



HAL
open science

PARAMETER ESTIMATION FOR FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES: NON-ERGODIC CASE

Rachid Belfadli, Khalifa Es-Sebaiy, Youssef Ouknine

► **To cite this version:**

Rachid Belfadli, Khalifa Es-Sebaiy, Youssef Ouknine. PARAMETER ESTIMATION FOR FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES: NON-ERGODIC CASE. 2011. hal-00569387

HAL Id: hal-00569387

<https://hal.science/hal-00569387v1>

Preprint submitted on 27 Feb 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PARAMETER ESTIMATION FOR FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES: NON-ERGODIC CASE

Rachid Belfadli¹ Khalifa Es-Sebaïy² Youssef Ouknine³

¹ Polydisciplinary Faculty of Taroudant, University Ibn Zohr, Taroudant, Morocco.
belfadli@gmail.com

² Institut de Mathématiques de Bourgogne, Université de Bourgogne, Dijon, France.
khalifasbai@gmail.com

³ Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University,
2390 Marrakesh, Morocco.
ouknine@ucam.ac.ma

February 25, 2011

Abstract

We consider the parameter estimation problem for the non-ergodic fractional Ornstein-Uhlenbeck process defined as $dX_t = \theta X_t dt + dB_t$, $t \geq 0$, with a parameter $\theta > 0$, where B is a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$. We study the consistency and the asymptotic distributions of the least squares estimator $\hat{\theta}_t$ of θ based on the observation $\{X_s, s \in [0, t]\}$ as $t \rightarrow \infty$.

Key words and phrases: Parameter estimation, Non-ergodic fractional Ornstein-Uhlenbeck process, Young integral.

2000 Mathematics Subject Classification: 62F12, 60G18, 60G15.

1 Introduction

We consider the Ornstein-Uhlenbeck process $X = \{X_t, t \geq 0\}$ given by the following linear stochastic differential equation

$$X_0 = 0; \quad dX_t = \theta X_t dt + dB_t, \quad t \geq 0, \quad (1)$$

where B is a fractional Brownian motion of Hurst index $H > \frac{1}{2}$ and $\theta \in (-\infty, \infty)$ is an unknown parameter. An interesting problem is to estimate the parameter θ when one observes the whole trajectory of X . First, let us recall some results in the case when B is a standard Brownian motion. In this special case, the parameter estimation for θ has been well studied by using the classical maximum likelihood method or by using the trajectory fitting method. If $\theta < 0$ (ergodic case), the maximum likelihood estimator (MLE) of θ is asymptotically normal (see Liptser and Shiryaev [9], Kutoyants [8]). If $\theta > 0$ (non-ergodic case), the MLE of θ is asymptotically Cauchy (see Basawa and Scott [3], Dietz and Kutoyants [4]). Recently, in a more general context, several authors extended this study to some generalizations of Ornstein-Uhlenbeck process driven by Brownian motion (for instance, Barczy and Pap [2]). Similar properties of the asymptotic behaviour of MLE has also been obtained with respect to the trajectory fitting estimators (see Dietz and Kutoyants [4]).

When B is replaced by an α -stable Lévy motion in the equation (1), Hu and Long [6] discussed the parameter estimation of θ in both the ergodic and the non-ergodic cases. They used the trajectory fitting method combined with the weighted least squares technique.

Now, let us consider a parameter estimation problem of the parameter θ for the fractional Ornstein-Uhlenbeck process X of (1).

In the case $\theta < 0$ (corresponding to the ergodic case), Hu and Nualart [7] studied the parameter estimation for θ by using the least squares estimator (LSE) defined as

$$\widehat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \quad t \geq 0. \quad (2)$$

This LSE is obtained by the least squares technique, that is, $\widehat{\theta}_t$ (formally) minimizes

$$\theta \mapsto \int_0^t |\dot{X}_s + \theta X_s|^2 ds.$$

To obtain the consistency of the LSE $\widehat{\theta}_t$, the authors of [7] are forced to consider $\int_0^t X_s dX_s$ as a Skorohod integral rather than Young integral in the definition (2). Assuming $\int_0^t X_s dX_s$ is a Skorohod integral and $\theta < 0$, they proved the strong consistence of $\widehat{\theta}_t$ if $H \geq \frac{1}{2}$, and that the LSE $\widehat{\theta}_t$ of θ is asymptotically normal if $H \in [\frac{1}{2}, \frac{3}{4})$. Their proof of the central limit theorem is based on the fourth moment theorem of Nualart and Peccati [12].

In this paper, our purpose is to study the non-ergodic case corresponding to $\theta > 0$. More precisely, we shall estimate θ by the LSE $\widehat{\theta}_t$ defined in (2), where in our case, the integral $\int_0^t X_s dX_s$ is interpreted as a Young integral. Indeed in that case, we have $\widehat{\theta}_t = \frac{X_t^2}{2 \int_0^t X_s^2 ds}$ which converges almost surely to θ , as t tends to infinity (see Theorem 1). Moreover, it turned out that the path-wise approach is the preferred way to simulate numerically an estimator $\widehat{\theta}_t$. Our technics used in this work are inspired from the recent paper by Es-Sebaïy and Nourdin [5].

