



HAL
open science

Above threshold ionization of atomic hydrogen in ns states with up to four excess photons

E Karule, A Gailitis

► **To cite this version:**

E Karule, A Gailitis. Above threshold ionization of atomic hydrogen in ns states with up to four excess photons. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 2010, 43 (6), pp.65601. 10.1088/0953-4075/43/6/065601 . hal-00569890

HAL Id: hal-00569890

<https://hal.science/hal-00569890>

Submitted on 25 Feb 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Above threshold ionization of atomic hydrogen in ns states with up to four excess photons

E. Karule[†] and A.Gailitis[‡] §

[†] Institute of Physics and Spectroscopy, University of Latvia, Raina blvd. 19, Riga, LV-1586 Latvia

[‡] Institute of Physics, University of Latvia, Salaspils-1, LV-2169 Latvia

E-mail: karule@latnet.lv

Abstract. In a high intensity laser field an atom can absorb more photons than the minimum necessary for ionization. It is known as above threshold ionization (ATI). Theoretically it is the most difficult case to handle as we have to consider transitions in continuum. To study ATI we use the perturbation theory and the Green's function formalism. We have derived the modified two-term Coulomb Green's function (CGF) Sturmian expansion. In each term explicit summation over all intermediate states is carried out. The transition amplitude may be obtained in a closed form. The generalized cross sections are evaluated for the photoionization of atomic hydrogen in ns states with up to four excess photons. Calculations are performed in the wide range of wavelengths for linear and circular polarization. In the cases for which data is available our results agree very well with previous ones.

PACS numbers: 32.80Fb, 32.80.Rm, 31.15.xp, 31.15.-p

Submitted to: *J. Phys. B: At. Mol. Opt. Phys.*

1. Introduction

The study of photoionization processes in intense laser fields is a subject of interest in atomic physics and astrophysics. In the case of ionization in strong laser fields the total number of absorbed photons N may exceed the minimum number of photons K necessary for ionization. Hence $N = K + S$, where S is the number of excess photons. $K = 1$ and $S = 0$ refers to the single photoionization but $S > 1$ refers to the situation when ATI takes place. We investigate ATI of atomic hydrogen by the circularly and linearly polarized radiation for $S = 1, 2, 3, 4$.

The first who calculated transition rates for ATI of H ($N = 2, S = 1$) were Zernik and Klopfenstein (1965). They solved the first-order non-homogeneous differential equation. Klarsfeld (1969) employed the integral representation of CGF to calculate the transition rates for ATI of H ($N = 2, S = 1$). In such a way it was not possible to calculate transition amplitudes in the whole energy region. Karule (1971) proposed to use CGF Sturmian expansion (Hostler 1970) to study multiphoton ionization (MPI) and calculated transition rates of MPI for H in $1s$ state for N up to 16 (Karule, 1975). In the case of ionization with excess photons all intermediate states belong to continuum. The continuum wave functions are oscillating, not exponentially decreasing as the discrete state wave functions. Therefore CGF Sturmian expansion, which converges for negative energies of intermediate states, for positive energies diverges. Karule (1978, 1985) adjusted the CGF Sturmian expansion for the calculation of ATI ($N = 2, S = 1$) of atomic hydrogen in nl ($1 \leq n \leq 9, 0 \leq l \leq n - 1$) states by the analytic continuation of transition amplitude. The same technique was applied also for the ATI of H in $1s$ state by Karule (1988) for $3 \leq N \leq 9, S = 1$. The two-photon ATI of H by circularly polarized radiation of H in ns states ($1 \leq n \leq 6$) is studied by (Klarsfeld and Maquet 1979, 1980), resuming the divergent series via Pade-approximants techniques. Aymar and Crance (1980) solved the system of nonhomogeneous differential equations and calculated transition rates for the photoionization with one excess photon for H in $1s$ state in the case of linear and circular polarization. Two-photon ATI of atomic hydrogen is investigated also in papers of Fainstein *et al* (1984), Broad (1984) and Gao and Starace (1988), Krylovetsky *et al* (2001). Shakeshaft (1986a, 1986b) calculated the generalized cross sections for the ionization of atomic hydrogen in the ground state by the circularly polarized radiation ($K = 1, S = 1, 2, 3$) solving the system of non-homogeneous differential equations numerically. Karule and Pratt (1991) derived the transformed CGF Sturmian expansion, which is similar to the modified CGF proposed in this paper and evaluated the transition rates for two-photon ATI of H in the ground state.

2. Transition amplitudes and the modified CGF Sturmian expansion

In the framework of perturbation theory the transition probability in a unity of time for $N > 1$ has nonlinear dependence on intensity and for an unimode laser reads

$$W^N = \hat{\sigma}^N I^N (\hbar\omega)^{-N}. \quad (1)$$

The generalized cross section by the definition is

$$\begin{aligned} \hat{\sigma}^N [cm^{2N} s^{N-1}] &= Q I^{1-N} (\hbar\omega)^{N-1} \\ &= Q I^{1-N} [13.605 \times 1.60219 \times 10^{-12} (\lambda R_H)^{-1}]^{N-1} \end{aligned} \quad (2)$$

where $Q I^{1-N}$ is the ionization rate in $W^{1-N} cm^{2N}$, Q is the total cross section for N photon ionization, I is the intensity of radiation in $W cm^{-2}$, λ is the wavelength in nm , $\lambda = 10^7 [2R_H \omega]^{-1}$, ω is the energy of a photon in a.u., $\hbar = h/2\pi$ is the Planck constant and R_H is the Rydberg constant. In our calculations we assign the Rydberg constant the value that gives the best agreement with the experimental observations for the hydrogen atom $R_H = 109\,677.58$.

