PARAMETER ESTIMATION FOR FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES: NON-ERGODIC CASE

Rachid Belfadli, Khalifa Es-Sebaiy, Youssef Ouknine

PARAMETER ESTIMATION FOR FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES: NON-ERGODIC CASE

Rachid Belfadli[1] Khalifa Es-Sebaiy[2] Youssef Ouknine[3]

[1] Polydisciplinary Faculty of Taroudant, University Ibn Zohr, Taroudant, Morocco.
belfadli@gmail.com

[2] Institut de Mathématiques de Bourgogne, Université de Bourgogne, Dijon, France.
khalifasbai@gmail.com

[3] Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University,
2390 Marrakesh, Morocco.
ouknine@ucam.ac.ma

February 25, 2011

Abstract

We consider the parameter estimation problem for the non-ergodic fractional Ornstein-Uhlenbeck process defined as $dX_t = \theta X_t dt + dB_t,\ t \geq 0$, with a parameter $\theta > 0$, where B is a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$. We study the consistency and the asymptotic distributions of the least squares estimator $\widehat{\theta}_t$ of θ based on the observation $\{X_s,\ s \in [0,t]\}$ as $t \to \infty$.

Key words and phrases: Parameter estimation, Non-ergodic fractional Ornstein-Uhlenbeck process, Young integral.

2000 Mathematics Subject Classification: 62F12, 60G18, 60G15.

1 Introduction

We consider the Ornstein-Uhlenbeck process $X = \{X_t, t \geq 0\}$ given by the following linear stochastic differential equation

$$X_0 = 0; \quad dX_t = \theta X_t dt + dB_t, \quad t \geq 0, \tag{1}$$

where B is a fractional Brownian motion of Hurst index $H > \frac{1}{2}$ and $\theta \in (-\infty, \infty)$ is an unknown parameter. An interesting problem is to estimate the parameter θ when one observes the whole trajectory of X. First, let us recall some results in the case when B is a standard Brownian motion. In this special case, the parameter estimation for θ has been well studied by using the classical maximum likelihood method or by using the trajectory fitting method. If $\theta < 0$ (ergodic case), the maximum likelihood estimator (MLE) of θ is asymptotically normal (see Liptser and Shiryaev [9], Kutoyants [8]). If $\theta > 0$ (non-ergodic case), the MLE of θ is asymptotically Cauchy (see Basawa and Scott [3], Dietz and Kutoyants [4]). Recently, in a more general context, several authors extended this study to some generalizations of Ornstein-Uhlenbeck process driven by Brownian motion (for instance, Barczy and Pap [2]). Similar properties of the asymptotic behaviour of MLE has also been obtained with respect to the trajectory fitting estimators (see Dietz and Kutoyants [4]).

When B is replaced by an α-stable Lévy motion in the equation (1), Hu and Long [6] discussed the parameter estimation of θ in both the ergodic and the non-ergodic cases. They used the trajectory fitting method combined with the weighted least squares technique.

Now, let us consider a parameter estimation problem of the parameter θ for the fractional Ornstein-Uhlenbeck process X of (1).

In the case $\theta < 0$ (corresponding to the ergodic case), Hu and Nualart [7] studied the parameter estimation for θ by using the least squares estimator (LSE) defined as

$$\widehat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \quad t \geq 0. \tag{2}$$

This LSE is obtained by the least squares technique, that is, $\widehat{\theta}_t$ (formally) minimizes

$$\theta \longmapsto \int_0^t \left| \dot{X}_s + \theta X_s \right|^2 ds.$$

To obtain the consistency of the LSE $\widehat{\theta}_t$, the authors of [7] are forced to consider $\int_0^t X_s dX_s$ as a Skorohod integral rather than Young integral in the definition (2). Assuming $\int_0^t X_s dX_s$ is a Skorohod integral and $\theta < 0$, they proved the strong consistence of $\widehat{\theta}_t$ if $H \geq \frac{1}{2}$, and that the LSE $\widehat{\theta}_t$ of θ is asymptotically normal if $H \in [\frac{1}{2}, \frac{3}{4})$. Their proof of the central limit theorem is based on the fourth moment theorem of Nualart and Peccati [12].

In this paper, our purpose is to study the non-ergodic case corresponding to $\theta > 0$. More precisely, we shall estimate θ by the LSE $\widehat{\theta}_t$ defined in (2), where in our case, the integral $\int_0^t X_s dX_s$ is interpreted as a Young integral. Indeed in that case, we have $\widehat{\theta}_t = \frac{X_t^2}{2\int_0^t X_s^2 ds}$ which converges almost surely to θ, as t tends to infinity (see Theorem 1). Moreover, it turned out that the path-wise approach is the preferred way to simulate numerically an estimator $\widehat{\theta}_t$. Our technics used in this work are inspired from the recent paper by Es-Sebaiy and Nourdin [5].

