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Interplay between entanglement and entropy in two-qubit systems

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Abstract. We study the exact entanglement and entropy dynamics of two qubits interacting with a common zero-temperature non-Markovian reservoir. It is a commonly held view that entanglement loss due to environmental decoherence is accompanied by loss of purity of the state of the system. We demonstrate that such intuitive picture does not always apply: the deterioration of entanglement and purity do not necessarily come together; i.e., revivals of entanglement can be accompanied by deterioration of purity. To complete our investigation on entanglement-mixedness interplay we consider the case of initial mixed states and study how the entanglement dynamics and its revivals are related to both the initial purity and the initial entanglement.

1. Introduction

Entanglement and purity are fundamental properties determining the usefulness of a quantum state in quantum information protocols. Entangled states play a central role in quantum key distribution, superdense coding, quantum teleportation and quantum error correction [1]. Purity of quantum states crucially determines the efficiency of certain applications based on quantum theory, e.g., it characterizes the security of quantum key distribution protocols [2]. The determinant role of these quantities has motivated research in the field of quantum estimation theory with the aim of envisaging protocols for optimal estimation of purity and entanglement [3].

However, realistic quantum systems are never completely isolated from their surroundings. The inevitable interaction between a system and its environment leads to decoherence phenomena and loss of quantumness [4]. So in realistic situation the state of a quantum system is mixed. That is why special status is accorded to the so-called maximally entangled mixed states (MEMS), i.e., states exhibiting the maximum amount of entanglement for a given degree of mixedness. The classification of bipartite MEMS and the study of their applications has received a lot of interest [5], representing an important step in the comprehension of the trade-off between entanglement and mixedness. On the other hand, the dynamical relation between entanglement and mixedness for specific exemplary cases has been far-less investigated [6, 7, 8, 9] especially in non-Markovian system [10].

It is a commonly held view that, whenever a small system interacts with the environment, the entanglement deteriorates with decreasing purity and the state goes toward a statistical mixture. However, this quite intuitive picture does not always apply. We demonstrate for two qubits interacting with a common zero temperature non-Markovian reservoir that the dynamical interplay between entanglement and mixedness depends dramatically on the initial state of the two qubits. The study of the dynamics of such a physical system, being exactly solvable, gives us the chance to investigate analytically the connection between entanglement and mixedness when sudden death/birth and revivals of entanglement occur.

The phenomenon of entanglement sudden death (ESD), consisting in the complete disappearance of bipartite entanglement in a finite time, is believed to set a limit on the life-time and usability of entanglement for practical purposes. Hence, lots of efforts have been done in order to understand the conditions under which ESD occurs [11, 12, 13, 14, 15, 16, 17, 18, 19]. In [19] we have studied the entanglement time evolution when two qubits are prepared in a Bell-like state with two excitations. Here, we extend our investigation to include the entanglement dynamics for initially mixed states. In particular we study a class of states having an "X"-structure density matrix, namely the extended Werner-like states (EWL). This class of states plays a crucial role in many applications of quantum information theory, such as teleportation [20] and quantum key distribution [21]. The extension of our analytical results to the case of initial mixed states also allows us to study how the entanglement dynamics and its

revivals are related to the purity and the amount of entanglement of the initial state.

The paper is organized as follows. In Sec. II we review the exactly solvable model of two qubits interacting with a common Lorentzian structured reservoir. In Sec. III we focus on the entanglement-mixedness interplay comparing concurrence and von Neumann entropy dynamics for initial Bell-like states; we also prove that sudden death of entanglement can never occur if the qubits are initially in a mixed state having at most one excitation. In Sec. IV we present the entanglement time evolution of EWL states, and compare our results with those obtained in [22] for independent structured reservoirs. Finally, we summarize our results in Sec. V.

2. The model

In this section we describe the model used to study the dynamics of two two-level systems (qubits) interacting with a common zero-temperature bosonic reservoir. Our approach is non-Markovian and non-perturbative, i.e., it does not rely on either the Born or the Markov approximations [19].

The Hamiltonian of the system, in the rotating wave approximation, is given by $H = H_0 + H_{\rm int}$,

$$H_0 = \omega_0(\sigma_+^A\sigma_-^A + \sigma_+^B\sigma_-^B) + \sum_k \omega_k a_k^\dagger a_k, \tag{1}$$

$$H_{\rm int} = (\sigma_+^A + \sigma_+^B)\sum_k g_k a_k + \text{h.c.}, \tag{2}$$

where $\sigma_\pm^A$ and $\sigma_\pm^B$ are the Pauli raising and lowering operators for qubit A and B respectively, ω_0 is the Bohr frequency of the two identical qubits, a_k and $a_k^\dagger$, ω_k and g_k are the annihilation and creation operators, the frequency and the coupling constant of the field mode k, respectively.

In Ref. [19] we observed that the dynamics of two qubits which are identical and equally coupled with the same reservoir can be conveniently described in the basis $\{|00\rangle, |+\rangle, |-\rangle, |11\rangle\}$ with $|+\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$ and $|-\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$ the super-radiant and sub-radiant states, respectively. The advantage of this basis is that sub-radiant state turns out to be decoupled from the other states and the reservoir, as a consequence it does not decay. Therefore the two qubits can be effectively described by a three-state system $\{|00\rangle, |+\rangle, |11\rangle\}$ coupled to the reservoir in ladder configuration, and the decoupled sub-radiant state.

In order to solve the dynamics of the two qubits one needs to specify the properties of the environment. In Ref. [19] we assumed that the two qubits interact resonantly with a non-Markovian Lorentzian structured reservoir, such as the electromagnetic field inside a lossy cavity [23], having spectral distribution

$$J(\omega) = \frac{\Omega^2}{2\pi}\frac{\Gamma}{(\omega-\omega_0)^2 + (\Gamma/2)^2}, \tag{3}$$

where Γ is the width of the Lorentzian function and Ω the coupling strength. There we showed that the dynamics can be exactly solved by means of the pseudomode approach. The pseudomodes are auxiliary variables defined from the properties of the spectral distribution, allowing us to derive an exact Markovian master equation for the extended system comprised of the system of interest and the pseudomode degrees of freedom. Only one pseudomode is associated to a Lorentzian spectral distribution and the master equation describing the extended system two-qubits-plus-pseudomode is

$$\frac{\partial \tilde{\rho}}{\partial t} = -i[V, \tilde{\rho}] - \frac{\Gamma}{2}(a^\dagger a \tilde{\rho} + \tilde{\rho} a^\dagger a - 2a\tilde{\rho}a^\dagger), \tag{4}$$

with

$$V = \sqrt{2}\Omega(a|+\rangle\langle 0| + a^\dagger|0\rangle\langle +| + a|2\rangle\langle +| + a^\dagger|+\rangle\langle 2|), \tag{5}$$

where $\tilde{\rho}$ is the density matrix of the extended system in the interaction picture and V the effective coupling between qubits and pseudomode. The parameters Γ and Ω defining the spectral distribution in the pseudomode approach describe, respectively, the pseudomode decay rate and the coupling constant between the pseudomode and the ladder system. We solve the master equation in Eqs. (**??**) and (5) using the Laplace transform technique [19], (in Appendix we present the analytic solution for the qubits density matrix elements for a widely used class of initial states.)