The organization of our paper is as follows. Section 2 contains the presentation of the basic tools that we will need throughout the paper: fractional Brownian motion, Malliavin derivative, Skorohod integral, Young integral and the link between Young and Skorohod integrals. The aim of Section 3 is twofold. Firstly, we prove when $H > \frac{1}{2}$ the strong consistence of the LSE $\widehat{\theta}_t$, that is, $\widehat{\theta}_t$ converges almost surely to θ , as t goes to infinity. Secondly, we investigate the asymptotic distribution of our estimator $\widehat{\theta}_t$ in the case $H > \frac{1}{2}$. We obtain that (see Theorem 5)

$$e^{\theta t} (\widehat{\theta}_t - \theta) \xrightarrow{\text{law}} 2\theta \mathcal{C}(1) \quad \text{as } t \rightarrow \infty,$$

with $\mathcal{C}(1)$ the standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$; $x \in \mathbb{R}$.

2 Preliminaries

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For more complete presentation on the subject, see [11], [1] and [10].

The fractional Brownian motion $(B_t, t \geq 0)$ with Hurst parameter $H \in (0, 1)$, is defined as a centered Gaussian process starting from zero with covariance

$$R_H(t, s) = E(B_t B_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

We assume that B is defined on a complete probability space (Ω, \mathcal{F}, P) such that \mathcal{F} is the sigma-field generated by B . By Kolmogorov's continuity criterion and the fact

$$E(B_t - B_s)^2 = |s - t|^{2H}; \quad s, t \geq 0,$$

we deduce that B has Hölder continuous paths of order $H - \varepsilon$, for all $\varepsilon \in (0, H)$.

Fix a time interval $[0, T]$. We denote by \mathcal{H} the canonical Hilbert space associated to the fractional Brownian motion B . That is, \mathcal{H} is the closure of the linear span \mathcal{E} generated by the indicator functions $1_{[0,t]}$, $t \in [0, T]$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle = R_H(t, s).$$

The application $\varphi \in \mathcal{E} \rightarrow B(\varphi)$ is an isometry from \mathcal{E} to the Gaussian space generated by B and it can be extended to \mathcal{H} .

If $H > \frac{1}{2}$ the elements of \mathcal{H} may be not functions but distributions of negative order (see [13]). Therefore, it is of interest to know significant subspaces of functions contained in it.

Let $|\mathcal{H}|$ be the set of measurable functions φ on $[0, T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := H(2H - 1) \int_0^T \int_0^T |\varphi(u)||\varphi(v)||u - v|^{2H-2} dudv < \infty.$$

Note that, if $\varphi, \psi \in |\mathcal{H}|$,

$$E(B(\varphi)B(\psi)) = H(2H - 1) \int_0^T \int_0^T \varphi(u)\psi(v)|u - v|^{2H-2} dudv.$$

It follows actually from [13] that the space $|\mathcal{H}|$ is a Banach space for the norm $\|\cdot\|_{|\mathcal{H}|}$ and it is included in \mathcal{H} . In fact,

$$L^2([0, T]) \subset L^{\frac{1}{H}}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

Let $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ be the class of infinitely differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that f and all its partial derivatives are bounded. We denote by \mathcal{S} the class of smooth cylindrical random variables F of the form

$$F = f(B(\varphi_1), \dots, B(\varphi_n)), \quad (3)$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $\varphi_1, \dots, \varphi_n \in \mathcal{H}$.

The derivative operator D of a smooth and cylindrical random variable F of the form (3) is defined as the \mathcal{H} -valued random variable

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_n)) \varphi_i(t)$$

In this way the derivative DF is an element of $L^2(\Omega; \mathcal{H})$. We denote by $D^{1,2}$ the closure of \mathcal{S} with respect to the norm defined by

$$\|F\|_{1,2}^2 = E(\|F\|^2) + E(\|DF\|_{\mathcal{H}}^2).$$

The divergence operator δ is the adjoint of the derivative operator D . Concretely, a random variable $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $Dom \delta$ if

$$E |\langle DF, u \rangle_{\mathcal{H}}| \leq c \|F\|_{L^2(\Omega)}$$

for every $F \in \mathcal{S}$. In this case $\delta(u)$ is given by the duality relationship

$$E(F\delta(u)) = E \langle DF, u \rangle_{\mathcal{H}}$$

for any $F \in D^{1,2}$. We will make use of the notation

$$\delta(u) = \int_0^T u_s \delta B_s, \quad u \in Dom(\delta).$$

In particular, for $h \in \mathcal{H}$, $B(h) = \delta(h) = \int_0^T h_s \delta B_s$.

For every $n \geq 1$, let \mathcal{H}_n be the n th Wiener chaos of B , that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$ where H_n is the n th Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n!H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\otimes n}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\otimes n}} = \frac{1}{\sqrt{n!}}\|\cdot\|_{\mathcal{H}^{\otimes n}}$) and \mathcal{H}_n . For every $f, g \in \mathcal{H}^{\otimes n}$ the following multiplication formula holds

$$E(I_n(f)I_n(g)) = n!\langle f, g \rangle_{\mathcal{H}^{\otimes n}}.$$

Finally, It is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces \mathcal{H}_n . That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where the $f_n \in \mathcal{H}^{\otimes n}$ are uniquely determined by F .