The organization of our paper is as follows. Section 2 contains the presentation of the basic tools that we will need throughout the paper: fractional Brownian motion, Malliavin derivative, Skorohod integral, Young integral and the link between Young and Skorohod integrals. The aim of Section 3 is twofold. Firstly, we prove when $H > \frac{1}{2}$ the strong consistence of the LSE $\widehat{\theta}_t$, that is, $\widehat{\theta}_t$ converges almost surely to θ, as t goes to infinity. Secondly, we investigate the asymptotic distribution of our estimator $\widehat{\theta}_t$ in the case $H > \frac{1}{2}$. We obtain that (see Theorem 5)

$$e^{\theta t}\left(\widehat{\theta}_t - \theta\right) \xrightarrow{\texttt{law}} 2\theta\mathcal{C}(1) \quad \text{as } t \longrightarrow \infty,$$

with $\mathcal{C}(1)$ the standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$; $x \in \mathbb{R}$.

2 Preliminaries

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For more complete presentation on the subject, see [11], [1] and [10].
The fractional Brownian motion $(B_t, t \geq 0)$ with Hurst parameter $H \in (0,1)$, is defined as a centered Gaussian process starting from zero with covariance

$$R_H(t,s) = E(B_t B_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We assume that B is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}$ is the sigma-field generated by B. By Kolmogorov's continuity criterion and the fact

$$E\left(B_t - B_s\right)^2 = |s-t|^{2H};\ s,\ t \geq\ 0,$$

we deduce that B has Hölder continuous paths of order $H-\varepsilon$, for all $\varepsilon \in (0,H)$.

Fix a time interval $[0,T]$. We denote by $\mathcal{H}$ the canonical Hilbert space associated to the fractional Brownian motion B. That is, $\mathcal{H}$ is the closure of the linear span $\mathcal{E}$ generated by the indicator functions $1_{[0,t]},\ t \in [0,T]$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle = R_H(t,s).$$

The application $\varphi \in \mathcal{E} \longrightarrow B(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by B and it can be extended to $\mathcal{H}$.
If $H > \frac{1}{2}$ the elements of $\mathcal{H}$ may be not functions but distributions of negative order (see [13]). Therefore, it is of interest to know significant subspaces of functions contained in it.
Let $|\mathcal{H}|$ be the set of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathcal{H}|} := H(2H-1)\int_0^T \int_0^T |\varphi(u)||\varphi(v)||u-v|^{2H-2} du dv < \infty.$$

Note that, if $\varphi,\ \psi \in |\mathcal{H}|$,

$$E(B(\varphi)B(\psi)) = H(2H-1)\int_0^T \int_0^T \varphi(u)\psi(v)|u-v|^{2H-2} du dv.$$

It follows actually from [13] that the space $|\mathcal{H}|$ is a Banach space for the norm $\|.\|_{|\mathcal{H}|}$ and it is included in $\mathcal{H}$. In fact,

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

Let $\mathrm{C}_b^\infty(\mathbb{R}^n, \mathbb{R})$ be the class of infinitely differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that f and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of smooth cylindrical random variables F of the form

$$F = f(B(\varphi_1), ..., B(\varphi_n)), \tag{3}$$

where $n \geq 1$, $f \in \mathrm{C}_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $\varphi_1, ..., \varphi_n \in \mathcal{H}$.
The derivative operator D of a smooth and cylindrical random variable F of the form (3) is defined as the $\mathcal{H}$-valued random variable

$$D_t F = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(B(\varphi_1), ..., B(\varphi_n))\varphi_i(t)$$

In this way the derivative DF is an element of $L^2(\Omega; \mathcal{H})$. We denote by $D^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|^2_{1,2} = E(\|F\|^2) + E(\|DF\|^2_{\mathcal{H}}).$$

The divergence operator δ is the adjoint of the derivative operator D. Concretely, a random variable $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $Dom\delta$ if

$$E\left|\langle DF, u\rangle_{\mathcal{H}}\right| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}$. In this case $\delta(u)$ is given by the duality relationship

$$E(F\delta(u)) = E\left\langle DF, u\right\rangle_{\mathcal{H}}$$

for any $F \in D^{1,2}$. We will make use of the notation

$$\delta(u) = \int_0^T u_s \delta B_s, \quad u \in Dom(\delta).$$

In particular, for $h \in \mathcal{H}$, $B(h) = \delta(h) = \int_0^T h_s \delta B_s$.

For every $n \geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of B , that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$ where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n!H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot n}$ (equipped with the modified norm $\|.\|_{\mathcal{H}^{\odot n}} = \frac{1}{\sqrt{n!}}\|.\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$. For every $f, g \in \mathcal{H}^{\odot n}$ the following multiplication formula holds

$$E\left(I_n(f)I_n(g)\right) = n!\langle f, g\rangle_{\mathcal{H}^{\otimes n}}.$$

Finally, It is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_n$. That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where the $f_n \in \mathcal{H}^{\odot n}$ are uniquely determined by F.