The ionization rate for H in the $n_0 l_0$ state may be written as follows

$$Q I^{1-N} = 4\pi^2 \alpha a_0 \omega I_0^{1-N} \sum_l |T^{(N)}(n_0 l_0, El | \omega)|^2 \quad (3)$$

where $\alpha = 1/137.037$ is the fine structure constant, $a_0 = 5.2917 \times 10^{-9} cm$ is the Bohr radius, $I_0 = 14.038 \times 10^{16} W cm^{-2}$, $T^{(N)}$ is the transition amplitude for N photon ionization.

The calculation of the generalized cross section reduces to the evaluation of the transition amplitude. In the dipole approximation radial component of the transition amplitude for ATI of atomic hydrogen in $n_0 l_0$ states is given by

$$\begin{aligned} T_{rad}^{(N)} &= \int_0^\infty dr_N r_N^3 R_{El}(r_N) \prod_{j=1}^{N-1} \int_0^\infty dr_j r_j^3 \\ &\quad \times G_{L_j}(r_j, r_{j+1}; \Omega_j) R_{n_0 l_0}(r_1) \end{aligned} \quad (4)$$

where N is the number of photons, $R_{n_0 l_0}$, R_{El} are the atomic hydrogen wave functions of the initial and final state, G_{L_j} is the radial part of CGF, n_0 is the principal quantum number, l_0, L_j, l are the orbital moments, $E_0 = (-2n_0^2)^{-1}$, $\Omega_j = E_0 + j\omega = k_j^2/2 > 0$ and $E = E_0 + N\omega = \kappa^2/2$ are the energies of the initial, intermediate and the final state. For linearly polarized radiation the following selection rules are applicable for each photon step $L_{j+1} = L_j \pm 1$. In the case of the circularly polarized radiation $L_{j+1} = L_j + 1$. Since angular momentum is strictly positive L_j cannot be zero when $L_{j+1} = L_j - 1$.

The radial component of transition amplitude for the single photoionization ($N = 1$, $S = 0$) has the form $T_{rad}^{(1)} = \int_0^\infty dr r^3 R_{El}(r) R_{n_0 l_0}(r)$ (Bethe and Salpeter 1977).

The different representations of CGF and its application to the calculation of multiphoton processes are discussed by Faisal (1987) and in the review paper of Maquet *et al* (1998). One of the radial components of CGF, which is given by Hostler and Pratt

(1963), may be written as follows

$$G_L(r, r'; \Omega) = -\frac{p\Gamma(L+1-p)}{(2L+1)!rr'} \quad (5)$$

$$\times M_{p,L+1/2}(2r_{<}/p)W_{p,L+1/2}(2r'_{>}/p)$$

where $p = 1/\sqrt{-2\Omega} = -i/k$, $r_{<} = \min[r, r']$, $r_{>} = \max[r, r']$. $M_{p,L+1/2}$ and $W_{p,L+1/2}$ are the regular and irregular Whittaker functions (Erdelyi *et al* 1953). We have to use different expressions for G_L dependent on which of the radial variables is greater or lesser. Thus (5) is difficult to apply for $N > 2$. Most often is used the CGF Sturmian expansion (Hostler 1970)

$$G_L(r, r'; \Omega) = \sum_{n=L+1}^{\infty} \frac{S_{nL}(2r/p)S_{nL}(2r'/p)}{1-n/p} \quad \Omega < 0. \quad (6)$$

The radial components of the Sturmian functions read

$$S_{nL}(2r/p) = -(2/p)^{L+1}r^L[(n-L)_{2L+1}]^{-1/2}$$

$$\times \exp(-r/p)L_{n-L-1}^{2L+1}(2r/p)$$

where L_{n-L-1}^{2L+1} are Laguerre polynomials (Erdelyi *et al* 1953), $(n-L)_{2L+1}$ is a Pochhammer symbol $(a)_n = a(a+1)\dots(a+n-1)$, $a_0 = 1$.

Sturmian functions (Rotenberg 1962, 1970) are the charge eigenfunctions. The radial components of the Sturmian functions are solutions of the second order differential equation

$$\left[\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{L(L+1)}{r^2} + \frac{2Z}{r} - \Omega \right] S_{nL}(2r/p)$$

$$= -\frac{2Z_n}{r} S_{nL}(2r/p) \quad (7)$$

where Z is the charge, $Z_n = n/p$ are the charge eigenvalues, n is a positive integer.

The Sturmian functions provide a denumerable basis set that is complete with respect to square integrable functions (Rotenberg 1970). If $Z = 1$ and $p = n$ the Sturmian functions resemble the radial eigenfunctions of the bound states of atomic hydrogen by the factor n . For $\Omega < 0$ values of p are real and the Sturmian wave functions are exponentially decreasing. Thus convergence of CGF Sturmian expansion for $\Omega < 0$ is secured.

Sturmian functions for $\Omega > 0$ have complex exponentials, which are periodic. Hence Sturmian functions have an oscillatory behavior and CGF Sturmian expansion (6) for $\Omega > 0$ diverges. Thus we get nonconvergent transition amplitudes.

