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Adaptive wavelet deconvolution for strongly mixing sequences

Christophe Chesneau

Abstract: This paper studies the estimation of a density in the convolution density model from weakly dependent observations. The ordinary smooth case is considered. Adopting the minimax approach under the mean integrated square error over Besov balls, we explore the performances of two wavelet estimators: a standard linear one based on projections and a new non-linear one based on a hard thresholding rule. In particular, under strong mixing conditions, we prove that our hard thresholding estimator attains a particular rate of convergence: the optimal one in the *i.i.d.* case up to a logarithmic term.

Key words and phrases: Deconvolution, Strongly mixing, Adaptivity, Wavelets, Hard thresholding.

AMS 2000 Subject Classifications: 62G07, 62G20.

1 Introduction

Let $(Y_v)_{v \in \mathbb{Z}}$ be a strictly stationary process such that, for any $v \in \mathbb{Z}$,

$$Y_v = X_v + \epsilon_v, \tag{1.1}$$

$(X_v)_{v \in \mathbb{Z}}$ are identically distributed random variables and $(\epsilon_v)_{v \in \mathbb{Z}}$ are identically distributed random variables, also independent of $(X_v)_{v \in \mathbb{Z}}$. The density of X_0 is unknown and denoted f . The one of ϵ_0 is known, denoted g and satisfies the ordinary smooth case i.e. the Fourier coefficients of g have

Christophe Chesneau, Université de Caen, LMNO, Campus II, Science 3, 14032, Caen, France

a polynomial decrease (see Section 2). The goal is to estimate f when only n random variables Y_1, \dots, Y_n of $(Y_v)_{v \in \mathbb{Z}}$ are observed.

When $(Y_v)_{v \in \mathbb{Z}}$ are *i.i.d.*, (1.1) becomes the standard convolution density model studies in many papers. See e.g. [1], [9], [21], [10], [11], [4], [7] and [14]. Since the *i.i.d.* assumption is not realistic in some applications, several authors have investigated the dependent case. We refer to e.g. [17, 18], [5], [13] and [24]. In particular, under strong mixing conditions on $(Y_v)_{v \in \mathbb{Z}}$, van Zanten and Zareba [24] have developed a new linear wavelet estimator for f in (1.1). Taking the mean integrated square error (MISE) over Besov balls, [24, Theorems 3.1 and 3.2] show that it attains a sharp rate of convergence. However, this rate is deteriorated by the considered dependence condition and it is slower than the optimal one related to the *i.i.d.* case.

In this paper, we provide two complementary contributions to the wavelet estimation of f in the strong mixing case:

1. We extend [24, Theorems 3.1 and 3.2]. More precisely, we prove that the linear wavelet estimator developed by [10] attains the standard rate of convergence i.e. the optimal one in the *i.i.d.* case. It is constructed from the Daubechies wavelet basis instead of the Meyer wavelet basis as in [24]. The properties of this basis allows us to apply sharp probabilistic inequalities which improve the performance of the considered linear wavelet estimator.
2. We treat the adaptive estimation of f . To the best our knowledge, this has not been addressed earlier via wavelets and for the ordinary smooth case (the supersmooth case has been done by [24]). We develop a new wavelet hard thresholding estimator and prove that it attains a sharp rate of convergence, close to the one attained by our linear wavelet estimator. The difference is an extra logarithmic term. And only this logarithmic term is deteriorated by the dependence of the observations.

The rest of the paper is organized as follows. Section 2 is devoted to the assumptions on (1.1) (strong mixing, ordinary smooth case, ...). In Section

3, we present wavelets and Besov balls. The considered wavelet estimators (linear and hard thresholding) are defined in Section 4. Our upper bounds results are set in Section 5. The proofs are postponed in Section 6.

2 Assumptions on the model

2.1 Assumptions on f and g

We suppose that the support of f is included in $[-\Omega, \Omega]$ and that there exists a constant $C > 0$ such that

$$\sup_{x \in \mathbb{R}} f(x) \leq C < \infty. \quad (2.1)$$

We define the Fourier transform of an integrable function h by

$$\mathcal{F}(h)(x) = \int_{-\infty}^{\infty} h(y) e^{-ixy} dy, \quad x \in \mathbb{R}.$$

The notation $\overline{\cdot}$ will be used for the complex conjugate.

The ordinary smooth case on g is considered: we assume that there exist three constants $C > 0$, $c > 0$ and $\delta > 1$ such that, for any $x \in \mathbb{R}$,

- the Fourier transform of g satisfies

$$|\mathcal{F}(g)(x)| \geq \frac{c}{(1+x^2)^{\delta/2}}, \quad (2.2)$$

- for any $\ell \in \{0, 1, 2\}$, the ℓ -th derivative of the Fourier transform of g satisfies

$$\lim_{x \rightarrow 0} |(\mathcal{F}(g)(x))^{(\ell)}| \leq C, \quad \lim_{x \rightarrow \infty} |x|^{\delta+\ell} |(\mathcal{F}(g)(x))^{(\ell)}| = C. \quad (2.3)$$

These assumptions control the decay of the Fourier coefficients of g and, a fortiori, the smoothness of g . They are similar to those considered in [10, (B2) and (B3)].

A simple example is the Laplace density: $g(x) = (1/2)e^{-|x|}$, $x \in \mathbb{R}$. We have $\mathcal{F}(g)(x) = 1/(1+x^2)$, $x \in \mathbb{R}$, so g satisfies (2.2) and (2.3) with $\delta = 2$.

