\frac{m\delta}{2}, \frac{(m+1)\delta}{2}]$ can be obtained by using interpolation inequality (2.2). To get the truth of $(E)_{\frac{(m+1)\delta}{2},N}$, it suffices to prove

$$\|\Phi_{\rho,N}D^\alpha u\|_{\frac{(m+1)\delta}{2}+n+1} \leq \frac{\tilde{C}_{27} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s(m+1)\delta}{2}} \tag{4.15}$$

and

$$\|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^\alpha u\|_{\frac{m\delta}{2}+n+1} \leq \frac{\tilde{C}_{28} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s(m+1)\delta}{2}}. \tag{4.16}$$

First, we repeat the procedure in which (4.9) is deduced from the validity of $(E)_{0,N}$, then we use the estimate of $(E)_{\frac{m\delta}{2},N}$ to get

$$\|[\mathcal{P},\ \Phi_{\rho,N}D^\alpha]u\|_{\frac{(m-1)\delta}{2}+n+1} \leq \frac{\tilde{C}_{29} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s(m+1)\delta}{2}}.$$

Similar to the arguments as (4.12) to get

$$\|\mathcal{P}\Phi_{\rho,N}D^\alpha u\|_{\frac{(m-1)\delta}{2}+n+1} \leq \frac{\tilde{C}_{30} L^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s(m+1)\delta}{2}}. \tag{4.17}$$

This together with the subelliptic estimate

$$\|\Phi_{\rho,N}D^\alpha u\|_{\frac{(m+1)\delta}{2}+n+1} \leq C(\Omega)\big\{\|\mathcal{P}\Phi_{\rho,N}D^\alpha u\|_{\frac{(m-1)\delta}{2}+n+1} + \|\Phi_{\rho,N}D^\alpha u\|_{n+1}\big\},$$

yields the required estimate (4.15). Moreover we can deduce that

$$\|\Phi_{\rho,N}\widetilde{\Lambda}^\sigma D^\alpha u\|_{\frac{m\delta}{2}+n+1} \leq \tilde{C}_{31}\big\{\|\mathcal{P}\Phi_{\rho,N}D^\alpha u\|_{\frac{(m-1)\delta}{2}+n+1} + \|\Phi_{\rho,N}D^\alpha u\|_{\frac{(m+1)\delta}{2}+n+1}\big\}.$$

In fact we have shown that the above inequality for $m = 0$ in Step 4 of the proof of Lemma 4.5, and the validity of the above inequality for general m can be deduced similarly without any additional difficulty. Consequently, the required estimate (4.16) follows from (4.17) and (4.15). Thus the proof of Lemma 4.6 is completed. □

Recall that the constants C_1, C_2, C_3 in Lemma 4.4, Lemma 4.5 and Lemma 4.6 depend only on $s, \sigma, n, \|u\|_{H^{n+6}(W)}$ and $\|a\|_{C^{2n+2}(\Omega)}$. Now take L in such a way that $L > \max\Big\{C_1,\, C_2,\, C_3,\, 2^{s+1}B_a,\, B_f,\, 4^{n+3}(\|u\|_{H^{n+6}(W)}+1)\Big\}$. Then by the above three Lemmas, we get the truth of $(E)_{r,N}$ for any $r \in [0,\ 1]$. This complete the proof of Proposition 4.1.

5. Gevrey regularity of nonlinear equation

In this section, $\mathcal{C}_j, j \geq 4$, will be used to denote suitable constants depending only on σ, the Gevrey index s, the dimension n and the Gevrey constants of the functions a, F. The existence and the Sobolev regularity of weak solutions for nonlinear Cauchy problems was proved in [15]. Now let $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ be a weak solution of (1.3). We first prove $u \in C^\infty(\mathbb{R}^{2n+1})$, and we need the following stability results by nonlinear composition (see for example [22]).

Lemma 5.1. *Let $F(t,x,v,q) \in C^\infty(\mathbb{R}^{2n+1}\times\mathbb{R})$ and $r \geq 0$. If $u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$, then $F(\cdot,u(\cdot)) \in H^r_{loc}(\mathbb{R}^{2n+1})$.*

In fact, if $u_1, u_2 \in H^r(\mathbb{R}^{2n+1}) \cap L^\infty(\mathbb{R}^{2n+1})$, then

$$\|u_1u_2\|_r \leq C_n\{\|u_1\|_{L^\infty}\|u_2\|_r + \|u_2\|_{L^\infty}\|u_1\|_r\}.$$

Thus if $r > (2n+1)/2$, the Sobolev embedding theorem implies that

$$\|u_1u_2\|_r \leq C\|u_1\|_r\|u_2\|_r. \tag{5.1}$$

Suppose that $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ is a weak solution of (1.3). Then by the subelliptic estimate (2.4), one has

$$\|\psi_1 u\|_{r+\delta} \leq C\{\ \|\psi_2 F(\cdot,u(\cdot))\|_r + \|\psi_2 u\|_r\ \}, \tag{5.2}$$

where $\psi_1, \psi_2 \in C^\infty_0(\mathbb{R}^{2n+1})$ and $\psi_2 = 1$ in the support of ψ_1. Combining Lemma 5.1 and the above subelliptic estimate (5.2), we have $u \in H^\infty_{loc}(\mathbb{R}^{2n+1})$ by standard iteration. We state this result in the following Proposition:

Proposition 5.2. *Let $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ be a weak solution of (1.3). Then $u \in C^\infty(\mathbb{R}^{2n+1})$.*

In this section we keep the same notations that we have set up in the previous sections. We prove the Gevrey regularity of the smooth solution u of Equation (1.3) on Ω. Set $W = 2\Omega = \big\{(t,x);\ \Big(t^2+|x|^2\Big)^{1/2} < 2\big\} \times \{v \in \mathbb{R}^n,\ |v| < 2\}$ and

$$M = \max_{(t,x,v)\in\bar{W}} |u(t,x,v)|.$$

Let $\{M_j\}$ be a sequence of positive coefficients. We say that it satisfies the monotonicity condition if there exists $B_0 > 0$ such that for any $j \in \mathbb{N}$,

$$\frac{j!}{i!(j-i)!}M_iM_{j-i} \leq B_0M_j,\ (i = 1,2,\cdots,j). \tag{5.3}$$

Let $\|u\|_{C^k(\Omega)}$ be the classic Hörder norm, that is, $\|u\|_{C^k(\Omega)} = \sum_{j=0}^k \left\|D^ju\right\|_{L^\infty(\Omega)}$.

We study now the stability of the Gevrey regularity by the non linear composition, which is an analogue of Lemma 1 in Friedman's work [11].

Lemma 5.3. *Let $N > n+2$ and $0 < \rho < 1$ be given. Let $\{M_j\}$ be a positive sequence satisfying the monotonicity condition (5.3) and that for some constant $\mathcal{C}_n$ depending only on n,*

$$\left(\frac{N}{\rho}\right)^{n+2} M_{N-n-2} \le \mathcal{C}_n M_{N-2}; \qquad M_j \ge \rho^{-j}, \quad j \ge 2. \tag{5.4}$$

Suppose that there exists $\mathcal{C}_4 > 1$, depending only on the Gevrey constant of F, such that:

1) the function $F(t,x,v;q)$ satisfies the following conditions: $\big\|F\big\|_{C^{n+2}(\bar\Omega\times[-M,M])} \le \mathcal{C}_4$ and for any k,l with $k+l \ge 1$,

$$\big\|D^\gamma_{t,x,v} D^l_q F\big\|_{C^{n+2}(\bar\Omega\times[-M,M])} \le \mathcal{C}_4^{k+l} M_{k-2} M_{l-2}, \quad \forall\ |\gamma| = k, \tag{5.5}$$

where we assume $M_{-j} = 1$ for nonnegative integer j.

2) the smooth function $g(t,x,v)$ satisfies the following conditions: $\|g\|_{L^\infty(\bar W)} \le M$ and

$$\|D^j g\|_{C^{n+3}(\bar W)} \le H_0, \quad 0 \le j \le 1, \tag{5.6}$$

and for any $0 < \rho < 1$ and any $j, 2 \le j \le N$, one has

$$\|\Phi_{\rho,j} D^\gamma g\|_{\nu+n+1} \le H_1^{j-2} M_{j-2}, \quad \forall\ |\gamma| = j, \tag{5.7}$$

where ν is a real number satisfying $-1/2 < \nu \le 1$, and $H_0, H_1 \ge 1$, $H_1 \ge \big(4^{n+2}\mathcal{C}_4 H_0\big)^2$.

Then there exists $\mathcal{C}_5 > 1$, depending only the Gevrey constant of F and the dimension n, such that for all $\rho, 0 < \rho < 1$, and all $\alpha \in \mathbb{N}^{2n+1}$ with $|\alpha| = N$,

$$\big\|\Phi_{\rho,N} D^\alpha \big[F\big(\cdot, g(\cdot)\big)\big]\big\|_{\nu+n+1} \le \mathcal{C}_5 H_0^2 H_1^{N-2} M_{N-2}. \tag{5.8}$$

Proof. In the proof, we use C_n to denote constants which depend only on n and may be different in different contexts. In the following, for each ρ, we always denote

$$\tilde\rho = \frac{(N-1)\rho}{N}, \qquad \tilde{\tilde\rho} = \frac{(N-2)\rho}{N}.$$

Observe that for ρ, $\tilde\rho$, $\tilde{\tilde\rho}$, we have the relation (4.2). Since $\Phi_{\tilde\rho,3} = 1$ in the support of $\Phi_{\rho,N}$, then by Lemma 3.1, one has

$$\begin{aligned}
&\big\|\Phi_{\rho,N} D^\alpha [F(\cdot, g(\cdot))]\big\|_{\nu+n+1} = \big\|\Phi_{\rho,N}\Phi_{\tilde\rho,3} D^\alpha [F(\cdot, g(\cdot))]\big\|_{\nu+n+1} \\
&\le C_n \Big\{ \big\|\Phi_{\tilde\rho,3} D^\alpha [F(\cdot, g(\cdot))]\big\|_{\nu+n+1} + \left(\frac{N}{\rho}\right)^{n+1+\nu} \big\|\Phi_{\tilde\rho,3} D^\alpha [F(\cdot, g(\cdot))]\big\|_0 \Big\} \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