In present paper we develop the modified CGF Sturmian expansion to calculate multiphoton processes with excess photons $S \geq 1$. We divide the CGF Sturmian expansion in two terms. In each term summation over intermediate states is carried out explicitly (in Appendix A we provide a detailed description of transformation). Thereafter the modified CGF may be written as follows

$$G_L(r, r'; \Omega) = G_L^A(r, r'; \Omega) + G_L^B(r, r'; \Omega) \quad \Omega > 0 \quad (8)$$

where

$$G_L^A(r, r'; \Omega) = -\frac{p\Gamma(L+1-p)}{(2L+1)!rr'} M_{p,L+1/2}(2r/p) W_{p,L+1/2}(2r'/p). \quad (9a)$$

$$\begin{aligned} G_L^B(r, r'; \Omega) &= r^L r'^{-L-2} p(L+1-p)^{-1} \exp(-r/p) \exp(r'/p) \\ &\times \sum_{n=1}^{\infty} [(L+1-p)_n (2r/p)^n] / [n!(2L+2)_n] \\ &\times \sum_{m=1}^{n-1} [(2L+2)_m (-2r'/p)^{-m}] / [(L+2-p)_m \Gamma(-m)]. \end{aligned} \quad (9b)$$

The expression for G_L^A (9a) is similar to (5) but G_L^A has an advantage as r and r' have not to be interchanged dependent on which of the radial variables is greater or lesser. The first term of the modified CGF is complex and its imaginary part is equal to the imaginary part of the total CGF, the second term is real

$$\begin{aligned} \text{Im } G_L^A(r, r'; \Omega) &= \pi R_{\Omega L}(r) R_{\Omega L}(r') \\ [G_L^B(r, r'; \Omega)]^* &= G_L^B(r, r'; \Omega) \end{aligned} \quad (10)$$

where $R_{\Omega L}$ is the Coulomb wave function for an intermediate state, $[G_L^B]^*$ is the complex conjugate of G_L^B .

The modified CGF is not convergent for $r' = 0$, but using it we get the convergent transition amplitude in a closed form. The radial integrals, which involve G_L^A and the atomic wave functions, are easily executed (Appendix B).

The terms of transition amplitude which involve G_L^B and the atomic wave functions must be real. We decompose R_{El} into ingoing f_{kl} and outgoing f_{kl}^* waves. The convergence of the radial integrals, which involve R_{El} and G_L^B , is ensured replacing R_{El} by $2\text{Re } f_{kl}$. The factor $[\Gamma(-m)]^{-1}$ cancels integrating over the radial variables (more detailed description of the integration over radial variables is given in Appendix B).

The application of modified CGF Sturmian expansion (8) may be extended to investigate free-free transitions. The modified CGF Sturmian expansion proposed by Karule and Pratt (1991) also can be applied to study free-free transitions but in this case expressions for some integrals are more complicated.

In papers of Karule (1978, 1985, 1988) CGF Sturmian expansion (6) was used to evaluate the transition amplitude. The obtained transition amplitude was divided by the denominator $(L+1-p+j)$ and summation over intermediate states was carried out explicitly. Thus the series in terms of polynomials was changed to the series in terms of Gauss hypergeometric functions. The convergence of transition amplitude was secured by the transformation of Gauss hypergeometric functions (Erdelyi *et al* 1953). In such a way it is not possible to make the summation over intermediate states for more than a single excess photon. Hence the technique used in papers (Karule 1978, 1985, 1988) is not applicable to calculate ATI of H for $S > 1$. It turns out to be not valid to study ATI of H in Rydberg states ($n > 10$) and free-free transitions.

3. Results and conclusion

The modified CGF is used to calculate the generalized cross sections for the photoionization of atomic hydrogen in the ns states accompanied by the absorption of up to four excess photons ($S = 0, 1, 2, 3, 4$). We treat the cases in which the radiation field is linearly and circularly polarized. The single photoionization threshold is $\lambda/n^2 = 91.1nm$. We have carried out calculations in the wide range of wavelengths $10nm \leq \lambda/n^2 \leq 90nm$.

For linear polarization the following selection rules are applicable for each photon step: $L_{j+1} = L_j \pm 1$. Hence for two-photon ionization of atomic hydrogen in ns states by linearly polarized radiation we have two different final angular momenta $l = 0, 2$ and two possible tracks, for $N = 3$ $l = 1, 3$ we have three possible tracks, for $N = 4$, $l = 0, 2, 4$ six possible tracks, for $N = 5$, $l = 1, 3, 5$ nine possible tracks. For circular polarization $L_{j+1} = L_j + 1$. Thus for the ionization of H in ns state we have only one track with the final angular momentum $l = N$.

The total generalized cross section for the ionization by circularly polarized radiation is given by

$$\hat{\sigma}_c^N = C_N \sum_{\tau} \hat{\sigma}_{\tau l}^N, \quad \hat{\sigma}_l^N = \sum_{\tau} \hat{\sigma}_{\tau l}^N \quad (11)$$

where $\hat{\sigma}_l^N$ is the total generalized cross section for the linear polarization. $\hat{\sigma}_{\tau l}^N$ is the cross section for a track τ . Summation runs over all possible tracks. $C_N = (2N - 1)!!/N!$.

Hence C_N is the maximum theoretically possible value for the ratio $\hat{\sigma}_c^N/\hat{\sigma}_l^N$, which may be reached if only one track with $L_{j+1} = L_j + 1$ (the single track for the circularly polarized radiation) contributes significantly to the total cross section for the linear polarization. C_N increases with N . $[\hat{\sigma}_c^2/\hat{\sigma}_l^2]_{max} = 3/2$, $[\hat{\sigma}_c^3/\hat{\sigma}_l^3]_{max} = 5/2$, $[\hat{\sigma}_c^4/\hat{\sigma}_l^4]_{max} = 4.25$, $[\hat{\sigma}_c^5/\hat{\sigma}_l^5]_{max} = 7.875$. Calculated by us ratio $\hat{\sigma}_c^N/\hat{\sigma}_l^N$ for the ATI of H in $1s, 2s$ states increases with wavelength but never reaches its maximum value (Figures 1,2). $\hat{\sigma}_c^2/\hat{\sigma}_l^2 \Rightarrow 3/2$ holds for the ATI of H in high Rydberg states with $l_0 = n_0 - 1$ (will be published).

