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# Gradient-based controllers for continuous Petri nets

**Dimitri Lefebvre, Edouard Leclercq, Fabrice Druaux**

Université Le Havre – GREAH, 25 rue P. Lebon, 76063 Le Havre, France

IUT Le Havre – GMP, Place R. Schuman, 76600 Le Havre, France

**Philippe Thomas**

Centre de Recherche en Automatique de Nancy (CRAN-UMR 7039), Nancy-University,  
CNRS

ENSTIB 27 rue du Merle Blanc, B.P. 1041, 88051 Epinal cedex 9 France, France

**Corresponding author: Pr. Dimitri Lefebvre**

Université Le Havre – GREAH, 25 rue P. Lebon, 76063 Le Havre, France

dimitri.lefebvre@univ-lehavre.fr

***Abstract:** This paper is about the control design of hybrid dynamical systems modelled with Petri nets. For this purpose, continuous Petri nets with variable speeds are investigated and described as piecewise bilinear state space representations. In this context, the marking vector is considered as a state space vector, subsets of places are defined as the model outputs, and the transitions are divided into non controllable ones and controllable ones that correspond to the model inputs. Gradient-based controllers are proposed and discussed in order to adapt the maximal firing frequencies of the controllable transitions according to desired trajectories of the output markings.*

## 1. Introduction

Petri nets (PN) are useful for the study of discrete event systems (DES) and hybrid dynamical systems (HDS) (Cassandras 1993, Zaytoon *et al.* 1998) because they combine, in a comprehensive way, intuitive graphical representations and powerful analytic expressions (Brams 1983, Brauer *et al.* 1986, Murata 1989). As a consequence, a lot of results based on PN theory have been established for the control design of DES and HDS. One of the most famous approach concern the supervisory control where the system and the controller are considered as DES (Ramadge *et al.* 1987, Giua *et al.* 1994, Uzam *et al.* 1999). Hybrid approaches were also developed in order to combine discrete and continuous signals (Krogh

*et al.* 1996, Bemporad *et al.* 1999). At last, continuous approaches were inspired from continuous flow models and continuous Petri nets.

The motivation to use continuous Petri nets is either to model the continuous part of HDS, or to work out a continuous approximation of DES in order to avoid the complexity associated to the exponential growth of states. Flow control design has been developed with different classes of controllers (Silva *et al.* 2003): constrained state feedback (Amrah *et al.* 1996, Lefebvre 1999), fuzzy control (Ghabri 1995), linear programming (Hanzalek 2003), and optimal control (Egilmez *et al.* 1994, Sharifnia A. 1988, 1994). This paper concerns another continuous Petri net approach where the proposed controllers are inspired from the neural network adaptation algorithms (Widrow *et al.* 1990, Thomas 1997) and based on input-output sensitivity functions. For this purpose, continuous Petri nets with variable speeds (VCPN) are investigated and described as piecewise bilinear state space representations where the places marking stands for the state space vector. The system outputs are defined as the marking of subsets of places, and the system inputs correspond to the maximal firing frequencies of a selection of controllable transitions. The main contributions are to investigate the input-output sensitivity of the PN model from structural and also functional points of view. On the one hand, a characterization of input – output structural sensitivity is defined and worked out in a systematic way. On the other hand, sensitivity functions are defined and processed thanks to numerical algorithms that are detailed. As a consequence, gradient-based controllers are proposed in order to adapt the controllable maximal firing frequencies according to desired trajectories of the output markings. The advantages of the gradient-based controllers are pointed out in case of multi-inputs and multi-outputs (MIMO) models for discrete and hybrid systems.

The paper is divided into 5 sections. The section 2 is about PN and VCPN. The section 3 concerns the structural analysis that provides useful results concerning the input-output structural sensitivity. The section 4 is about the design of gradient-based controllers. Various examples of VCPN are proposed in section 5 in order to discuss the proposed results and to compare the gradient-based controllers with proportional and bang-bang ones.

## 2. Petri nets

A Petri net (PN) with  $n$  places and  $p$  transitions is defined as  $\langle P, T, \text{Pre}, \text{Post}, M_0 \rangle$  where  $P = \{P_i\}_{i=1, \dots, n}$  is a not empty finite set of places,  $T = \{T_j\}_{j=1, \dots, p}$  is a not empty finite set of transitions, such that  $P \cap T = \emptyset$  (Brams 1983, Murata 1989).  $IN$  is defined as the set of integer numbers.  $\text{Pre}: P \times T \rightarrow IN$  is the pre-incidence application:  $\text{Pre}(P_i, T_j)$  is the weight of

the arc from place  $P_i$  to transition  $T_j$  and  $W_{PR} = (w_{ij}^{PR})_{i=1,\dots,n, j=1,\dots,p} \in \mathbb{N}^{n \times p}$  with  $w_{ij}^{PR} = \text{Pre}(P_i, T_j)$  is the pre-incidence matrix. Post:  $P \times T \rightarrow \mathbb{N}$  is the post-incidence application:  $\text{Post}(P_i, T_j)$  is the weight of the arc from transition  $T_j$  to place  $P_i$  and  $W_{PO} = (w_{ij}^{PO})_{i=1,\dots,n, j=1,\dots,p} \in \mathbb{N}^{n \times p}$  with  $w_{ij}^{PO} = \text{Post}(P_i, T_j)$  is the post-incidence matrix. The PN incidence matrix  $W$  is defined as  $W = W_{PO} - W_{PR} \in \mathbb{N}^{n \times p}$ . Let us also define  $M = (m_i)_{i=1,\dots,n} \in \mathbb{N}^n$  as the marking vector and  $M_0 \in \mathbb{N}^n$  as the initial marking vector.  ${}^\circ T_j$  (resp  $T_j^\circ$ ) stands for the pre-set (resp. post-set) places of  $T_j$ . Firing sequences are defined as an ordered series of transitions that are successively fired from marking  $M$  to marking  $M'$ . Such a sequence is represented by its characteristic vector  $X = (x_j)_{j=1,\dots,p} \in \mathbb{N}^p$  where  $x_j$  stands for the number of  $T_j$  firings. The marking  $M'$  is related to the marking  $M$  and to the firing sequence  $X$  according to the relation (1):

$$M' = M + W.X. \quad (1)$$

When two transitions  $T_j$  and  $T_{j'}$  have a common place in the pre-set, the PN presents a structural conflict. The conflict becomes an effective one if there are not enough tokens in the common place to fire both transitions. PN theory does not solve the conflicts. Conflicts are solved according to a decision maker that completes the PN models and that is not considered in this study. In autonomous PN without conflict, the enabling degree of each transition  $T_j$ , related to the marking  $M$ , is given by equation (2):

$$x_j = \min_{P_i \in {}^\circ T_j} \left( \text{fix} \left( \frac{m_i}{w_{ij}^{PR}} \right) \right), \quad (2)$$

where  $\text{fix}(\cdot)$  stands for the integer part of  $(\cdot)$ .

## 2.1. Continuous Petri nets

Continuous PN are a particular class of timed PN, deduced from T - timed Petri nets (TPN) (Ramchandani 1973) to provide a continuous approximation of DES behaviour (Alla *et al.* 1999, David *et al.* 1992). A continuous PN with  $n$  places and  $p$  transitions is defined as  $\langle \text{PN}, X_{max} \rangle$  where PN is a Petri net and  $X_{max} = (x_{maxj})_{j=1,\dots,p} \in \mathbb{R}^{+p}$  is the vector of maximal firing frequencies with  $\mathbb{R}^+$  the set of non-negative real numbers. The marking  $m_i(t)$  of each place  $P_i$ ,  $i = 1, \dots, n$ , at time  $t$  has a non-negative real value and each transition firing is a continuous flow in continuous PN. In fact, the transition  $T_j$  is fired with a frequency  $x_j(t)$  less

than the maximal frequency  $x_{max\ j}$ . Let us define  $X(t) = (x_j(t))_{j=1,\dots,p} \in \mathbb{R}^{+p}$  as the firing frequencies vector at time  $t$ . The marking evolution is given by the differential system (3):

$$\frac{dM(t)}{dt} = W.X(t). \quad (3)$$

Among the existing models of continuous PN, continuous PN with variable speeds (VCPN), and continuous PN with constant speeds (CCPN) were proved to give good approximations of TPN. In the next sections of this paper, VCPN are preferred because of their interesting properties (David *et al.* 1992): no effective conflict occurs with VCPN; components of the marking vector are continuous functions of the time; components of the firing frequencies vector  $X(t)$  depend continuously on the marking of the places according to equations (4) and (5):

$$x_j(t) = x_{max\ j} \cdot \mu_j(t), \quad (4)$$

with:

$$\mu_j(t) = \min_{P_i \in {}^oT_j} \left( \frac{m_i(t)}{w_{ij}^{PR}} \right). \quad (5)$$

Let us also notice that other models of continuous PN were investigated as differential Petri nets (Demongodin *et al.* 1998) or hybrid PN (Zaytoon *et al.* 1998) as illustrated by the example in the next section.

## 2.2. Continuous PN models of hybrid systems

Continuous PN are suitable to approximate DES or to model the continuous part of HDS (Balduzzi *et al.* 2000, Zaytoon *et al.* 1998) as illustrated with the example in figure 1, modelled with the hybrid PN in figure 2.

[Insert figure 1 about here]

The places  $P_1$  and  $P_2$  are continuous and the markings  $m_1$  and  $m_2$  stand respectively for the height of liquid in tank 1 and tank 2 according to (6):

$$\begin{aligned} S_1 \cdot \dot{m}_1 &= x_1 - x_2 - x_3 \\ S_2 \cdot \dot{m}_2 &= x_2 + x_3 - x_4 \end{aligned} \quad (6)$$

where  $S_1$  and  $S_2$  stand for the sections of tank 1 and tank 2. The transitions  $T_1$  to  $T_4$  are continuous which firing represents respectively the input flow ( $T_1$ ), the output flow ( $T_4$ ) and the flows through the pipes A ( $T_2$ ) and B ( $T_3$ ) according to (7):

$$\begin{aligned}
 x_1 &= D \\
 x_2 &= \alpha_2 \cdot \sqrt{m_1 - m_2} \\
 x_3 &= \alpha_3 \cdot \sqrt{\sup(m_1, h) - \sup(m_2, h)} \\
 x_4 &= \alpha_4 \cdot \sqrt{m_2}
 \end{aligned} \tag{7}$$

where  $D$ ,  $\alpha_2$ ,  $\alpha_3$  and  $\alpha_4$  are related to the system specifications and it is assumed that  $m_1 \geq m_2$ . The discrete part of the PN (places  $P_3$  and  $P_4$  and transitions  $T_5$  and  $T_6$ ) stands for the controller. A token in  $P_3$  means that valve  $V_1$  is open and  $V_2$  is closed. On the contrary, a token in  $P_4$  means that valve  $V_2$  is open and  $V_1$  is closed. The arcs from  $P_1$  to  $T_5$  and from  $P_2$  to  $T_6$  are test arcs (the value of the places  $P_1$  and  $P_2$  is not changed by firing the transitions  $T_5$  and  $T_6$ ). The goal of the controller is to open  $V_1$  and close  $V_2$  when  $m_2 < N_2$  and to open  $V_2$  and close  $V_1$  when  $m_1 > N_1$ . Such a discrete control design results in a cyclic steady state for the continuous variables  $m_1$  and  $m_2$  (figure 17). On the contrary, our approach described in sections 3 and 4 results in continuous control design useful to reach desired levels or to track reference trajectories (figure 18).