Fix $T > 0$. Let $f, g : [0, T] \longrightarrow \mathbb{R}$ are Hölder continuous functions of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha + \beta > 1$. Young [14] proved that the Riemann-Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta \in (\frac{1}{2}, 1)$ and $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function of class $\mathcal{C}^1$, the integrals $\int_0^{\cdot} \frac{\partial \phi}{\partial f}(f_u, g_u)df_u$ and $\int_0^{\cdot} \frac{\partial \phi}{\partial g}(f_u, g_u)dg_u$ exist in the Young sense and the following change of variables formula holds:

$$\phi(f_t, g_t) = \phi(f_0, g_0) + \int_0^t \frac{\partial \phi}{\partial f}(f_u, g_u)df_u + \int_0^t \frac{\partial \phi}{\partial g}(f_u, g_u)dg_u, \quad 0 \leq t \leq T. \tag{4}$$

As a consequence, if $H > \frac{1}{2}$ and $(u_t,\ t \in [0, T])$ be a process with Hölder paths of order $\alpha > 1 - H$, the integral $\int_0^T u_s dB_s$ is well-defined as Young integral. Suppose moreover that for any $t \in [0, T]$, $u_t \in D^{1,2}$, and

$$P\left(\int_0^T \int_0^T |D_s u_t||t - s|^{2H-2} dsdt < \infty\right) = 1.$$

Then, by [1], $u \in Dom\delta$ and for every $t \in [0, T]$,

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + H(2H - 1)\int_0^t \int_0^t D_s u_r |s - r|^{2H-2} drds. \tag{5}$$

In particular, when φ is a non-random Hölder continuous function of order $\alpha > 1 - H$, we obtain

$$\int_0^T \varphi_s dB_s = \int_0^T \varphi_s \delta B_s = B(\varphi). \tag{6}$$

In addition, for all $\varphi,\ \psi \in |\mathcal{H}|$,

$$E\left(\int_0^T \varphi_s dB_s \int_0^T \psi_s dB_s\right) = H(2H - 1)\int_0^T \int_0^T \varphi(u)\psi(v)|u - v|^{2H-2} dudv. \tag{7}$$

3 Asymptotic behavior of the least squares estimator

Throughout this paper we assume $H \in (\frac{1}{2}, 1)$ and $\theta > 0$. Let us consider the equation (1) driven by a fractional Brownian motion B with Hurst parameter H and θ is the unknown parameter to be estimated from the observation X. The linear equation (1) has the following explicit solution:

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dB_s, \qquad t \geq 0, \tag{8}$$

where the integral $\int_0^t e^{-\theta s} dB_s$ is a Young integral.
Let us introduce the following process

$$\xi_t := \int_0^t e^{-\theta s} dB_s, \qquad t \geq 0.$$

By using the equation (1) and (8) we can write the LSE $\widehat{\theta}_t$ defined in (2) as follows

$$\widehat{\theta}_t = \theta + \frac{\int_0^t X_s dB_s}{\int_0^t X_s^2 ds} = \theta + \frac{\int_0^t e^{\theta s} \xi_s dB_s}{\int_0^t e^{2\theta s} \xi_s^2 ds}. \tag{9}$$

3.1 Consistency of the estimator LSE

The following theorem proves the strong consistency of the LSE $\widehat{\theta}_t$.

Theorem 1 *Assume $H \in (\frac{1}{2}, 1)$, then*

$$\widehat{\theta}_t \longrightarrow \theta \quad \textit{almost surely}$$

as $t \longrightarrow \infty$.

For the proof of Theorem 1 we need the following two lemmas.

Lemma 2 *Suppose that $H > \frac{1}{2}$. Then*

i) For all $\varepsilon \in (0, H)$, the process ξ admits a modification with $(H-\varepsilon)$-Hölder continuous paths, still denoted ξ in the sequel.

ii) $\xi_t \longrightarrow \xi_\infty := \int_0^\infty e^{-\theta r} dB_r$ almost surely and in $L^2(\Omega)$ as $t \longrightarrow \infty$.

Lemma 3 *Let $H > \frac{1}{2}$. Then, as $t \to \infty$,*

$$e^{-2\theta t} \int_0^t X_s^2 ds = e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds \longrightarrow \frac{\xi_\infty^2}{2\theta} \ \textit{almost surely.}$$

Proof of Lemma 2. We prove the point i). We have, for every $0 \leq s < t$,

$$\begin{aligned} E\left(\xi_t - \xi_s\right)^2 &= E\left(\int_s^t e^{-\theta r} dB_r\right)^2 \\ &= H(2H-1) \int_s^t \int_s^t e^{-\theta u} e^{-\theta v} |u-v|^{2H-2} du dv \\ &\leq H(2H-1) \int_s^t \int_s^t |u-v|^{2H-2} du dv \\ &= E\left(B_t - B_s\right)^2 = |t-s|^{2H}. \end{aligned}$$

Thus, by applying the Kolmogorov-Centsov theorem to the centered gaussian process ξ we deduce i).
Concerning the second point ii), we first notice that the integral $\xi_\infty = \int_0^\infty e^{-\theta r} dB_r$ is well defined.