In principle this approach can be suitably used to study the non-resonant case in which the spectral distribution of the environment is peaked at a different frequency than the transition frequency of the qubits. However, the price to pay is that the unitary part of the master equation, describing the coupling between the pseudomode and the qubits, becomes explicitly time-dependent. This of course greatly complicates the derivation of the analytic solution of the dynamics. The off-resonant case, indeed, does not present new qualitative features with respect to those we highlight in this paper, i.e. the relationship between loss of entanglement and purity.

In this work we are mainly interested in the time-evolution of entanglement and mixedness in the effects that the non-Markovian reservoir induces on the correlation between the two qubits. To quantify entanglement we use the Wootters concurrence [24], defined as $C(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where $\{\lambda_i\}$ are the eigenvalues of the matrix $R = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B)$, with ρ^* the complex conjugate of ρ and $\sigma_y^{A/B}$ are the Pauli matrices for atoms A and B. This quantity attains its maximum value of 1 for maximally entangled states and vanishes for separable states.

We focus now on the dynamics of initial "X" states. We use the method presented in Ref. [19] (we just briefly recalled) to calculate the time evolution of the elements of the pseudomode-plus-qubits density matrix. By tracing out the pseudomode degree of freedom we obtain the time evolution of the density matrix of the two qubits

$$\rho(t) = \begin{pmatrix} a(t) & 0 & 0 & w(t) \\ 0 & b(t) & z(t) & 0 \\ 0 & z^*(t) & c(t) & 0 \\ w^*(t) & 0 & 0 & d(t) \end{pmatrix}, \tag{6}$$

which is written in the basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$. Due to the structure of the differential equations derived from the master equation in (4) and (5), the "X" form of the state is preserved during the evolution. In the Appendix we present the analytical solution in the Laplace transform space for a particular type of "X" state.

For this class of states the concurrence assumes a simple analytic expression

$$C(t) = \max\{0, C_1(t), C_2(t)\}, \tag{7}$$

where

$$C_1(t) = 2|w(t)| - 2\sqrt{b(t)c(t)}, \tag{8}$$
$$C_2(t) = 2|z(t)| - 2\sqrt{a(t)d(t)}. \tag{9}$$

We notice that coherences give a positive contribution to $C_1(t)$ and $C_2(t)$ and so to concurrence, while the negative parts involve populations only.

In the next section we will also consider the evolution of the mixedness of the two qubit state, quantified by the von Neumann entropy

$$S(\rho) = -\mathrm{Tr}\{\rho(t)\log_N(\rho(t))\}. \tag{10}$$

The von Neumann entropy is equal to zero for pure states, and attains its maximum value 1 (equal to $\log_N N$ with N the dimension of the Hilbert space) for a maximally mixed state. Alternatively, to quantify the amount of mixedness in a quantum state one can use purity, defined us

$$P(\rho) = \mathrm{Tr}\{\rho^2\} \tag{11}$$

The purity can range between unity, corresponding to a completely pure state, and $1/d$ (with d the dimension of the density matrix) for a maximally mixed state.

All the physical quantities we consider in this work (and the associated plots) have been calculated directly from the knowledge of the density matrix elements of Eq. (6) in the Laplace transform space. The initial conditions on the state are directly inserted in those analytic solutions, which are then anti-transformed and manipulated to calculate the quantity under investigation.

3. Entanglement-mixedness dynamical interplay

Here, we study and compare the dynamics of concurrence and von Neumann entropy of two qubits prepared in the Bell-like states

$$|\Phi\rangle = \alpha|10\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|01\rangle, \tag{12}$$

and

$$|\Psi\rangle = \alpha|00\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|11\rangle. \tag{13}$$

Our aim is to understand the interplay between these two different quantum properties, in particular when peculiar phenomena such as ESD or "entanglement sudden birth"(ESB) [18, 19, 25, 26], and revivals of entanglement occur.

The entanglement dynamics of two qubits in a Lorentzian structured reservoir, prepared in a Bell-like state with two excitations as in (13), has been presented in [19]. The evolution of entanglement for a Bell-like state as in (12) has been studied also in [27]. For both the Bell-like states in (12) and (13) the entanglement dynamics is the result of two combined effects: the backaction of the non-Markovian reservoir and the reservoir-mediated interaction between the qubits. The memory effects due to the non-Markovianity of the reservoir causes oscillations in entanglement dynamics which in fact are not present in the Markovian case [15]. The sharing of the reservoir plays also a special role. Indeed, the common reservoir provides an effective coupling between the qubits, and consequently creates quantum correlations between them. As a result, qubits prepared in a factorized state can become entangled due to the interaction with the common reservoir, in contrast with the independent non-Markovian reservoirs case [16, 17].

The results in [27] show that ESD does not occur for a Bell-like state with one excitation as in (12) for any value of α^2, even when the dipolar interaction between the qubits is included [28]. Actually, a straightforward calculation shows that for every pure or mixed state of the qubits containing at most one excitation, ESD and ESB cannot take place. In fact, the density matrix describing a generic mixed state with maximum one excitation, written in the same basis of (6), has the form

$$\rho(t) = \begin{pmatrix} a(t) & j(t) & k(t) & 0 \\ j^*(t) & b(t) & z(t) & 0 \\ k^*(t) & z^*(t) & c(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

The expression of the concurrence, for any value of the parameters, is

$$C(t) = \max\{0, 2|z(t)|\}. \tag{15}$$

Here concurrence is directly given by the coherence between the $|10\rangle$ and $|01\rangle$ states. Since the coherence vanishes in asymptotic way, there cannot be ESD for any generic state with maximum one excitation. Analogously, entanglement can be smoothly generated but it cannot suddenly appear. This result does not depend on the degree of purity of the state. This is true as long as the form of the density matrix in (14) is maintained. On the other hand, if some population is transferred to the two excitations state then ESD can appear. This is the case of two qubits in a Bell state interacting with a non-RWA common reservoir [29].

3.1. Entanglement and von Neumann entropy time evolution

We will now try to understand how environmental decoherence affects the entanglement and the purity of the state. One could intuitively expect that the interaction with the environment not only deteriorates the entanglement content but also destroys purity. Stated another way, one could think that the loss of entanglement is related to the transition from an entangled quantum superposition such as those given by equations