$\hat{\sigma}_c^3/\hat{\sigma}_l^3 > \hat{\sigma}_c^4/\hat{\sigma}_l^4 > \hat{\sigma}_c^5/\hat{\sigma}_l^5$ for the ionization of H in $1s$ and $2s$ states (Figures 1,2) in the whole range of λ/n^2 . It may be explained by the rapidly growing number of tracks with growing N in the case of linear polarization. All tracks contribute to the total cross section. $\hat{\sigma}_c^2/\hat{\sigma}_l^2 > \hat{\sigma}_c^3/\hat{\sigma}_l^3$ holds in the whole region of λ in the case of ionization from $1s$ state (Figure 1). For the ATI of H in $2s$ state $\hat{\sigma}_c^3/\hat{\sigma}_l^3 > \hat{\sigma}_c^2/\hat{\sigma}_l^2$ holds in the region $\lambda/n^2 \geq 80nm$ (Figure 2). In Figure 3 we show $\hat{\sigma}_c^N/\hat{\sigma}_l^N$ versus principal quantum number for $\lambda/n^2 = 80nm$. These curves have maxima for $N = 2, S = 1$ at $n = 2$, for $N = 3, S = 2$ at $n = 4$, for $N = 4, S = 3$ at $n = 6$ and for $N = 5, S = 4$ at $n = 8$. The position of maxima is only slightly dependent on the wavelength. At $60nm$ we get maxima for $S = 1, 2, 3, 4$ correspondingly at $n = 3, 5, 8, 10$. As the principal quantum number increases beyond 10 the ratio $\hat{\sigma}_c^N/\hat{\sigma}_l^N$ decreases for all N .

In the case of linear polarization the number of possible tracks for angular momenta increases rapidly with each photon step. Therefore the time spent to calculate cross

sections is growing very rapidly with N . Programming is done in Fortran language and all computations are performed on desktop PC. To test the accuracy of results calculations are performed also using a complex conjugate of CGF ($[G_L^A]^*$, $[G_L^B]^*$) and $f_{\kappa l}^*$. Smoothness of the curves for $\hat{\sigma}_c^N/\hat{\sigma}_l^N$ also serves to test the accuracy of results.

In the cases for which data is available results are compared with previous ones. In the cases of linear and circular polarization $\hat{\sigma}_l$, $\hat{\sigma}_c$ are compared with data of Karule (1978) for $N = 2$, $S = 1$ and Klarsfeld and Maquet (1980) for $N = 2$, $S = 1$ and $N = 3$, $S = 2$ (Tables 1,2). In Table 2 the calculated generalized cross sections $\hat{\sigma}_c$ for $N = 3$, $S = 2$ and $N = 4$, $S = 3$ are given together with those calculated by Shakeshaft (1986a,b) for the circularly polarized radiation. Results of present calculations are in a good agreement with results of previous calculations.

Perturbation theory may be used when the intensity of the radiation field is moderate (up to $10^{11} - 10^{13} Wcm^{-2}$). The applicability of perturbation theory to the description of atomic systems in intense laser fields (intensity $I > 10^{13} Wcm^2$) is a subject of continuing concern. Calculated by us AC Stark shift of the ground state of atomic hydrogen using the nonperturbative Floquet method (Karule and Gailitis, 2004) shows a good agreement with calculations carried out in dipole approximation up to an intensity of $10^{15} Wcm^{-2}$. In the case of photoionization we have no resonances thus our calculations of generalized cross sections for ATI of atomic hydrogen in the ground state also may be valid up to an intensity $I = 10^{15} Wcm^{-2}$.

In Figure 4 we present the generalized cross sections for the photoionization $N = 1 + S$ ($S = 0, 1, 2, 3, 4$) of atomic hydrogen in ns states ($n = 1, 5, 15, 30$) by linearly polarized radiation dependent on the energy of ejected electron $E = -(2n^2)^{-1} + N\omega$. The generalized cross sections for linearly polarized radiation are decreasing linearly dependent on the energy of ejected electrons. In the case of ionization of H_2 by linearly polarized radiation experimental measurements of electron yield dependent on the energy of electrons up to 20eV have similar behavior except the region of low energies (Blaga *et al* , 2009).

Appendix A. The transformation of CGF Sturmian expansion

In this section we derive the modified CGF Sturmian expansion which can be used for $\Omega > 0$. We write the CGF (6) expansion over Laguerre polynomials and start with the power series expansion for one of Laguerre polynomials

$$\begin{aligned}
G_L &= -[2^{2L+2} r^L r'^L \exp(r/p)]/[p^{2L+1} (2L+1)!] \\
&\times \exp(-r'/p) \sum_{j=0}^{\infty} L_j^{2L+1}(2r'/p)(L+1-p+j)^{-1} \\
&\times \sum_{n=0}^{\infty} [(2L+2+j)_n (-2r/p)^n]/[(2L+2)_n n!].
\end{aligned} \tag{A.1}$$

Dividing $(2L+2+j)_n$ by the denominator $(L+1-p+j)$ yields

$$\begin{aligned}
(2L+2+j)_n (L+1-p+j)^{-1} &= (L+1+p)_n [(L+1-p+j)^{-1} \\
&+ \sum_{m=1}^{n-1} (2L+2+j)_m / (p+L+1)_{m+1}].
\end{aligned} \tag{A.2}$$

Let us use the equality

$$(2L+2)_j (2L+2+j)_m = (2L+2)_m (2L+2+m)_j \tag{A.3}$$

Substituting expressions (A.2) and (A.3) into equation (A.1) we arrive at two-term CGF.