In fact,

$$\begin{aligned}
& H(2H-1)\int_0^\infty\int_0^\infty e^{-\theta r}e^{-\theta s}|r-s|^{2H-2}drds \\
= \;& 2H(2H-1)\int_0^\infty ds e^{-\theta s}\int_0^s dr e^{-\theta r}(s-r)^{2H-2} \\
= \;& 2H(2H-1)\int_0^\infty ds e^{-2\theta s}\int_0^s du e^{\theta u}u^{2H-2} \\
= \;& 2H(2H-1)\int_0^\infty du e^{\theta u}u^{2H-2}\int_u^\infty ds e^{-2\theta s} \\
= \;& \frac{H(2H-1)}{\theta}\int_0^\infty e^{-\theta u}u^{2H-2}du \\
= \;& \frac{H(2H-1)}{\theta^{2H}}\Gamma(2H-1) = \frac{H\Gamma(2H)}{\theta^{2H}} < \infty, \qquad (10)
\end{aligned}$$

with Γ denotes the classical Gamma function. Moreover, ξ_t converges to ξ_∞ in $L^2(\Omega)$. Indeed,

$$\begin{aligned}
E\left[(\xi_t-\xi_\infty)^2\right] &= H(2H-1)\int_t^\infty\int_t^\infty e^{-\theta r}e^{-\theta s}|r-s|^{2H-2}drds \\
&= 2H(2H-1)\int_t^\infty ds e^{-\theta s}\int_t^s dr e^{-\theta r}(s-r)^{2H-2} \\
&= 2H(2H-1)\int_t^\infty dv e^{-2\theta s}\int_0^{s-t} du e^{\theta u}u^{2H-2} \\
&= 2H(2H-1)e^{-2\theta t}\int_0^\infty dv e^{-2\theta v}\int_0^v du e^{\theta u}u^{2H-2} \\
&= 2H(2H-1)e^{-2\theta t}\int_0^\infty du e^{\theta u}u^{2H-2}\int_u^\infty dv e^{-2\theta v} \\
&= \frac{H(2H-1)}{\theta}e^{-2\theta t}\int_0^\infty e^{-\theta u}u^{2H-2}du \\
&= \frac{H\Gamma(2H)}{\theta^{2H}}e^{-2\theta t} \\
&\to 0 \text{ as } t\to\infty.
\end{aligned}$$

Now, let us show that $\xi_t \longrightarrow \xi_\infty$ almost surely as $t\to\infty$. By using Borel-Cantelli lemma, it is sufficient to prove that, for any $\varepsilon>0$

$$\sum_{n\geq 0} P\left(\sup_{n\leq t\leq n+1}\left|\int_t^\infty e^{-\theta s}dB_s\right| > \varepsilon\right) < \infty. \qquad (11)$$

For this purpose, let $\frac{1}{2}<\alpha<H$. As in the proof of [Theorem 4, [1]], we can write for every $t>0$

$$\int_t^\infty e^{-\theta s}dB_s = c_\alpha^{-1}\int_t^\infty dB_s e^{-\theta s}\left(\int_t^s dr(s-r)^{-\alpha}(r-t)^{\alpha-1}\right),$$

with $c_\alpha = \int_t^s (s-r)^{-\alpha}(r-t)^{\alpha-1}dr = \beta(\alpha,1-\alpha)$, where β is the Beta function.
By Fubini's stochastic theorem (see for example [11]), we have

$$\int_t^\infty e^{-\theta s}dB_s = c_\alpha^{-1}\int_t^\infty dr(r-t)^{\alpha-1}\left(\int_r^\infty dB_s e^{-\theta s}(s-r)^{-\alpha}\right).$$

Cauchy-Schwarz's inequality implies that

$$\begin{aligned}
&\left|\int_t^\infty e^{-\theta s}dB_s\right|^2 \\
\leq\ & c_\alpha^{-2}\left(\int_t^\infty (r-t)^{2(\alpha-1)}e^{-\theta(r-t)}dr\right)\left(\int_t^\infty dre^{-\theta(r-t)}\left|\int_r^\infty dB_s e^{-\theta s}(s-r)^{-\alpha}e^{\theta(r-t)}\right|^2\right) \\
=\ & \frac{c_\alpha^{-2}\Gamma(2\alpha-1)}{\theta^{2\alpha-1}}e^{-2\theta t}\int_t^\infty dre^{-\theta(r-t)}\left|\int_r^\infty dB_s(s-r)^{-\alpha}e^{-\theta(s-r)}\right|^2
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup_{n\leq t\leq n+1}\left|\int_t^\infty e^{-\theta s}dB_s\right|^2 \\
&\leq \frac{c_\alpha^{-2}\Gamma(2\alpha-1)}{\theta^{2\alpha-1}}e^{-2\theta n}e^{\theta}\int_n^\infty dre^{-\theta(r-n)}\left|\int_r^\infty dB_s(s-r)^{-\alpha}e^{-\theta(s-r)}\right|^2
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&E\left(\left|\int_r^\infty (s-r)^{-\alpha}e^{-\theta(s-r)}dB_s\right|^2\right) \\
=\ & H(2H-1)\int_r^\infty dv(v-r)^{-\alpha}e^{-\theta(v-r)}\int_r^\infty du(u-r)^{-\alpha}e^{-\theta(u-r)}|u-v|^{2H-2} \\
=\ & H(2H-1)\int_0^\infty dvv^{-\alpha}e^{-\theta v}\int_0^\infty duu^{-\alpha}e^{-\theta u}|u-v|^{2H-2} \\
=\ & 2H(2H-1)\int_0^\infty dvv^{-\alpha}e^{-\theta v}\int_0^v duu^{-\alpha}e^{-\theta u}(v-u)^{2H-2} \\
=\ & 2H(2H-1)\int_0^\infty dvv^{-\alpha}e^{-\theta v}\int_0^v du(v-u)^{-\alpha}e^{-\theta(v-u)}u^{2H-2} \\
\leq\ & 2H(2H-1)\int_0^\infty dvv^{-\alpha}e^{-\theta v}\int_0^v du(v-u)^{-\alpha}u^{2H-2} \\
=\ & 2H(2H-1)\int_0^\infty dvv^{2H-2\alpha-1}e^{-\theta v}\int_0^1 duu^{2H-2}(1-u)^{-\alpha} \\
=\ & 2H(2H-1)\frac{\Gamma(2H-2\alpha)\beta(2H-1,1-\alpha)}{\theta^{2H-2\alpha}} := C_1(\alpha,H,\theta)<\infty.
\end{aligned}$$