2.2 Dependence assumptions

1. *Strongly mixing case.* For any $m \in \mathbb{Z}$, we define the m -th strongly mixing coefficient of $(Y_v)_{v \in \mathbb{Z}}$ by

$$a_m = \sup_{(A,B) \in \mathcal{F}_{-\infty,0}^Y \times \mathcal{F}_{m,\infty}^Y} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where, for any $u \in \mathbb{Z}$, $\mathcal{F}_{-\infty,u}^Y$ is the σ -algebra generated by \dots, Y_{u-1}, Y_u and $\mathcal{F}_{u,\infty}^Y$ is the σ -algebra generated by Y_u, Y_{u+1}, \dots

We formulate the two following assumptions :

- there exist two constants $\gamma \in (0, 1)$ and $C > 0$ such that

$$\sum_{m=1}^{\infty} m^\gamma a_m^\gamma \leq C. \quad (2.4)$$

- there exist three constants $\gamma > 0$, $c > 0$ and $\theta > 0$ such that, for any $m \in \mathbb{Z}$,

$$a_m \leq \gamma \exp(-c|m|^\theta). \quad (2.5)$$

Such strongly mixing conditions are satisfied by some GARCH processes. See e.g. [23], [8], [20] and [2]. Remark that, for (2.5), the standard *i.i.d.* case corresponds to $\theta \rightarrow \infty$.

Naturally, (2.5) implies (2.4).

2. Let q be the density of Y_0 and, for any $m \in \mathbb{Z}$, $q_{(Y_0, Y_m)}$ be the one of (Y_0, Y_m) . We suppose that there exists a constant $C > 0$ such that

$$\sup_{m \in \mathbb{Z}} \sup_{(x,y) \in [0,1]^2} |q_{(Y_0, Y_m)}(x, y) - q(x)q(y)| \leq C. \quad (2.6)$$

Assumptions (2.4) and (2.6) are similar to those used in [17].

3 Wavelets and Besov balls

Let $N \in \mathbb{N}^*$, and ϕ and ψ be the Daubechies wavelets dbN . We chose N such that $\phi \in \mathcal{C}^v$ and $\psi \in \mathcal{C}^v$ for $v > 1 + \delta$ where δ refers to (2.2) and (2.3). Set

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Then there exists an integer τ and a set of consecutive integers Λ_j with a length proportional to 2^j such that, for any integer $\ell \geq \tau$, the collection

$$\mathcal{B} = \{\phi_{\ell,k}(\cdot), k \in \Lambda_\ell; \psi_{j,k}(\cdot); j \in \mathbb{N} - \{0, \dots, \ell - 1\}, k \in \Lambda_j\}$$

is an orthonormal basis of $\mathbb{L}^2([-\Omega, \Omega]) = \{h : [-\Omega, \Omega] \rightarrow \mathbb{R}; \int_{-\Omega}^{\Omega} h^2(x)dx < \infty\}$. We refer to [3] and [16].

Note that this wavelet basis satisfies [10, (A2) and (A3)].

For any integer $\ell \geq \tau$, any $h \in \mathbb{L}^2([-\Omega, \Omega])$ can be expanded on \mathcal{B} as

$$h(x) = \sum_{k \in \Lambda_\ell} \alpha_{\ell,k} \phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty} \sum_{k \in \Lambda_j} \beta_{j,k} \psi_{j,k}(x),$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ are the wavelet coefficients of h defined by

$$\alpha_{j,k} = \int_{-\Omega}^{\Omega} h(x) \phi_{j,k}(x) dx, \quad \beta_{j,k} = \int_{-\Omega}^{\Omega} h(x) \psi_{j,k}(x) dx. \quad (3.1)$$

Let $M > 0$, $s > 0$, $p \geq 1$ and $r \geq 1$. A function h belongs to $B_{p,r}^s(M)$ if and only if there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients (3.1) satisfy

$$2^{\tau(1/2-1/p)} \left(\sum_{k \in \Lambda_\tau} |\alpha_{\tau,k}|^p \right)^{1/p} + \left(\sum_{j=\tau}^{\infty} \left(2^{j(s+1/2-1/p)} \left(\sum_{k \in \Lambda_j} |\beta_{j,k}|^p \right)^{1/p} \right)^r \right)^{1/r} \leq M^*.$$

In this expression, s is a smoothness parameter and p and r are norm parameters. Besov balls contain the Hölder and Sobolev balls. See e.g. [19] and [16].

Details and results on wavelets in nonparametric estimation can be found in [12].

4 Estimators

For any integer $j \geq \tau$, $k \in \Lambda_j$, $v \in \{\phi, \psi\}$ and $y \in \mathbb{R}$, set

$$(\mathcal{K}v)_{j,k}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{\mathcal{F}(v_{j,k})}(x)}{\mathcal{F}(g)(x)} e^{-ixy} dx. \quad (4.1)$$

We estimate the unknown wavelet coefficients $\alpha_{j,k} = \int_{-\Omega}^{\Omega} f(x)\phi_{j,k}(x)dx$ and $\beta_{j,k} = \int_{-\Omega}^{\Omega} f(x)\psi_{j,k}(x)dx$ by respectively

$$\hat{\alpha}_{j,k} = \frac{1}{n} \sum_{v=1}^n (\mathcal{K}\phi)_{j,k}(Y_v), \quad \hat{\beta}_{j,k} = \frac{1}{n} \sum_{v=1}^n (\mathcal{K}\psi)_{j,k}(Y_v). \quad (4.2)$$

Statistical properties of these estimators are given in Propositions 6.1, 6.2 and 6.3 below.

Suppose that (2.2) and (2.3) are satisfied and $f \in B_{p,r}^s(M)$ with $p \geq 2$. We define the linear wavelet estimator \hat{f}^L by

$$\hat{f}^L(x) = \sum_{k \in \Lambda_{j_0}} \hat{\alpha}_{j_0,k} \phi_{j_0,k}(x), \quad (4.3)$$

where j_0 is the integer such that

$$\frac{1}{2} n^{1/(2s+2\delta+1)} < 2^{j_0} \leq n^{1/(2s+2\delta+1)}.$$

This estimator is the one in [10, eq (4)] with *i.i.d.* Y_1, \dots, Y_n .