The proof will be completed if we can show that there exists a constant $\mathcal{E}$ depending only the Gevrey constant of F and the dimension n, such that

$$\mathcal{I}_1 \le \mathcal{E} H_0^2 H_1^{N-2} M_{N-2}. \tag{5.9}$$

Indeed, choose a multi-index $\tilde{\alpha} \leq \alpha$ such that $|\tilde{\alpha}| = n$. Then

$$\begin{aligned}\mathcal{I}_2 &= C_n \left(\frac{N}{\rho}\right)^{n+1+\nu} \left\|\Phi_{\tilde{\rho},3} D^{\tilde{\alpha}} \Phi_{\tilde{\tilde{\rho}},3} D^{\alpha-\tilde{\alpha}}[F(\cdot, g(\cdot))]\right\|_0 \\ &\leq C_n \left(\frac{N}{\rho}\right)^{n+1+\nu} \left\|\Phi_{\tilde{\tilde{\rho}},3} D^{\alpha-\tilde{\alpha}}[F(\cdot, g(\cdot))]\right\|_n \\ &\leq C_n \left(\frac{N}{\rho}\right)^{n+2} \left\|\Phi_{\tilde{\tilde{\rho}},3} D^{\alpha-\tilde{\alpha}}[F(\cdot, g(\cdot))]\right\|_{\nu+n+1}.\end{aligned}$$

Assuming that (5.9) holds, then by virtue of the condition (5.4), we have

$$\begin{aligned}\mathcal{I}_2 \leq C_n \left(\frac{N}{\rho}\right)^{n+2} \left\|\Phi_{\tilde{\tilde{\rho}},3} D^{\alpha-\tilde{\alpha}}[F(\cdot, g(\cdot))]\right\|_{\nu+n+1} &\leq C_n \left(\frac{N}{\rho}\right)^{n+2} \mathcal{E} H_0^2 H^{N-n-2} M_{N-n-2} \\ &\leq C_n \mathcal{E} H_0^2 H^{N-2} M_{N-2}.\end{aligned}$$

With (5.9), the conclusion follows at once.

The rest is devoted to the proof of (5.9). By Faa di Bruno' formula, $\Phi_{\tilde{\rho},3} D^{\alpha}[F(\cdot, g(\cdot))]$ is the linear combination of terms of the form

$$\Phi_{\tilde{\rho},3} \left(D^{\beta}_{t,x,v} \partial_q^l F\right)(\cdot, g(\cdot)) \cdot \prod_{j=1}^{l} D^{\gamma_j} g, \tag{5.10}$$

where $|\beta| + l \leq |\alpha|$ and $\gamma_1 + \gamma_2 + \cdots + \gamma_l = \alpha - \beta$, and if $\gamma_i = 0$, $D^{\gamma_i} g$ doesn't appear in (5.10).

Next we estimate the Sobolev norm of the form (5.10). Take a function $\Psi \in C_0^{\infty}(W)$ such that $\Psi = 1$ in Ω. Note that $n + 1 + \nu > (2n+1)/2$. We apply (5.1) to compute

$$\begin{aligned}&\left\|\Phi_{\tilde{\rho},3} \left(D^{\beta}_{t,x,v} \partial_q^l F\right)(\cdot, g(\cdot)) \cdot \prod_{j=1}^{l} D^{\gamma_j} g\right\|_{\nu+n+1} \\ &\leq \left\|\Phi_{\tilde{\rho},3} \left(D^{\beta}_{t,x,v} \partial_q^l F\right)(\cdot, g(\cdot))\right\|_{\nu+n+1} \cdot \prod_{j=1}^{l} \left\|\Psi_j D^{\gamma_j} g\right\|_{\nu+n+1},\end{aligned}$$

where Ψ_j is given by setting $\Psi_j = \Psi$ if $|\gamma_j| = 1$, and $\Psi_j = \Phi_{\tilde{\tilde{\rho}},|\gamma_j|}$ if $|\gamma_j| \geq 2$. Moreover a direct computation yields

$$\begin{aligned}&\left\|\Phi_{\tilde{\rho},3} \left(D^{\beta}_{t,x,v} \partial_q^l F\right)(\cdot, g(\cdot))\right\|_{\nu+n+1} \leq \left\|\Phi_{\tilde{\rho},3} \left(D^{\beta}_{t,x,v} \partial_q^l F\right)(\cdot, g(\cdot))\right\|_{n+2} \\ &\leq C_n H_0 \left\{\sup \left|D^{n+2} \Phi_{\tilde{\rho},3}\right| \cdot \left\|D^{\beta}_{t,x,v} \partial_q^l F\right\|_{C(\bar{\Omega}\times[-M,M])} + \left\|D^{\beta}_{t,x,v} \partial_q^l F\right\|_{C^{n+2}(\bar{\Omega}\times[-M,M])}\right\} \\ &\leq C_n H_0 \left\{\left(\frac{3}{\rho}\right)^{n+2} \left\|D^{\beta}_{t,x,v} \partial_q^l F\right\|_{C(\bar{\Omega}\times[-M,M])} + \left\|D^{\beta}_{t,x,v} \partial_q^l F\right\|_{C^{n+2}(\bar{\Omega}\times[-M,M])}\right\}\end{aligned}$$