Combining this with the fact that $\int_n^\infty e^{-\theta(r-n)}dr = \frac{1}{\theta}$, we obtain

$$E\left(\sup_{n\leq t\leq n+1}\left|\int_t^\infty e^{-\theta s}dB_s\right|^2\right) \leq C_2(\alpha,H,\theta)e^{-2\theta n},$$

with

$$C_2(\alpha,H,\theta) = \frac{c_\alpha^{-2}\Gamma(2\alpha-1)e^{\theta}}{\theta^{2\alpha}}C_1(\alpha,H,\theta).$$

Consequently,

$$\begin{aligned}
\sum_{n\geq 0}P\left(\sup_{n\leq t\leq n+1}\left|\int_t^\infty e^{-\theta s}dB_s\right|>\varepsilon\right) &\leq \varepsilon^{-2}\sum_{n\geq 0}E\left(\sup_{n\leq t\leq n+1}\left|\int_t^\infty e^{-\theta s}dB_s\right|^2\right) \\
&\leq \varepsilon^{-2}C_2(\alpha,H,\theta)\sum_{n\geq 0}e^{-2\theta n}<\infty.
\end{aligned}$$

This finishes the proof of the claim (11), and thus the proof of Lemma 2. ■

Proof of Lemma 3. Using (10), we have

$$E[\xi_\infty^2] = \frac{H\Gamma(2H)}{\theta^{2H}} < \infty.$$

Hence $\xi_\infty \backsim \mathcal{N}(0, \frac{H\Gamma(2H)}{\theta^{2H}})$ and this implies that

$$P(\xi_\infty = 0) = 0. \tag{12}$$

The continuity of ξ entails that, for every $t > 0$

$$\int_0^t e^{2\theta s}\xi_s^2 ds \geq \int_{\frac{t}{2}}^t e^{2\theta s}\xi_s^2 ds \geq \frac{t}{2}e^{\theta t}\left(\inf_{\frac{t}{2}\leq s\leq t}\xi_s^2\right) \text{ almost surely.} \tag{13}$$

Furthermore, the continuity of ξ and the point ii) of Lemma 2 yield

$$\lim_{t\to\infty}\left(\inf_{\frac{t}{2}\leq s\leq t}\xi_s^2\right) = \xi_\infty \text{ almost surely.}$$

Combining this last convergence with (13) and (12), we deduce that

$$\lim_{t\to\infty}\int_0^t e^{2\theta s}\xi_s^2 ds = \infty \text{ almost surely.}$$

Hence, we can use L'Hôspital's rule and we obtain

$$\lim_{t\to\infty}\frac{\int_0^t e^{2\theta s}\xi_s^2 ds}{e^{2\theta t}} = \lim_{t\to\infty}\frac{\xi_t^2}{2\theta} = \frac{\xi_\infty^2}{2\theta} \quad \text{almost surely.}$$

This completes the proof of Lemma 3. ■

Proof of Theorem 1. Using the change of variable formula (4), we conclude that

$$\frac{1}{2}e^{2\theta t}\xi_t^2 = \theta\int_0^t e^{2\theta s}\xi_s^2 ds + \int_0^t e^{\theta s}\xi_s dB_s$$

Hence

$$\widehat{\theta}_t - \theta = \frac{\int_0^t e^{\theta s}\xi_s dB_s}{\int_0^t e^{2\theta s}\xi_s^2 ds} = \frac{\xi_t^2}{2e^{-2\theta t}\int_0^t e^{2\theta s}\xi_s^2 ds} - \theta.$$

Combining this with Lemma 2 and Lemma 3, we deduce that $\widehat{\theta}_t \to \theta$ almost surely as $t \longrightarrow \infty$. ■

3.2 Asymptotic distribution of the estimator LSE

This paragraph is devoted to the investigation of asymptotic distribution of the LSE $\widehat{\theta}_t$ of θ. We start with the following lemma.