Suppose that (2.2), (2.3) and (2.5) are satisfied. We define the hard thresholding estimator \hat{f}^H by

$$\hat{f}^H(x) = \sum_{k=0}^{2^\tau-1} \hat{\alpha}_{\tau,k} \phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \hat{\beta}_{j,k} \mathbb{I}_{\{|\hat{\beta}_{j,k}| \geq \kappa \lambda_j\}} \psi_{j,k}(x), \quad (4.4)$$

where, for any random event \mathcal{A} , $\mathbb{I}_{\mathcal{A}}$ is the indicator function on \mathcal{A} , j_1 is the integer satisfying

$$\frac{1}{2} \left(\frac{n}{(\ln n)^{1+1/\theta}} \right)^{1/(2\delta+1)} < 2^{j_1} \leq \left(\frac{n}{(\ln n)^{1+1/\theta}} \right)^{1/(2\delta+1)},$$

κ is a large enough constant (the one in Proposition 6.3 below) and

$$\lambda_j = 2^{\delta j} \sqrt{\frac{(\ln n)^{1+1/\theta}}{n}}. \quad (4.5)$$

Note that, if $\theta \rightarrow \infty$ (the *i.i.d.* case), \widehat{f}^H becomes the hard thresholding estimator considered by [10].

5 Upper bounds

Theorem 5.1 (Upper bound for \widehat{f}^L) Consider (1.1) under the assumptions (2.1), (2.2), (2.3), (2.4) and (2.6). Suppose that $f \in B_{p,r}^s(M)$ with $s > 0$, $p \geq 2$ and $r \geq 1$. Let \widehat{f}^L be (4.3). Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_{-\Omega}^{\Omega} \left(\widehat{f}^L(x) - f(x) \right)^2 dx \right) \leq C n^{-2s/(2s+2\delta+1)}.$$

The rate of convergence $n^{-2s/(2s+2\delta+1)}$ is the optimal one in the minimax sense when Y_1, \dots, Y_n are *i.i.d.* (see [10, Theorem 2]). Theorem 5.1 proves that our wavelet linear estimator \widehat{f}^L attains this rate without deterioration. This extends the results [24, Theorems 3.1 and 3.2].

Theorem 5.2 (Upper bound for \widehat{f}^H) Consider (1.1) under the assumptions (2.1), (2.2), (2.3), (2.5) (implying (2.4)) and (2.6). Let \widehat{f}^H be (4.4). Suppose that $f \in B_{p,r}^s(M)$ with $r \geq 1$, $\{p \geq 2 \text{ and } s > 0\}$ or $\{p \in [1, 2) \text{ and } s > (2\delta + 1)/p\}$. Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_{-\Omega}^{\Omega} \left(\widehat{f}^H(x) - f(x) \right)^2 dx \right) \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}.$$

Theorem 5.2 shows that, besides being adaptive, \widehat{f}^H attains a rate of convergence close to the one of \widehat{f}^L . The only difference is the logarithmic term $(\ln n)^{(1+1/\theta)(2s/(2s+1))}$.

Note that, if we restrict our study to the independent case i.e. $\theta \rightarrow \infty$, $((\ln n)^{1+1/\theta}/n)^{2s/(2s+2\delta+1)} = (\log n/n)^{2s/(2s+2\delta+1)}$ and this is the standard one for the corresponding hard thresholding estimator. See [10, Theorem 7].

6 Proofs

In this section, C denotes any constant that does not depend on j , k and n . Its value may change from one term to another and may depend on ϕ or ψ .

6.1 Auxiliary results

We will need [10, Lemma 6]. The proof is based on some properties of \mathcal{B} , the ordinary smooth conditions (2.2), (2.3) and a double integration by parts.

Lemma 6.1 ([10]) *Suppose that (2.2) and (2.3) hold. Consider the wavelet basis \mathcal{B} defined in Section 3. For any integer $j \geq \tau$, $k \in \Lambda_j$, $v \in \{\phi, \psi\}$ and $y \in \mathbb{R}$, let $(\mathcal{K}v)_{j,k}(y)$ be (4.1). Then there exists a constant $C > 0$ such that*

$$|(\mathcal{K}v)_{j,k}(y)| \leq C 2^{j(1/2+\delta)} \frac{1}{(1 + |2^j y - k|)^2}.$$

Proposition 6.1 *Consider (1.1) under the assumptions (2.1), (2.2), (2.3), (2.4) and (2.6). For any integer $j \geq \tau$ such that $2^j \leq n$ and $k \in \Lambda_j$, let $\alpha_{j,k}$ be the wavelet coefficient (3.1) of f and $\hat{\alpha}_{j,k}$ be (4.2). Then there exists a constant $C > 0$ such that*

$$\mathbb{E} \left(|\hat{\alpha}_{j,k} - \alpha_{j,k}|^2 \right) \leq C 2^{2\delta j} \frac{1}{n}.$$

This inequality holds for $\hat{\beta}_{j,k}$ (4.2) instead of $\hat{\alpha}_{j,k}$ and $\beta_{j,k}$ instead of $\alpha_{j,k}$.