Fix $T > 0$. Let $f, g : [0, T] \rightarrow \mathbb{R}$ are Hölder continuous functions of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha + \beta > 1$. Young [14] proved that the Riemann-Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta \in (\frac{1}{2}, 1)$ and $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function of class \mathcal{C}^1 , the integrals $\int_0^t \frac{\partial \phi}{\partial f}(f_u, g_u) df_u$ and $\int_0^t \frac{\partial \phi}{\partial g}(f_u, g_u) dg_u$ exist in the Young sense and the following change of variables formula holds:

$$\phi(f_t, g_t) = \phi(f_0, g_0) + \int_0^t \frac{\partial \phi}{\partial f}(f_u, g_u) df_u + \int_0^t \frac{\partial \phi}{\partial g}(f_u, g_u) dg_u, \quad 0 \leq t \leq T. \quad (4)$$

As a consequence, if $H > \frac{1}{2}$ and $(u_t, t \in [0, T])$ be a process with Hölder paths of order $\alpha > 1 - H$, the integral $\int_0^T u_s dB_s$ is well-defined as Young integral. Suppose moreover that for any $t \in [0, T]$, $u_t \in D^{1,2}$, and

$$P\left(\int_0^T \int_0^T |D_s u_t| |t - s|^{2H-2} ds dt < \infty\right) = 1.$$

Then, by [1], $u \in \text{Dom} \delta$ and for every $t \in [0, T]$,

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + H(2H - 1) \int_0^t \int_0^t D_s u_r |s - r|^{2H-2} dr ds. \quad (5)$$

In particular, when φ is a non-random Hölder continuous function of order $\alpha > 1 - H$, we obtain

$$\int_0^T \varphi_s dB_s = \int_0^T \varphi_s \delta B_s = B(\varphi). \quad (6)$$

In addition, for all $\varphi, \psi \in |\mathcal{H}|$,

$$E\left(\int_0^T \varphi_s dB_s \int_0^T \psi_s dB_s\right) = H(2H - 1) \int_0^T \int_0^T \varphi(u)\psi(v)|u - v|^{2H-2} dudv. \quad (7)$$

3 Asymptotic behavior of the least squares estimator

Throughout this paper we assume $H \in (\frac{1}{2}, 1)$ and $\theta > 0$. Let us consider the equation (1) driven by a fractional Brownian motion B with Hurst parameter H and θ is the unknown parameter to be estimated from the observation X . The linear equation (1) has the following explicit solution:

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dB_s, \quad t \geq 0, \quad (8)$$

where the integral $\int_0^t e^{-\theta s} dB_s$ is a Young integral.
Let us introduce the following process

$$\xi_t := \int_0^t e^{-\theta s} dB_s, \quad t \geq 0.$$

By using the equation (1) and (8) we can write the LSE $\widehat{\theta}_t$ defined in (2) as follows

$$\widehat{\theta}_t = \theta + \frac{\int_0^t X_s dB_s}{\int_0^t X_s^2 ds} = \theta + \frac{\int_0^t e^{\theta s} \xi_s dB_s}{\int_0^t e^{2\theta s} \xi_s^2 ds}. \quad (9)$$

3.1 Consistency of the estimator LSE

The following theorem proves the strong consistency of the LSE $\widehat{\theta}_t$.

Theorem 1 *Assume $H \in (\frac{1}{2}, 1)$, then*

$$\widehat{\theta}_t \longrightarrow \theta \text{ almost surely}$$

as $t \longrightarrow \infty$.

For the proof of Theorem 1 we need the following two lemmas.

Lemma 2 *Suppose that $H > \frac{1}{2}$. Then*

i) For all $\varepsilon \in (0, H)$, the process ξ admits a modification with $(H - \varepsilon)$ -Hölder continuous paths, still denoted ξ in the sequel.

ii) $\xi_t \longrightarrow \xi_\infty := \int_0^\infty e^{-\theta r} dB_r$ almost surely and in $L^2(\Omega)$ as $t \longrightarrow \infty$.

Lemma 3 *Let $H > \frac{1}{2}$. Then, as $t \rightarrow \infty$,*

$$e^{-2\theta t} \int_0^t X_s^2 ds = e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds \longrightarrow \frac{\xi_\infty^2}{2\theta} \text{ almost surely.}$$

Proof of Lemma 2. We prove the point *i*). We have, for every $0 \leq s < t$,

$$\begin{aligned} E(\xi_t - \xi_s)^2 &= E\left(\int_s^t e^{-\theta r} dB_r\right)^2 \\ &= H(2H - 1) \int_s^t \int_s^t e^{-\theta u} e^{-\theta v} |u - v|^{2H-2} dudv \\ &\leq H(2H - 1) \int_s^t \int_s^t |u - v|^{2H-2} dudv \\ &= E(B_t - B_s)^2 = |t - s|^{2H}. \end{aligned}$$

Thus, by applying the Kolmogorov-Centsov theorem to the centered gaussian process ξ we deduce *i*).

Concerning the second point *ii*), we first notice that the integral $\xi_\infty = \int_0^\infty e^{-\theta r} dB_r$ is well defined.