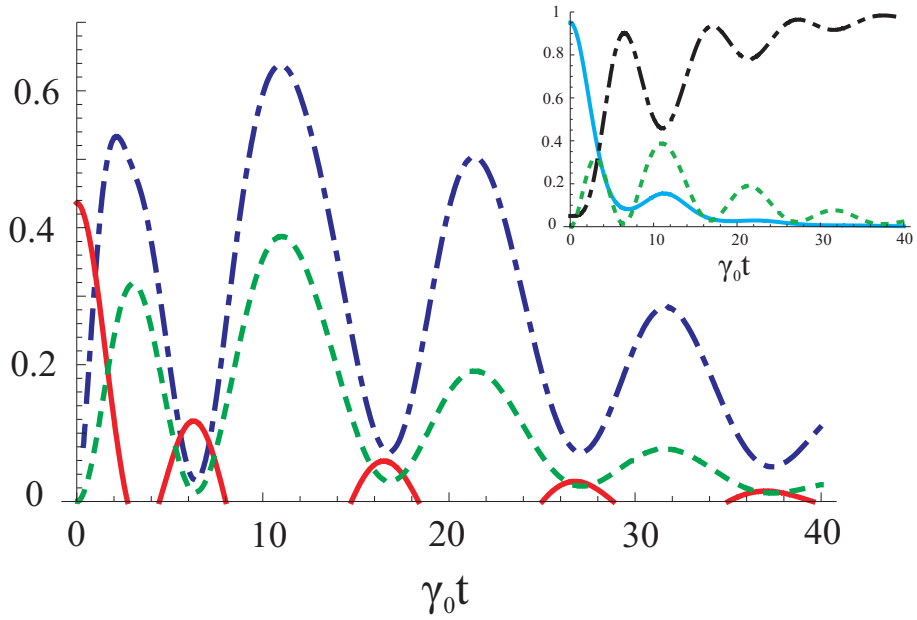

Figure 1. (Color online) Dynamics in a common Lorentzian structured reservoir as a function of scaled time for two atoms prepared in the Bell-like state Ψ with $\alpha^2 = 1/20$ and $\theta = 0$. Solid red line is concurrence; dotted-dashed blue line is von Neumann entropy; dotted green line is the population of the super-radiant state $\rho_{++}(t)$. In the inset: dotted green line is the super-radiant state population; dotted-dashed black line is the ground state population; solid light blue line is the excited state $|11\rangle$ population.

(12) and (13), to the corresponding statistical mixture. In fact, as we will show in the following, this intuitive picture holds for two-photon Bell-like states and for those states containing the maximally excited state $|11\rangle$. In such cases the concurrence and the von Neumann entropy dynamics are clearly connected. However, we have discovered that the dynamical relation between entanglement and entropy is far less trivial for Bell-like states with one excitation.

We mentioned in the previous section that the state of the two qubits is effectively equivalent to a four-state system, in which three states $\{|00\rangle,\ |+\rangle,\ |11\rangle\}$ interact in ladder configuration with the electromagnetic field, and the fourth state, the sub-radiant state $|-\rangle$, is completely decoupled from both the other states and the electromagnetic field.

We first look at the evolution of the von Neumann entropy of the qubit pair when the state is initially prepared in a Bell-like state of the form (12) and (13). We begin by considering the Ψ state dynamics as a function of the dimension-less time $\gamma_0 t$ where the parameter $\gamma_0 = 4\Omega^2/\Gamma$ is the Markovian decay rate of the atoms, i.e., the inverse of the atomic relaxation time in the Markovian limit. It is particularly interesting to see how the degree of mixedness of the state evolves when the qubits undergo ESD. Since ESD occurs when $\alpha^2 \leq 1/4$ [19], we choose $\alpha^2 = 1/20$ and we set $\theta = 0$ for the sake of convenience. In figure 1 we notice that ESD appears for high value of mixedness of the system, i.e., concurrence and von Neumann entropy oscillate out of phase. Stated

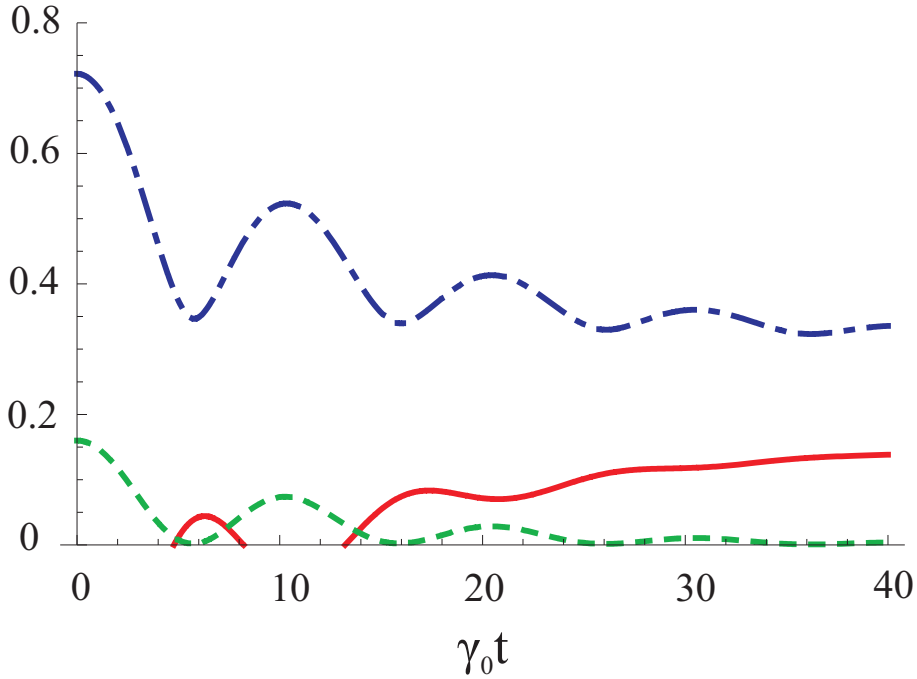

Figure 2. (Color online) Dynamics in a common Lorentzian structured reservoir as a function of scaled time for two atoms prepared in the mixed state (16) with $\alpha^2 = 0.75$. Solid red line is concurrence; dotted-dashed blue line is von Neumann entropy; dotted green line is super-radiant state population.

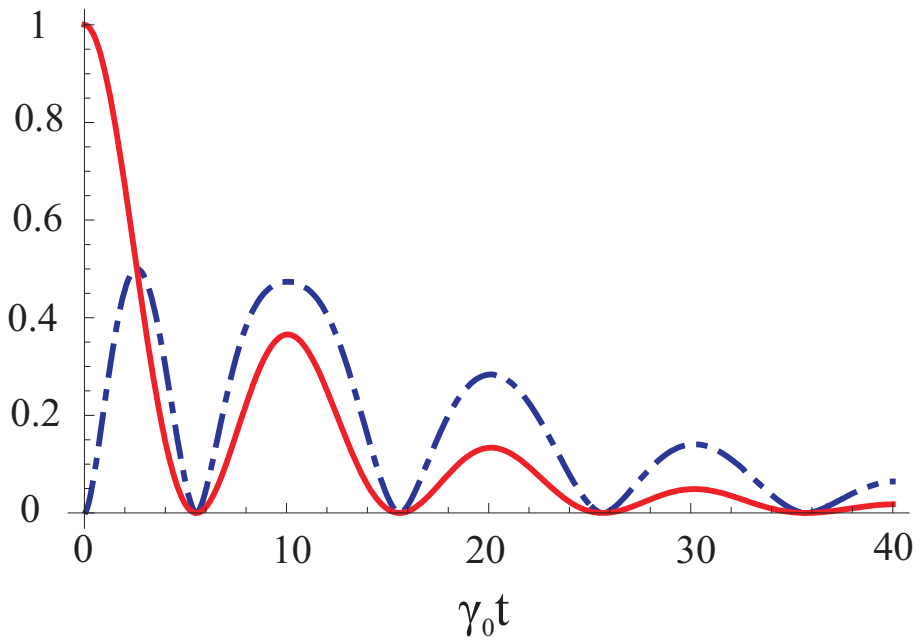

Figure 3. (Color online) Dynamics in a common Lorentzian structured reservoir as a function of scaled time for two atoms prepared in the Bell-like state Φ with $\alpha^2 = 1/2$ and $\theta = 0$. Solid red line is concurrence and dotted-dashed blue line is von Neumann entropy. The super-radiant state population overlaps with concurrence.