Proof of Proposition 6.1. First of all, let us prove that $\hat{\alpha}_{j,k}$ is an unbiased estimator for $\alpha_{j,k}$. Since X_0 and ϵ_0 are independent, we have $\mathbb{E}(e^{-ixY_0}) = \mathbb{E}(e^{-ixX_0}) \mathbb{E}(e^{-ix\epsilon_0}) = \mathcal{F}(f)(x) \mathcal{F}(g)(x)$. This combined with the Fubini theorem and the Parseval-Plancherel theorem yield

$$\begin{aligned} \mathbb{E}(\hat{\alpha}_{j,k}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{\mathcal{F}(\phi_{j,k})(x)}}{\mathcal{F}(g)(x)} \mathbb{E}(e^{-ixY_0}) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathcal{F}(\phi_{j,k})(x)} \mathcal{F}(f)(x) dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\mathcal{F}(\phi_{j,k})(x)} \mathcal{F}(f)(x) dx = \int_{-\Omega}^{\Omega} f(x) \phi_{j,k}(x) dx \\ &= \alpha_{j,k}. \end{aligned} \tag{6.1}$$

Therefore

$$\mathbb{E}(|\hat{\alpha}_{j,k} - \alpha_{j,k}|^2) = \mathbb{V}(\hat{\alpha}_{j,k}) \quad (6.2)$$

and, by a standard covariance decomposition,

$$\begin{aligned} \mathbb{V}(\hat{\alpha}_{j,k}) &= \frac{1}{n^2} \sum_{v=1}^n \sum_{\ell=1}^n \mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_v), (\mathcal{K}\phi)_{j,k}(Y_\ell)) \\ &= \frac{1}{n} \mathbb{V}((\mathcal{K}\phi)_{j,k}(Y_0)) + \frac{2}{n^2} \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_v), (\mathcal{K}\phi)_{j,k}(Y_\ell)) \\ &\leq \frac{1}{n} \mathbb{V}((\mathcal{K}\phi)_{j,k}(Y_0)) + \frac{2}{n^2} \left| \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_v), (\mathcal{K}\phi)_{j,k}(Y_\ell)) \right|. \end{aligned} \quad (6.3)$$

Let us bound the first term in (6.3). Since X_0 and ϵ_0 are independent, the density of Y_0 is $q(x) = (f \star g)(x) = \int_{-\infty}^{\infty} g(t)f(x-t)dt$, $x \in \mathbb{R}$. By (2.1) and the fact that g is a density, we have $\sup_{x \in \mathbb{R}} q(x) \leq C \int_{-\infty}^{\infty} g(y)dy = C$. This with Lemma 6.1 and the change of variables $u = 2^j x - k$ imply that

$$\begin{aligned} \mathbb{V}((\mathcal{K}\phi)_{j,k}(Y_0)) &\leq \mathbb{E}(|(\mathcal{K}\phi)_{j,k}(Y_0)|^2) = \int_{-\infty}^{\infty} |(\mathcal{K}\phi)_{j,k}(y)|^2 q(y) dy \\ &\leq C \int_{-\infty}^{\infty} |(\mathcal{K}\phi)_{j,k}(y)|^2 dy \leq C 2^{j(1+2\delta)} \int_{-\infty}^{\infty} \frac{1}{(1+|2^j y - k|)^4} dy \\ &= C 2^{2\delta j} \int_{-\infty}^{\infty} \frac{1}{(1+|u|)^4} du \leq C 2^{2\delta j}. \end{aligned} \quad (6.4)$$

It follows from the stationarity of $(Y_v)_{v \in \mathbb{Z}}$ that

$$\begin{aligned} &\left| \sum_{v=2}^n \sum_{\ell=1}^{v-1} \mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_v), (\mathcal{K}\phi)_{j,k}(Y_\ell)) \right| \\ &= \left| \sum_{m=1}^n (n-m) \mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m)) \right| \\ &\leq n \sum_{m=1}^n |\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))|. \end{aligned} \quad (6.5)$$

We can write

$$n \sum_{m=1}^n |\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))| = \mathbf{T}_1 + \mathbf{T}_2, \quad (6.6)$$

where

$$\mathbf{T}_1 = n \sum_{m=1}^{2^j-1} |\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))|$$

and

$$\mathbf{T}_2 = n \sum_{m=2^j}^n |\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))|.$$

Let us now bound \mathbf{T}_1 and \mathbf{T}_2 .

Upper bound for \mathbf{T}_1 . Using (2.6), Lemma 6.1 and doing the change of variables $u = 2^j y - k$, we obtain

$$\begin{aligned} & |\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))| \\ &= \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (q_{(Y_0, Y_m)}(x, y) - q(x)q(y)) (\mathcal{K}\phi)_{j,k}(x) (\mathcal{K}\phi)_{j,k}(y) dx dy \right| \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |q_{(Y_0, Y_m)}(x, y) - q(x)q(y)| |(\mathcal{K}\phi)_{j,k}(x)| |(\mathcal{K}\phi)_{j,k}(y)| dx dy \\ &\leq C \left(\int_{-\infty}^{\infty} |(\mathcal{K}\phi)_{j,k}(y)| dy \right)^2 \leq C \left(2^{j(1/2+\delta)} \int_{-\infty}^{\infty} \frac{1}{(1 + |2^j y - k|)^2} dy \right)^2 \\ &= C 2^{-j} 2^{2\delta j} \left(\int_{-\infty}^{\infty} \frac{1}{(1 + |u|)^2} du \right)^2 = C 2^{-j} 2^{2\delta j}. \end{aligned}$$

Therefore

$$\mathbf{T}_1 \leq C n 2^{2\delta j} 2^{-j} 2^j = C 2^{2\delta j} n. \quad (6.7)$$

Upper bound for \mathbf{T}_2 . By the Davydov inequality for strongly mixing processes (see [6]), for the considered $\gamma \in (0, 1)$ in (2.4), it holds that

$$\begin{aligned} |\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))| &\leq 10 a_m^\gamma \left(\mathbb{E} \left(|(\mathcal{K}\phi)_{j,k}(Y_0)|^{2/(1-\gamma)} \right) \right)^{1-\gamma} \\ &\leq 10 a_m^\gamma (\sup_{y \in \mathbb{R}} |(\mathcal{K}\phi)_{j,k}(y)|)^{2\gamma} \left(\mathbb{E} \left(|(\mathcal{K}\phi)_{j,k}(Y_0)|^2 \right) \right)^{1-\gamma}. \end{aligned}$$