In fact,

$$\begin{aligned}
& H(2H-1) \int_0^\infty \int_0^\infty e^{-\theta r} e^{-\theta s} |r-s|^{2H-2} dr ds \\
&= 2H(2H-1) \int_0^\infty ds e^{-\theta s} \int_0^s dr e^{-\theta r} (s-r)^{2H-2} \\
&= 2H(2H-1) \int_0^\infty ds e^{-2\theta s} \int_0^s du e^{\theta u} u^{2H-2} \\
&= 2H(2H-1) \int_0^\infty du e^{\theta u} u^{2H-2} \int_u^\infty ds e^{-2\theta s} \\
&= \frac{H(2H-1)}{\theta} \int_0^\infty e^{-\theta u} u^{2H-2} du \\
&= \frac{H(2H-1)}{\theta^{2H}} \Gamma(2H-1) = \frac{H\Gamma(2H)}{\theta^{2H}} < \infty,
\end{aligned} \tag{10}$$

with Γ denotes the classical Gamma function. Moreover, ξ_t converges to ξ_∞ in $L^2(\Omega)$. Indeed,

$$\begin{aligned}
E [(\xi_t - \xi_\infty)^2] &= H(2H-1) \int_t^\infty \int_t^\infty e^{-\theta r} e^{-\theta s} |r-s|^{2H-2} dr ds \\
&= 2H(2H-1) \int_t^\infty ds e^{-\theta s} \int_t^s dr e^{-\theta r} (s-r)^{2H-2} \\
&= 2H(2H-1) \int_t^\infty dv e^{-2\theta s} \int_0^{s-t} du e^{\theta u} u^{2H-2} \\
&= 2H(2H-1) e^{-2\theta t} \int_0^\infty dv e^{-2\theta v} \int_0^v du e^{\theta u} u^{2H-2} \\
&= 2H(2H-1) e^{-2\theta t} \int_0^\infty du e^{\theta u} u^{2H-2} \int_u^\infty dv e^{-2\theta v} \\
&= \frac{H(2H-1)}{\theta} e^{-2\theta t} \int_0^\infty e^{-\theta u} u^{2H-2} du \\
&= \frac{H\Gamma(2H)}{\theta^{2H}} e^{-2\theta t} \\
&\rightarrow 0 \text{ as } t \rightarrow \infty.
\end{aligned}$$

Now, let us show that $\xi_t \rightarrow \xi_\infty$ almost surely as $t \rightarrow \infty$. By using Borel-Cantelli lemma, it is sufficient to prove that, for any $\varepsilon > 0$

$$\sum_{n \geq 0} P \left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} dB_s \right| > \varepsilon \right) < \infty. \tag{11}$$

For this purpose, let $\frac{1}{2} < \alpha < H$. As in the proof of [Theorem 4, [1]], we can write for every $t > 0$

$$\int_t^\infty e^{-\theta s} dB_s = c_\alpha^{-1} \int_t^\infty dB_s e^{-\theta s} \left(\int_t^s dr (s-r)^{-\alpha} (r-t)^{\alpha-1} \right),$$

with $c_\alpha = \int_t^s (s-r)^{-\alpha} (r-t)^{\alpha-1} dr = \beta(\alpha, 1-\alpha)$, where β is the Beta function. By Fubini's stochastic theorem (see for example [11]), we have

$$\int_t^\infty e^{-\theta s} dB_s = c_\alpha^{-1} \int_t^\infty dr (r-t)^{\alpha-1} \left(\int_r^\infty dB_s e^{-\theta s} (s-r)^{-\alpha} \right).$$

Cauchy-Schwarz's inequality implies that

$$\begin{aligned}
& \left| \int_t^\infty e^{-\theta s} dB_s \right|^2 \\
& \leq c_\alpha^{-2} \left(\int_t^\infty (r-t)^{2(\alpha-1)} e^{-\theta(r-t)} dr \right) \left(\int_t^\infty dr e^{-\theta(r-t)} \left| \int_r^\infty dB_s e^{-\theta s} (s-r)^{-\alpha} e^{\theta(r-t)} \right|^2 \right) \\
& = \frac{c_\alpha^{-2} \Gamma(2\alpha-1)}{\theta^{2\alpha-1}} e^{-2\theta t} \int_t^\infty dr e^{-\theta(r-t)} \left| \int_r^\infty dB_s (s-r)^{-\alpha} e^{-\theta(s-r)} \right|^2
\end{aligned}$$

Thus,

$$\begin{aligned}
& \sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} dB_s \right|^2 \\
& \leq \frac{c_\alpha^{-2} \Gamma(2\alpha-1)}{\theta^{2\alpha-1}} e^{-2\theta n} e^\theta \int_n^\infty dr e^{-\theta(r-n)} \left| \int_r^\infty dB_s (s-r)^{-\alpha} e^{-\theta(s-r)} \right|^2
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& E \left(\left| \int_r^\infty (s-r)^{-\alpha} e^{-\theta(s-r)} dB_s \right|^2 \right) \\
& = H(2H-1) \int_r^\infty dv (v-r)^{-\alpha} e^{-\theta(v-r)} \int_r^\infty du (u-r)^{-\alpha} e^{-\theta(u-r)} |u-v|^{2H-2} \\
& = H(2H-1) \int_0^\infty dv v^{-\alpha} e^{-\theta v} \int_0^\infty du u^{-\alpha} e^{-\theta u} |u-v|^{2H-2} \\
& = 2H(2H-1) \int_0^\infty dv v^{-\alpha} e^{-\theta v} \int_0^v du u^{-\alpha} e^{-\theta u} (v-u)^{2H-2} \\
& = 2H(2H-1) \int_0^\infty dv v^{-\alpha} e^{-\theta v} \int_0^v du (v-u)^{-\alpha} e^{-\theta(v-u)} u^{2H-2} \\
& \leq 2H(2H-1) \int_0^\infty dv v^{-\alpha} e^{-\theta v} \int_0^v du (v-u)^{-\alpha} u^{2H-2} \\
& = 2H(2H-1) \int_0^\infty dv v^{2H-2\alpha-1} e^{-\theta v} \int_0^1 du u^{2H-2} (1-u)^{-\alpha} \\
& = 2H(2H-1) \frac{\Gamma(2H-2\alpha)\beta(2H-1, 1-\alpha)}{\theta^{2H-2\alpha}} := C_1(\alpha, H, \theta) < \infty.
\end{aligned}$$