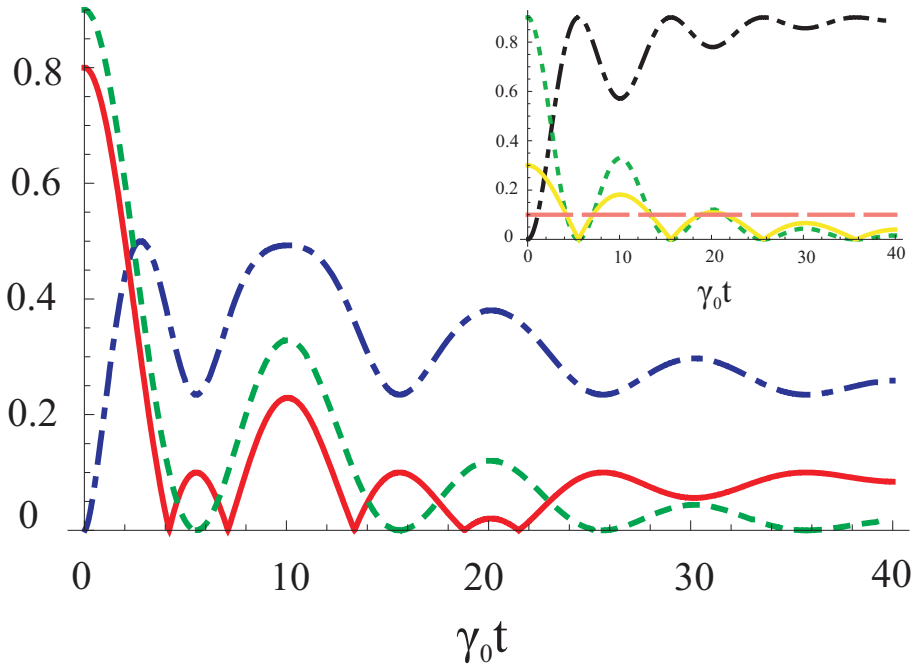

Figure 4. (Color online) Dynamics in a common Lorentzian structured reservoir as a function of scaled time for two atoms prepared in the Bell-like state Φ with $\alpha^2 = 1/5$ and $\theta = 0$. Solid red line is concurrence; dotted-dashed blue line is von Neumann entropy; dotted green line is super-radiant state population. In the inset: dotted green line is super-radiant state population; dotted-dashed black line is ground state population; dashed pink line is sub-radiant state population; solid yellow line is absolute value of the coherence between super-radiant and sub-radiant states.

another way, the revivals of entanglement appear roughly in correspondence of the minima of the von Neumann entropy, when the state becomes purer. Since the initial state does not contain any sub-radiant state component, the only states involved are $|00\rangle$, $|+\rangle$, $|11\rangle$. For this particular initial conditions the super-radiant state population explains the link between entropy and concurrence. In fact, for the chosen initial state $\rho_{++}(t) = 2\sqrt{b(t)c(t)}$ and the time-dependent part of the concurrence is $2|w(t)| - \rho_{++}(t)$. The von Neumann entropy has a more complicated analytical expression, however it follows closely the time evolution of $\rho_{++}(t)$. We can grasp the main features of the dynamics of mixedness by considering the analytical expression of the linear entropy, $S_L(\rho) = 1 - \mathrm{Tr}[\rho^2] = 2(\rho_{++}(t) - \rho_{++}^2(t) + a(t)d(t) - |w(t)|)$. When the initial state $\rho(0)$ is such that $\langle 11|\rho|11\rangle \neq 0$ the flow of probability from the $|11\rangle$ component to the super-radiant component destroys entanglement, even if the super-radiant state is a maximally entangled state. Indeed, whenever the population $\rho_{++}(t)$ reaches its relative maxima, the state attains a maximum value of mixedness, the time-dependent part of the concurrence $(2|w(t)| - \rho_{++}(t))$ becomes negative and entanglement disappears. On the other hand, whenever the population of the super-radiant state reaches a minimum, the population of the $|11\rangle$ excited state and the $|00\rangle$ ground state have their maxima (see inset in figure 1), and the system goes toward the initial Bell-like state $|\Psi\rangle$. As a

consequence the system becomes purer and entanglement is partially recovered.

The same kind of dynamical relation between entanglement and entropy exists if the qubits are prepared in a state of the form

$$\rho(0) = (\alpha^2|0_A\rangle\langle 0_A| + (1-\alpha^2)|1_A\rangle\langle 1_A|) \tag{16}$$

$$\otimes\, (\alpha^2|0_B\rangle\langle 0_B| + (1-\alpha^2)|1_B\rangle\langle 1_B|). \tag{17}$$

The dynamics of this state in a common non-Markovian reservoir is characterized by ESB and revivals of disentanglement, as we have shown in [19]. In figure 2 we see how these interesting features are related to the degree of purity of the state. As we have seen before, the positions of maxima and minima of the entropy and of the super-radiant state population match. The sudden creation of entanglement happens roughly when the entropy hits its first minimum. Entanglement is again lost when the amount of mixedness increases, and it reappears again when the entropy reaches another minimum.

A completely different dynamics characterizes the system when the qubits are prepared in a Bell-like state as the one of (12). In fact in this case there is no ESD, and revivals of entanglement are not necessarily related to a reduction in mixedness. To start with consider as initial state the super-radiant Bell $|+\rangle$ state. As shown in figure 3 revivals of entanglement are always accompanied by increase of entropy, so the intuitive picture fails. Moreover the zeroes of entanglement and entropy coincide. For those times, in fact, the system goes into the ground state which is pure and factorized. When some population returns in the super-radiant state entanglement is recovered, and the state is again mixed. This examples certainly shows that loss of entanglement is not simply due to a transition from quantum superposition to statistical mixture.

A more complex situation appears for a generic one-photon Bell-like state. When $\alpha^2 \neq 1/2$ and/or $\theta \neq 0$ the initial state is a superposition of super-radiant and sub-radiant states. Although the sub-radiant component does not evolve in time, being decoupled from the super-radiant and ground states, its presence affects the entanglement and entropy dynamics. For the sake of convenience we choose $\alpha^2 = 1/5$ and $\theta = 0$. In figure 4 we see that entropy still follows closely the time evolution of the super-radiant state population, having its relative minima in the same positions of the zeroes of $\rho_{++}(t)$. On the contrary, entanglement has new relative maxima when the population of the super-radiant state is zero. This is due to the presence of the sub-radiant state. In fact, in this case both the super-radiant and the sub-radiant states contribute to the total entanglement. Thus there are two different sets of entanglement maxima, those associated with the maxima of the super-radiant state population, and those associated to the sub-radiant state. Entanglement is zero whenever the population of the super-radiant state $\rho_{++}(t)$, the population of the sub-radiant state $\rho_{--}(t)$, and the absolute value of the coherence between super-radiant and sub-radiant states $\rho_{+-}(t)$, are equal. This can be explained having in mind equations (22) in the Appendix, where $b(t)$, $c(t)$ and $z(t)$ are written as a function of those quantities. Indeed when $\rho_{++}(t)$, $\rho_{--}(t)$ and $|\rho_{+-}(t)|$ have equal value, then $z(t)$ is equal to zero, irrespective of the sign of $\rho_{+-}(t)$. Specifically, this happens in correspondence to the 1st, 4th and

5th zeroes of entanglement, where $b(t) = 0$ and $c(t) = 2\rho_{--} = k$. In this case the state of the system becomes $(1-k)|00\rangle\langle 00| + k|01\rangle\langle 01|$, which is clearly not entangled. Similarly, for the 2nd and 3rd zeroes, $b(t) = k$, $c(t) = 0$ and the non-entangled state is $(1-k)|00\rangle\langle 00| + k|10\rangle\langle 10|$.