In the last inequality, we have used (3.1). Without loss of generality we may assume $|\beta| \geq n + 2$. Then we may choose $\tilde{\beta} \leq \beta$ such that $\left|\tilde{\beta}\right| = |\beta| - (n+2)$. Thus by (5.4), (5.5) and the monotonicity condition (5.3), one has

$$\left\|D^{\beta}_{t,x,v} \partial_q^l F\right\|_{C^{n+2}(\bar{\Omega}\times[-M,M])} \leq M_{|\beta|-2} M_{l-2},$$

and

$$\begin{aligned}\left(\frac{3}{\rho}\right)^{n+2}\left\|D^{\beta}_{t,x,v}\partial^{l}_{q}F\right\|_{C(\bar{\Omega}\times[-M,M])} &\leq \left(\frac{3}{\rho}\right)^{n+2}\left\|D^{\tilde{\beta}}_{t,x,v}\partial^{l}_{q}F\right\|_{C^{n+2}(\bar{\Omega}\times[-M,M])}\\ &\leq 3^{n+2}M_{n+2}M_{|\tilde{\beta}|-2}M_{l-2}\\ &\leq 3^{n+2}M_{|\beta|-2}M_{l-2}.\end{aligned}$$

Hence,

$$\left\|\Phi_{\tilde{\rho},3}\left(D^{\beta}_{t,x,v}\partial^{l}_{q}F\right)(\cdot,g(\cdot))\right\|_{\nu+n+1}\leq C_nH_0M_{|\beta|-2}M_{l-2}.$$

Hence

(5.11)

$$\|\Phi_{\tilde{\rho},3}\left(D^{\beta}_{t,x,v}\partial^{l}_{q}F\right)(\cdot,g(\cdot))\prod_{j=1}^{l}D^{\gamma_j}g\|_{\nu+n+1}\leq C_nH_0M_{|\beta|-2}M_{l-2}\prod_{j=1}^{l}\left\|\Psi_jD^{\gamma_j}g\right\|_{\nu+n+1},$$

By virtue of (5.6)-(5.7) and (5.10)-(5.11), the situation is now similar to [11]. In fact, we work with the Sobolev norm, and we shall follow the idea of [11] to prove (5.9). First we define the polynomial functions w, X_1, X_2 in $\mathbb{R}$ as follows:

$$w=w(y)=H_0\left(y+\sum_{j=2}^{N}\frac{H_1^{j-2}M_{j-2}y^j}{j!}\right),\quad y\in\mathbb{R};$$

$$X_1(w)=1+\mathcal{C}_4w+\sum_{j=2}^{N}\frac{\mathcal{C}_4^jM_{j-2}w^j}{j!};$$

$$X_2(y)=1+\mathcal{C}_4y+\sum_{j=2}^{N}\frac{\mathcal{C}_4^jM_{j-2}y^j}{j!},\quad y\in\mathbb{R}.$$

By the conditions (5.6) and (5.7), we have

$$\left\|\Psi_jD^jg\right\|_{\nu+n+1}\leq\frac{d^jw(y)}{dy^j}\Big|_{y=0},\quad\forall\ 1\leq j\leq N;$$

Define $X(y,w)=X_1(w)X_2(y)$. Then by virtue of (5.5), it follows

$$M_{k-2}M_{l-2}\leq\frac{\partial^{k+l}X(y,w)}{\partial y^k\partial w^l}\Big|_{(y,w)=(0,0)},\quad\forall\ 2\leq k,\ l\leq N.$$

By (5.11) and the above two inequalities, we get that for all α, $|\alpha|=N$,

$$\mathcal{I}_1=C_n\left\|\Phi_{\tilde{\rho},3}D^{\alpha}[F(\cdot,g(\cdot))]\right\|_{\nu+n+1}\leq C_nH_0\frac{d^N}{dy^N}X\left(y,w(y)\right)\Big|_{y=0}.$$

Hence, the proof of (5.9) will be complete if we show that,

$$\frac{d^N}{dy^N}\left(X_1\left(w(y)\right)X_2(y)\right)\Big|_{y=0}\leq 72\mathcal{C}_4H_0H_1^{N-2}M_{N-2}. \tag{5.12}$$

To prove the above inequality, we need to treat $X_j^{(k)}(0):=\frac{d^k}{dy^k}X_j(y)\Big|_{y=0}$, $0\leq k\leq N$, $j=1,2$. We say $w(y)\ll h(y)$ if the following relation holds:

$$w^{(j)}(0)\leq h^{(j)}(0),\quad 0\leq j\leq N.$$

Observe that

$$w(y) \ll w(y) = H_0 \left(y + \sum_{j=2}^{N} \frac{H_1^{j-2} M_{j-2} y^j}{j!} \right).$$

We can prove that

$$w^2(y) \ll 35 H_0^2 \left(y^2 + \sum_{j=3}^{N} \frac{H_1^{j-3} M_{j-3} y^j}{(j-1)!} \right). \tag{5.13}$$

In fact, a direct verification shows that

$$w^2(y) = H_0^2 \left\{ y^2 + \sum_{j=3}^{N} \Big[\frac{2H_1^{j-3} M_{j-3}}{(j-1)!} + \sum_{i=2}^{j-2} \frac{H_1^{j-4} M_{i-2} M_{j-i-2}}{i!(j-i)!} \Big] y^j \right\} + O(y^{N+1}).$$