Combining this with the fact that $\int_n^\infty e^{-\theta(r-n)} dr = \frac{1}{\theta}$, we obtain

$$E \left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} dB_s \right|^2 \right) \leq C_2(\alpha, H, \theta) e^{-2\theta n},$$

with

$$C_2(\alpha, H, \theta) = \frac{c_\alpha^{-2} \Gamma(2\alpha-1) e^\theta}{\theta^{2\alpha}} C_1(\alpha, H, \theta).$$

Consequently,

$$\begin{aligned}
\sum_{n \geq 0} P \left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} dB_s \right| > \varepsilon \right) & \leq \varepsilon^{-2} \sum_{n \geq 0} E \left(\sup_{n \leq t \leq n+1} \left| \int_t^\infty e^{-\theta s} dB_s \right|^2 \right) \\
& \leq \varepsilon^{-2} C_2(\alpha, H, \theta) \sum_{n \geq 0} e^{-2\theta n} < \infty.
\end{aligned}$$

This finishes the proof of the claim (11), and thus the proof of Lemma 2. ■

Proof of Lemma 3. Using (10), we have

$$E[\xi_\infty^2] = \frac{H\Gamma(2H)}{\theta^{2H}} < \infty.$$

Hence $\xi_\infty \sim \mathcal{N}(0, \frac{H\Gamma(2H)}{\theta^{2H}})$ and this implies that

$$P(\xi_\infty = 0) = 0. \quad (12)$$

The continuity of ξ entails that, for every $t > 0$

$$\int_0^t e^{2\theta s} \xi_s^2 ds \geq \int_{\frac{t}{2}}^t e^{2\theta s} \xi_s^2 ds \geq \frac{t}{2} e^{\theta t} \left(\inf_{\frac{t}{2} \leq s \leq t} \xi_s^2 \right) \text{ almost surely.} \quad (13)$$

Furthermore, the continuity of ξ and the point ii) of Lemma 2 yield

$$\lim_{t \rightarrow \infty} \left(\inf_{\frac{t}{2} \leq s \leq t} \xi_s^2 \right) = \xi_\infty^2 \text{ almost surely.}$$

Combining this last convergence with (13) and (12), we deduce that

$$\lim_{t \rightarrow \infty} \int_0^t e^{2\theta s} \xi_s^2 ds = \infty \text{ almost surely.}$$

Hence, we can use L'Hôpital's rule and we obtain

$$\lim_{t \rightarrow \infty} \frac{\int_0^t e^{2\theta s} \xi_s^2 ds}{e^{2\theta t}} = \lim_{t \rightarrow \infty} \frac{\xi_t^2}{2\theta} = \frac{\xi_\infty^2}{2\theta} \text{ almost surely.}$$

This completes the proof of Lemma 3. ■

Proof of Theorem 1. Using the change of variable formula (4), we conclude that

$$\frac{1}{2} e^{2\theta t} \xi_t^2 = \theta \int_0^t e^{2\theta s} \xi_s^2 ds + \int_0^t e^{\theta s} \xi_s dB_s$$

Hence

$$\widehat{\theta}_t - \theta = \frac{\int_0^t e^{\theta s} \xi_s dB_s}{\int_0^t e^{2\theta s} \xi_s^2 ds} = \frac{\xi_t^2}{2e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds} - \theta.$$

Combining this with Lemma 2 and Lemma 3, we deduce that $\widehat{\theta}_t \rightarrow \theta$ almost surely as $t \rightarrow \infty$. ■

3.2 Asymptotic distribution of the estimator LSE

This paragraph is devoted to the investigation of asymptotic distribution of the LSE $\widehat{\theta}_t$ of θ . We start with the following lemma.