3.2. Dynamics in the concurrence-vs-purity phase space

To summarize and better understand the entanglement-mixedness dynamical interplay we look at our results from a different perspective. Figure 5 shows the time-evolution in the entanglement-purity phase diagram for different class of initial states of the two qubits. In particular, figure 5 a) shows the parametric plot of concurrence and purity for a Bell-like state with two excitation Ψ with $\alpha^2 = 1/20$. In figure 5 b) the state of preparation of the system is a Bell-like state with one excitation Φ with $\alpha^2 = 1/5$ and $\theta = 0$. The two plots are qualitatively very different.

The dynamical feature we noticed from the analysis of figure 1 is even more apparent in figure 5 a): deterioration, and revivals as well, of entanglement and purity come together during the dynamics of a Ψ state. The initial moment of the time evolution corresponds to the upper corner of the C-P diagram, the system evolves toward smaller entanglement and purity, till sudden death of entanglement occurs; afterwards entanglement and purity revive accordingly. We plot the curve up to the time 30 $\gamma_0 t$, similar structures appear later on in the dynamics until the system reaches its pure and separable asymptotic state, the ground state (at the lower right corner of the CP diagram).

Analogously, in part b) of figure 5 the upper right corner of the C-P diagram corresponds to the starting time of the dynamics. Entanglement and purity decrease together until at ($P = 0.5$, $C = 0.3$) a change of behavior occurs: after that while concurrence is still decreasing purity starts to increase. Once more a sudden change in the dynamical interplay has place at the point when concurrence becomes zero and entanglement and purity behave again in the same way. As time passes the system starts travelling back on the same curve, therefore there is a period of time for which while purity is decreasing concurrence increases. This is a clear sign of non-Markovianity. In fact, as pointed out by Ziman and Buzek [7] the C-P line associated to a Markovian process must be non-increasing respect to concurrence, this prevents the presence of loops in the diagram even though allows the growth of purity when concurrence is decreasing. Here, revivals of concurrence can be accompanied by decrement of purity, and the system travels partially back and forth on the same curve till it reaches its asymptotic state which is a mixed non separable state, statistical mixture of the ground and sub-radiant state. To the best of our knowledge this type of dynamics was not yet been observed.

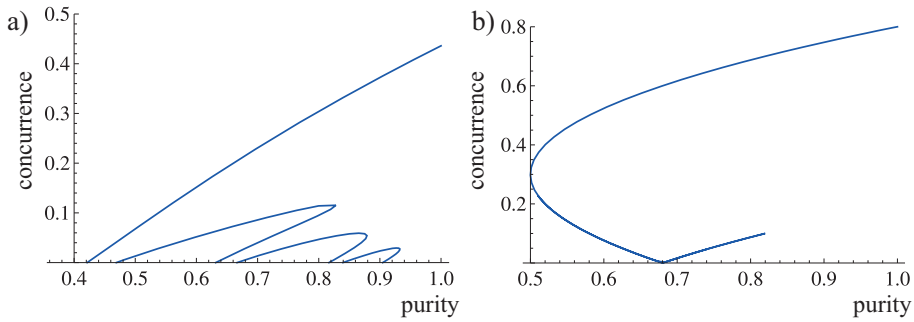

Figure 5. (Color online) Parametric plot of the time evolution of concurrence and purity of two qubits when initially prepared in: a) a Bell-like state with two excitation Ψ with $\alpha^2 = 1/20$; b) a Bell-like state with one excitation Φ with $\alpha^2 = 1/5$ and $\theta = 0$.

4. Werner states

In this section we study the two qubits entanglement dynamics of extended Werner-like states in a common zero-temperature Lorentzian structured reservoir. Our goal here is to study how, starting from an initial state that is not perfectly pure, as in realistic experimental conditions, affects the entanglement dynamics, and in particular the occurrence of ESD and ESB phenomena.

4.1. Terminology and previous works

The standard two-qubit Werner state, introduced in 1989 by Werner [30], is defined as

$$\rho_W = r|-\rangle\langle-| + \frac{1-r}{4}\mathbb{I}, \tag{18}$$

where $|-\rangle$ is the singlet state. In [30] Werner demonstrated that while pure entangled states always violate the Bell inequality, mixed entangled states might not. The Werner state is the first entangled state to be proven to satisfy all Bell inequalities [30]. The generalized or Werner-like states are defined as

$$\rho_{WL} = r|M\rangle\langle M| + \frac{1-r}{4}\mathbb{I} \tag{19}$$

with $|M\rangle$ one of the four maximally entangled Bell states. For a given r, Werner and Werner-like states exhibit the same entanglement. A further generalization is the extended Werner-like states (EWL), containing a non-maximally entangled state part, which are defined as

$$\rho^{\Phi}_{EWL} = r|\Phi\rangle\langle\Phi| + \frac{1-r}{4}\mathbb{I}, \tag{20}$$

where $|\Phi\rangle = \alpha|10\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|01\rangle$, and

$$\rho^{\Psi}_{EWL} = r|\Psi\rangle\langle\Psi| + \frac{1-r}{4}\mathbb{I}, \tag{21}$$

with $|\Psi\rangle = \alpha|00\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|11\rangle$.

The dynamics of entanglement of two qubits prepared in Werner, Werner-like or extended Werner-like states has attracted a lot of attention. The appearance of ESD has been demonstrated for two qubits prepared in a Werner-like state and interacting with Markovian independent reservoirs [31, 32] or with independent noisy channels [33]. Violation of Bell inequality has been examined in independent thermal reservoirs [34, 35]. Entanglement has been studied in a Markovian common thermal reservoir [36], in the presence of collective dephasing [37] and in the maximal noise limit [38]. Finally, the dynamics of an EWL in two independent Lorentzian structured reservoirs has been investigated in [22].

Moreover, Werner and Werner-like states have been used so far in many applications in quantum information processes such as teleportation [20] and entanglement teleportation [39]. The experimental preparation and characterization of the Werner states have also been widely investigated. Werner states are prepared via spontaneous parametric down-conversion [40] or using a universal source of entanglement [41], and used in ancilla-assisted process tomography [42] and secure quantum key distribution [21].

4.2. Entanglement dynamics

We evaluate the dynamics of the entanglement of EWL states as a function of the initial amount of mixedness, controlled by the purity parameter r, and as a function of the initial degree of entanglement measured by α^2.

The study of the evolution of this kind of states allows to infer how the degree of purity of the initial state influences the entanglement dynamics. We also compare our results with those obtained in [22] for two qubits in two independent Lorentzian structured reservoirs.