In both terms we have the infinite summation over the intermediate states j

$$\begin{aligned}
G_L &= -[2^{2L+2} r^L r'^L \exp(-r'/p)]/[p^{2L+1} (2L+1)!] \\
&\times \{ \exp(-r/p) F(L+1-p, 2L+2; 2r/p) \\
&\times \sum_{j=0}^{\infty} L_j^{2L+1}(2r'/p)/(L+1-p+j) \\
&+ \exp(r/p) \sum_{n=1}^{\infty} [(L+1+p)_n (-2r/p)^n]/[(2L+2)_n n!] \\
&\times \sum_{m=1}^{n-1} [(2L+2)_m / (L+1+p)_{m+1}] \\
&\times \sum_{j=0}^{\infty} [(2L+2+m)_j L_j^{2L+1}(2r'/p)]/[(2L+2)_j] \}.
\end{aligned} \tag{A.4}$$

The summation over all intermediate states in the first term of (A.4) (Erdelyi *et al* 1953) yields

$$\sum_{j=0}^{\infty} L_j^{2L+1}(2r/p)/(L+1-p+j) = \Gamma(L+1-p) \Psi(L+1-p, 2L+2; 2r/p). \tag{A.5}$$

Thus we obtain the first term of the modified CGF (9a). In the second term of CGF (A.4) summation over intermediate states is carried out as follows (Erdelyi *et al* 1953)

$$\sum_{j=0}^{\infty} [(2L+2+m)_j L_j^{2L+1}(2r/p)/\Gamma(2L+2+j)]$$

$$= (2r/p)^{-2L-2-m} / \Gamma(-m). \quad (\text{A.6})$$

We get the second term in the form

$$\begin{aligned} [G_L^B(r, r'; \Omega)]^* &= -r^L r'^{-L-2} p (L+1+p)^{-1} \exp(r/p) \exp(-r'/p) \\ &\times \sum_{n=1}^{\infty} [(L+1+p)_n (-2r/p)^n] / [n! (2L+2)_n] \quad \Omega > 0 \\ &\times \sum_{m=1}^{n-1} [(2L+2)_m (p/2r')^{-m}] / [(L+2+p)_m \Gamma(-m)]. \end{aligned} \quad (\text{A.7})$$

$[G_L^B]^*$ (A.7) and G_L^B (9b) are the complex conjugates of one another but both of them are real. To get the convergent expressions for transition amplitudes we have to use G_L^B (9b) if as the first term of CGF is used G_L^A (9a).

In the paper Karule and Pratt (1991) to carry out summation over intermediate states we introduced a variable X in the second term of CGF. Hence the summation over all intermediate states (Erdelyi *et al* 1953) yields

$$\begin{aligned} &\sum_{j=0}^{\infty} (2L+2+m)_j L_j^{2L+1} (2r/p) X^j / (2L+2)_j \\ &= (1-X)^{-2L-2-m} F(2L+2+m, 2L+2; 2r(1-X^{-1})/p) \\ &\quad \lim X \Rightarrow 1. \end{aligned} \quad (\text{A.8})$$

In this case the second term of CGF has a hypergeometric function. We can go to the limit $X \Rightarrow 1$ after the integration over the radial variables in transition amplitude. Both modified CGF Sturmian expansions may be used to study ATI of H and the application may be extended to study free-free transitions. The second term of the modified CGF in a form (9b) is simpler. Hence it is better suited for computation since transition amplitudes can be calculated more easily and rapidly.

Appendix B. The evaluation of the transition rate for $N=2$, $S=1$

The modified CGF has two terms. Hence in the case two-photon ionization ($N=2$) radial component of the transition amplitude also has two terms

$$T_{rad}(n_0 l_0, El; \Omega) = T^A(n_0 l_0, El; \Omega) + T^B(n_0 l_0, El; \Omega) \quad (\text{B.1})$$

where $l_0 = 0$, $L = 1$ and $l = 0, 2$.

In both terms the integration runs over two radial variables. To ensure the convergence of T^B we decompose $R_{El}(r)$ into ingoing $f_{\kappa l}$ and outgoing $f_{\kappa l}^*$ waves

$$R_{El} = f_{\kappa l} + f_{\kappa l}^* \quad \kappa = \sqrt{2E}. \quad (\text{B.2})$$

The ingoing wave has the form

$$\begin{aligned} f_{\kappa l} &= C_{El} (-2r)^{l+1} \exp(-i\kappa r - \pi/\kappa) / \Gamma(l+1 - i/\kappa) \\ &\times \Psi(l+1 + i/\kappa, 2l+2; 2i\kappa r) \end{aligned} \quad (\text{B.3})$$

$$C_{El} = [1 - \exp(-2\pi/\kappa)]^{-1} \prod_{s=1}^l (1 + \kappa^2 s^2).$$

The wave functions of atomic hydrogen and the term G_L^B (9b) of the modified CGF are real. Hence T^B also is real and may be written as follows

$$T^B = 2\text{Re} \int_0^\infty dr r^3 f_{\kappa l}(r) \int_0^\infty dr' r'^3 G_L^B(r, r') R_{n_0 l_0}(r'). \quad (\text{B.4})$$

The integral over the radial variable r' is easily to execute and it ensures the cancelation of $\Gamma(-m)$

$$\begin{aligned} I_1 &= [\Gamma(-m)]^{-1} \int_0^\infty dr \exp(ik - 1/n_0) r^{l-L+1-m} \\ &\times F(l_0 + 1 - n_0, 2l_0 + 2; 2r/n_0) \\ &= (-m)_{l_0-L+2} (1/n_0 - ik)^{L-l_0+m-2} \\ &\times F(l_0 + 1 - n_0, l_0 - L + 2 - m; 2l_0 + 2; 1 + x_1) \end{aligned} \quad (\text{B.5})$$

where $x_1 = (1/n_0 - ik)/(1/n_0 + ik)$.

The other integral has the form

$$\begin{aligned} I_2 &= \int_0^\infty dr \exp(-ik - ik) r^{L+l+n+3} \\ &\times \psi(l + 1 + i/k, 2l + 2; 2ikr). \end{aligned} \quad (\text{B.6})$$

To carry out this integral we use a formula given by Erdelyi *et al* (1953).