Lemma 4 *Suppose that $H > \frac{1}{2}$. Then, for every $t \geq 0$, we have*

$$\begin{aligned} \int_0^t dB_s e^{\theta s} \int_0^s dB_r e^{-\theta r} &= \int_0^t dB_s e^{\theta s} \int_0^t dB_r e^{-\theta r} - \int_0^t \delta B_s e^{-\theta s} \int_0^s \delta B_r e^{\theta r} \\ &\quad - H(2H - 1) \int_0^t ds e^{-\theta s} \int_0^s dr e^{\theta r} |s - r|^{2H-2}. \end{aligned}$$

Proof. Let $t \geq 0$. By the change of variable formula (4)

$$\int_0^t dB_s e^{\theta s} \int_0^s dB_r e^{-\theta r} = \int_0^t dB_s e^{\theta s} \int_0^t dB_r e^{-\theta r} - \int_0^t dB_s e^{-\theta s} \int_0^s dB_r e^{\theta r}.$$

On the other hand, according to (5) and (6),

$$\begin{aligned} & \int_0^t dB_s e^{-\theta s} \int_0^s dB_r e^{\theta r} \\ &= \int_0^t \delta B_s e^{-\theta s} \int_0^s \delta B_r e^{\theta r} + H(2H-1) \int_0^t ds e^{-\theta s} \int_0^s dr e^{\theta r} |s-r|^{2H-2}, \end{aligned}$$

which completes the proof. ■

Theorem 5 Let $H > \frac{1}{2}$ be fixed. Then, as $t \rightarrow \infty$,

$$e^{\theta t} (\hat{\theta}_t - \theta) \xrightarrow{\text{law}} 2\theta C(1),$$

with $C(1)$ the standard Cauchy distribution.

In order to prove Theorem 5 we need the following two lemmas.

Lemma 6 Fix $H > \frac{1}{2}$. Let F be any $\sigma\{B\}$ -measurable random variable such that $P(F < \infty) = 1$. Then, as $t \rightarrow \infty$,

$$\left(F, e^{-\theta t} \int_0^t e^{\theta s} dB_s \right) \xrightarrow{\text{law}} \left(F, \frac{\sqrt{H\Gamma(2H)}}{\theta^H} N \right),$$

where $N \sim \mathcal{N}(0, 1)$ is independent of B .

Lemma 7 Let $H > \frac{1}{2}$. Then, as $t \rightarrow \infty$,

$$e^{-\frac{\theta t}{2}} \int_0^t \delta B_s e^{-\theta s} \int_0^s \delta B_r e^{\theta r} \rightarrow 0 \quad \text{in } L^2(\Omega), \quad (14)$$

and

$$e^{-\frac{\theta t}{2}} \int_0^t ds e^{-\theta s} \int_0^s dr e^{\theta r} |s-r|^{2H-2} \rightarrow 0. \quad (15)$$

Proof of Lemma 6. For any $d \geq 1$, $s_1 \dots s_d \in [0, \infty)$, we shall prove that, as $t \rightarrow \infty$,

$$\left(B_{s_1}, \dots, B_{s_d}, e^{-\theta t} \int_0^t e^{\theta s} dB_s \right) \xrightarrow{\text{law}} \left(B_{s_1}, \dots, B_{s_d}, \frac{\sqrt{H\Gamma(2H)}}{\theta^H} N \right) \quad (16)$$

which is enough to lead to the desired conclusion. Because the left-hand side in the previous convergence is a Gaussian vector (see proof of [Lemma 7, [5]]), to get (16) it is sufficient to check the convergence of its covariance matrix. Let us first compute the limiting variance of

$e^{-\theta t} \int_0^t e^{\theta s} dB_s$ as $t \rightarrow \infty$. We have

$$\begin{aligned}
E \left[\left(e^{-\theta t} \int_0^t e^{\theta s} dB_s \right)^2 \right] &= H(2H-1)e^{-2\theta t} \int_0^t \int_0^t e^{\theta s} e^{\theta r} |s-r|^{2H-2} dr ds \\
&= 2H(2H-1)e^{-2\theta t} \int_0^t ds e^{\theta s} \int_0^s dr e^{\theta r} |s-r|^{2H-2} \\
&= 2H(2H-1)e^{-2\theta t} \int_0^t ds e^{2\theta s} \int_0^s dr e^{-\theta r} r^{2H-2} \\
&= 2H(2H-1)e^{-2\theta t} \int_0^t dr e^{-\theta r} r^{2H-2} \int_r^t ds e^{2\theta s} \\
&= \frac{H(2H-1)}{\theta} \left(\int_0^t r^{2H-2} e^{-\theta r} dr - e^{-2\theta t} \int_0^t r^{2H-2} e^{\theta r} dr \right) \\
&\rightarrow \frac{H\Gamma(2H)}{\theta^{2H}} \text{ as } t \rightarrow \infty,
\end{aligned}$$

because $e^{-2\theta t} \int_0^t r^{2H-2} e^{\theta r} dr \leq e^{-\theta t} \int_0^t r^{2H-2} dr = \frac{t^{2H-1}}{(2H-1)e^{\theta t}} \rightarrow 0$ as $t \rightarrow \infty$.
Thus,

$$\lim_{t \rightarrow \infty} E \left[\left(e^{-\theta t} \int_0^t e^{\theta s} dB_s \right)^2 \right] = \frac{H\Gamma(2H)}{\theta^{2H}}.$$

Hence, to finish the proof it remains to check that, for all fixed $s \geq 0$,

$$\lim_{t \rightarrow \infty} E \left(B_s \times e^{-\theta t} \int_0^t e^{\theta v} dB_v \right) = 0.$$

Indeed, for $s < t$,

$$\begin{aligned}
&E \left(B_s \times e^{-\theta t} \int_0^t e^{\theta v} dB_v \right) \\
&= H(2H-1)e^{-\theta t} \int_0^t dv e^{\theta v} \int_0^s du |u-v|^{2H-2} \\
&= H(2H-1)e^{-\theta t} \int_0^s dv e^{\theta v} \int_0^s du |u-v|^{2H-2} + H(2H-1)e^{-\theta t} \int_s^t dv e^{\theta v} \int_0^s du (v-u)^{2H-2} \\
&= H(2H-1)e^{-\theta t} \int_0^s dv e^{\theta v} \int_0^s du |u-v|^{2H-2} + H e^{-\theta t} \int_s^t e^{\theta v} (v^{2H-1} - (v-s)^{2H-1}) dv \\
&:= I_t + J_t.
\end{aligned}$$

It's clear that $I_t \rightarrow 0$ as $t \rightarrow \infty$.