Figures 6, 7 and 8 show the entanglement dynamics as a function of the dimensionless quantity $\gamma_0 t$ (with $\gamma_0 = 4\Omega^2/\Gamma$ the Markovian decay rate of the atoms) and of the purity parameter r, for the two EWL states in (20) and (21). In figure 6 the qubits are initially in the state given by (20) with $\alpha^2 = 1/2$ and $\theta = 0$. When $r = 1$ the qubits are prepared in the super-radiant state, entanglement exhibits oscillations and ESD is never present. However, whenever a little amount of mixedness is added, periods of finite-time disentanglement immediately occur. This is due to the appearance of some population in the excited state $|11\rangle$. As a consequence the oscillating part of the concurrence $\{2|z(t)| - 2\sqrt{a(t)d(t)}\}$ can become negative. Due to the non-Markovianity of the system and to the effective coupling provided by the common reservoir, ESD regions are followed by revivals, and for long times entanglement reaches a stationary value. When $r < 1/3$ the state is initially factorized, but as time passes, because of the reservoir-mediated interaction, entanglement between the qubits is suddenly created. The non-Markovianity of the reservoir enriches the dynamics causing revivals of disentanglement. Eventually the entanglement reaches a non-zero stationary value. The amount of entanglement that has been created depends on the population of the

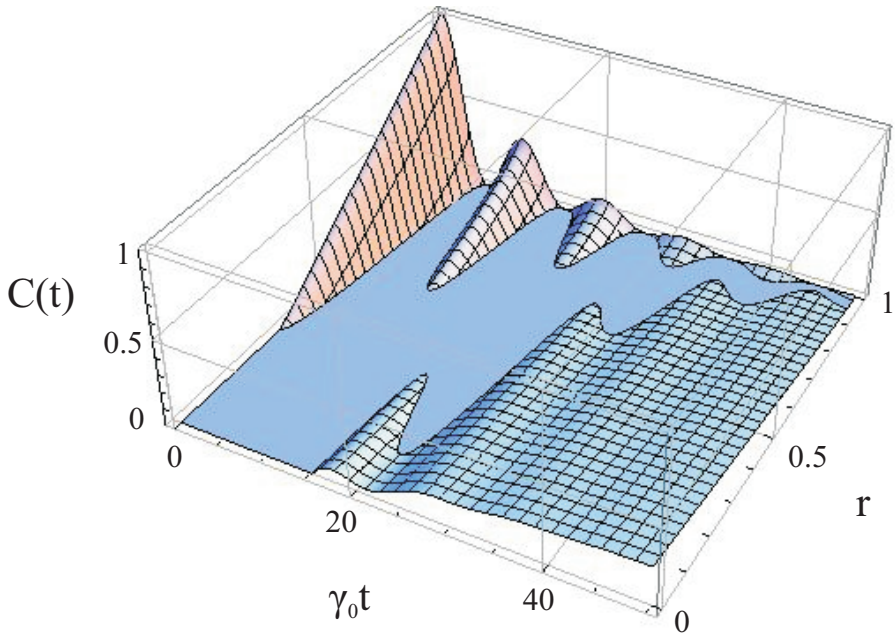

Figure 6. (Color online) Concurrence as a function of scaled time and r for two atoms prepared in the Werner state (20) (Φ) with $\alpha^2 = 1/2$ and $\theta = 0$.

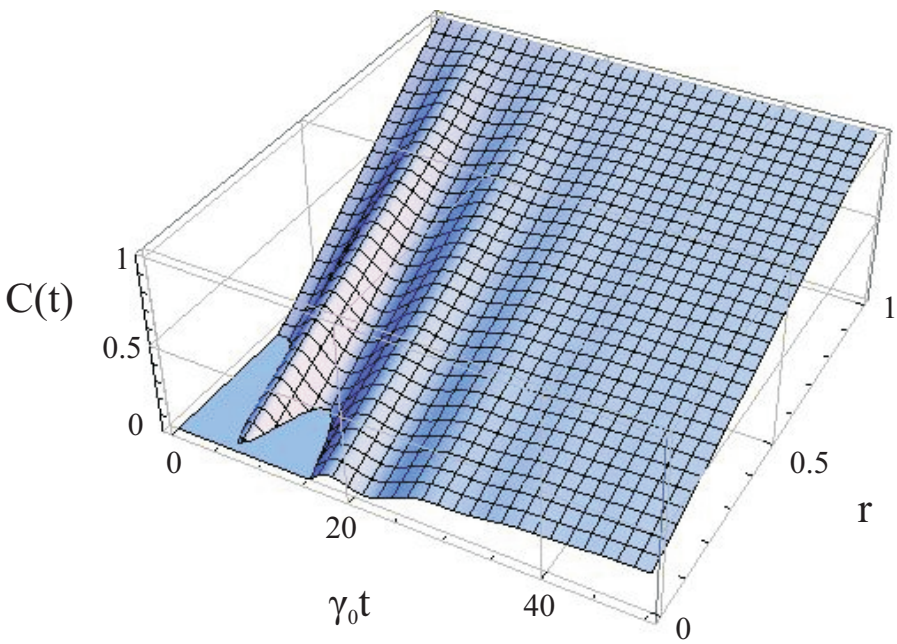

Figure 7. (Color online) Concurrence as a function of scaled time and r for two atoms prepared in the Werner state (20) (Φ) with $\alpha^2 = 1/2$ and $\theta = \pi$.

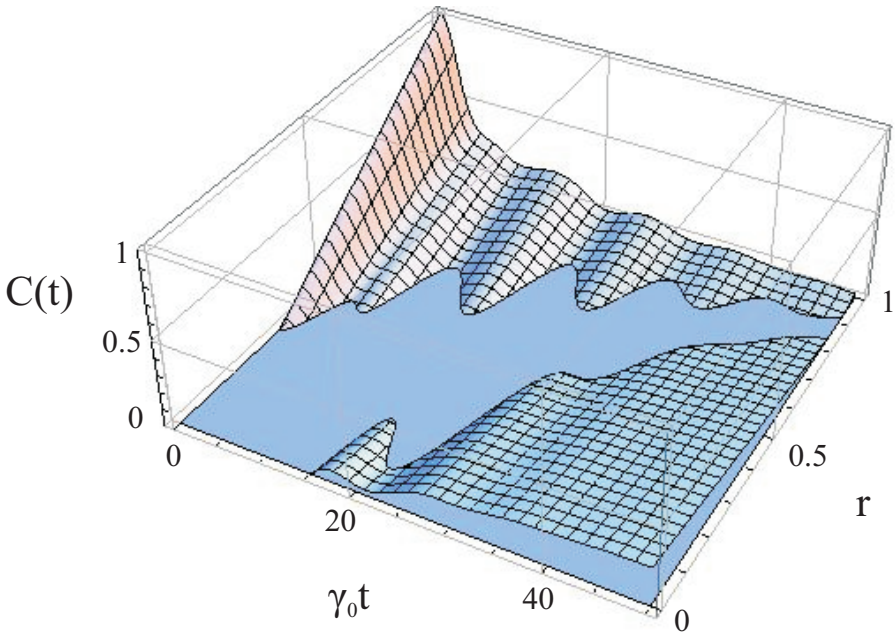

Figure 8. (Color online) Concurrence as a function of scaled time and r for two atoms prepared in the Werner state (21) (Ψ) with $\alpha^2 = 1/2$ and $\theta = 0$.

sub-radiant state, which carries the entanglement.