Since $\{M_j\}$ satisfies the monotonicity condition (5.3), we compute

$$\sum_{i=2}^{j-2} \frac{H_1^{j-4} M_{i-2} M_{j-i-2}}{i!(j-i)!} \le \frac{4H_1^{j-4} M_{j-4}}{(j-4)! j^2} \sum_{i=2}^{j-2} \frac{j^2}{i^2 (j-i)^2} \le \frac{32 H_1^{j-3} M_{j-3}}{(j-1)!}.$$

Combing these, we obtain (5.13). Inductively, we have the following relations:

$$w^i(y) \ll 35^{i-1} H_0^i \left(y^i + \sum_{j=i+1}^{N} \frac{H_1^{j-i-1} M_{j-i-1} y^j}{(j-i+1)!} \right), \quad 2 \le i \le N-1;$$

$$w^N(y) \ll 35^N H_0^N y^N.$$

Thus by the definition of X_1, it follows that

$$X_1(y) = X_1(w(y)) \ll 1 + \mathcal{C}_4 H_0 y + \left(H_0 M_0/2 + 35 \mathcal{C}_4^2 M_0 H_0^2/2 \right) y^2$$
$$+ \sum_{j=3}^{N} \left(\frac{H_0 H_1^{j-2} M_{j-2}}{j!} + \frac{35^{j-1} \mathcal{C}_4^j H_0^j M_{j-2}}{j!} + \sum_{i=2}^{j-1} \frac{35^{i-1} \mathcal{C}_4^i H_0^i H_1^{j-i-1} M_{i-2} M_{j-i-1}}{i!(j-i+1)!} \right) y^j.$$

This gives

$$X_1(0) = 1, \quad X_1'(0) \le \mathcal{C}_4 H_0, \quad X_1^{(2)}(0) \le H_0 M_0 + 35 \mathcal{C}_4^2 M_0 H_0^2,$$

and moreover for $j \ge 3$,

$$X_1^{(j)}(0) \le \mathcal{C}_4 H_0 H_1^{j-2} M_{j-2} + 35^{j-1} \mathcal{C}_4^j H_0^j M_{j-2} + \sum_{i=2}^{j-1} \frac{j! 35^{i-1} \mathcal{C}_4^i H_0^i H_1^{j-i-1} M_{i-2} M_{j-i-1}}{i!(j-i+1)!}.$$

Observe that $H_1 \ge (35 \mathcal{C}_4 H_0)^2$, and hence $X_1^{(2)} \le 2 \mathcal{C}_4 H_0 H_1 M_0$, and for $j \ge 3$,

$$X_1^{(j)}(0) \le 2 C_4 H_0 H_1^{j-2} M_{j-2} + \frac{4 C_4 (j-2)! H_0 H_1^{j-2} M_{j-3}}{(j-3)!} \sum_{i=2}^{j-1} \frac{j^2}{i^2 (j-i)^2}$$
$$\le 6 C_4 H_0 H_1^{j-2} M_{j-2}.$$

On the other hand, it is clear that

$$X_2(0) = 1, \quad X_2'(0) \le \mathcal{C}_4, \quad X_2^{(j)}(0) \le \mathcal{C}_4^j M_{j-2}, \ 2 \le j \le N.$$

By virtue of the above relations, we have, for $H_1 \geq (35\mathcal{C}_4 H_0)^2$,

$$\frac{d^N}{dy^N} X\left(y, w(y)\right)\Big|_{y=0} = \sum_{j=0}^{N} \frac{N!}{j!(N-j)!} X_1^{(j)}(0) X_2^{(N-j)}(0)$$

$$\leq \mathcal{C}_4^N M_{N-2} + \mathcal{C}_4^N N H_0 M_{N-3} + 2N(N-1)\mathcal{C}_4^{N-1} H_0 H_1 M_0 M_{N-4} + 6\mathcal{C}_4^2 H_0 H_1^{N-3} M_{N-3}$$

$$+ 6\mathcal{C}_4 H_0 H_1^{N-2} M_{N-2} + 6C_4 \sum_{j=3}^{N-2} \frac{N!}{j!(N-j)!} H_0 H_1^{j-2} M_{j-2} \mathcal{C}_4^{N-j} M_{N-j-2}$$

$$\leq 72 C_4 H_0 H_1^{N-2} M_{N-2}.$$

This gives (5.12), and hence (5.9). This completes the proof of Lemma 5.3. □

Now starting from the smooth solution u, we prove the Gevrey regularity result as follows:

Proposition 5.4. *Let $\delta = \max\left\{\frac{\sigma}{4},\ \frac{\sigma}{2} - \frac{1}{6}\right\}$, and let $s \geq \frac{2}{\delta}$ be a real number. Let $W = 2\Omega = \left\{(t,x,v);\ (\frac{t}{2}, \frac{x}{2}, \frac{v}{2}) \in \Omega\right\}$. Suppose that $u \in C^\infty(\bar{W})$ is a solution of (1.3) where $a(t,x,v) \in G^s(\bar{\Omega}), a > 0$ and $F(t,x,v,q) \in G^s(\bar{\Omega} \times [-M, M])$. Then there exits a constant R such that for any $r \in [0,1]$ and any $N \in \mathbb{N}$, $N \geq 4$,*

$$(E)'_{r,N} \qquad \begin{aligned} &\|\Phi_{\rho,N} D^\alpha u\|_{r+n+1} + \|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \\ &\leq \frac{R^{|\alpha|-1}}{\rho^{(s+n)(|\alpha|-3)}} \left((|\alpha|-3)!\right)^s \left(\frac{N}{\rho}\right)^{sr} \end{aligned}$$

holds for all α, $|\alpha| = N$ and all $0 < \rho < 1$. Thus, $u \in G^s(\Omega)$.