The first part of transition amplitude may be written as follows

$$\begin{aligned} T^A &= -\Gamma(L + 1 - p) p^{-1} \int_0^\infty dr r^2 R_{El}(r) M_{p, L+1/2}(r) \\ &\times \int_0^\infty dr' r'^2 W_{p, L+1/2}(r') R_{n_0 l_0}(r'). \end{aligned} \quad (\text{B.7})$$

The integral over the radial variable r involves two confluent hypergeometric functions

$$\begin{aligned} I_3 &= \int_0^\infty dr \exp(-ikr - ikr) r^{L+l+3} \\ &\times F(a, 2L + 2; 2ikr) F(b, 2l + 2; 2ikr) \end{aligned} \quad (\text{B.8})$$

where $\kappa = \sqrt{2E}$, $k = \sqrt{2\Omega}$, $a = L + 1 - p$, $b = l + 1 - q$. Gordon (1929) has given a closed form for such integrals, which differ for $l = L + 1$ and $l = L - 1$ (Landau and Lifshitz 1965). We integrate (B.8) using the expression derived by Karule (1990), which may be applied for $l = L \pm 1$

$$\begin{aligned} I_3 &= \Gamma(L + l + 4) / [i(k + \kappa)]^{L+l+4} x_2^a (-x_2)^{-b} \\ &\times \sum_{m=0}^{L-l+2} [(l - L - 2)_m (b)_m (1 + x_2^{-1})^m] / [m!(2l + 2)_m] \\ &\times F_1(a, L - l - 2 - m, L + 3 + q; 2L + 2; 1 - x_2^{-1}, z) \end{aligned} \quad (\text{B.9})$$

where $x_2 = (\kappa - k)/(\kappa + k)$, $|x_2| < 1$, $z = 1 - x_2^{-2} > 1$. F_1 is Appell hypergeometric function (Erdelyi *et al* 1953) of two variables. Appell function by definition is the series

in terms of Gauss hypergeometric functions

$$F_1(\alpha, \beta', \beta, \gamma; x, y) = \sum_{n=0}^{\infty} \frac{\alpha_n \beta'_n x^n}{n! \gamma_n} F(\alpha + n, \beta, \gamma + n; y). \quad (\text{B.10})$$

Appell function in the equation (B.9) has an argument $|z| > 1$ and therefore diverge. Transforming the Appell hypergeometric function (Erdelyi *et al* 1953) we get the convergent expression for the integral I_3

$$F_1(a, L - l - 2 - m, L + 3 + q; 2L + 2; 1 - x_2^{-1}, z) \quad (\text{B.11})$$

$$= x_2^{2a} F_1(a, L - l - 2 - m, b + m; 2L + 2; 1 - x_2, z_2)$$

where $z_2 = 1 - x_2^2$.

The integral I_3 was carried out also in another way

$$I_3 = 2C \operatorname{Re} \int_0^{\infty} dr \exp(-ikr - i\kappa r) r^{L+l+3} \quad (\text{B.12})$$

$$\times F(a, 2L + 2; 2ikr) \Psi(b, 2l + 2; 2i\kappa r)$$

where $C = (2l + 1)!(-1)^{l+1} \exp(-\pi/\kappa) [\Gamma(l + 1 + q)]^{-1}$.

The second integral of T_L^A involves two different confluent hypergeometric functions (Erdelyi *et al* 1953)

$$I_4 = \int_0^{\infty} dr \exp(-r/n_0 - r/p) r^{L+l_0+3} \quad (\text{B.13})$$

$$\times F(a_0, 2l_0 + 2, 2r/n_0) \Psi(a, 2L + 2; 2r/p)$$

where $a_0 = l_0 + 1 - n_0$ is a negative integer. One of the confluent hypergeometric functions is a polynomial, which we expand in the power series. Thus the integral simplifies. In such a way, to carry out this integral, we may follow up the same course as for integral I_2 (Erdelyi *et al* 1953). For ATI with $N > 2$ and in the case of free-free transitions there are integrals

$$I_5 = \int_0^{\infty} dr \exp(-r/p_j - r/p_{j+1}) r^{L_j+L_{j+1}+3} \quad (\text{B.14})$$

$$\times F(a_j, 2L_j + 2, 2r/p_j) \Psi(a_{j+1}, 2L_{j+1} + 2; 2r/p_{j+1})$$

where $a_j = L_j + 1 - p_j$ and $a_{j+1} = L_{j+1} + 1 - p_{j+1}$.

The integrals I_3 (B.12) and I_5 (B.14) may be carried out in a closed form as the series in terms of convergent Gauss functions.

Only infinite summation makes evaluation of the term with G_L^B more complicated.

Acknowledgment

This work was supported by the Latvian Council of Sciences under the grant No 09.1205 and we are especially grateful to the Taiwan National Science Council for support under Taiwan-Baltic project (1-23/172).