If the qubits are prepared in the state given by (20) with $\alpha^2 = 1/2$ and $\theta = \pi$, the dynamics is completely different compared to the previous $\theta = 0$ case, as shown in figure 7. In fact when $r = 1$ the qubits are prepared in the sub-radiant state and concurrence does not evolve, being at any time equal to 1. When the mixed part is present little oscillations appear, but eventually the concurrence attains the stationary value $(1+3r)/4$. When $r < 1/3$ the initial state is factorized, entanglement is suddenly created, a revival of disentanglement is present, and again after some oscillations concurrence reaches its stationary value.

When $\alpha^2 \neq 1/2$ and/or $\theta \neq 0, \pi$ the non-mixed part of the initial state is a superposition of super-radiant and sub-radiant states. Thus the entanglement dynamics depends on the weights of those two states. As a result there is a wide variety of entanglement dynamics in between the two asymptotic behaviors described above.

In figure 8 the qubits are prepared in the state given by (21) with $\alpha^2 = 1/2$ and $\theta = 0$. Note however that the results are independent of the choice of the relative phase θ. For $r = 1$ the initial state is the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$. For this initial condition not only there is no ESD, but also concurrence vanishes only for infinite time. When an increasing amount of mixedness is present in the initial state, finite-time disentanglement appears. ESD is then followed by revivals and, as expected, a certain amount of entanglement is preserved. Analogously to the other EWL state, for $r < 1/3$ the initial state is factorized. For the reasons mentioned above, entanglement is suddenly created, momentarily deteriorated, and it finally goes to the stationary value

$r/4$, coinciding with the population of the sub-radiant state. For this type of EWL state different choices of α lead to the same qualitative behavior of the entanglement dynamics.

The results we just described for two qubits in a common structured reservoir are quite different from those presented in [22] for two qubits in two independent Lorentzian structured reservoirs. First of all, if two uncoupled qubits interact with two independent reservoirs, entanglement cannot be created from a factorized state. Hence, the ESB region, characterizing the dynamics in a common reservoir, is not present in the results of [22], for any of the EWL states. Moreover, for qubits prepared in the initial state of (21), we notice that the reservoir-mediated interaction between the qubits keeps the value of concurrence higher compared to the two independent reservoirs case.

The crucial difference between the common and independent reservoirs cases is that for qubits in two independent reservoirs the decoherence-free sub-radiant state does not exist. When the qubits are in two independent reservoirs, the entanglement dynamics of the $|+\rangle$ and $|-\rangle$ states are the same. As a consequence the relative phase θ in (12) and (20) does not affect the results. This is definitely in contrast with the results we present in figures 6 and 7, showing two completely different asymptotic behavior when changing the relative phase θ of a π factor.

5. Conclusive Remarks

In this paper we have investigated the dynamical relation between entanglement and mixedness in a system of two qubits interacting with a common zero temperature non-Markovian reservoir. We have used the exactly solvable model presented in [19], in the Appendix we attach the analytical solution in the Laplace transform space in the case of extended-Werner-like states. All the plots we have presented in this work have been derived by manipulation of the exact analytic solution in the Laplace transform space.

Our results show that decoherence induced by the environment does not necessarily affect in the same way the entanglement content and the purity of the state. Nevertheless, for qubits prepared in two-photon Bell-like states or in states comprising the maximally excited state $|11\rangle$, loss of entanglement is generally accompanied by increasing of mixedness. ESD seems to appear when the state becomes highly mixed, whereas revivals of entanglement are associated to minima of the entropy, where the state becomes purer. Moreover when starting from a factorized state, sudden birth of entanglement occurs for lower values of mixedness, while revivals of disentanglement are accompanied by peaks of the entropy.

The dynamical connection between entanglement and mixedness becomes much more complex for initial Bell-like state with one excitation. However, by looking at the dynamics of the population of the super-radiant and sub-radiant states, one sees two sets of maxima of entanglement can be identified. The super-radiant component mainly carries the entanglement when it is maximally populated; when its population is zero, the entanglement is associated to the sub-radiant component. When the populations of

these states are equal, the entanglement vanishes.

The dynamical interplay between the entanglement content and the purity of the state of the two qubits can be more clearly seen in the concurrence-versus-purity phase diagram. There, it is apparent that when starting from a Bell-like state with two excitations entanglement and purity behave accordingly, decreasing or reviving at the same time. For an initial Bell-like state with one excitation their trade-off is far less trivial, the diagram shows loops (the system traveling back and forth over the same curve) and revivals of entanglement can take place in correspondence of decrement of purity. Those are further signs of the non-Markovianity of the system [7].

We have also considered the entanglement dynamics of a particular class of mixed states, the extended Werner-like states. We have demonstrated that the amount of purity of the initial state plays a key role in the entanglement dynamics, controlling the appearance of ESD and ESB phenomena. As expected, for stronger non-Markovian conditions, the dynamics exhibits stronger and longer lasting entanglement oscillations as well as an increasing number of dark periods and revivals as well. On the other hand, in the Markovian regime, no oscillations are present, however the basic features of the dynamics, and so the ESD and ESB regions, are still present.

Entanglement and purity are two different physical quantities characterizing the degree of non-classicality of a quantum state. Understanding their interplay is important not only from an applicative but also from a fundamental point of view. We believe that the present results contribute to the comprehension of their dynamical relations.

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Appendix

Here we present the exact analytic solution for two qubits interacting with a Lorentzian structured reservoir, when the qubits are prepared in an EWL state as in (20) and (21). We provide the expressions for the density matrix element in (6) as a function of the solution $\tilde{\rho}_{ij}$ of the pseudomode master equation [43] in (5) and (6) of [19]

$$\begin{aligned}
a(t) &= \tilde{\rho}_{aa}(t) + \tilde{\rho}_{bb}(t) + \tilde{\rho}_{cc}(t), \\
b(t) &= \frac{\tilde{\rho}_{++}(t) + \tilde{\rho}_{--}(t) + \tilde{\rho}_{+-}(t) + \tilde{\rho}_{-+}(t)}{2}, \\
c(t) &= \frac{\tilde{\rho}_{++}(t) + \tilde{\rho}_{--}(t) - \tilde{\rho}_{+-}(t) - \tilde{\rho}_{-+}(t)}{2}, \\
z(t) &= \frac{\tilde{\rho}_{++}(t) - \tilde{\rho}_{--}(t) - \tilde{\rho}_{+-}(t) + \tilde{\rho}_{-+}(t)}{2},
\end{aligned} \tag{22}$$

$$d(t) = 1 - a(t) - b(t) - c(t) = \tilde{\rho}_{ff}(t),$$
$$w(t) = \tilde{\rho}_{af}(t),$$
$$\tilde{\rho}_{++}(t) = \tilde{\rho}_{dd}(t) + \tilde{\rho}_{ee}(t).$$

The subscripts a, b and c refer to the states in which both the qubits are in the ground state, and the pseudomode has 0, 1 or 2 excitations respectively. In d and e the atomic system is in the super-radiant state, and the pseudomode has 0 or 1 excitations. f is the state having 0 excitations in the pseudomode and both the qubits in their excited states.