Remark 5.1. Here the Gevrey constant L of u is determined by the Gevrey constants B_a and B_F of the functions a, F, and depends only on $s, \sigma, n, \|u\|_{H^{n+6}(W)}$ and $\|a\|_{C^{2n+2}(\Omega)}$.

Proof. We prove the estimate $(E)'_{r,N}$ by induction on N. We shall follow the same procedure as that in the proof of Proposition 4.1. First, the truth of $(E)'_{r,4}$ can be deduced by the same argument as that in the proof of $(E)_{r,4}$ in the previous section.

Let now $N > 4$ and assume that $(E)'_{r,N-1}$ holds for any $r \in [0,1]$. We need to prove the truth of $(E)'_{r,N}$ for $0 \leq r \leq 1$. In the following discussion, we fix N and for each $0 < \rho < 1$, define $\tilde{\rho} = \frac{N-1}{N}\rho$, $\tilde{\tilde{\rho}} = \frac{N-2}{N}\rho$. Let $\Phi_{\rho,N}$ be the cutoff function which satisfies the property (3.1).

First, the same argument as the proof of Lemma 4.4 yields

(5.14)

$$\|\Phi_{\rho,N} D^\alpha u\|_{n+1} + \|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{-\frac{\delta}{2}+n+1} \leq \frac{C_1 R^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}} \left((|\alpha|-3)!\right)^s, \quad \forall\, 0 < \rho < 1.$$

Next we prove, for all $r, 0 < r \leq \frac{\delta}{2}$,

(5.15)

$$\|\Phi_{\rho,N} D^\alpha u\|_{r+n+1} + \|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1} \leq \frac{\mathcal{C}_6 R^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}} \left((|\alpha|-3)!\right)^s (N/\rho)^{sr}.$$

Observe that we need only to show the above inequality in the case when $r = \frac{\delta}{2}$, that is

$$(5.16)\quad \|\Phi_{\rho,N} D^\alpha u\|_{\frac{\delta}{2}+n+1} + \|\Phi_{\rho,N} \widetilde{\Lambda}^\sigma D^\alpha u\|_{n+1} \leq \frac{\mathcal{C}_6 R^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}} \left((|\alpha|-3)!\right)^s (N/\rho)^{\frac{s\delta}{2}},$$

and the truth of (5.15) for general $r \in]0, \frac{\delta}{2}[$ follows by the interpolation inequality (2.2).

To prove (5.16), we first show the following inequality

$$\|\mathcal{P}\Phi_{\rho,N}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1} \leq \frac{\mathcal{C}_7 R^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s\delta}{2}}. \tag{5.17}$$

In fact,

$$\begin{aligned}\|\mathcal{P}\Phi_{\rho,N}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1} &\leq \|[\mathcal{P},\ \Phi_{\rho,N}D^{\alpha}]u\|_{-\frac{\delta}{2}+n+1} + \|\Phi_{\rho,N}D^{\alpha}\mathcal{P}u\|_{-\frac{\delta}{2}+n+1}\\ &\leq \|[\mathcal{P},\ \Phi_{\rho,N}D^{\alpha}]u\|_{-\frac{\delta}{2}+n+1} + \|\Phi_{\rho,N}D^{\alpha}[F(\cdot,u(\cdot))]\|_{-\frac{\delta}{2}+n+1}.\end{aligned}$$

Since there is no nonlinear form involved in the first term of the right-hand side of the above inequality, the same argument as in the proof of (4.9) gives that

$$\|[\mathcal{P},\ \Phi_{\rho,N}D^{\alpha}]u\|_{-\frac{\delta}{2}+n+1} \leq \frac{\mathcal{C}_8 R^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^s (N/\rho)^{\frac{s\delta}{2}}, \tag{5.18}$$

Thus we need only to treat the second term $\|\Phi_{\rho,N}D^{\alpha}[F(\cdot,u(\cdot))]\|_{-\frac{\delta}{2}+n+1}$. The smoothness of u gives

$$\|D^j u\|_{C^{n+3}(\bar{W})} \leq \|u\|_{C^{n+5}(\bar{W})}, \qquad 0 \leq j \leq 2, \tag{5.19}$$

and by the induction hypothesis, for any $3 \leq j < N$ and any $0 < \rho < 1$,

$$\begin{aligned}\|\Phi_{\rho,j}D^{\beta}u\|_{-\frac{\delta}{2}+n+1} \leq \|\Phi_{\rho,j}D^{\beta}u\|_{n+1} &\leq \frac{C_1 R^{j-2}}{\rho^{(s+n)(j-3)}}\big((j-3)!\big)^s\\ &\leq \frac{C_1 R^{j-2}}{\rho^{(s+n)(j-3)}}\big((j-3)!\big)^s (j/\rho)^{\frac{s\delta}{2}}, \qquad \forall\ \beta,\ |\beta| = j,\end{aligned} \tag{5.20}$$