[Insert figure 2 about here]

Another example (figure 5) to motivate the use of continuous PN for the modelling and control of HDS will also be considered in the next sections.

### 2.3. Piecewise bilinear state space representation for VCPN

Due to the commutation function « min » and to the products between marking vector and maximal firing frequencies vector, VCPN models are not linear but piecewise bilinear systems (Amrah *et al.* 1996, Lefebvre 1998, Lefebvre *et al.* 2003a, Lefebvre *et al.* 2004). In order to bring VCPN models in the usual state space representation, let us introduce  $U(t) \in \mathbb{R}^{+d}$  as the VCPN input vector at time  $t$  and  $Y(t) \in \mathbb{R}^{+q}$  as the VCPN output vector at same instant.

The input vector  $U(t)$  is defined as the maximal firing frequencies of the controllable transitions. For this purpose, the set of transitions  $T$  is divided into 2 disjoint subsets  $T_C$ , and

$T_{NC}$  such that  $T = T_C \cup T_{NC}$ .  $T_C$  is the subset of the controllable transitions, and  $T_{NC}$  is the uncontrollable transitions subset. An obvious case is given by  $T_C = T$  and  $T_{NC} = \emptyset$ , but in many cases, not all transitions are controllable. For instance, the transitions  $T_2$  and  $T_3$  in the figure 2, correspond to the flows through the pipes A and B that are not controllable in the sense that these pipes have no valve. As a consequence, let us define  $X_C(t) = (x_j(t))_{T_j \in T_C} \in \mathbb{R}^{+d}$  and  $X_{NC}(t) = (x_j(t))_{T_j \in T_{NC}} \in \mathbb{R}^{+p-d}$  according to (8):

$$D^{-1}.X(t) = \begin{pmatrix} X_C(t) \\ X_{NC}(t) \end{pmatrix}, \quad (8)$$

with  $D \in \mathbb{R}^{p \times p}$  a suitable permutation matrix (i.e.  $D$  is the matrix of a bijective mapping from  $T$  to  $T$  that clusters the set of transitions into controllable and uncontrollable ones). The controllable inputs vector  $U(t) = X_{max\ C}(t) \in \mathbb{R}^{+d}$  corresponds to the maximal firing frequencies of the transitions to be controlled. The input vector is constrained  $0 \leq U(t) \leq U_{max}$  in order to limit the firing frequencies in a non negative bounded interval. The uncontrollable maximal firing frequencies  $X_{max\ NC}$  are supposed to be constant according to the VCPN models.

The output vector  $Y(t) = Q.M(t) \in \mathbb{R}^{+e}$  is composed of a selection of subnets markings that are observable. For this purpose, let us define  $Q = (q_{ki})_{k=1,\dots,e, i=1,\dots,n} \in \mathbb{R}^{e \times n}$  as a positive observation matrix (i.e.  $Q$  is the matrix of a constant projector, each row corresponds to a positive weighted sum of the PN places marking). As a consequence, observation may concern not only the marking of some individual places but also the global marking of subsets with several places. The goal of the controller is to drive  $Y(t)$  according to some reference trajectories in the output space. Equation (3) can be rewritten as:

$$\begin{aligned} \frac{dM(t)}{dt} &= W_C.X_C(t) + W_{NC}.X_{NC}(t) \\ Y(t) &= Q.M(t) \end{aligned} \quad (9)$$

with  $W_C = (w_{C\ ij})_{i=1,\dots,n, j=1,\dots,d} \in \mathbb{R}^{n \times d}$  and  $W_{NC} = (w_{NC\ ij})_{i=1,\dots,n, j=1,\dots,p-d} \in \mathbb{R}^{n \times (p-d)}$  such that  $(W_C / W_{NC}) = W.D$ .

Several phases occur in the VCPN behaviour (Zehrouni *et al.* 1995). Each phase  $\varphi$  is active between two successive commutations of the “min” operators in (5) and corresponds to a particular configuration of these operators characterised by the  $p$  clustering functions  $f_j$ :

$$\forall T_j \in T, \quad f_j: \mathbb{R}^{+n} \rightarrow \{1, \dots, n\}$$

$$M(t) \rightarrow f_j(M(t)) = k \text{ such that } m_k(t) = \mu_j(t). \quad (10)$$

Each function  $f_j$  specifies the place in the preset of  $T_j$  which has the minimal marking. During each phase  $\varphi$ , a constant relationship between the components of vectors  $X_C(t)$  and  $M(t)$  and also between  $X_{NC}(t)$  and  $M(t)$  occurs. This relation can be expressed under scalar form by using the functions  $f_j$  or under vectorial form by using the set of vectors  $A_j(\varphi) \in \{0,1\}^{1 \times n}$  and  $B_j(\varphi) \in \{0,1\}^{1 \times n}$  which are constant during each phase but which may varied from one phase to another:

$$\begin{aligned} x_{Cj}(t) &= u_j(t) \cdot m_{f_j}(t) \\ &= u_j(t) \cdot A_j(\varphi) \cdot M(t), \quad j = 1, \dots, d \\ x_{NCj}(t) &= x_{\max NCj} \cdot m_{f_j}(t) \\ &= x_{\max NCj} \cdot B_j(\varphi) \cdot M(t) \quad j = 1, \dots, p-d \end{aligned} \quad (11)$$

Equation (8) can be rewritten under scalar form:

$$\begin{aligned} \frac{m_i(t)}{dt} &= \sum_{j=1}^d w_{Cij} \cdot u_j(t) \cdot m_{f_j}(t) + \sum_{j=1}^{p-d} w_{NCij} \cdot x_{\max NCj} \cdot m_{f_j}(t) \\ y_k(t) &= \sum_{i=1}^n q_{ki} \cdot m_i(t) \end{aligned} \quad (12)$$

or under vectorial form:

$$\begin{aligned} \frac{dM(t)}{dt} &= \left( \sum_{j=1}^d u_j(t) \cdot W_{Cj} \cdot A_j(\varphi) + \sum_{j=1}^{p-d} x_{\max NCj} \cdot W_{NCj} \cdot B_j(\varphi) \right) \cdot M(t) \\ Y(t) &= Q \cdot M(t), \end{aligned} \quad (13)$$

where  $W_{Cj}$  denotes the  $j^{\text{th}}$  column of matrix  $W_C$  and  $W_{NCj}$  denotes the  $j^{\text{th}}$  column of matrix  $W_{NC}$ . Equations (12) and (13) are piecewise bilinear representations of the VCPN (3) (Mohler 1973). Each phase is characterised by a set of matrices  $W_{Cj} \cdot A_j(\varphi) \in IN^{n \times n}$  associated to the controllable transitions and  $W_{NCj} \cdot B_j(\varphi) \in IN^{n \times n}$  associated to the uncontrollable ones.

The design of gradient-based controllers for VCPN includes structural and functional aspects:

- The structural analysis is necessary to determine which inputs act on a given output. It is also useful to know which outputs are sensitive with respect to the variations of a given input. In section 3, structural sensitivity is defined and structural analysis is discussed.

- The functional analysis consists to adapt the usual gradient algorithm in order to drive the VCPN outputs near the desired marking. In section 4, sensitivity functions are defined and worked out to design gradient - based controllers.

### 3. Structural analysis

The structural analysis provides qualitative results useful to study the controllability of PN models (Brams 1983, Brauer *et al.* 1986, David *et al.* 1992, Murata 1989, 1977).

#### 3.1. W-sensitivity

This section concerns the structural sensitivity, referred as  $W$ -sensitivity in the next sections, of the outputs with respect to the variations of the PN inputs. The  $W$ -sensitivity depends only on the structure of the PN models. As a consequence, the  $W$ -sensitivity analysis provides controllability properties that are required for the control design of PN and that will be used in section 4. This study is based on the  $W$ -sensitivity of the places and transitions with respect to the PN firing conditions (Lefebvre 2002, Lefebvre *et al.* 2003b).

**Definition 3.1** : The node  $N$  (i.e. transition  $T_j \in T$  or place  $P_i \in P$ ) is  $W$ -sensitive with respect to the transition  $T_k \in T$  if the firings of  $T_k$  could influence the variable attached to  $N$  (i.e. the marking  $m_i$  of place  $P_i$  or the firing  $x_j$  of transition  $T_j$ ). In this case there exists a causality relationship from transition  $T_k$  to node  $N$ .

The  $W$ -sensitivity of the outputs with respect to the variations of the PN inputs is defined as a consequence.

**Definition 3.2** : The output  $y_i$  is  $W$ -sensitive with respect to the input  $u_k$  if a variation of  $u_k$  (i.e. the firings of transition  $T_k \in T_C$ ) could influence  $y_i$ . In this case there exists a causality relationship from input  $u_k$  to output  $y_i$ .

The causality relationships can be worked out with the pre and post incidence matrices, according to the theorem 3.1.

**Theorem 3.1** : The output  $y_i$  is  $W$ -sensitive with respect to the input  $u_k$  if and only if there exists an integer  $r \in [0, \min(n, p)]$  such that equation (14) holds:

$$C_i^T \cdot Q \cdot (W_{PR} + W_{PO}) \cdot (W_{PR})^T \cdot (W_{PR} + W_{PO}) \cdot D \cdot \begin{pmatrix} I_d \\ 0_{p-d} \end{pmatrix} \cdot B_k \neq 0 \quad (14)$$

with  $I_d \in \mathbb{R}^{d \times d}$  the identity matrix,  $0_{p-d} \in \mathbb{R}^{(p-d) \times d}$  the zeros matrix,  $B_k = (b_j^k) \in \{0, 1\}^d$  such that  $b_j^k = 0$  if  $k \neq j$  and  $b_k^k = 1$ , and  $C_i = (c_j^i) \in \{0, 1\}^q$  such that  $c_j^i = 0$  if  $i \neq j$  and  $c_i^i = 1$ .

**Proof :** A perturbation of the firing conditions of transition  $T_k$  yields a deviation of the places marking next to  $T_k$  ( ${}^\circ T_k \cup T_k^\circ$ ) from its true value. This deviation is likely to change the firing of the downstream transitions ( $({}^\circ T_k \cup T_k^\circ)^\circ$ ). In fact, the initial perturbation could propagate in the PN according to the following rules.

1) A perturbation of the firing conditions of transition  $T_k$  yields a deviation of the  $T_k$  - input and  $T_k$  - output places marking ( ${}^\circ T_k \cup T_k^\circ$ ) from its true value. But the perturbation could influence the firing conditions of  $T_j$  only if the  $T_j$  - input places ( ${}^\circ T_j$ ) marking is modified.

[Insert figure 3 about here]

2) A deviation of the marking of the place  $P_i$  influences the firing conditions of the  $P_i$  - downstream transitions ( $P_i^\circ$ ). But the marking of the place  $P_i$  has a structural sensitivity with respect to the  $P_i$  - upstream and  $P_i$  - downstream transitions ( ${}^\circ P_i \cup P_i^\circ$ ).