Lemma 4 *Suppose that $H > \frac{1}{2}$. Then, for every $t \geq 0$, we have*

$$\begin{aligned}\int_0^t dB_s e^{\theta s}\int_0^s dB_r e^{-\theta r} &= \int_0^t dB_s e^{\theta s}\int_0^t dB_r e^{-\theta r} - \int_0^t \delta B_s e^{-\theta s}\int_0^s \delta B_r e^{\theta r} \\ &\quad -H(2H-1)\int_0^t ds e^{-\theta s}\int_0^s dr e^{\theta r}|s-r|^{2H-2}.\end{aligned}$$

Proof. Let $t \geq 0$. By the change of variable formula (4)

$$\int_0^t dB_s e^{\theta s} \int_0^s dB_r e^{-\theta r} = \int_0^t dB_s e^{\theta s} \int_0^t dB_r e^{-\theta r} - \int_0^t dB_s e^{-\theta s} \int_0^s dB_r e^{\theta r}.$$

On the other hand, according to (5) and (6),

$$\begin{aligned} & \int_0^t dB_s e^{-\theta s} \int_0^s dB_r e^{\theta r} \\ = & \int_0^t \delta B_s e^{-\theta s} \int_0^s \delta B_r e^{\theta r} + H(2H-1) \int_0^t ds e^{-\theta s} \int_0^s dr e^{\theta r} |s-r|^{2H-2}, \end{aligned}$$

which completes the proof. ■

Theorem 5 *Let $H > \frac{1}{2}$ be fixed. Then, as $t \longrightarrow \infty$,*

$$e^{\theta t} \left(\widehat{\theta}_t - \theta \right) \xrightarrow{law} 2\theta \mathcal{C}(1),$$

with $\mathcal{C}(1)$ the standard Cauchy distribution.

In order to prove Theorem 5 we need the following two lemmas.

Lemma 6 *Fix $H > \frac{1}{2}$. Let F be any $\sigma\{B\}$-measurable random variable such that $P(F < \infty) = 1$. Then, as $t \longrightarrow \infty$,*

$$\left(F, e^{-\theta t} \int_0^t e^{\theta s} dB_s \right) \xrightarrow{law} \left(F, \frac{\sqrt{H\Gamma(2H)}}{\theta^H} N \right),$$

where $N \sim \mathcal{N}(0,1)$ is independent of B.

Lemma 7 *Let $H > \frac{1}{2}$. Then, as $t \to \infty$,*

$$e^{-\frac{\theta t}{2}} \int_0^t \delta B_s e^{-\theta s} \int_0^s \delta B_r e^{\theta r} \longrightarrow 0 \quad \text{in } L^2(\Omega), \tag{14}$$

and

$$e^{-\frac{\theta t}{2}} \int_0^t ds e^{-\theta s} \int_0^s dr e^{\theta r} |s-r|^{2H-2} \longrightarrow 0. \tag{15}$$

Proof of Lemma 6. For any $d \geq 1$, $s_1 \ldots s_d \in [0, \infty)$, we shall prove that, as $t \longrightarrow \infty$,

$$\left(B_{s_1}, \ldots, B_{s_d}, e^{-\theta t} \int_0^t e^{\theta s} dB_s \right) \xrightarrow{\text{law}} \left(B_{s_1}, \ldots, B_{s_d}, \frac{\sqrt{H\Gamma(2H)}}{\theta^H} N \right) \tag{16}$$

which is enough to lead to the desired conclusion. Because the left-hand side in the previous convergence is a Gaussian vector (see proof of [Lemma 7, [5]]), to get (16) it is sufficient to check the convergence of its covariance matrix. Let us first compute the limiting variance of

$e^{-\theta t}\int_0^t e^{\theta s}dB_s$ as $t \longrightarrow \infty$. We have

$$\begin{aligned}
E\left[\left(e^{-\theta t}\int_0^t e^{\theta s}dB_s\right)^2\right] &= H(2H-1)e^{-2\theta t}\int_0^t\int_0^t e^{\theta s}e^{\theta r}|s-r|^{2H-2}drds \\
&= 2H(2H-1)e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dre^{\theta r}|s-r|^{2H-2} \\
&= 2H(2H-1)e^{-2\theta t}\int_0^t dse^{2\theta s}\int_0^s dre^{-\theta r}r^{2H-2} \\
&= 2H(2H-1)e^{-2\theta t}\int_0^t dre^{-\theta r}r^{2H-2}\int_r^t dse^{2\theta s} \\
&= \frac{H(2H-1)}{\theta}\left(\int_0^t r^{2H-2}e^{-\theta r}dr - e^{-2\theta t}\int_0^t r^{2H-2}e^{\theta r}dr\right) \\
&\to \frac{H\Gamma(2H)}{\theta^{2H}} \quad \text{as } t\to\infty,
\end{aligned}$$

because $e^{-2\theta t}\int_0^t r^{2H-2}e^{\theta r}dr \le e^{-\theta t}\int_0^t r^{2H-2}dr = \frac{t^{2H-1}}{(2H-1)e^{\theta t}} \to 0$ as $t\to\infty$.
Thus,

$$\lim_{t\to\infty} E\left[\left(e^{-\theta t}\int_0^t e^{\theta s}dB_s\right)^2\right] = \frac{H\Gamma(2H)}{\theta^{2H}}.$$

Hence, to finish the proof it remains to check that, for all fixed $s \ge 0$,

$$\lim_{t\to\infty} E\left(B_s \times e^{-\theta t}\int_0^t e^{\theta v}dB_v\right) = 0.$$