The analytic expressions are given in the Laplace transform space. In the following we provide the solutions for a EWL state of the form (20):

$$\tilde{\rho}_{aa}(t) = \mathcal{L}^{-1}\Big\{ -\frac{1}{4s}(-1 + r + \frac{64\Gamma^2\Omega^4(-1+r)l(s)}{j(s)k(s)} - \frac{8\Gamma\Omega^2(1 + r + 4\alpha\sqrt{1-\alpha^2}r\cos\theta)}{j(s)})\Big\}, \tag{23}$$

$$\tilde{\rho}_{bb}(t) = \mathcal{L}^{-1}\Big\{\frac{2\Omega^2}{j(s)}(1 + r - \frac{8\Gamma\Omega^2(-1+r)l(s)}{k(s)} + 4\alpha\sqrt{1-\alpha^2}r\cos\theta)\Big\}, \tag{24}$$

$$\tilde{\rho}_{cc}(t) = \mathcal{L}^{-1}\Big\{-\frac{48\Omega^4(-1+r)(\Gamma+s)}{k(s)}\Big\}, \tag{25}$$

$$\begin{aligned}\tilde{\rho}_{dd}(t) = \mathcal{L}^{-1}\Big\{&\frac{1}{4j(s)}(-\frac{1}{k(s)}(8\Gamma\Omega^2(-1+r)(6\Gamma^5 + 31\Gamma^4 s + 20\Gamma^3(-2\Omega^2 + 3s^2)\\ &+5\Gamma^2 s(-20\Omega^2 + 11s^2) + 8\Gamma(56\Omega^4 - \frac{17}{2}\Omega^2 s^2 + 3s^4)\\ &+4(120\Omega^4 s - 2\Omega^2 s^3 + s^5))) + (8\Omega^2 + (\Gamma+s)(\Gamma+2s))\\ &(1 + r + 4\alpha\sqrt{1-\alpha^2}r\cos\theta))\Big\},\end{aligned} \tag{26}$$

$$\tilde{\rho}_{ee}(t) = \mathcal{L}^{-1}\Big\{-\frac{2\Omega^2(-1+r)(6\Gamma^3 + 8\Gamma\Omega^2 + 13\Gamma^2 s + 12\Omega^2 s + 9\Gamma s^2 + 2s^3)}{k(s)}\Big\}, \tag{27}$$

$$\begin{aligned}\tilde{\rho}_{ff}(t) = \mathcal{L}^{-1}\Big\{&-\frac{1}{4k(s)}((-1+r)(6\Gamma^5 + 31\Gamma^4 s + 4\Gamma^3(38\Omega^2 + 15s^2)\\ &+\Gamma^2 s(412\Omega^2 + 55s^2) + 8\Gamma(40\Omega^4 + \frac{91}{2}\Omega^2 s^2 + 3s^4)\\ &+4(72\Omega^4 s + 26\Omega^2 s^3 + s^5)))\Big\},\end{aligned} \tag{28}$$

$$\tilde{\rho}_{+-}(t) = \mathcal{L}^{-1}\Big\{\frac{r(\Gamma+2s)(-1 + 2\alpha^2 + 2i\alpha\sqrt{1-\alpha^2}\sin\theta))}{2(4\Omega^2 + s(\Gamma+2s)}\Big\}, \tag{29}$$

$$\tilde{\rho}_{-+}(t) = \tilde{\rho}^*_{+-}(t), \tag{30}$$

$$\tilde{\rho}_{--}(t) = \frac{r(1 - 2\alpha\sqrt{1-\alpha^2}\cos\theta)}{2} + \frac{1-r}{4}, \tag{31}$$

$$\tilde{\rho}_{af}(t) = 0. \tag{32}$$

The solutions for both the EWL states are written as functions of $k(s)$, $j(s)$, $l(s)$ defined as

$$k(s) = (16\Gamma\Omega^2 + 2\Gamma^2 s + 24\Omega^2 s + 3\Gamma s^2 + s^3)$$
$$(3\Gamma^3 + 28\Gamma\Omega^2 + 11\Gamma^2 s + 24\Omega^2 s + 12\Gamma s^2 + 4s^3), \tag{33}$$

$$j(s) = (\Gamma + 2s)(8\Omega^2 + s(\Gamma + s)), \tag{34}$$

$$l(s) = (6\Gamma^3 + 31\Gamma^2 s + \Gamma(56\Omega^2 + 45s^2) + 20(3\Omega^2 s + s^3)). \tag{35}$$

Here we present the solutions for a EWL state of the form (21):

$$\tilde{\rho}_{aa}(t) = \mathcal{L}^{-1}\Big\{-\frac{1}{4s}(-1 + r - 4\alpha^2 r + \frac{8\Gamma\Omega^2(-1+r)}{j(s)} + \frac{64\Gamma^2\Omega^4(-1 + (-3 + 4\alpha^2)r)l(s)}{j(s)k(s)})\Big\}, \tag{36}$$

$$\tilde{\rho}_{bb}(t) = \mathcal{L}^{-1}\Big\{\frac{2\Omega^2}{j(s)}(1 - r - \frac{8\Gamma\Omega^2(-1 + (-3 + 4\alpha^2)r)l(s)}{k(s)})\Big\}, \tag{37}$$

$$\tilde{\rho}_{cc}(t) = \mathcal{L}^{-1}\Big\{-\frac{48\Omega^4(-1 + (-3 + 4\alpha^2)r)(\Gamma + s)}{k(s)}\Big\}, \tag{38}$$

$$\tilde{\rho}_{dd}(t) = \mathcal{L}^{-1}\Big\{-\frac{1}{4j(s)}\Big((8\Omega^2 + (\Gamma + s)(\Gamma + 2s))(-1 + r) + \frac{1}{k(s)}(8\Gamma\Omega^2(-1 + (-3 + 4\alpha^2)r)(6\Gamma^5 + 31\Gamma^4 s + 20\Gamma^3(-2\Omega^2 + 3s^2) + 5\Gamma^2 s(-20\Omega^2 + 11s^2) + 8\Gamma(56\Omega^4 - \frac{17}{2}\Omega^2 s^2 + 3s^4) + 4(120\Omega^4 s - 2\Omega^2 s^3 + s^5)))\Big)\Big\}, \tag{39}$$

$$\tilde{\rho}_{ee}(t) = \mathcal{L}^{-1}\Big\{-\frac{2\Omega^2}{k(s)}(-1 + (-3 + 4\alpha^2)r)(6\Gamma^3 + 8\Gamma\Omega^2 + 13\Gamma^2 s + 12\Omega^2 s + 9\Gamma s^2 + 2s^3)\Big\}, \tag{40}$$

$$\tilde{\rho}_{ff}(t) = \mathcal{L}^{-1}\Big\{-\frac{1}{4k(s)}((-1 + (-3 + 4\alpha^2)r)(6\Gamma^5 + 31\Gamma^4 s + 4\Gamma^3(38\Omega^2 + 15s^2) + \Gamma^2 s(412\Omega^2 + 55s^2) + 8\Gamma(40\Omega^4 + \frac{91}{2}\Omega^2 s^2 + 3s^4) + 4(72\Omega^4 s + 26\Omega^2 s^3 + s^5)))\Big\}, \tag{41}$$

$$\tilde{\rho}_{af}(t) = \mathcal{L}^{-1}\Big\{\frac{\alpha r\sqrt{1-\alpha^2}((8\Omega^2 + (\Gamma + s)(\Gamma + 2s))}{4\Gamma\Omega^2 + \Gamma^2 s + 12\Omega^2 + 3\Gamma s^2 + 2s^3}\Big\}, \tag{42}$$

$$\tilde{\rho}_{--}(t) = \tilde{\rho}_{+-}(t) = \tilde{\rho}_{-+}(t) = 0. \tag{43}$$

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