References

- Aymar M and Crance M 1980 *J. Phys.B: At. Mol. Phys.* **13** L287
- Bethe H A and Salpeter E E 1977 *Quantum mechanics of One and Two-Electron Atoms* (N.Y.: Plenum Press)
- Blaga C I, Catoire F, Colosimo P, Paulus G G, Muller H G, Agostini A, DiMauro L F 2009 *Nature Physics* **5** 335
- Broad J T 1984 *Phys. Rev. A* **31** 1494
- Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions vols 1,2* (McGraw-Hill, New York)
- Fainstein A G, Manakov N L and Marmo S I 1984 *Phys. Lett. A* **104** 347
- Faisal F H M 1987 *Theory of Multiphoton Processes* (New York: Plenum Press)
- Gao Bo and Starace A F 1988 *Phys. Rev. Lett.* **61** 404
- Gordon W 1929 *Ann. Phys. Leipzig* **2** 1031
- Hostler L, and Pratt R H 1963 *Phys. Rev. Lett.* **10** 469
- Hostler L 1970 *J. Math. Lett.* **11** 2966
- Karule E 1971 *J. Phys B: At. Mol. Phys.* **4** L67
- Karule E 1975 *Atomnīje procesi*, (Riga, Zinatne) 5
- Karule E 1978 *J. Phys.B: At. Mol. Phys.* **11** 441
- Karule E 1985 *J. Phys.B: At. Mol. Phys.* **18** 2207
- Karule E 1988 *J.Phys.B: At. Mol. Phys.* **21** 1997
- Karule E 1990 *J.Phys.A: Math. Gen.* **23** 1969
- Karule E and Pratt R 1991 *J.Phys.B: At. Mol. Opt. Phys.* **24** 1585
- Karule E and Gailitis A 2004 *Appl. Phys. B* **78** 817
- Klarsfeld S 1969 *Lett. Nuovo Cimento* **2** 548
- Klarsfeld S and Maquet A 1979 *J.Phys. B: At. Mol. Phys.* **12** L553
- Klarsfeld S and Maquet A 1980 *Phys. Lett. A* **78** 40
- Krylovetsky A A, Manakov N L and Marmo S I 2001 *Zh. Eksp. Teor. Phys.* **119** 45 [2001 *Sov. Phys.-JETP* **92** 37]
- Landau L D and Lifshitz E M 1965 *Quantum Mechanics* 2nd ed. (N.Y. Pergamon Press)
- Maquet A, Veniard V and Marian T A 1998 *J.Phys.B: At. Mol.Opt. Phys.* **31** 3743
- Monakov N L, Maquet A, Marmo S I, Veniard V and Ferrante G 1999 *J. Phys. B: At. Mol. Phys.* **32** 3747
- Rotenberg M 1962 *Ann. Phys. NY* **11** 262
- Rotenberg M 1970 *Adv. Atom. Mol. Phys.* **6** 233
- Shakeshaft R 1986a *Phys. Rev. A* **34** 244
- Shakeshaft R 1986b *Phys. Rev. A* **34** 5119
- Zernik W and Klopfenstein R W 1965 *J. Math. Phys.* **6** 262

Tables and table captions

Table 1. The generalized cross sections $\hat{\sigma}_{1s}(cm^{2N} s^{N-1})$ for the ionization of atomic hydrogen in the state $1s$ with excess photons $S = 1, 2, 3, 4$ by the linearly polarized radiation. The total number of photons is $N = S + 1$. a) Karule(1978), b) Klarsfeld and Maquet(1980)

$\lambda(nm)$	$S = 1$	$S = 2$	$S = 3$	$S = 4$
10	8.12-57 8.12-57(a)	7.73-93	7.21-129	3.72-163
20	2.99-55 2.99-55(a)	1.15-90 1.14-90(b)	4.28-126	4.21-160
30	2.43-54 2.43-54(a)	2.10-89	1.73-124	2.41-158
40	1.07-53 1.07-53(a)	1.64-88 1.63-88(b)	2.36-123	4.12-157
50	3.37-53 3.37-53(a)	8.00-88	1.76-122	3.68-156
60	8.07-53 8.07-53(a)	2.92-87 2.90-87(b)	9.09-122	2.21-155
70	1.94-52 1.94-52(a)	8.77-87	3.63-121	1.03-154
80	3.92-52 3.92-52(a)	2.28-86 2.27-86(b)	1.20-120	4.07-154
90	7.35-52 7.35-52(a)	5.34-86	3.48-120	1.44-153

Table 2. Generalized cross sections $\hat{\sigma}_{1s}$ ($cm^{2N} s^{N-1}$) for the ionization of atomic hydrogen in the state $1s$ with excess photons $S = 1, 2, 3, 4$ by the circularly polarized radiation. The total number of photons is $N = S + 1$.
a) Karule(1978), b) Klarsfeld and Maquet(1980),
c) Shakeshaft(1986a,b).

$\lambda(nm)$	$S = 1$	$S = 2$	$S = 3$	$S = 4$
10	6.00-57 6.00-57(a)	3.65-93	2.04-129	4.95-167
20	2.42-55 2.42-55(a)	6.33-91 6.29-91(b)	1.47-126	2.23-163
30	3.22-54 3.22-54(a)	1.33-89	7.15-125	2.99-161
40	1.01-53 1.01-53(a) 1.0 -53(c)	1.17-88 1.16-88(b) 1.3 -88(c)	1.15-123	9.87-160
50	3.39-53 3.39-53(a) 3.4 -53(c)	6.44-88 6.4 -88(c)	1.01-122 1.1 -122(c)	1.55-158
60	9.22-53 9.22-53(a) 9.2 -53(c)	2.63-87 2.62-87(b) 2.6 -87(c)	6.09-122 6.2 -122(c)	1.56-157
70	2.17-52 2.17-52(a) 2.2 -52(c)	8.74-87 8.7 -87(c)	2.81-121 2.9 -121(c)	1.16-156
80	4.58-52 4.58-52(a) 4.6 -52(c)	2.51-86 2.62-86(b) 2.5 -86(c)	1.07-120 1.1 -120(c)	6.95-156
90	8.95-52 8.95-52(a)	6.42-86	3.53-120	3.50-155