Indeed, for $s < t$,

$$\begin{aligned}
& E\left(B_s \times e^{-\theta t}\int_0^t e^{\theta v}dB_v\right) \\
= \;& H(2H-1)e^{-\theta t}\int_0^t dve^{\theta v}\int_0^s du|u-v|^{2H-2} \\
= \;& H(2H-1)e^{-\theta t}\int_0^s dve^{\theta v}\int_0^s du|u-v|^{2H-2} + H(2H-1)e^{-\theta t}\int_s^t dve^{\theta v}\int_0^s du(v-u)^{2H-2} \\
= \;& H(2H-1)e^{-\theta t}\int_0^s dve^{\theta v}\int_0^s du|u-v|^{2H-2} + He^{-\theta t}\int_s^t e^{\theta v}(v^{2H-1}-(v-s)^{2H-1})dv \\
:= \;& I_t + J_t.
\end{aligned}$$

It's clear that $I_t \to 0$ as $t\to\infty$.
Using integration by parts, the therm J_t can be written as

$$\begin{aligned}
& J_t \\
= \;& He^{-\theta t}\int_s^t e^{\theta v}(v^{2H-1}-(v-s)^{2H-1})dv \\
= \;& He^{-\theta t}\left(\frac{e^{\theta t}}{\theta}[t^{2H-1}-(t-s)^{2H-1}] - \frac{e^{\theta s}}{\theta}s^{2H-1} + \frac{2H-1}{\theta}\int_s^t e^{\theta v}[(v-s)^{2H-2}-v^{2H-2}]dv\right) \\
\le \;& \frac{H}{\theta}[t^{2H-1}-(t-s)^{2H-1}] + \frac{H(2H-1)}{\theta}e^{-\theta t}\int_s^t e^{\theta v}(v-s)^{2H-2}dv \\
:= \;& J_t^1 + J_t^2.
\end{aligned}$$

Since $H<1$, $J_t^1=\frac{H}{\theta}[t^{2H-1}-(t-s)^{2H-1}]\to 0$ as $t\to\infty$.
On the other hand,

$$\begin{aligned}
J_t^2 &= \frac{H(2H-1)}{\theta}e^{-\theta t}\int_s^t e^{\theta v}(v-s)^{2H-2}dv\\
&= \frac{H(2H-1)e^{\theta s}}{\theta}e^{-\theta t}\int_0^{t-s} e^{\theta u}u^{2H-2}du\\
&\le \frac{H(2H-1)e^{\theta s}}{\theta}e^{-\theta t}\int_0^{t} e^{\theta u}u^{2H-2}du\\
&= \frac{H(2H-1)e^{\theta s}}{\theta}\int_0^{t} e^{-\theta v}(t-v)^{2H-2}dv\\
&= \frac{H(2H-1)e^{\theta s}}{\theta}t^{2H-1}\int_0^{1} e^{-\theta tu}(1-u)^{2H-2}du
\end{aligned}$$

Fix $u\in(0,1)$. The function $t\in[0,\infty)\mapsto t^{2H-1}e^{-\theta tu}$ attains its maximum at $t=\frac{2H-1}{\theta u}$. Then

$$\sup_{t\ge 0}(t^{2H-1}e^{-\theta tu})=ce^{-\frac{2H-1}{u}}u^{1-2H}\le cu^{1-2H},$$

with $c=\left(\frac{2H-1}{\theta}\right)^{2H-1}$. In addition, $\int_0^1 u^{1-2H}(1-u)^{2H-2}du<\infty$, and for any $u\in(0,1)$,

$$t^{2H-1}e^{-\theta tu}(1-u)^{2H-2}\to 0 \text{ as } t\to\infty.$$

Therefore, using the dominated convergence theorem, we obtain that J_t^2 converges to 0 as $t\to\infty$. Thus, we deduce the desired conclusion. ■

Proof of Lemma 7. Let us prove the convergence (14). We have

$$\begin{aligned}
& e^{-\theta t}E\left(\int_0^t \delta B_s e^{-\theta s}\int_0^s \delta B_r e^{\theta r}\right)^2\\
= \;& e^{-\theta t}E\left(\int_0^t\int_0^s e^{-\theta|s-r|}\delta B_r\delta B_s\right)^2\\
= \;& e^{-\theta t}E\left(\frac{1}{2}I_2(e^{-\theta|s-r|}1_{[0,t]^2})\right)^2\\
= \;& \frac{H^2(2H-1)^2}{2}e^{-\theta t}\int_{[0,t]^4} e^{-\theta|v-s|}e^{-\theta|u-r|}|v-u|^{2H-2}|s-r|^{2H-2}dudvdrds\\
\le \;& \frac{H^2(2H-1)^2}{2}e^{-\theta t}\int_{[0,t]^4} |v-u|^{2H-2}|s-r|^{2H-2}dudvdrds\\
= \;& \frac{1}{2}[E(B_t^2)]^2e^{-\theta t}=\frac{1}{2}t^{4H}e^{-\theta t}\\
\to \;& 0 \text{ as } t\to\infty.
\end{aligned}$$

For the convergence (15), we have

$$\begin{aligned}
& H(2H-1)e^{-\frac{\theta t}{2}}\int_0^t dse^{-\theta s}\int_0^s dre^{\theta r}|s-r|^{2H-2}\\
\le \;& H(2H-1)e^{-\frac{\theta t}{2}}\int_0^t ds\int_0^s dr|s-r|^{2H-2}\\
= \;& \frac{t^{2H}}{2}e^{-\frac{\theta t}{2}}\\
\to \;& 0 \text{ as } t\to\infty.
\end{aligned}$$