Similarly, by (5.14), we have for any $0 < \rho < 1$,

$$\|\Phi_{\rho,N}D^{\alpha}u\|_{-\frac{\delta}{2}+n+1} \leq \frac{C_1 R^{N-2}}{\rho^{(s+n)(N-3)}}\big((N-3)!\big)^s (N/\rho)^{\frac{s\delta}{2}}, \qquad \forall\ \alpha,\ |\alpha| = N. \tag{5.21}$$

Since $F \in G^s(\bar{\Omega}\times[-M,M])$, then

$$\|D^k_{t,x,v}\partial^l_q F\|_{C^{n+2}(\bar{\Omega}\times[-M,M])} \leq B_F^{k+l}\big((k-3)!\big)^s\big((l-3)!\big)^s, \quad k,l \geq 3. \tag{5.22}$$

Define M_j, H_0, H_1 by setting

$$H_1 = R;\ \ H_0 = \|u\|_{C^{n+3}(\bar{W})} + 1;\ \ M_0 = 1;\ \ M_j = \frac{\big((j-1)!\big)^s}{\rho^{(s+n)(j-1)}}((j+2)/\rho)^{\frac{s\delta}{2}},\ \ j \geq 1.$$

We can choose R large enough such that $H_1 = R \geq (4^{n+1}B_F H_0)^2$. Then (5.19)-(5.22) can be rewritten as

$$\|D^j u\|_{C^{n+3}(\bar{W})} \leq H_0, \quad 0 \leq j \leq 1, \tag{5.23}$$

$$\|\Phi_{\rho,j}D^{\gamma}u\|_{-\frac{\delta}{2}+n+1} \leq H_0 H_1^{j-2} M_{j-2}, \quad \forall\ 0 < \rho < 1,\ \forall\ |\gamma| = j, \quad 2 \leq j \leq N, \tag{5.24}$$

$$\|D^k_{t,x,v}\partial^l_q F\|_{C^{n+2}(\bar{\Omega}\times[-M,M])} \leq B_F^{k+l} M_{k-2}M_{l-2}, \quad k,l \geq 2. \tag{5.25}$$

For each j, note that $s \geq \frac{2}{\delta}$. Hence we compute

(5.26)

$$\begin{aligned}\frac{j!}{i!(j-i)!}M_iM_{j-i} &= \frac{j!}{i(j-i)}\big((i-1)!\big)^{s-1}\big((j-i-1)!\big)^{s-1}\rho^{-(s+n)(i-1)}\rho^{-(s+n)(j-i-1)}\\ &\quad\times(i+2)^{\frac{s\delta}{2}}(j-i+2)^{\frac{s\delta}{2}}\rho^{-s\delta}\\ &\leq j!\big((j-2)!\big)^{s-1}\rho^{-(s+n)(j-2)}(j+2)^{\frac{s\delta}{2}}(j+2)^{\frac{s\delta}{2}}\rho^{-s\delta}\\ &\leq \frac{j(j+2)^{\frac{s\delta}{2}}}{(j-1)^{s-1}}\big((j-1)!\big)^{s}\rho^{-(s+n)(j-1)}(j+2)^{\frac{s\delta}{2}}\rho^{-\frac{s\delta}{2}}\rho^{s+n-\frac{s\delta}{2}}\\ &\leq \frac{j(j+2)^{\frac{s\delta}{2}}}{(j-1)^{s-1}}M_j\\ &\leq \widetilde{C}_sM_j.\end{aligned}$$

In the last inequality we used the fact that $s-1 \geq 1+\frac{s\delta}{2}$, where $\widetilde{C}_s$ is a constant depending only on s. Moreover, it is easy to verify that, $M_j \geq \rho^{-(s+n)(j-1)} \geq \rho^{-j}$ for each $j \geq 2$, and

$$\begin{aligned}\left(\frac{N}{\rho}\right)^{n+2}M_{N-n-2} &= \left(\frac{N}{\rho}\right)^{n+2}\frac{\big((N-n-3)!\big)^{s}}{\rho^{(s+n)(N-n-3)}}((N-n)/\rho)^{\frac{s\delta}{2}}\\ &\leq \mathcal{C}_n\frac{\big((N-1)!\big)^{s}}{\rho^{(s+n)(N-1)}}((N+2)/\rho)^{\frac{s\delta}{2}} = \mathcal{C}_nM_{N-2}.\end{aligned}$$

Thus $\{M_j\}$ satisfies the monotonicity condition (5.3) and the condition (5.4). By virtue of (5.23)-(5.26), we can use Lemma 5.3 with $\nu = -\frac{\delta}{2} > -\frac{1}{2}$ to obtain

$$\begin{aligned}\|\Phi_{\rho,N}D^\alpha[F(\cdot,u(\cdot))]\|_{-\frac{\delta}{2}+n+1} &\leq \mathcal{C}_5H_0^2H_1^{|\alpha|-2}M_{|\alpha|-2}\\ &\leq 2\mathcal{C}_5\left(1+\|u\|^2_{C^{n+3}(\bar{W})}\right)\frac{R^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^{s}(N/\rho)^{\frac{s\delta}{2}}.\end{aligned}$$

This along with (5.18) yields (5.17), if we choose $\mathcal{C}_7 = \mathcal{C}_8 + 2\mathcal{C}_5\left(1+\|u\|^2_{C^{n+3}(\bar{W})}\right)$. By virtue of (5.17), we can repeat the discussion as in Step 4 in the previous section. This gives (5.16), and hence (5.15).