Using integration by parts, the term J_t can be written as

$$\begin{aligned}
&J_t \\
&= H e^{-\theta t} \int_s^t e^{\theta v} (v^{2H-1} - (v-s)^{2H-1}) dv \\
&= H e^{-\theta t} \left(\frac{e^{\theta t}}{\theta} [t^{2H-1} - (t-s)^{2H-1}] - \frac{e^{\theta s}}{\theta} s^{2H-1} + \frac{2H-1}{\theta} \int_s^t e^{\theta v} [(v-s)^{2H-2} - v^{2H-2}] dv \right) \\
&\leq \frac{H}{\theta} [t^{2H-1} - (t-s)^{2H-1}] + \frac{H(2H-1)}{\theta} e^{-\theta t} \int_s^t e^{\theta v} (v-s)^{2H-2} dv \\
&:= J_t^1 + J_t^2.
\end{aligned}$$

Since $H < 1$, $J_t^1 = \frac{H}{\theta}[t^{2H-1} - (t-s)^{2H-1}] \rightarrow 0$ as $t \rightarrow \infty$.
On the other hand,

$$\begin{aligned}
J_t^2 &= \frac{H(2H-1)}{\theta} e^{-\theta t} \int_s^t e^{\theta v} (v-s)^{2H-2} dv \\
&= \frac{H(2H-1)e^{\theta s}}{\theta} e^{-\theta t} \int_0^{t-s} e^{\theta u} u^{2H-2} du \\
&\leq \frac{H(2H-1)e^{\theta s}}{\theta} e^{-\theta t} \int_0^t e^{\theta u} u^{2H-2} du \\
&= \frac{H(2H-1)e^{\theta s}}{\theta} \int_0^t e^{-\theta v} (t-v)^{2H-2} dv \\
&= \frac{H(2H-1)e^{\theta s}}{\theta} t^{2H-1} \int_0^1 e^{-\theta t u} (1-u)^{2H-2} du
\end{aligned}$$

Fix $u \in (0, 1)$. The function $t \in [0, \infty) \mapsto t^{2H-1} e^{-\theta t u}$ attains its maximum at $t = \frac{2H-1}{\theta u}$. Then

$$\sup_{t \geq 0} (t^{2H-1} e^{-\theta t u}) = c e^{-\frac{2H-1}{u}} u^{1-2H} \leq c u^{1-2H},$$

with $c = \left(\frac{2H-1}{\theta}\right)^{2H-1}$. In addition, $\int_0^1 u^{1-2H} (1-u)^{2H-2} du < \infty$, and for any $u \in (0, 1)$,

$$t^{2H-1} e^{-\theta t u} (1-u)^{2H-2} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Therefore, using the dominated convergence theorem, we obtain that J_t^2 converges to 0 as $t \rightarrow \infty$. Thus, we deduce the desired conclusion. ■

Proof of Lemma 7. Let us prove the convergence (14). We have

$$\begin{aligned}
&e^{-\theta t} E \left(\int_0^t \delta B_s e^{-\theta s} \int_0^s \delta B_r e^{\theta r} \right)^2 \\
&= e^{-\theta t} E \left(\int_0^t \int_0^s e^{-\theta|s-r|} \delta B_r \delta B_s \right)^2 \\
&= e^{-\theta t} E \left(\frac{1}{2} I_2(e^{-\theta|s-r|} 1_{[0,t]^2}) \right)^2 \\
&= \frac{H^2(2H-1)^2}{2} e^{-\theta t} \int_{[0,t]^4} e^{-\theta|v-s|} e^{-\theta|u-r|} |v-u|^{2H-2} |s-r|^{2H-2} dudvdrds \\
&\leq \frac{H^2(2H-1)^2}{2} e^{-\theta t} \int_{[0,t]^4} |v-u|^{2H-2} |s-r|^{2H-2} dudvdrds \\
&= \frac{1}{2} [E(B_t^2)]^2 e^{-\theta t} = \frac{1}{2} t^{4H} e^{-\theta t} \\
&\rightarrow 0 \text{ as } t \rightarrow \infty.
\end{aligned}$$

For the convergence (15), we have

$$\begin{aligned}
&H(2H-1) e^{-\frac{\theta t}{2}} \int_0^t ds e^{-\theta s} \int_0^s dr e^{\theta r} |s-r|^{2H-2} \\
&\leq H(2H-1) e^{-\frac{\theta t}{2}} \int_0^t ds \int_0^s dr |s-r|^{2H-2} \\
&= \frac{t^{2H}}{2} e^{-\frac{\theta t}{2}} \\
&\rightarrow 0 \text{ as } t \rightarrow \infty.
\end{aligned}$$