This finishes the proof. ■

Proof of the theorem 5. By combining (9) and Lemma 4, we can write,

$$\begin{aligned} e^{\theta t}\left(\widehat{\theta}_t - \theta\right) &= \frac{e^{\theta t}\int_0^t dB_s e^{\theta s}\int_0^s dB_r e^{-\theta r}}{\int_0^t e^{2\theta s}\xi_s^2 ds} \\ &= \frac{\xi_t \xi_\infty}{e^{-2\theta t}\int_0^t e^{2\theta s}\xi_s^2 ds} \times \frac{e^{-\theta t}\int_0^t e^{\theta s} dB_s}{\xi_\infty} \\ &\quad - \frac{e^{-\theta t}\int_0^t \delta B_s e^{-\theta s}\int_0^s \delta B_r e^{\theta r}}{e^{-2\theta t}\int_0^t e^{2\theta s}\xi_s^2 ds} \\ &\quad - H(2H-1)\frac{e^{-\theta t}\int_0^t ds e^{-\theta s}\int_0^s dr e^{\theta r}|s-r|^{2H-2}}{e^{-2\theta t}\int_0^t e^{2\theta s}\xi_s^2 ds} \\ &:= A_t^\theta \times B_t^\theta - C_t^\theta - D_t^\theta. \end{aligned}$$

Using Lemme 2 and Lemma 3, we obtain that

$$A_t^\theta \longrightarrow 2\theta \text{ almost surely as } t \longrightarrow \infty.$$

According to Lemma 6, we deduce

$$B_t^\theta \xrightarrow{\text{law}} \frac{\sqrt{H\Gamma(2H)}}{\theta^H}\frac{N}{\xi_\infty} \text{ as } t \longrightarrow \infty.$$

Moreover,

$$\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\frac{N}{\xi_\infty} \overset{\text{law}}{=} \mathcal{C}(1),$$

because $\frac{\theta^H \xi_\infty}{\sqrt{H\Gamma(2H)}} \backsim \mathcal{N}(0,1)$ and $N \sim \mathcal{N}(0,1)$ are independent.
Thus, by Slutsky's theorem, we conclude that

$$A_t^\theta \times B_t^\theta \xrightarrow{\text{law}} 2\theta\mathcal{C}(1) \text{ as } t \longrightarrow \infty.$$

On the other hand, it follows from Lemma 3 and Lemma 7, that

$$C_t^\theta \xrightarrow{\text{prob.}} 0 \text{ as } t \longrightarrow \infty,$$

and

$$D_t^\theta \longrightarrow 0 \text{ almost surely as } t \longrightarrow \infty.$$

Finally, by combining the previous convergences, the proof of Theorem 5 is done. ■

Acknowledgments. The authors would like to thank Ivan Nourdin for many valuable discussions on the subject. We warmly thank him for proving that J_t converges to zero (see Proof of Lemma 6).

References

[1] E. Alòs and D. Nualart. *Stochastic integration with respect to the fractional Brownian motion. Stoch. stoch. reports 75, no. 3, (2003), 129-152.*

[2] M. Barczy and G. Pap. *Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes. J. Statist. Plan. Infer. 140, no. 6, (2010), 1576-1593.*

[3] I.V. Basawa and D.J. Scott. *Asymptotic optimal inference for non-ergodic models. Lecture Notes in Statistics, 17, Springer, New York, (1983).*

[4] H.M. Dietz and YU.A. Kutoyants. *Parameter estimation for some non-recurrent solutions of SDE. Statistics and Decisions 21(1), (2003), 29-46.*

[5] K. Es-Sebaiy, I. Nourdin. *Parameter estimation for α-fractional bridges. (2011), Submitted. http://arxiv.org/abs/1101.5790v1.*

[6] Y. Hu and H. Long. *Parameter estimation for Ornstein-Uhlenbeck processes driven by α-stable Lévy motions. Commun. Stoch. Anal. 1, (2007), 175-192.*

[7] Y. Hu and D. Nualart. *Parameter estimation for fractional Ornstein-Uhlenbeck processes.* Statistics and Probability Letters 80, (2010), 1030-1038.

[8] Yu. A. Kutoyants. *Statistical Inference for Ergodic Diffusion Processes. Springer, Berlin, Heidelberg, (2004).*

[9] R. S. Liptser and A. N. Shiryaev. *Statistics of Random Processes: II Applications. Second Edition, Applications of Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, (2001).*

[10] I. Nourdin. *An invitation to fractional Brownian motion. Doctoral course. http://www.iecn.u-nancy.fr/∼nourdin/lectures-fbm.pdf, (2010).*

[11] D. Nualart. *The Malliavin calculus and related topics. Springer-Verlag, Berlin, second edition, (2006).*

[12] D. Nualart and G. Peccati. *Central limit theorems for sequences of multiple stochastic integrals. Ann. Probab. 33 (1), (2005), 177-193.*

[13] V. Pipiras and M.S. Taqqu. *Integration questions related to fractional Brownian motion Probab. Theory Rel. Fields, 118, no. 2, (2000), 251-291.*

[14] L. C. Young. *An inequality of the Hölder type connected with Stieltjes integration. Acta Math. 67, (1936), 251-282.*