[Insert figure 4 about here]

The characterisation of the neighbourhood in PN results from the algebraic properties of the post and pre incidence matrices:

- The position of the non-zero entries of the  $j^{\text{th}}$  column in  $W_{PR}$  (resp. in  $W_{PO}$ ) corresponds to the  $T_j$  - input places (resp.  $T_j$  - output places).
- The position of the non-zero entries of the  $i^{\text{th}}$  row in  $W_{PR}$  (resp. in  $W_{PO}$ ) corresponds to the  $P_i$  - downstream transitions (resp.  $P_i$  - upstream transitions).
- The position of the non-zero entries of the  $j^{\text{th}}$  column in  $W_{PR} + W_{PO}$  (resp. the  $i^{\text{th}}$  row in  $W_{PR} + W_{PO}$ ) corresponds to the places (resp. transitions) next to  $T_j$  (resp.  $P_i$ ).

The set of places that are structurally sensitive with respect to the firing conditions of  $T_k \in T_C$  is worked out with a recursive algorithm. The position of the non-zero entries of the  $k^{\text{th}}$  column in  $W_{PR} + W_{PO}$  corresponds to the places next to  $T_k$ . The position of the non-zero entries of the  $k^{\text{th}}$  column in  $(W_{PR})^T \cdot (W_{PR} + W_{PO})$  corresponds to the downstream transitions

next to the places next to  $T_k$ . The position of the non-zero entries of the  $k^{\text{th}}$  column in  $(W_{PR} + W_{PO}) \cdot (W_{PR})^T \cdot (W_{PR} + W_{PO})$  corresponds to the places next to the downstream transitions next to the places next to  $T_k$ , and so on. When the PN has  $n$  places and  $p$  transitions, the structural sensitivity analysis of the places and transitions is completed in a finite number of steps no larger than  $\min(n, p)$ . The output  $y_i$  is  $W$ -sensitive with respect to  $T_k \in T_C$  if at least one place of the subnet  $y_i$  is sensitive with respect to  $T_k$ . The permutation matrix  $D$  and projector  $Q$  are used to limit the neighbourhood characterization to the controllable transitions and output subnets.

**Definition 3.3 :** The matrix  $S_W = (s_W(y_i, u_k))_{i=1, \dots, q, k=1, \dots, d} \in \{[0, \min(n, p)] \cup \infty\}^{q \times d}$  is defined as the input-output  $W$ -sensitivity matrix where  $s_W(y_i, u_k)$  is given by equation (15):

$$s_W(y_i, u_k) = \min_{r \in [0, \min(n, p)] \cup \infty} \left\{ C_i^T \cdot Q \cdot ((W_{PR} + W_{PO}) \cdot (W_{PR})^T)^r \cdot (W_{PR} + W_{PO}) \cdot D \cdot \begin{pmatrix} I_d \\ 0_{p-d} \end{pmatrix} \cdot B_k \neq 0 \right\} \quad (15)$$

$s_W(y_i, u_k)$  equals either infinity if  $y_i$  is not  $W$ -sensitive with respect to the input  $u_k$ , and no causality relationship exists from  $u_k$  to  $y_i$ , or the number of intermediate places in the shortest causality relationship (Lefebvre *et al.* 2003b) from  $u_k$  to  $y_i$  if  $y_i$  is  $W$ -sensitive with respect to the input  $u_k$ . In this last case,  $s_W(y_i, u_k)$  is named the  $W$ -sensitivity rank of  $y_i$  with respect to  $u_k$ . The  $W$ -sensitivity matrix provides immediate results about the causality relationships in PN, as explained in theorem 3.2:

**Theorem 3.2:**

The set of outputs (resp. rank -  $r$  outputs) that are  $W$ -sensitive with respect to the input  $u_k$  is given by the position of the finite entries (resp. entries with value  $r$ ) of the  $k^{\text{th}}$  column in matrix  $S_W$ .

The set of inputs (resp. rank -  $r$  inputs) whose firing conditions are likely to influence the output  $y_i$  is given by the position of the finite entries (resp. entries with value  $r$ ) of the  $i^{\text{th}}$  row in matrix  $S_W$ .

**Proof :** the proof of theorem 3.2 is obvious and results from definition 3.2 and theorem 3.1.

As a conclusion, let us notice that the causality relationships do not coincide with directed paths (see, for example, system B and equation (19):  $s_W(P_1, T_2) = 1$ , but the shortest directed

path from  $T_2$  to  $P_1$  is of length 2). Let us also emphasize the fact that the  $W$ -sensitivity is not restricted to a specific class of PN. In fact the  $W$ -sensitivity concerns all classes of PN that result from the basic relationship (1).

### 3.2. Examples

In order to illustrate the  $W$  – sensitivity analysis, the following examples are proposed.

The VCPN B with the marking vector  $M(t) = (m''_0(t), m''_1(t), m''_2(t), m_1(t), m_2(t), m'_1(t), m'_2(t))^T$  shown in figure 5 is the model of a manufacturing process with 2 machines  $M_1$  and  $M_2$  corresponding to the 2 transitions  $T_1$  and  $T_2$ . Machines are fed by buffers with limited capacities corresponding to the subsets of places  $\{P_1, P'_1\}$  and  $\{P_2, P'_2\}$  (Amrah *et al.* 1996).

[Insert figure 5 about here]

The maximal capacities  $C_1$  and  $C_2$  of the buffers correspond to the initial marking  $m_1(0) + m'_1(0) = C_1$  and  $m_2(0) + m'_2(0) = C_2$ . Pieces enter in the system by firing  $T_0$ . The number of pieces that are simultaneously processed by each machine is bounded by the marking of the places  $P''_0$ ,  $P''_1$ , and  $P''_2$ . (i.e. an initial marking  $m''_i(0) = 1$ ,  $i = 1, \dots, 3$  stands for single servers and  $m''_i(0) > 1$  stands for multi servers). The continuous behaviour of this system is given by equation (16):

$$\begin{aligned} \dot{m}_1(t) &= x_{\max 0} \cdot \min(m''_0(t), m'_1(t)) - x_{\max 1} \cdot \min(m''_1(t), m_1(t), m'_2(t)) \\ \dot{m}_2(t) &= x_{\max 1} \cdot \min(m''_1(t), m_1(t), m'_2(t)) - x_{\max 2} \cdot \min(m''_2(t), m_2(t)) \\ m'_i(t) &= C_i - m_i(t) & i = 1, 2 \\ m''_j(t) &= m''_j(0) & j = 1, 2, 3 \end{aligned} \quad (16)$$

The set of controllable transitions and the set of outputs subnets depend on system specifications. Let us first consider the system (16) as a single input – multi outputs one and assume that  $T_C = \{T_0\}$  and  $T_{NC} = \{T_1, T_2\}$ , then  $u(t) = x_{\max 0}(t)$ . Let us also assume that  $y_1(t) = m_1(t)$ , and  $y_2(t) = m_2(t)$ . Other specifications are discussed in the following. The VCPN (16) can be written as a scalar form (17) by using the functions  $f_j$  defined as in (10):

$$\begin{aligned} \dot{m}_1(t) &= u(t) \cdot m_{f_0}(t) - x_{\max 1} \cdot m_{f_1}(t) \\ \dot{m}_2(t) &= x_{\max 1} \cdot m_{f_1}(t) - x_{\max 2} \cdot m_{f_2}(t) \\ y_1(t) &= m_1(t), & y_2(t) &= m_2(t) \end{aligned} \quad (17)$$

or as a vectorial form (18):

$$\begin{aligned} \dot{M}(t) &= (u(t) \cdot W_{C1} \cdot A_1(\varphi) + x_{\max 1} \cdot W_{NC1} \cdot B_1(\varphi) + x_{\max 2} \cdot W_{NC2} \cdot B_2(\varphi)) \cdot M(t) \\ Y(t) &= Q \cdot M(t) \end{aligned} \quad (18)$$

with  $W_{C1} = (0, 0, 0, 1, 0, -1, 0)^T$ ,  $W_{NC1} = (0, 0, 0, -1, 1, 1, -1)^T$ ,  $W_{NC2} = (0, 0, 0, 0, -1, 0, 1)^T$ ,  $Q = ((0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0)^T; (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0)^T)^T$ . The row vectors  $A_1(\varphi)$ ,  $B_1(\varphi)$ , and  $B_2(\varphi)$  depend of the current phase (table 1). For example, if  $M = M_0$ , these row vectors are  $A_1(\varphi) = (1, 0, 0, 0, 0, 0, 0)$ ,  $B_1(\varphi) = (0, 0, 0, 1, 0, 0, 0)$ , and  $B_2(\varphi) = (0, 0, 0, 0, 1, 0, 0)$ .

[Insert table 1 about here]

The  $W$ -sensitivity matrix of the PN places with respect to the transitions is given by  $S_W(P, T)$  and the  $W$ -sensitivity matrix of the PN outputs with respect to the single input is given by  $S_W(Y, U)$  according to equation (19):

$$S_W(P, T) = \begin{pmatrix} T_0 & T_1 & T_2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} P''_0 \\ P''_1 \\ P''_2 \\ P_1 \\ P_2 \\ P'_1 \\ P'_2 \end{matrix} \quad S_W(Y, U) = \begin{matrix} u \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix} \begin{matrix} y_1 \\ y_2 \end{matrix} \quad (19)$$

The  $W$ -sensitivity matrix  $S_W(P, T)$  shows that the marking of each place depends on the firing of all transitions: the content of each intermediate buffer depends of the production rate of upstream but also downstream machines. These causality relationships concern the immediate neighbourhood when  $S_W(P_i, T_j) = 0$  or non immediate neighbourhood when  $S_W(P_i, T_j) > 0$ . The same conclusions can be driven concerning the input – output  $W$ -sensitivity matrix  $S_W(Y, U)$ .

The table 2 provides the  $W$ -sensitivity matrices of the output(s) with respect to the input(s) for several set of controllable transitions and several output subnets.

[Insert table 2 about here]

The investigation of the input – output causality relationships is useful in order to design efficient control. For instance, if the controller goal is that the content of the first intermediate buffer reaches a desired level or tracks a desired trajectory, it is more convenient to control the input transition  $T_0$  ( $S_W(y = m_1, u = x_{max\ 0}) = 0$ ), than the transition  $T_2$  ( $S_W(y = m_1, u = x_{max\ 2}) = 1$ ). Such a conclusion will be confirmed in section 4.