By Lemma 6.1, we have

$$\sup_{y \in \mathbb{R}} |(\mathcal{K}\phi)_{j,k}(y)| \leq C 2^{j(1/2+\delta)} \sup_{y \in \mathbb{R}} \frac{1}{(1 + |2^j y - k|)^2} \leq C 2^{j(1/2+\delta)}$$

and, by (6.4),

$$\mathbb{E} \left(|(\mathcal{K}\phi)_{j,k}(Y_0)|^2 \right) \leq C 2^{2\delta j}.$$

Therefore

$$|\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))| \leq C2^{2\delta j}2^{\gamma j}a_m^\gamma.$$

By (2.4), we have

$$\begin{aligned} \mathbf{T}_2 &\leq Cn2^{2\delta j}2^{\gamma j} \sum_{m=2^j}^n a_m^\gamma \leq Cn2^{2\delta j} \sum_{m=2^j}^n m^\gamma a_m^\gamma \\ &\leq Cn2^{2\delta j} \sum_{m=1}^{\infty} m^\gamma a_m^\gamma \leq Cn2^{2\delta j}. \end{aligned} \quad (6.8)$$

It follows from (6.6), (6.7) (6.8) that

$$n \sum_{m=1}^n |\mathbb{C}((\mathcal{K}\phi)_{j,k}(Y_0), (\mathcal{K}\phi)_{j,k}(Y_m))| \leq Cn2^{2\delta j}. \quad (6.9)$$

Therefore, combining (6.2), (6.3), (6.4), (6.5) and (6.9), we obtain

$$\mathbb{E}(|\hat{\alpha}_{j,k} - \alpha_{j,k}|^2) \leq C2^{2\delta j} \frac{1}{n}.$$

The proof of Proposition 6.1 is complete. □

Proposition 6.2 *Consider (1.1) under the assumptions (2.1), (2.2), (2.3), (2.4) and (2.6). For any integer $j \geq \tau$ such that $2^j \leq n$ and $k \in \Lambda_j$, let $\beta_{j,k}$ be the wavelet coefficient (3.1) of f and $\hat{\beta}_{j,k}$ be (4.2). Then there exists a constant $C > 0$ such that*

$$\mathbb{E} \left(\left| \hat{\beta}_{j,k} - \beta_{j,k} \right|^4 \right) \leq C2^{j(1+4\delta)} \frac{1}{n}.$$

Proof of Proposition 6.2. Using Lemma 6.1, we obtain

$$|\hat{\beta}_{j,k}| \leq \sup_{y \in \mathbb{R}} |(\mathcal{K}\psi)_{j,k}(y)| \leq C2^{j(1/2+\delta)} \sup_{y \in \mathbb{R}} \frac{1}{(1 + |2^j y - k|)^2} \leq C2^{j(1/2+\delta)}.$$

By (2.1), we have $|\beta_{j,k}| \leq C$. Therefore

$$|\hat{\beta}_{j,k} - \beta_{j,k}| \leq |\hat{\beta}_{j,k}| + |\beta_{j,k}| \leq C(2^{j(1/2+\delta)} + 1) \leq C2^{j(1/2+\delta)}. \quad (6.10)$$

It follows from (6.10) and Proposition 6.1 that

$$\mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^4 \right) \leq C 2^{j(1+2\delta)} \mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^2 \right) \leq C 2^{j(1+4\delta)} \frac{1}{n}.$$

The proof of Proposition 6.2 is complete. \square

Proposition 6.3 *Consider (1.1) under the assumptions (2.1), (2.2), (2.3), (2.5) and (2.6). For any integer $j \geq \tau$ such that $2^j \leq n/(\ln n)^{1+1/\theta}$ and $k \in \Lambda_j$, let $\beta_{j,k}$ be the wavelet coefficient (3.1) of f , $\widehat{\beta}_{j,k}$ be (4.2) and λ_j be (4.5). Then there exist two constants $\kappa > 0$ and $C > 0$ such that*

$$\mathbb{P} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right| \geq \kappa \lambda_j / 2 \right) \leq C \frac{1}{n^4}.$$

Proof of Proposition 6.3. We will use a Bernstein inequality for exponentially strongly mixing process. The proof can be found in [22] and [15].

Lemma 6.2 ([22] and [15]) *Let $\gamma > 0$, $c > 0$, $\theta > 1$ and $(Y_i)_{i \in \mathbb{Z}}$ be a stationary process such that, for any $m \in \mathbb{Z}$, the associated m -th strongly mixing coefficient (2.5) satisfies $a_m \leq \gamma \exp(-c|m|^\theta)$. Let $n \in \mathbb{N}^*$, $h : \mathbb{R} \rightarrow \mathbb{C}$ be a measurable function and, for any $i \in \mathbb{Z}$, $U_i = h(Y_i)$. We assume that $\mathbb{E}(U_1) = 0$ and there exists a constant $M > 0$ satisfying $|U_1| \leq M$. Then, for any $m \in \{1, \dots, n\}$ and $\lambda > 4mM/n$, we have*

$$\mathbb{P} \left(\left| \frac{1}{n} \sum_{i=1}^n U_i \right| \geq \lambda \right) \leq 4 \exp \left(- \frac{\lambda^2 n}{m(64\mathbb{E}(U_1^2) + 8\lambda M/3)} \right) + 4\gamma \frac{n}{m} \exp(-cm^\theta).$$

We can write

$$\widehat{\beta}_{j,k} - \beta_{j,k} = \frac{1}{n} \sum_{v=1}^n U_v$$

where, for any $v \in \{1, \dots, n\}$,

$$U_v = (\mathcal{K}\psi)_{j,k}(Y_v) - \beta_{j,k}.$$

So

$$\mathbb{P}\left(|\widehat{\beta}_{j,k} - \beta_{j,k}| \geq \kappa\lambda_j/2\right) = \mathbb{P}\left(\left|\frac{1}{n} \sum_{v=1}^n U_v\right| \geq \kappa\lambda_j/2\right),$$

where U_1, \dots, U_n are identically distributed, depend on $(Y_v)_{v \in \mathbb{Z}}$ satisfying (2.5),

- by (6.1), $\mathbb{E}(U_1) = 0$,
- by (6.4), $\mathbb{E}(|U_1|^2) = \mathbb{V}((\mathcal{K}\psi)_{j,k}(Y_1)) \leq C2^{2\delta j}$,
- in a similar fashion to (6.10), $|U_1| \leq 2^{j(1/2+\delta)}$.