Similarly, we can prove that for any r with $\frac{\delta}{2} \leq r \leq \delta$,

$$\|D^\alpha u\|_{r+n+1,\Omega_\rho} + \|\widetilde{\Lambda}^\sigma D^\alpha u\|_{r-\frac{\delta}{2}+n+1,\Omega_\rho} \leq \frac{\mathcal{C}_9R^{|\alpha|-2}}{\rho^{(s+n)(|\alpha|-3)}}\big((|\alpha|-3)!\big)^{s}(N/\rho)^{sr}.$$

Inductively, for any $m \in \mathbb{N}$ with $\frac{m\delta}{2} < 1+\frac{\delta}{2}$, the above inequality still holds for any r with $\frac{(m-1)\delta}{2} \leq r \leq \frac{m\delta}{2}$. Hence, for r with $0 \leq r \leq 1$, we obtain the truth of $(E)'_{r,N}$. This completes the proof of Proposition 5.4. □

References

[1] R.Alexandre, Y.Morimoto, S.Ukai, C.-J.Xu, T.Yang, Uncertainty principle and kinetic equations, *J. Funct. Anal.* **255** (2008), 2013-2066

[2] R. Alexandre, L. Desvillettes, C. Villani, B. Wennberg, Entropy dissipation and long-range interactions, *Arch. Rational Mech. Anal.* **152** (2000) 327-355.

[3] R. Alexandre, M. Safadi, Littlewood Paley decomposition and regularity issues in Boltzmann equation homogeneous equations. I. Non cutoff and Maxwell cases, *Math. Models Methods Appl. Sci.* **15** (2005) 907-920.

[4] F. Bouchut, Hypoelliptic regularity in kinetic equations. *J. Math. Pure Appl.* **81** (2002), 1135-1159.
[5] H. Chen, W.-X. Li, C.-J. Xu, Gevrey hypoellipticity for linear and non-linear Fokker-Planck equations, *J. Diff. Equ.* **246** (2009), 320-339.
[6] H. Chen, L.Rodino, General theory of PDE and Gevrey class. *General theory of partial differential equations and microlocal analysis*(Trieste 1995), Pitman Res. Notes in Math. Ser., **349**, Longman, Harlow, 6-81, (1996).
[7] M.Derridj, C.Zuily, Sur la régularité Gevrey des opérateurs de Hörmander. *J. Math. Pure Appl.* **52** (1973), 309-336.
[8] L. Desvillettes, B. Wennberg, Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff. *Comm. Partial Differential Equations* **29** (2004), no. 1-2, 133–155.
[9] L. Desvillettes, G. Furioli and E. Terraneo, Propagation of Gevrey regularity for solutions of Boltzmann equation for Maxwellian molecules, *Trans. Amer. Math. Soc.* **361** (2009) 1731-1747.
[10] M. Durand, Régularité Gevrey d'une classe d'opérateurs hypo-elliptiques. *J. Math. Pure Appl.* **57** (1978), 323-360.
[11] A.Friedman, On the Regularity of the solutions of Non-linear Elliptic and Parabolic Systems of Partial Differential Equations. *J. Math. Mech.* **7** (1958), 43-59.
[12] B. Helffer, F. Nier, Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. *Lecture Notes in Mathematics*, **1862** Springer-Verlag, Berlin, 2005.
[13] F. Hérau, F. Nier, Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.* **171** (2004), no. 2, 151–218.
[14] J.Kohn, Lectures on degenerate elliptic problems. Psedodifferential operators with applications, C.I.M.E., Bressanone 1977, 89-151(1978).
[15] Y.Morimoto, C.-J. Xu, Hypoellipticity for a class of kinetic equations. *J. Math. Kyoto Univ* **47** (2007), 129–152.
[16] Y. Morimoto and C.-J. Xu, Ultra-analytic effect of Cauchy problem for a class of kinetic equations, *to appear in "Journ. Diff. Equ."*
[17] Y.Morimoto, S.Ukai, C.-J.Xu, T.Yang, Regularity of solutions to the spatially homogeneous Boltzmann equation without Angular cutoff, *Discrete and Continuous Dynamical Systems - Series A* **24** (2009) 187-212.
[18] L.Rodino, *Linear partial differential operators in Gevrey class*. World Scientific, Singapore, 1993.
[19] F.Treves, *Introduction to Pseudodifferential and Fourier Integral Operators*. Plenum, New York, 1980.
[20] S. Ukai, Local solutions in Gevrey classes to the nonlinear Boltzmann equation without cutoff, *Japan J. Appl. Math.* **1**(1984), no. 1, 141–156.
[21] C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, *Arc. Rational Mech. Anal.,* **143**, 273–307, (1998).
[22] C.-J. Xu, Nonlinear microlocal analysis. *General theory of partial differential equations and microlocal analysis*(Trieste 1995), Pitman Res. Notes in Math. Ser., **349**, Longman, Harlow, 155-182, (1996).

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
E-mail address: chenhua@whu.edu.cn

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
E-mail address: wei-xi.li@whu.edu.cn

Université de Rouen, UMR 6085-CNRS, Mathématiques, Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray, France
and School of mathematics, Wuhan University, 430072, Wuhan, China
E-mail address: Chao-Jiang.Xu@univ-rouen.fr