The results obtained with the structural analysis can be more definitive as shown with a simple modification of the previous example. The system is changed in the sense that the intermediate buffers have an infinite capacity according to figure 6 and equation (20):

[Insert figure 6 about here]

$$\begin{aligned} \dot{m}_1(t) &= x_{max\ 0} \cdot m''_0(t) - x_{max\ 1} \cdot \min(m''_1(t), m_1(t)) \\ \dot{m}_2(t) &= x_{max\ 1} \cdot \min(m''_1(t), m_1(t)) - x_{max\ 2} \cdot \min(m''_2(t), m_2(t)) \\ m''_j(t) &= m''_j(0) \quad j = 1, 2, 3 \end{aligned} \tag{20}$$

With the same specification as previously, the transitions - places  $W$ -sensitivity matrix  $S'_W(P, T)$  and the input – output  $W$ -sensitivity matrix  $S'_W(Y, U)$  are given by (21):

$$S'_W(P_i, T_j) = \begin{pmatrix} T_0 & T_1 & T_2 \\ 0 & \infty & \infty \\ 1 & 0 & \infty \\ 2 & 1 & 0 \\ 0 & 0 & \infty \\ 1 & 0 & 0 \end{pmatrix} \begin{matrix} P''_0 \\ P''_1 \\ P''_2 \\ P_1 \\ P_2 \end{matrix} \neq S_W(P_i, T_j) \quad S'_W(Y, U) = \begin{matrix} u \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix} \begin{matrix} y_1 \\ y_2 \end{matrix} = S_W(Y, U) \tag{21}$$

The table 3 that provides the  $W$ -sensitivity matrices of the output(s) with respect to the input(s) for several sets of controllable transitions and several output subnets must be compared with table 2.

[Insert table 3 about here]

From table 3, it is obvious that transition  $T_2$  can no more be used to control the output  $y(t) = m_1(t)$ : there exist no causality relationship from  $T_2$  to  $P_1$  because of the infinite capacity buffer represented by  $P_2$ .

Another example of generalised VCPN is given by system C in figure 7 with the marking vector  $M(t) = (m'_1(t), m'_2(t), m'_3(t), m'_4(t), m'_5(t), m_1(t), m_2(t), m_3(t), m_4(t))^T$ . Weighted arcs  $T_2 \rightarrow P_1$  and  $P_1 \rightarrow T_1$  means that the flow of tokens that fire  $T_2$  to  $P_1$  is multiplied by 3 and the flow of tokens that fire  $T_1$  from  $P_1$  is divided by 2. As previously, places  $P'_1$  to  $P'_5$  limit the number of simultaneous firings of the transitions  $T_1$  to  $T_5$ . The set of controllable transitions is assumed to be given as  $T_C = \{T_4, T_5\}$  and the set of non controllable transitions is given by  $T_{NC} = \{T_1, T_2, T_3\}$ . In this case, the controllable transitions correspond to source transitions that represent the interface between the system and the “outside word” and the set of non controllable transitions corresponds to internal transitions that are assumed to behave according to their own dynamic. The maximal firing frequencies of the internal transitions is given by  $X_{max\ NC} = \{2, 1, 3\}$ . The output subnets are defined according to  $\{P_1, P_3\}$  and  $\{P_2, P_4\}$  (i.e.  $y_1(t) = m_1(t) + m_3(t)$  and  $y_2(t) = m_2(t) + m_4(t)$ ).

[Insert figure 7 about here]

This VCPN can be written as a scalar form (22) or as a vectorial form (23):

$$\begin{aligned}
 \dot{m}_1(t) &= u_1(t) \cdot m_{f_4}(t) + 3 \cdot x_{\max 2} \cdot m_{f_2}(t) - 2 \cdot x_{\max 1} \cdot m_{f_1}(t) \\
 \dot{m}_2(t) &= u_2(t) \cdot m_{f_5}(t) + x_{\max 3} \cdot m_{f_3}(t) - x_{\max 1} \cdot m_{f_1}(t) \\
 \dot{m}_3(t) &= x_{\max 1} \cdot m_{f_1}(t) - x_{\max 2} \cdot m_{f_2}(t) \\
 \dot{m}_4(t) &= x_{\max 1} \cdot m_{f_1}(t) - x_{\max 3} \cdot m_{f_3}(t)
 \end{aligned} \tag{22}$$

$$y_1(t) = m_1(t) + m_3(t)$$

$$y_2(t) = m_2(t) + m_4(t)$$

Let us mention that the functions  $f_4$  and  $f_5$  are constant and  $m_{f_4}(t) = m_{f_5}(t) = 1$ , because the controllable transitions are source transitions.

$$\begin{aligned}
 \dot{M}(t) &= (u_1(t) \cdot W_{C1} \cdot A_1(\varphi) + u_2(t) \cdot W_{C2} \cdot A_2(\varphi) \\
 &\quad + x_{\max 1} \cdot W_{NC1} \cdot B_1(\varphi) + x_{\max 2} \cdot W_{NC2} \cdot B_2(\varphi) + x_{\max 3} \cdot W_{NC3} \cdot B_3(\varphi)) \cdot M(t)
 \end{aligned} \tag{23}$$

$$Y(t) = Q \cdot M(t)$$

with  $W_{C1} = (0, 0, 0, 0, 0, 1, 0, 0, 0)^T$ ,  $W_{C2} = (0, 0, 0, 0, 0, 0, 1, 0, 0)^T$ ,  $W_{NC1} = (0, 0, 0, 0, 0, 0, -1, 1, 1)^T$ ,  $W_{NC2} = (0, 0, 0, 0, 0, 1, 0, -1, 0)^T$ ,  $W_{NC3} = (0, 0, 0, 0, 0, 0, 1, 0, -1)^T$ ,  $A_1(\varphi)$ ,  $A_2(\varphi)$ ,  $B_1(\varphi)$ ,  $B_2(\varphi)$ , and  $B_3(\varphi)$  that depend of the current phase, and  $Q = ((0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0)^T; (0\ 0\ 0$

$0 \ 0 \ 0 \ 1 \ 0 \ 1)^T)^T$ . The input-output  $W$ -sensitivity matrix (24) shows that both outputs are correlated according to the transition  $T_1$ .

$$S_w(Y, U) = \begin{matrix} & u_1 & u_2 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & y_1 \\ & & y_2 \end{matrix} \quad (24)$$

#### 4. Control design for VCPN

Flow control for VCPN was investigated by several authors (Amrah *et al.* 1996, Egilmez *et al.* 1994, Ghabri 1995, Hanzalek 2003, Lefebvre 1999, Silva *et al.* 2003). Such methods have provided interesting results but require strong conditions concerning the transitions to control and the places to observe. Moreover, the proposed results are often local ones, and are attached to a specific phase in the VCPN behaviour. This paper focus on another approach based on gradient method and inspired from neural networks. Gradient-based methods have been intensively investigated for the learning of neural networks (Widrow *et al.* 1990) and the identification of continuous systems (Ljung 1987, Thomas 1997) but only a few studies have concerned the hybrid and discrete event systems (Balduzzi *et al.* 2000). This approach takes advantages on the propagation of the gradient through the PN nodes in order to minimise the square of instantaneous error between desired and measured outputs by modifying the maximal firing frequencies of controllable transitions. Gradient algorithms perform the minimisation of a scalar cost function that evaluates the distance between the desired output  $Y_d(t)$  and the system output  $Y(t)$ . Let us assume that measurements of the desired output are obtained with a sampling period  $\Delta t$  during the time horizon  $H$ . As a consequence, the proposed controllers will be worked out in discrete time.

##### 4.1 Sensitivity functions

Gradient algorithms are based on the evaluation of sensitivity functions. Such functions are defined in continuous time for VCPN (definition 4.1) and will be worked out in discrete time (theorem 4.1) according to the sampling period  $\Delta t$  in order to be implemented in numerical controllers.

**Definition 4.1:** The scalar sensitivity function  $s_{\alpha\gamma}(t)$  for the output  $y_\alpha$  with respect to the input  $u_\gamma$  and the sensitivity function vector  $S_\alpha(t)$  for the output  $y_\alpha$  with respect to the input vector  $U$  are defined as:

$$s_{\alpha\gamma}(t) = \frac{\partial y_\alpha}{\partial u_\gamma} \in \mathbb{R}, \quad S_\alpha(t) = (s_{\alpha\gamma}(t))_{\gamma=1,\dots,d} = \left( \frac{\partial y_\alpha}{\partial u_\gamma} \right)_{\gamma=1,\dots,d} \in \mathbb{R}^d \quad (25)$$

**Theorem 4.1:** The numerical process given by equation (26) tends to the value of the scalar sensitivity function  $s_{\alpha\gamma}(t)$  worked out at time  $t = k \cdot \Delta t$ :

$$\delta s_{\alpha\gamma}(k) = \sum_{i=1}^n q_{\alpha i} \cdot \left( w_{i\gamma} \cdot \mu_\gamma(k) + \sum_{\substack{j=1 \\ j \neq \gamma}}^p w_{ij} \cdot x_{\max j} \cdot s_{fj\gamma}(k) \right) \cdot \Delta t \quad (26)$$

with  $\delta s_{\alpha\gamma}(k) = s_{\alpha\gamma}(k) - s_{\alpha\gamma}(k-1)$  and  $s_{\alpha\gamma}(0) = 0$ .

**Proof :** Let us first notice that the sensitivity functions can be formulated in terms of the VCPN marking and of the transitions maximal firing frequency at time  $t = k \cdot \Delta t$ :

$$s_{\alpha\gamma}(k) = \left( \frac{\partial y_\alpha}{\partial u_\gamma} \right)_{\substack{y_\alpha = y_\alpha(k, \Delta t) \\ u_\gamma = u_\gamma(k, \Delta t)}} = \sum_{i=1}^n q_{\alpha i} \cdot \left( \frac{\partial m_i}{\partial x_{\max \gamma}} \right)_{t=k \cdot \Delta t} \quad (27)$$

According to the equations (3), (4) and (5):

$$\frac{\partial}{\partial x_{\max \gamma}} \left( \frac{dm_i(t)}{dt} \right) = \sum_{j=1}^p w_{ij} \cdot \frac{\partial x_j(t)}{\partial x_{\max \gamma}} = w_{i\gamma} \cdot \mu_\gamma(t) + \sum_{\substack{j=1 \\ j \neq \gamma}}^p w_{ij} \cdot x_{\max j} \cdot \frac{\partial \mu_j(t)}{\partial x_{\max \gamma}} \quad (28)$$

Using the clustering functions defined as in (10), equation (28) results in (29):

$$\frac{\partial}{\partial x_{\max \gamma}} \left( \frac{dm_i(t)}{dt} \right) = w_{i\gamma} \cdot \mu_\gamma(t) + \sum_{\substack{j=1 \\ j \neq \gamma}}^p w_{ij} \cdot x_{\max j} \cdot s_{fj\gamma}(t) \quad (29)$$

The derivation with respect to time of the sensitivity functions (25) and the use of a first order numerical method leads to equation (26).