Lemma 6.2 applied with U_1, \dots, U_n , $\lambda = \kappa C\lambda_j$, $\lambda_j = 2^{\delta j}((\ln n)^{1+1/\theta}/n)^{1/2}$, $m = (u \ln n)^{1/\theta}$ with $u > 0$ (chosen later), $M = C2^{j(1/2+\delta)}$ and $2^j \leq n/(\ln n)^{1+1/\theta}$ gives

$$\begin{aligned} & \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^n U_i\right| \geq \kappa\lambda_j/2\right) \\ & \leq 4 \exp\left(-C \frac{\kappa^2 \lambda_j^2 n}{m(2^{2\delta j} + \kappa\lambda_j M)}\right) + 4\gamma \frac{n}{m} \exp(-cm^\theta) \\ & \leq 4 \exp\left(-C \frac{\kappa^2 2^{2\delta j} (\ln n)^{1+1/\theta}}{(u \ln n)^{1/\theta} (2^{2\delta j} + \kappa 2^{j/2} 2^{2\delta j} ((\ln n)^{1+1/\theta}/n)^{1/2})}\right) \\ & + 4\gamma \frac{n}{(u \ln n)^{1/\theta}} \exp(-cu \ln n) \\ & = 4 \exp\left(-C \frac{\kappa^2 \ln n}{u^{1/\theta} (1 + \kappa 2^{j/2} ((\ln n)^{1+1/\theta}/n)^{1/2})}\right) + 4\gamma \frac{1}{(u \ln n)^{1/\theta}} n^{1-cu} \\ & \leq C \left(n^{-C\kappa^2/(u^{1/\theta}(1+\kappa))} + n^{1-cu}\right). \end{aligned}$$

Therefore, for large enough κ and u , we have

$$\mathbb{P}\left(|\widehat{\beta}_{j,k} - \beta_{j,k}| \geq \kappa\lambda_j/2\right) \leq C \frac{1}{n^4}.$$

This ends the proof of Proposition 6.3. □

6.2 Proofs of the main results

Proof of Theorem 5.1. We expand the function f on \mathcal{B} as

$$f(x) = \sum_{k \in \Lambda_{j_0}} \alpha_{j_0,k} \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_{k \in \Lambda_j} \beta_{j,k} \psi_{j,k}(x),$$

where $\alpha_{j_0,k} = \int_{-\Omega}^{\Omega} f(x) \phi_{j_0,k}(x) dx$ and $\beta_{j,k} = \int_{-\Omega}^{\Omega} f(x) \psi_{j,k}(x) dx$.

Since \mathcal{B} is an orthonormal basis of $\mathbb{L}^2([-\Omega, \Omega])$, we have

$$\begin{aligned} \mathbb{E} \left(\int_{-\Omega}^{\Omega} \left(\widehat{f}^L(x) - f(x) \right)^2 dx \right) = \\ \sum_{k \in \Lambda_{j_0}} \mathbb{E} \left(|\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^2 \right) + \sum_{j=j_0}^{\infty} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2. \end{aligned}$$

Using Proposition 6.1 and the definitions of Λ_{j_0} and j_0 , we obtain

$$\sum_{k \in \Lambda_{j_0}} \mathbb{E} \left(|\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^2 \right) \leq C 2^{j_0} \frac{2^{2\delta j_0}}{n} \leq C n^{-2s/(2s+2\delta+1)}.$$

Since $p \geq 2$, we have $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$. Hence

$$\sum_{j=j_0}^{\infty} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2 \leq C 2^{-2j_0 s} \leq C n^{-2s/(2s+2\delta+1)}.$$

Therefore

$$\mathbb{E} \left(\int_{-\Omega}^{\Omega} \left(\widehat{f}^L(x) - f(x) \right)^2 dx \right) \leq C n^{-2s/(2s+2\delta+1)}.$$

Theorem 5.1 is proved □

Proof of Theorem 5.2. We expand the function f on \mathcal{B} as

$$f(x) = \sum_{k \in \Lambda_{\tau}} \alpha_{\tau,k} \phi_{\tau,k}(x) + \sum_{j=\tau}^{\infty} \sum_{k \in \Lambda_j} \beta_{j,k} \psi_{j,k}(x),$$

where $\alpha_{\tau,k} = \int_{-\Omega}^{\Omega} f(x) \phi_{\tau,k}(x) dx$ and $\beta_{j,k} = \int_{-\Omega}^{\Omega} f(x) \psi_{j,k}(x) dx$.

Since \mathcal{B} is an orthonormal basis of $\mathbb{L}^2([-\Omega, \Omega])$, we have

$$\mathbb{E} \left(\int_{-\Omega}^{\Omega} \left(\widehat{f}^H(x) - f(x) \right)^2 dx \right) = \mathbf{R} + \mathbf{S} + \mathbf{T}, \quad (6.11)$$

where

$$\mathbf{R} = \sum_{k \in \Lambda_\tau} \mathbb{E} \left(|\widehat{\alpha}_{\tau,k} - \alpha_{\tau,k}|^2 \right), \quad \mathbf{S} = \sum_{j=\tau}^{j_1} \sum_{k \in \Lambda_j} \mathbb{E} \left(\left| \widehat{\beta}_{j,k} \mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} - \beta_{j,k} \right|^2 \right)$$

and

$$\mathbf{T} = \sum_{j=j_1+1}^{\infty} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2.$$

Let us bound \mathbf{R} , \mathbf{T} and \mathbf{S} (by order of difficulty).