This finishes the proof. ■

Proof of the theorem 5. By combining (9) and Lemma 4, we can write,

$$\begin{aligned}
e^{\theta t} \left(\widehat{\theta}_t - \theta \right) &= \frac{e^{\theta t} \int_0^t dB_s e^{\theta s} \int_0^s dB_r e^{-\theta r}}{\int_0^t e^{2\theta s} \xi_s^2 ds} \\
&= \frac{\xi_t \xi_\infty}{e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds} \times \frac{e^{-\theta t} \int_0^t e^{\theta s} dB_s}{\xi_\infty} \\
&\quad - \frac{e^{-\theta t} \int_0^t \delta B_s e^{-\theta s} \int_0^s \delta B_r e^{\theta r}}{e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds} \\
&\quad - H(2H-1) \frac{e^{-\theta t} \int_0^t ds e^{-\theta s} \int_0^s dr e^{\theta r} |s-r|^{2H-2}}{e^{-2\theta t} \int_0^t e^{2\theta s} \xi_s^2 ds} \\
&:= A_t^\theta \times B_t^\theta - C_t^\theta - D_t^\theta.
\end{aligned}$$

Using Lemme 2 and Lemma 3, we obtain that

$$A_t^\theta \longrightarrow 2\theta \text{ almost surely as } t \longrightarrow \infty.$$

According to Lemma 6, we deduce

$$B_t^\theta \xrightarrow{\text{law}} \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \frac{N}{\xi_\infty} \text{ as } t \longrightarrow \infty.$$

Moreover,

$$\frac{\sqrt{H\Gamma(2H)}}{\theta^H} \frac{N}{\xi_\infty} \xrightarrow{\text{law}} \mathcal{C}(1),$$

because $\frac{\theta^H \xi_\infty}{\sqrt{H\Gamma(2H)}} \sim \mathcal{N}(0, 1)$ and $N \sim \mathcal{N}(0, 1)$ are independent.

Thus, by Slutsky's theorem, we conclude that

$$A_t^\theta \times B_t^\theta \xrightarrow{\text{law}} 2\theta \mathcal{C}(1) \text{ as } t \longrightarrow \infty.$$

On the other hand, it follows from Lemma 3 and Lemma 7, that

$$C_t^\theta \xrightarrow{\text{prob.}} 0 \text{ as } t \longrightarrow \infty,$$

and

$$D_t^\theta \longrightarrow 0 \text{ almost surely as } t \longrightarrow \infty.$$

Finally, by combining the previous convergences, the proof of Theorem 5 is done. ■

Acknowledgments. The authors would like to thank Ivan Nourdin for many valuable discussions on the subject. We warmly thank him for proving that J_t converges to zero (see Proof of Lemma 6).

References

- [1] E. Alòs and D. Nualart. *Stochastic integration with respect to the fractional Brownian motion. Stoch. stoch. reports* 75, no. 3, (2003), 129-152.
- [2] M. Barczy and G. Pap. *Asymptotic behavior of maximum likelihood estimator for time inhomogeneous diffusion processes. J. Statist. Plan. Infer.* 140, no. 6, (2010), 1576-1593.

- [3] I.V. Basawa and D.J. Scott. *Asymptotic optimal inference for non-ergodic models. Lecture Notes in Statistics, 17, Springer, New York, (1983).*
- [4] H.M. Dietz and YU.A. Kutoyants. *Parameter estimation for some non-recurrent solutions of SDE. Statistics and Decisions 21(1), (2003), 29-46.*
- [5] K. Es-Sebaiy, I. Nourdin. *Parameter estimation for α -fractional bridges. (2011), Submitted. <http://arxiv.org/abs/1101.5790v1>.*
- [6] Y. Hu and H. Long. *Parameter estimation for Ornstein-Uhlenbeck processes driven by α -stable Lévy motions. Commun. Stoch. Anal. 1, (2007), 175-192.*
- [7] Y. Hu and D. Nualart. *Parameter estimation for fractional Ornstein-Uhlenbeck processes. Statistics and Probability Letters 80, (2010), 1030-1038.*
- [8] Yu. A. Kutoyants. *Statistical Inference for Ergodic Diffusion Processes. Springer, Berlin, Heidelberg, (2004).*
- [9] R. S. Liptser and A. N. Shiryaev. *Statistics of Random Processes: II Applications. Second Edition, Applications of Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, (2001).*
- [10] I. Nourdin. *An invitation to fractional Brownian motion. Doctoral course. <http://www.iecn.u-nancy.fr/~nourdin/lectures-fbm.pdf>, (2010).*
- [11] D. Nualart. *The Malliavin calculus and related topics. Springer-Verlag, Berlin, second edition, (2006).*
- [12] D. Nualart and G. Peccati. *Central limit theorems for sequences of multiple stochastic integrals. Ann. Probab. 33 (1), (2005), 177-193.*
- [13] V. Pipiras and M.S. Taqqu. *Integration questions related to fractional Brownian motion Probab. Theory Rel. Fields, 118, no. 2, (2000), 251-291.*
- [14] L. C. Young. *An inequality of the Hölder type connected with Stieltjes integration. Acta Math. 67, (1936), 251-282.*