## 4.2 Gradient-based controllers

For the seek of simplicity, let us first consider the single output case. The instantaneous error is defined as  $\varepsilon(k, i) = y_d(k) - y(k, i)$ , where  $y_d(k)$  stands for the desired output at time  $t = k.\Delta t$ , and  $y(k, i)$  stands for the marking of the VCPN output  $y$  at time  $t = k.\Delta t$  obtained from the marking  $M(k-1)$  and the input vector  $U(k-1, i)$  according to a first order numerical method:

$$y(k, i) = y(k-1) + \Delta t. \left( \sum_{j=1}^d u_j(k-1, i) \cdot Q \cdot W_{Cj} \cdot A_j(\varphi) + \sum_{j=1}^{p-d} x_{\max_{NC} j} \cdot Q \cdot W_{NCj} \cdot B_j(\varphi) \right) \cdot M(k-1) \quad (30)$$

$U(k, i)$  is the updating of the input vector obtained after the  $i^{\text{th}}$  iteration of the gradient algorithm at time  $t = k.\Delta t$ . A maximal number of  $N$  iterations is considered, for each instant  $t = k.\Delta t$ , in order to work out the input  $U(k)$  in finite time. According to this truncation, we have  $U(k) = U(k, N)$ , and  $y(k) = y(k, N)$ . Let us consider the scalar cost function  $v(k, i)$ :

$$v(k, i) = \frac{1}{2} \varepsilon^2(k, i) \in \mathbb{R}. \quad (31)$$

Gradient-based methods result from the Taylor series expansion of the cost function  $v(k, i)$  in order to work out the optimal value of the input vector  $U(k, i)$ :

$$\begin{aligned} v(k, i+1) = & v(k, i) + \left( \frac{\partial v}{\partial U} \right)_{U=U(k, i)}^T \delta U(k, i) \\ & + \frac{1}{2} \cdot (\delta U(k, i))^T \left( \frac{\partial^2 v}{\partial U \cdot \partial U^T} \right)_{U=U(k, i)} \delta U(k, i) + o\left( \left| \delta U(k, i)^T \cdot \delta U(k, i) \right| \right) \end{aligned} \quad (32)$$

with  $\delta U(k, i) = U(k, i+1) - U(k, i)$ . The stationary condition results in:

$$\delta U(k, i) = - \left( \frac{\partial^2 v}{\partial U \cdot \partial U^T} \right)_{U=U(k, i)}^{-1} \cdot \left( \frac{\partial v}{\partial U} \right)_{U=U(k, i)} \quad (33)$$

with:

$$\left( \frac{\partial v}{\partial U} \right)_{U=U(k,i)} = -S(k) \cdot \mathcal{E}(k,i) \quad (34)$$

$$\left( \frac{\partial^2 v}{\partial U \cdot \partial U^T} \right)_{U=U(k,i)} \approx \left( \frac{\partial^2 v}{\partial U \cdot \partial U^T} \right)_{U=U(k-1)} \approx S(k) \cdot S(k)^T + \alpha I$$

$S(k)$  is the output sensitivity function vector of output  $y$  with respect to the input vector  $U$  defined as in (25) and worked out at time  $t = k \cdot \Delta t$  according to equation (26). Let us notice that the sensitivity functions do not depend on the iteration  $i$ .  $S(k)$  is computed a single time for each new measurement. Moreover, second order terms are usually neglected in equation (33), but an adaptive term  $\alpha I$  is added in order to approximate the inverse of the Hessian matrix when it is not regular or badly conditioned (Hagan *et al.* 1995). Thus, equation (33) results in the updating rule of the controller (35):

$$\begin{aligned} \delta U(k,i) &= (S(k) \cdot S(k)^T + \alpha I)^{-1} \cdot S(k) \cdot \mathcal{E}(k,i), \quad i = 0, \dots, N-1 \\ U(k,0) &= U(k-1) \end{aligned} \quad (35)$$

Let us point out two limit cases. When  $\alpha \gg 1$ , equation (35) corresponds to the gradient method (Van der Smagt *et al.* 1994):

$$\delta U(k,i) = \frac{1}{\alpha} \cdot S(k) \cdot \mathcal{E}(k,i). \quad (36)$$

When  $\alpha \ll 1$ , equation (35) corresponds to the Gauss-Newton method (Thomas 1997):

$$\delta U(k,i) = (S(k) \cdot S(k)^T)^{-1} \cdot S(k) \cdot \mathcal{E}(k,i). \quad (37)$$

The previous controller can be generalised in the multi-outputs case, by considering the scalar cost function (38):

$$v(k,i) = \frac{1}{2} \cdot \sum_{j=1}^q \mathcal{E}_j^2(k,i) \in \mathbb{R} \quad (38)$$

that results in the following updating rule for the controller:

$$\begin{aligned} \delta U(k,i) &= \left( \sum_{j=1}^q (S_j(k) \cdot S_j(k)^T + \alpha I) \right)^{-1} \cdot \left( \sum_{j=1}^q S_j(k) \cdot \mathcal{E}_j(k,i) \right) \\ U(k,0) &= U(k-1) \end{aligned} \quad (39)$$

where  $S_j(k)$  stands for sensitivity function vector of the output  $y_j$  with respect to the input vector  $U$  at time  $t = k \cdot \Delta t$ .

## 5. Examples

In order to illustrate the proposed controllers, let us first consider the system B modelled as a VCPN where the incidence matrices and parameters are defined as in section 3. The piecewise bilinear state space representation of system B is given by equation (18) with constrained input limited to 5 tokens /UT. The initial marking vector is  $M_0 = (1 \ 1 \ 1 \ 0 \ 0 \ 3 \ 3)$  and the parameter  $\alpha$  equals 0,1 in order to avoid the singularities in the Hessian approximation (34).

The figure 8 illustrates the equilibriums for system B in the plan  $(m_1, m_2)$  obtained for the same sets of controllable transitions :  $T_C = \{T_0\}$ ,  $T_C = \{T_2\}$ , or  $T_C = \{T_0, T_2\}$  that were investigated in section 3 (in black: without control, in light grey: with constant control, in dark grey: reachable equilibriums from the origin with constant control). Let us mention that the region of reachable equilibriums from the origin with constant control is always strictly included in the region of the system equilibriums obtained for all admissible initial markings  $0 \leq m_1(0) \leq 3$  and  $0 \leq m_2(0) \leq 3$ .

[Insert figure 8 about here]

Let us first consider the case  $T_C = \{T_0\}$ . The figure 9 points out the influence of the output matrix on the controller response: 3 scalar outputs are investigated  $y = m_1$ ,  $y = m_2$  and  $y = m_1 + m_2$  that correspond respectively to  $Q_1 = (0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0)$ ,  $Q_2 = (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0)$ ,  $Q_3 = (0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0)$ . In all cases, the objective of the controller is to drive the output of the system to the desired value  $y_d = 2$  tokens. The maximal number of iterations is  $N = 100$ . Concerning the output matrices  $Q_2$  and  $Q_3$  the desired value is rapidly reached with a good accuracy, but in case of output matrix  $Q_1$  some oscillations are observed. Such an input – output specification is not suitable with our approach because the marking of the unobservable place  $P_1$  is not considered in the calculation of the input firing frequency. As a consequence, the desired level is exceeded and oscillations arise due to the delay between the firing of  $T_0$  and the observation of  $P_2$  marking. In order to avoid the undesirable cumulative effects of the marking, the inputs and outputs of the systems must be preferred such that the sensitivity rank equals 0 (immediate neighbourhood) as shown in table 2.

[Insert figure 9 about here]

The speed of the algorithm increases as the maximal number of iterations in the gradient – based algorithm. The figure 10 illustrates the influence of the number of iterations  $N$  when the output matrix is given by  $Q = (0\ 0\ 0\ 1\ 0\ 0\ 0)$ . A small number ( $N = 2$  or  $N = 10$ ) results in a poor controller, that is not quick enough to correct the input when the desired level is reached. In this case the level is exceeded and there is nothing to do. A large number ( $N = 100$ ) compensates the slowness of the gradient algorithm.

[Insert figure 10 about here]

In figures 11 and 12, gradient-based controllers (full line) with  $T_C = \{T_0\}$ ,  $N = 100$  and  $Q = (0\ 0\ 0\ 1\ 1\ 0\ 0)$  are compared with usual proportional (dashed line) and bang-bang controllers (dotted line) when the desired output is the piecewise linear function of time given by equation (31) (figure 11):

$$\begin{cases} y_d = \frac{t}{3}, & 0 \leq t < 6 \\ y_d = \frac{1}{9}t + \frac{4}{3}, & t \geq 6 \end{cases} \quad (31)$$

or a  $C^0$  non linear function of time given by equation (32) (figure 12):

$$y_d = |\sin(t)| + \frac{t}{6} \quad (32)$$

All proposed controllers track the desired trajectories with an instantaneous error that does not exceed 0.5 token ( $\varepsilon < 0.2$  tokens for the proportional controller,  $\varepsilon < 0.4$  tokens for the bang-bang controller,  $\varepsilon < 0.05$  tokens for the gradient-based controller). Let us notice that the error of the gradient-based controller is smaller than the one of the proportional or the bang-bang controller. Furthermore, the input signal is very different according to the controller choice. The proportional controller uses the error signal as the input frequency (gain = 1). As a consequence, this controller is not suitable when the error signal presents a lot of variations. The bang-bang controller is defined as a series of commutations. The input flow of tokens is tuned according to the commutations frequency. For this reason, the interest of this controller depends strongly of the field of application. The most outstanding difference between the different classes of controllers is that only the gradient-based controller is able to learn the

growing rate of the output in the case of a linear trajectory and the cyclic behaviour of the output in the case of a non linear sinusoidal trajectory.

[Insert figures 11 and 12 about here]

Let us now consider the VCPN model of system C (figure 7) given as a MIMO piecewise bilinear state space representation (23) with inputs that correspond to  $T_C = \{T_4, T_5\}$  and outputs that correspond to the subsets  $\{P_1, P_3\}$  and  $\{P_2, P_4\}$ . The firing of controllable transitions is limited to 5 tokens /UT, the initial marking vector is  $M_0 = (1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0)^T$ , the parameter  $\alpha$  equals 0.1, and the number of iterations is limited to  $N = 100$ . The desired trajectories correspond to 2 piecewise linear trajectories given by equation (33):

$$\begin{cases} y_{d1} = \frac{5}{3}.t, & 0 \leq t < \frac{12}{5} \\ y_{d1} = \frac{5}{24}.t + \frac{7}{2}, & t \geq \frac{12}{5} \end{cases} \quad \begin{cases} y_{d2} = t, & 0 \leq t < 6 \\ y_{d2} = \frac{1}{3}.t + 4, & t \geq 6 \end{cases} \quad (33)$$

Only the outputs of the gradient-based controller are presented in figure 13. In fact the proportional and bang-bang controllers provide only poor results because the inputs and outputs of the system C are coupled thanks to the transition  $T_7$ . These controllers focus on one desired trajectory but cannot track simultaneously both ones. On the contrary, the gradient-based controller tracks simultaneously both trajectories with an instantaneous error that does not exceed 0,2 tokens. The input-output decomposition is obtained thanks to the sensitivity functions that evaluate for each output the relative influence of both inputs.

[Insert figure 13 about here]

The figure 15 illustrates the case of a non admissible output trajectory. The desired output signals are defined as previously but the incidence relationships of the VCPN have been modified according to figure 14 (system C’):

[Insert figure 14 about here]

System C is tokens consumer but system C’ is tokens producer: the free response ( $U(t) = 0$ ) of system C’ corresponds to increasing marking functions whereas the free

response of system C corresponds to a decreasing marking functions. After  $t = 4$  TU, we have  $u_1(t) = 0$ , but the number of tokens in subnet  $y_2$  increases more quickly than the desired output  $y_{d2}$  and the controller fails.