Using Proposition 6.1, we obtain

$$\mathbf{R} \leq C 2^{\tau(1+2\delta)} \frac{1}{n} \leq C \frac{1}{n} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \quad (6.12)$$

For $r \geq 1$ and $p \geq 2$, we have $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$. Therefore

$$\begin{aligned} \mathbf{T} &\leq C \sum_{j=j_1+1}^{\infty} 2^{-2js} \leq C 2^{-2j_1s} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2\delta+1)} \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \end{aligned}$$

For $r \geq 1$ and $p \in [1, 2)$, we have $B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$. Since $s > (2\delta + 1)/p$, we have $(s + 1/2 - 1/p)/(2\delta + 1) > s/(2s + 2\delta + 1)$. Hence

$$\begin{aligned} \mathbf{T} &\leq C \sum_{j=j_1+1}^{\infty} 2^{-2j(s+1/2-1/p)} \leq C 2^{-2j_1(s+1/2-1/p)} \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2(s+1/2-1/p)/(2\delta+1)} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \end{aligned}$$

Hence, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1, 2)$ and $s > (2\delta + 1)/p\}$, we have

$$\mathbf{T} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \quad (6.13)$$

Let us now investigate the upper bound for \mathbf{S} . We can write

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4, \quad (6.14)$$

where

$$\mathbf{S}_1 = \sum_{j=\tau}^{j_1} \sum_{k \in \Lambda_j} \mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} \mathbb{I}_{\{|\beta_{j,k}| < \kappa \lambda_j / 2\}} \right),$$

$$\mathbf{S}_2 = \sum_{j=\tau}^{j_1} \sum_{k \in \Lambda_j} \mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa \lambda_j\}} \mathbb{I}_{\{|\beta_{j,k}| \geq \kappa \lambda_j / 2\}} \right),$$

$$\mathbf{S}_3 = \sum_{j=\tau}^{j_1} \sum_{k \in \Lambda_j} \mathbb{E} \left(|\beta_{j,k}|^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| < \kappa \lambda_j\}} \mathbb{I}_{\{|\beta_{j,k}| \geq 2\kappa \lambda_j\}} \right)$$

and

$$\mathbf{S}_4 = \sum_{j=\tau}^{j_1} \sum_{k \in \Lambda_j} \mathbb{E} \left(|\beta_{j,k}|^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| < \kappa \lambda_j\}} \mathbb{I}_{\{|\beta_{j,k}| < 2\kappa \lambda_j\}} \right).$$

Let us analyze each term \mathbf{S}_1 , \mathbf{S}_2 , \mathbf{S}_3 and \mathbf{S}_4 in turn.

Upper bounds for \mathbf{S}_1 and \mathbf{S}_3 . We have

$$\left\{ |\widehat{\beta}_{j,k}| < \kappa \lambda_j, |\beta_{j,k}| \geq 2\kappa \lambda_j \right\} \subseteq \left\{ |\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2 \right\},$$

$$\left\{ |\widehat{\beta}_{j,k}| \geq \kappa \lambda_j, |\beta_{j,k}| < \kappa \lambda_j / 2 \right\} \subseteq \left\{ |\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2 \right\}$$

and

$$\left\{ |\widehat{\beta}_{j,k}| < \kappa \lambda_j, |\beta_{j,k}| \geq 2\kappa \lambda_j \right\} \subseteq \left\{ |\beta_{j,k}| \leq 2|\widehat{\beta}_{j,k} - \beta_{j,k}| \right\}.$$

So

$$\max(\mathbf{S}_1, \mathbf{S}_3) \leq C \sum_{j=\tau}^{j_1} \sum_{k \in \Lambda_j} \mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2\}} \right).$$

It follows from the Cauchy-Schwarz inequality, Propositions 6.2 and 6.3 and $2^j \leq 2^{j_1} \leq n$ that

$$\begin{aligned} & \mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2\}} \right) \\ & \leq \left(\mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^4 \right) \right)^{1/2} \left(\mathbb{P} \left(|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa \lambda_j / 2 \right) \right)^{1/2} \\ & \leq C \left(2^{j(1+4\delta)} \frac{1}{n} \right)^{1/2} \left(\frac{1}{n^4} \right)^{1/2} \leq C 2^{2\delta j} \frac{1}{n^2} \leq C 2^{2\delta j} \frac{(\ln n)^{1+1/\theta}}{n^2}. \end{aligned}$$

We have

$$\begin{aligned} \max(\mathbf{S}_1, \mathbf{S}_3) & \leq C \frac{(\ln n)^{1+1/\theta}}{n^2} \sum_{j=\tau}^{j_1} 2^{j(1+2\delta)} \leq C \frac{(\ln n)^{1+1/\theta}}{n^2} 2^{j_1(1+2\delta)} \\ & \leq C \frac{(\ln n)^{1+1/\theta}}{n} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \end{aligned} \quad (6.15)$$

Upper bound for \mathbf{S}_2 . Using Proposition 6.1, we obtain

$$\mathbb{E} \left(\left| \widehat{\beta}_{j,k} - \beta_{j,k} \right|^2 \right) \leq C 2^{2\delta j} \frac{1}{n} \leq C 2^{2\delta j} \frac{(\ln n)^{1+1/\theta}}{n}.$$

Hence

$$\mathbf{S}_2 \leq C \frac{(\ln n)^{1+1/\theta}}{n} \sum_{j=\tau}^{j_1} 2^{2\delta j} \sum_{k \in \Lambda_j} \mathbb{I}_{\{|\beta_{j,k}| > \kappa \lambda_j / 2\}}.$$

Let j_2 be the integer satisfying

$$\frac{1}{2} \left(\frac{n}{(\ln n)^{1+1/\theta}} \right)^{1/(2s+2\delta+1)} < 2^{j_2} \leq \left(\frac{n}{(\ln n)^{1+1/\theta}} \right)^{1/(2s+2\delta+1)}. \quad (6.16)$$