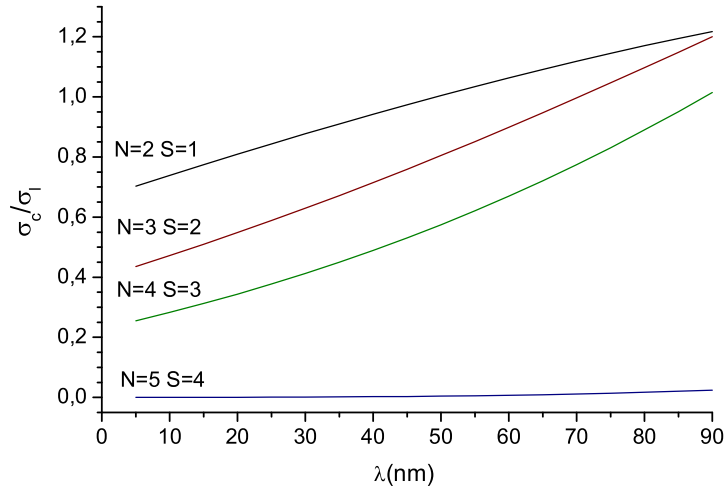


Figure 1. $\hat{\sigma}_c^N / \hat{\sigma}_l^N$ as a function of the wavelength for the atomic hydrogen in $1s$ state. N is the total number of photons, S is the number of excess photons.

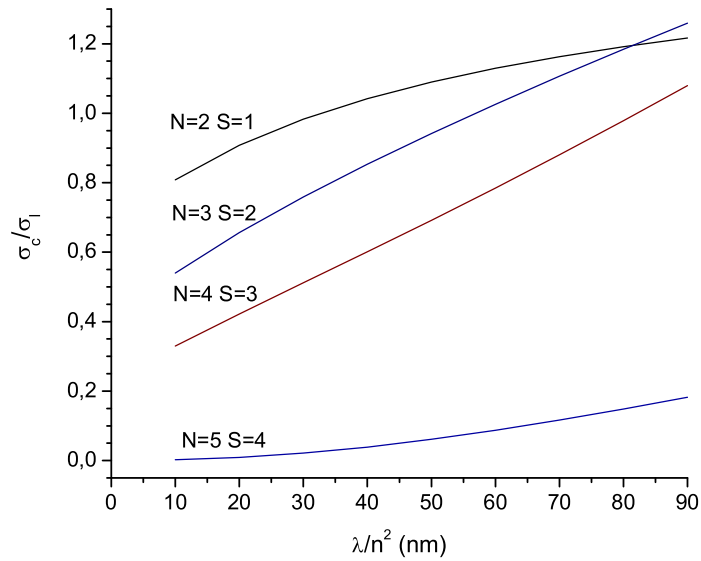


Figure 2. $\hat{\sigma}_c^N / \hat{\sigma}_l^N$ as a function of λ/n^2 ($n = 2$) for the atomic hydrogen in $2s$ state. N is the total number of photons, S is the number of excess photons.

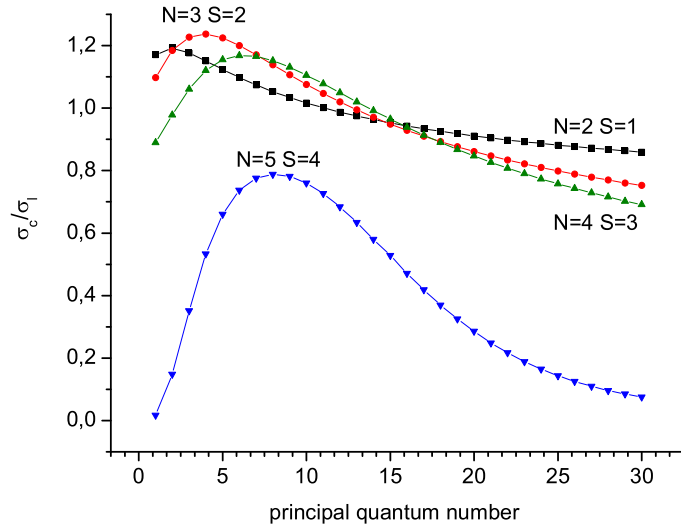


Figure 3. $\hat{\sigma}_c^N / \hat{\sigma}_l^N$ for the atomic hydrogen in ns states as a function of the principal quantum number for $\lambda/n^2 = 80nm$. N is the total number of photons, S is the number of excess photons.

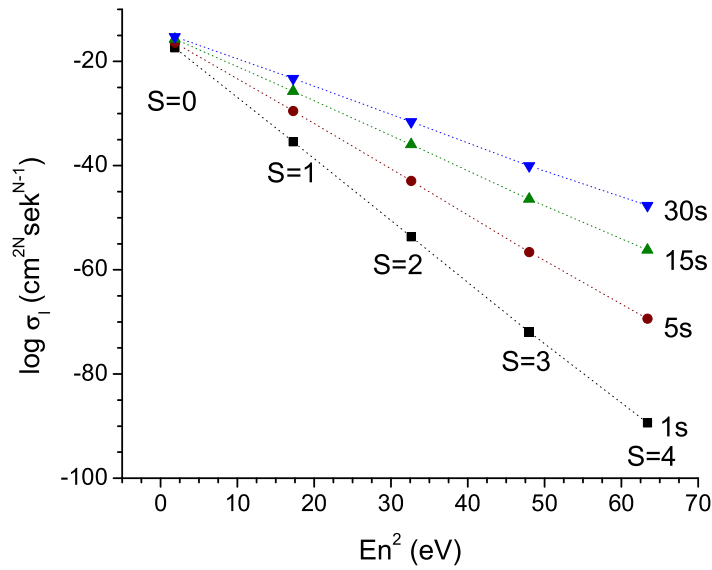


Figure 4. $\log \hat{\sigma}_l^N$ for the ionization of atomic hydrogen in $1s$, $5s$, $15s$ and $25s$ states by linearly polarized radiation as a function of the energy of ejected electron multiplied by n^2 for $\lambda/n^2 = 80nm$. S is the number of excess photons.