[Insert figure 15 about here]

At last, let us consider the hybrid PN model of the two tanks system A (figure 1) given as a MIMO non linear state space representation which inputs correspond to the maximal firing frequencies of the controllable transitions  $T_C = \{T_1, T_4\}$  and outputs correspond to the marking  $m_1$  and  $m_2$ . The controller is obtained according to an adaptation of the gradient based algorithm to non linear behaviours. As a consequence, the discrete part of the hybrid model become useless (figure 2). The firing of controllable transitions is limited according to the system specifications :  $0 \leq u_1 \leq D$ , and  $0 \leq u_2 \leq \alpha_4$ , with  $\alpha_2 = \alpha_3 = \alpha_4 = 1.6 \cdot 10^{-4} \text{ m}^{3/2} \cdot \text{s}^{-1}$ ,  $D = 1 \cdot 10^{-4} \text{ m}^3 \cdot \text{s}^{-1}$ ,  $h = 0.5 \text{ m}$ ,  $S_1 = S_2 = 0.0154 \text{ m}^2$ . The initial marking vector of the continuous part of the model is  $M_0 = (0, 0)$ ; the coefficient  $\alpha$  equals 0,1; the number of iterations is limited to  $N = 1$  (single step controller). The desired trajectories correspond to a periodical level for tank 1 and a constant level for tank 2. Simulation results for the gradient based controller are given in figure 16 (system outputs are in full line, and desired trajectories are in dotted line) and can be compared with the results obtained with the discrete control design (figure 17).

[Insert figures 16 and 17 about here]

Both controllers are different in terms of objective. Nevertheless, one can notice that the discrete controller is not suitable to track some reference trajectories or to reach some desired levels. On the contrary, with gradient – based controller, the desired level in tank 2 is reached and the reference trajectory in tank 1 is almost everywhere tracked after some transitory behaviours. But one can also notice that, due to system specifications, level 0.6 m cannot be reached in tank 1 when level in tank 2 is 0.4 m. In fact, the desired levels  $m_{1d} = 0.6 \text{ m}$  and  $m_{2d} = 0.4 \text{ m}$  do not belong to the set of equilibriums. At last, because of immediate causality relationships from  $T_1$  to  $P_1$  and from  $T_4$  to  $P_2$ , the gradient – based algorithm behaves like a proportional controller (i.e. the input – output sensitivity matrix tends to a diagonal one).

## 6. Conclusions

The continuous Petri net controllers that have been proposed in this paper are based on the evaluation of the input-output sensitivity functions. For this purpose, the structural sensitivity of PN models has been first investigated. Places to be observed and transitions to be controlled are obtained as a consequence. An explicit characterisation of the input-output sensitivity functions has also been proposed for VCPN models. At last, VCPN controllers have been designed that calculate the gradient of the outputs with respect to the input variations in order to adapt the maximal firing frequencies of the controllable transitions according to desired trajectories of the output markings. An adaptation of this algorithm for HDS and continuous non linear PN was also developed.

In our opinion, the method is not only suitable for trajectory tracking but also for complex behaviours learning. We will further investigate the combination of Petri nets and neural networks in order to design learning Petri nets (Hirasawa *et al.* 1998). These perspectives include not only the continuous Petri nets but also the autonomous and timed Petri nets. At last, the sensitivity analysis will also be adapted for the monitoring of hybrid systems modelled with Petri nets.

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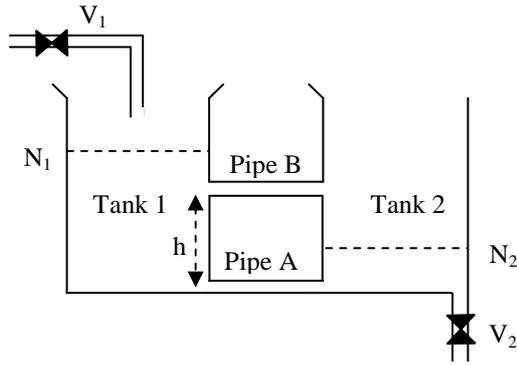


Figure 1: Two tanks system (system A)

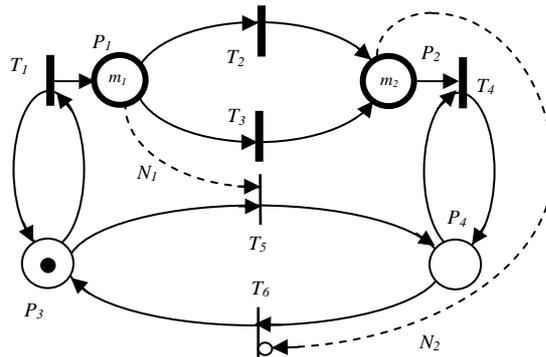
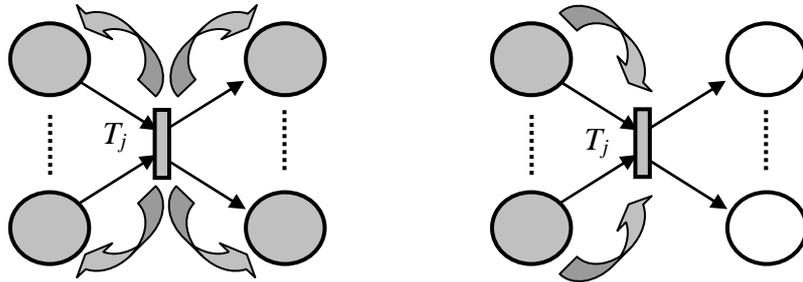
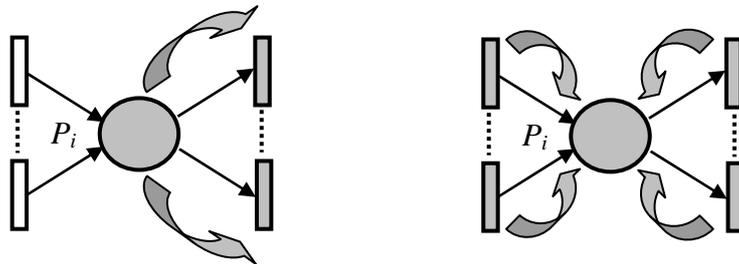


Figure 2: Hybrid PN of the two tanks system



a) Sensitivity of the places next to  $T_j$  with respect to the firing conditions of  $T_j$       b) Sensitivity of  $T_j$  with respect to the marking of the places next to  $T_j$

Figure 3: Propagation of the perturbation next to the transition  $T_j$



a) Sensitivity of the transitions next to  $P_i$  with respect to the marking of  $P_i$       b) Sensitivity of  $P_i$  with respect to the firing conditions of the transitions next to  $P_i$

Figure 4: Propagation of the perturbation next to the place  $P_i$

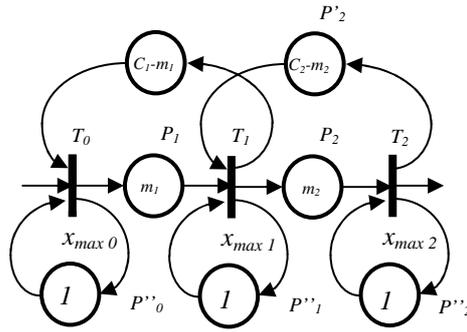


Figure 5: VCPN model of a manufacturing process (system B)

	$A_1(\varphi)$		$B_1(\varphi)$
$C_1 - m_1(t) \geq m''_0(0)$	(1, 0, 0, 0, 0, 0, 0)	$m_1(t) \geq C_2 - m_2(t) \geq m''_1(0)$	(0, 1, 0, 0, 0, 0, 0)
$m''_0(0) \geq C_1 - m_1(t)$	(0, 0, 0, 0, 0, 1, 0)	$C_2 - m_2(t) \geq m_1(t) \geq m''_1(0)$	
		$m''_1(0) \geq m_1(t) \geq C_2 - m_2(t)$	(0, 0, 0, 0, 0, 0, 1)
		$m_1(t) \geq m''_1(0) \geq C_2 - m_2(t)$	
		$C_2 - m_2(t) \geq m''_1(0) \geq m_1(t)$	(0, 0, 0, 1, 0, 0, 0)
		$m''_1(0) \geq C_2 - m_2(t) \geq m_1(t)$	
	$B_2(\varphi)$		
$m_2(t) \geq m''_2(0)$	(0, 0, 1, 0, 0, 0, 0)		
$m''_2(0) \geq m_2(t)$	(0, 0, 0, 0, 1, 0, 0)		

Table 1: Phases specification for system B

$U \setminus Y$	$Y(t) = m_2(t)$	$Y(t) = m_1(t)$	$Y(t) = m_1(t) + m_2(t)$	$Y(t) = (m_1(t), m_2(t))^T$
	$Q = (0\ 0\ 0\ 0\ 1\ 0\ 0)$	$Q = (0\ 0\ 0\ 1\ 0\ 0\ 0)$	$Q = (0\ 0\ 0\ 1\ 1\ 0\ 0)$	$Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$
$T_C = \{T_0\}$	$S_w(Y, U) = (1)$	$S_w(Y, U) = (0)$	$S_w(Y, U) = (0)$	$S_w(Y, U) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$T_C = \{T_2\}$	$S_w(Y, U) = (0)$	$S_w(Y, U) = (1)$	$S_w(Y, U) = (0)$	$S_w(Y, U) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$T_C = \{T_0, T_2\}$	$S_w(Y, U) = (1\ 0)$	$S_w(Y, U) = (0\ 1)$	$S_w(Y, U) = (0\ 0)$	$S_w(Y, U) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Table 2: Input – output  $W$  – sensitivity matrices for system B

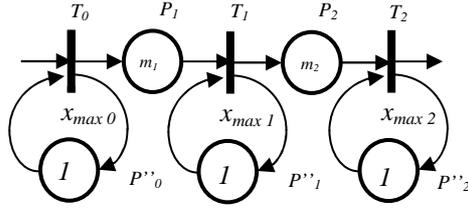


Figure 6: VCPN model of system B'

$U \setminus Y$	$Y(t) = m_2(t)$	$Y(t) = m_1(t)$	$Y(t) = m_1(t) + m_2(t)$	$Y(t) = (m_1(t), m_2(t))^T$
		$Q = (0\ 0\ 0\ 0\ 1)$	$Q = (0\ 0\ 0\ 1\ 0)$	$Q = (0\ 0\ 0\ 1\ 1\ 0\ 0)$
$T_C = \{T_0\}$	$S_W(Y, U) = (1)$	$S_W(Y, U) = (0)$	$S_W(Y, U) = (0)$	$S_W(Y, U) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$T_C = \{T_2\}$	$S_W(Y, U) = (0)$	$S_W(Y, U) = (\infty)$	$S_W(Y, U) = (0)$	$S_W(Y, U) = \begin{pmatrix} \infty \\ 0 \end{pmatrix}$
$T_C = \{T_0, T_2\}$	$S_W(Y, U) = (1\ 0)$	$S_W(Y, U) = (0\ \infty)$	$S_W(Y, U) = (0\ 0)$	$S_W(Y, U) = \begin{pmatrix} 0 & \infty \\ 1 & 0 \end{pmatrix}$

Table 3: Input – output  $W$  – sensitivity matrices for system B'