Then we have

$$\mathbf{S}_2 \leq \mathbf{S}_{2,1} + \mathbf{S}_{2,2},$$

where

$$\mathbf{S}_{2,1} = C \frac{(\ln n)^{1+1/\theta}}{n} \sum_{j=\tau}^{j_2} 2^{2\delta j} \sum_{k \in \Lambda_j} \mathbb{I}_{\{|\beta_{j,k}| > \kappa \lambda_j / 2\}}$$

and

$$\mathbf{S}_{2,2} = C \frac{(\ln n)^{1+1/\theta}}{n} \sum_{j=j_2+1}^{j_1} 2^{2\delta j} \sum_{k \in \Lambda_j} \mathbb{I}_{\{|\beta_{j,k}| > \kappa \lambda_j / 2\}}.$$

We have

$$\begin{aligned} \mathbf{S}_{2,1} &\leq C \frac{(\ln n)^{1+1/\theta}}{n} \sum_{j=\tau}^{j_2} 2^{j(1+2\delta)} \leq C \frac{(\ln n)^{1+1/\theta}}{n} 2^{j_2(1+2\delta)} \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \end{aligned}$$

For $r \geq 1$ and $p \geq 2$, since $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$, we obtain

$$\begin{aligned} \mathbf{S}_{2,2} &\leq C \frac{(\ln n)^{1+1/\theta}}{n} \sum_{j=j_2+1}^{j_1} 2^{2\delta j} \frac{1}{\lambda_j^2} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2 \leq C \sum_{j=j_2+1}^{\infty} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2 \\ &\leq C \sum_{j=j_2+1}^{\infty} 2^{-2js} \leq C 2^{-2j_2s} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \end{aligned}$$

For $r \geq 1$, $p \in [1, 2)$ and $s > (2\delta + 1)/p$, since $B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$ and $(2s + 2\delta + 1)(2 - p)/2 + (s + 1/2 - 1/p + \delta - 2\delta/p)p = 2s$, we have

$$\begin{aligned} \mathbf{S}_{2,2} &\leq C \frac{(\ln n)^{1+1/\theta}}{n} \sum_{j=j_2+1}^{j_1} 2^{2\delta j} \frac{1}{\lambda_j^p} \sum_{k \in \Lambda_j} |\beta_{j,k}|^p \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{j\delta(2-p)} 2^{-j(s+1/2-1/p)p} \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p+\delta-2\delta/p)p} \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \end{aligned}$$

So, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1, 2)$ and $s > (2\delta + 1)/p\}$, we have

$$\mathbf{S}_2 \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \quad (6.17)$$

Upper bound for \mathbf{S}_4 . We have

$$\mathbf{S}_4 \leq \sum_{j=\tau}^{j_1} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2 \mathbb{I}_{\{|\beta_{j,k}| < 2\kappa\lambda_j\}}.$$

Let j_2 be the integer (6.16). We have

$$\mathbf{S}_4 \leq \mathbf{S}_{4,1} + \mathbf{S}_{4,2},$$

where

$$\mathbf{S}_{4,1} = \sum_{j=\tau}^{j_2} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2 \mathbb{I}_{\{|\beta_{j,k}| < 2\kappa\lambda_j\}}, \quad \mathbf{S}_{4,2} = \sum_{j=j_2+1}^{j_1} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2 \mathbb{I}_{\{|\beta_{j,k}| < 2\kappa\lambda_j\}}.$$

We have

$$\begin{aligned} \mathbf{S}_{4,1} &\leq C \sum_{j=\tau}^{j_2} 2^j \lambda_j^2 = C \frac{(\ln n)^{1+1/\theta}}{n} \sum_{j=\tau}^{j_2} 2^{j(1+2\delta)} \leq C \frac{(\ln n)^{1+1/\theta}}{n} 2^{j_2(1+2\delta)} \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \end{aligned}$$

For $r \geq 1$ and $p \geq 2$, since $B_{p,r}^s(M) \subseteq B_{2,\infty}^s(M)$, we have

$$\mathbf{S}_{4,2} \leq \sum_{j=j_2+1}^{\infty} \sum_{k \in \Lambda_j} |\beta_{j,k}|^2 \leq C \sum_{j=j_2+1}^{\infty} 2^{-2js} \leq C 2^{-2j_2s} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}.$$

For $r \geq 1$, $p \in [1, 2)$ and $s > (2\delta + 1)/p$, since $B_{p,r}^s(M) \subseteq B_{2,\infty}^{s+1/2-1/p}(M)$

and $(2s + 2\delta + 1)(2 - p)/2 + (s + 1/2 - 1/p + \delta - 2\delta/p)p = 2s$, we have

$$\begin{aligned}
\mathbf{S}_{4,2} &\leq C \sum_{j=j_2+1}^{j_1} \lambda_j^{2-p} \sum_{k \in \Lambda_j} |\beta_{j,k}|^p \\
&= C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{(2-p)/2} \sum_{j=j_2+1}^{j_1} 2^{j\delta(2-p)} \sum_{k \in \Lambda_j} |\beta_{j,k}|^p \\
&\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{j\delta(2-p)} 2^{-j(s+1/2-1/p)p} \\
&\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p+\delta-2\delta/p)p} \\
&\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}.
\end{aligned}$$

So, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1, 2)$ and $s > (2\delta + 1)/p\}$, we have

$$\mathbf{S}_4 \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \quad (6.18)$$

It follows from (6.14), (6.15), (6.17) and (6.18) that

$$\mathbf{S} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}. \quad (6.19)$$

Combining (6.11), (6.12), (6.13) and (6.19), we have, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1, 2)$ and $s > (2\delta + 1)/p\}$,

$$\sup_{f \in B_{p,r}^s(M)} \mathbb{E} \left(\int_{-\Omega}^{\Omega} (\widehat{f}^H(x) - f(x))^2 dx \right) \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+2\delta+1)}.$$

The proof of Theorem 5.2 is complete. □

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