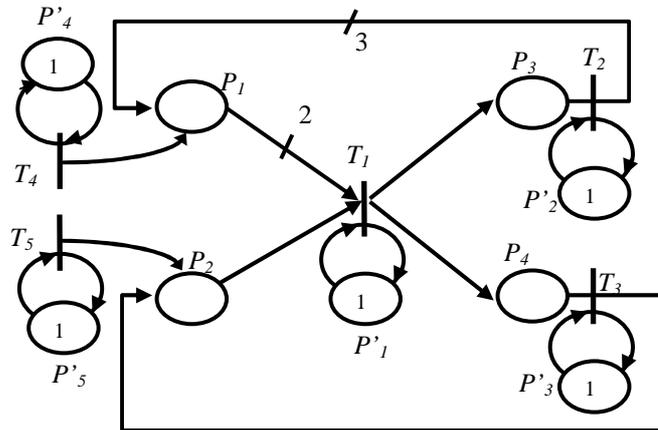


Figure 7: Closed loop process (system C)

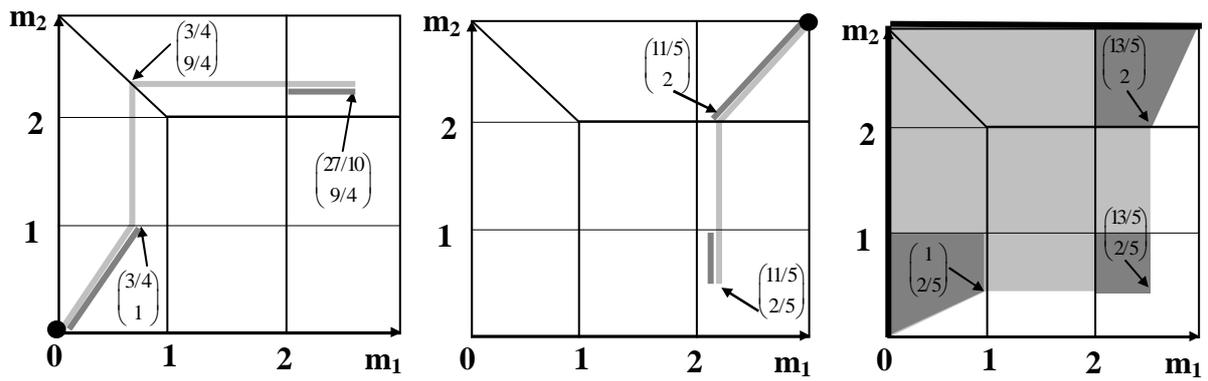


Figure 8: Sets of equilibriums a)  $U = x_{max0}$  b)  $U = x_{max2}$  c)  $U = (x_{max0}, x_{max2})$

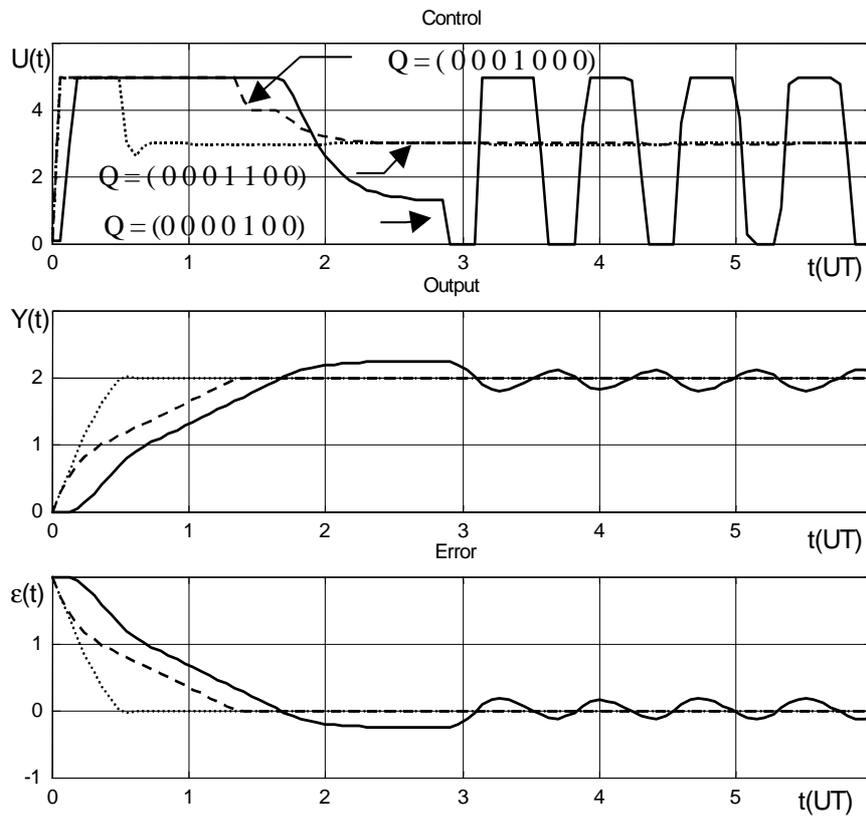


Figure 9: Influence of the output matrix

(full line :  $Q = (0\ 0\ 0\ 0\ 1\ 0\ 0)$ , dashed line:  $Q = (0\ 0\ 0\ 1\ 0\ 0\ 0)$ , dotted line  $Q = (0\ 0\ 0\ 1\ 1\ 0\ 0)$ )

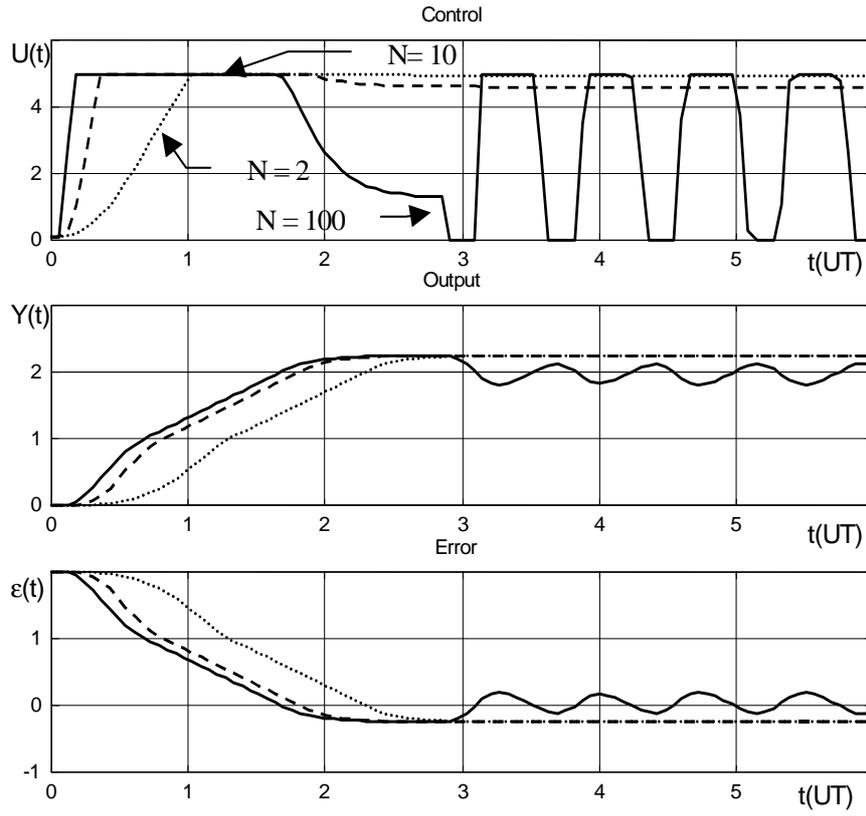


Figure 10: Influence of the iterations number  
(full line:  $N = 100$ , dashed line:  $N = 10$ , dotted line:  $N = 2$ )

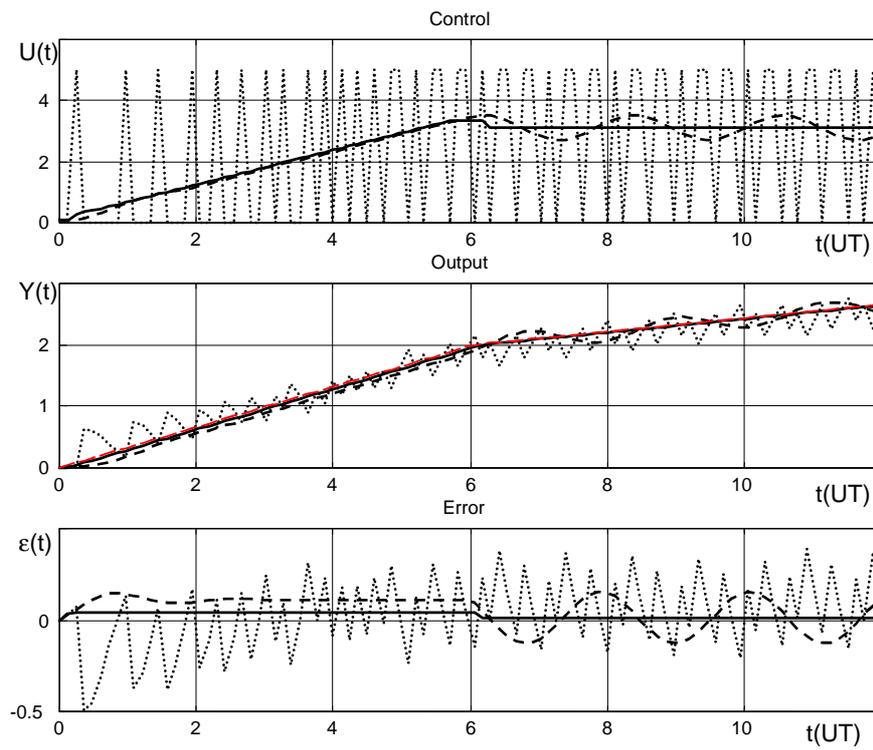


Figure 11: Piecewise linear trajectory

(full line: gradient-based controller, dashed line: proportional controller,  
dotted line: bang-bang controller)

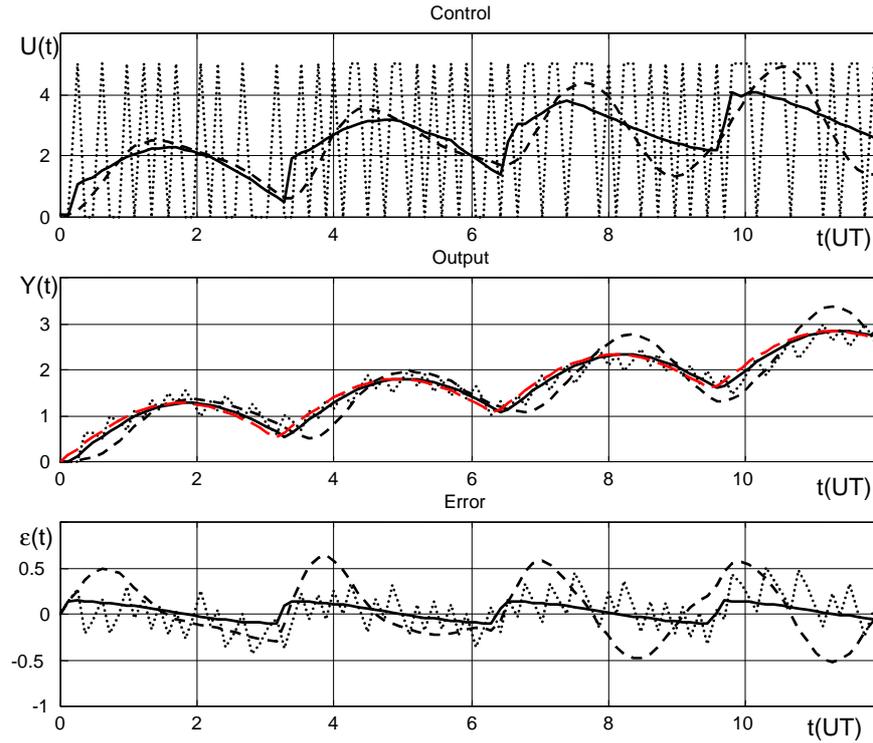


Figure 12: Non linear trajectory  
(full line: gradient-based controller, dashed line: proportional controller,  
dotted line: bang-bang controller)

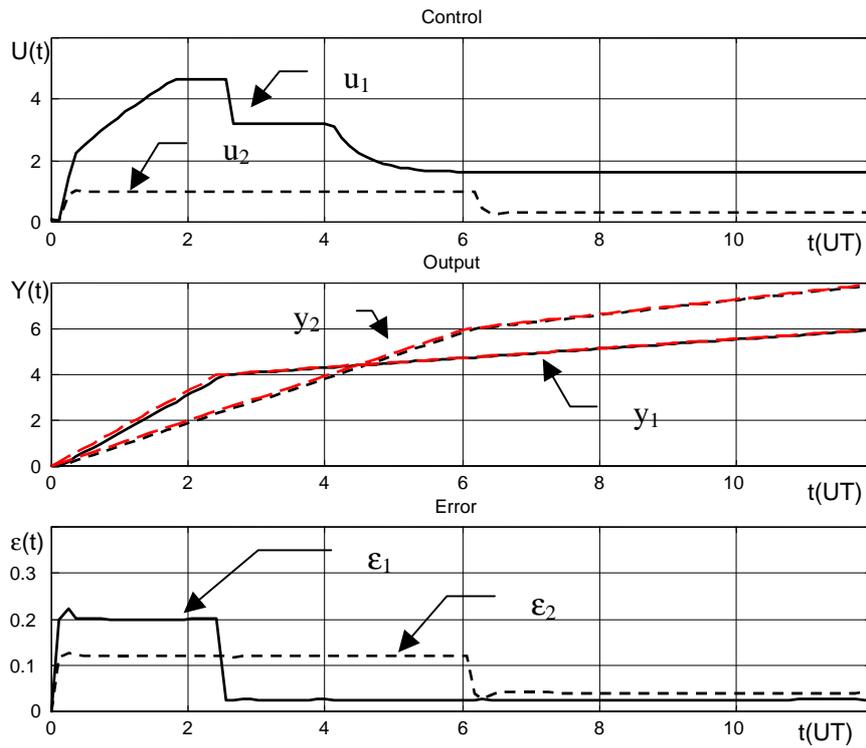


Figure 13: Control design of system C  
 (full line: first input, first output, dashed line: second input second output)

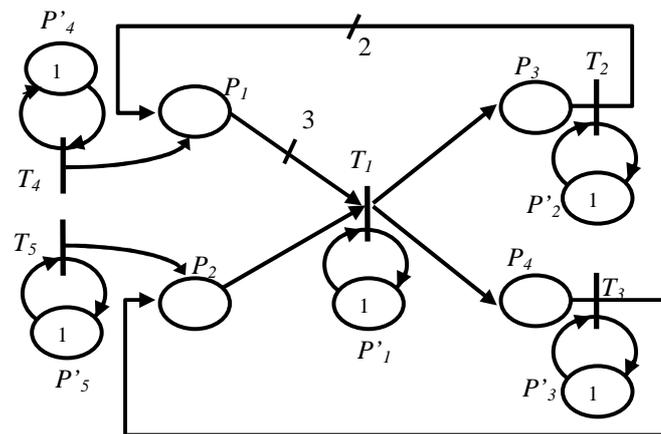


Figure 14: System C'

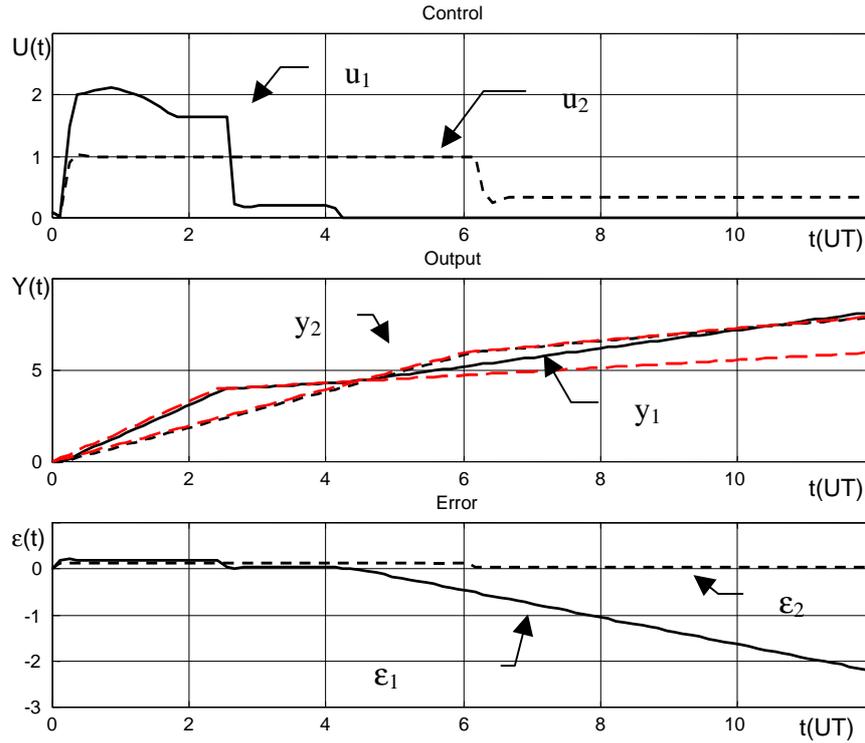


Figure 15: Control design of system C'

(full line: first input, first output, dashed line: second input second output)

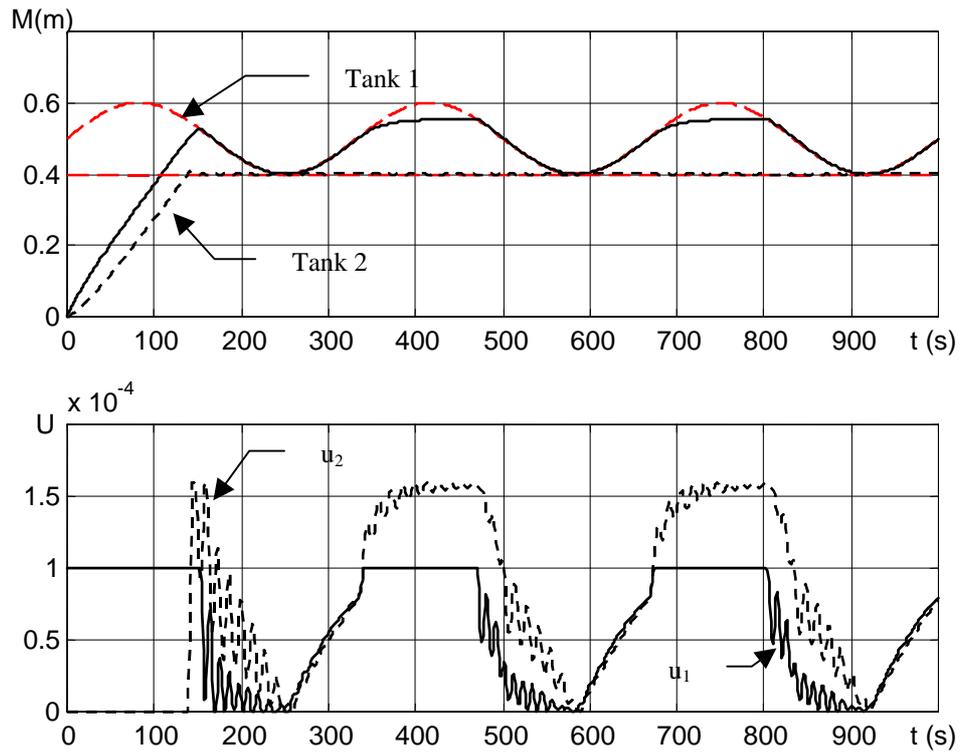


Figure 16: Gradient – based controller for 2 tanks system  
(full line: first input, first output, dashed line: second input second output)

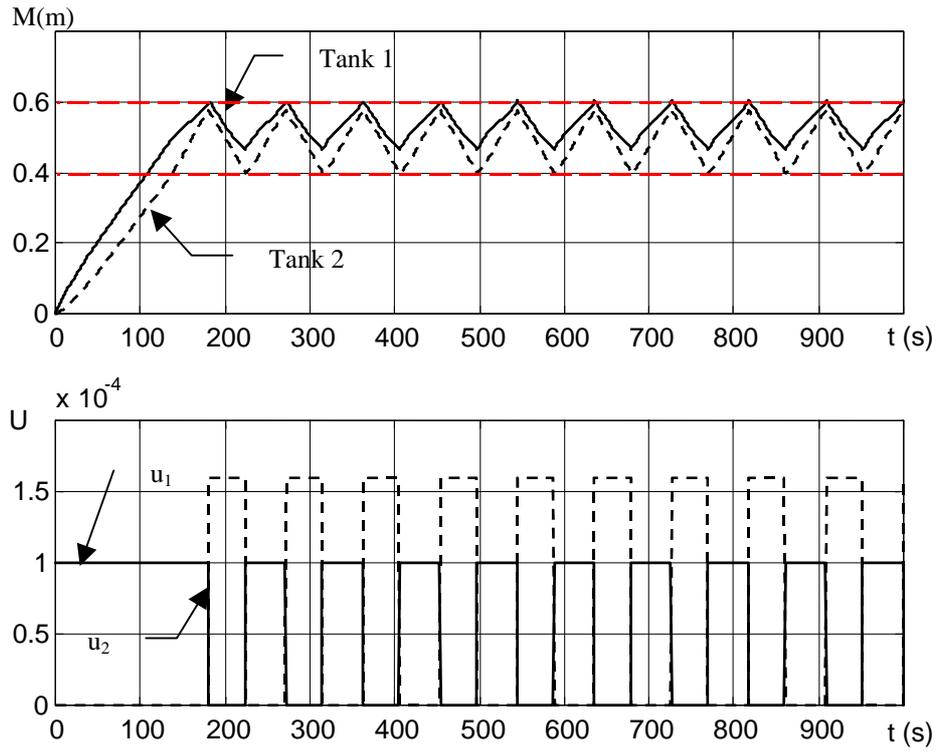


Figure 17: Discrete controller for 2 tanks system

(full line: first input, first output, dashed line: second input second output)

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