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# GLOBAL STABILITY FOR THE MULTI-CHANNEL GEL'FAND-CALDERÓN INVERSE PROBLEM IN TWO DIMENSIONS 

MATTEO SANTACESARIA


#### Abstract

We prove a global logarithmic stability estimate for the multi-channel Gel'fand-Calderón inverse problem on a two-dimensional bounded domain, i.e. the inverse boundary value problem for the equation $-\Delta \psi+v \psi=0$ on $D$, where $v$ is a smooth matrix-valued potential defined on a bounded planar domain $D$.


## 1. Introduction

The Schrödinger equation at zero energy

$$
\begin{equation*}
-\Delta \psi+v(x) \psi=0 \text { on } D \subset \mathbb{R}^{2} \tag{1.1}
\end{equation*}
$$

arises in quantum mechanics, acoustics and electrodynamics. The reconstruction of the complex-valued potential $v$ in equation (1.1) through the Dirichlet-to-Neumann operator is one of the most studied inverse problems (see [9], [8], [3], 10], (11], 12] and references therein).

In this article we consider the multi-channel two-dimensional Schrödinger equation, i.e. equation (1.1) with matrix-valued potentials and solutions; this case was already studied in [13, 12]. One of the motivations for studying the multi-channel equation is that it comes up as a 2 D -approximation for the 3D equation (see [12, Sec. 2]).

This paper is devoted to give a global stability estimate for this inverse problem in the multi-channel case, which is highly related to the reconstruction method of [12].

Let $D$ be an open bounded domain in $\mathbb{R}^{2}$ with $C^{2}$ boundary and $v \in$ $C^{1}\left(\bar{D}, M_{n}(\mathbb{C})\right)$, where $M_{n}(\mathbb{C})$ is the set of the $n \times n$ complex-valued matrices. The Dirichlet-to-Neumann map associated to $v$ is the operator $\Phi$ : $C^{1}\left(\partial D, M_{n}(\mathbb{C})\right) \rightarrow L^{p}\left(\partial D, M_{n}(\mathbb{C})\right), p<\infty$ defined by:

$$
\begin{equation*}
\Phi(f)=\left.\frac{\partial \psi}{\partial \nu}\right|_{\partial D} \tag{1.2}
\end{equation*}
$$

where $f \in C^{1}\left(\partial D, M_{n}(\mathbb{C})\right), \nu$ is the outer normal of $\partial D$ and $\psi$ is the $H^{1}\left(\bar{D}, M_{n}(\mathbb{C})\right)$-solution of the Dirichlet problem

$$
\begin{equation*}
-\Delta \psi+v(x) \psi=0 \text { on } D,\left.\quad \psi\right|_{\partial D}=f \tag{1.3}
\end{equation*}
$$

here we assume that
(1.4) 0 is not a Dirichlet eigenvalue for the operator $-\Delta+v$ in $D$.

The following inverse boundary value problem arises from this construction: given $\Phi$, find $v$.

This problem can be considered as the Gel'fand inverse boundary value problem for the multi-channel Schrödinger equation at zero energy (see [6], [9]) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see [4], [6]). Note also that we can think of this problem as a model for the monochromatic ocean tomography (e.g. see [2] for similar problems arising in this tomography).

In the case of complex-valued potentials the global injectivity of the map $v \rightarrow \Phi$ was firstly proved in [9] for $D \subset \mathbb{R}^{d}$ with $d \geq 3$ and in [3] for $d=2$ with $v \in L^{p}$ : in particular, these results were obtained by the use of global reconstructions developed in the same papers. The first global uniqueness result (along with an exact reconstruction method) for matrix-valued potentials was given in [12], which deals with $C^{1}$ matrix-valued potentials defined on a domain in $\mathbb{R}^{2}$. A global stability estimate for the Gel'fand-Calderón problem for $d \geq 3$ was found for the first time by Alessandrini in [1]; this result was recently improved in 10. In the two-dimensional case the first global stability estimate was given in 11.

In this paper we extend the results of [11] to the matrix-valued case; we do not discuss global results for special real-valued potentials arising from conductivities: for this case the reader is referred to the references given in (1), (3], [8], [9], (10], [17].

Our main result is the following:
Theorem 1.1. Let $D \subset \mathbb{R}^{2}$ be an open bounded domain with $C^{2}$ boundary, let $v_{1}, v_{2} \in C^{2}\left(\bar{D}, M_{n}(\mathbb{C})\right)$ be two matrix-valued potentials which satisfy (1.4), with $\left\|v_{j}\right\|_{C^{2}(\bar{D})} \leq N$ for $j=1,2$, and $\Phi_{1}, \Phi_{2}$ the corresponding Dirichlet-to-Neumann operators. For simplicity we assume also that $\left.v_{j}\right|_{\partial D}=0$ and $\left.\frac{\partial}{\partial \nu} v_{j}\right|_{\partial D}=0$ for $j=1,2$. Then there exists a constant $C=C(D, N, n)$ such that

$$
\begin{equation*}
\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq C\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)\right)^{-\frac{3}{4}}\left(\log \left(3 \log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)\right)\right)^{2} \tag{1.5}
\end{equation*}
$$

where $\|A\|$ denotes the norm of an operator $A: L^{\infty}\left(\partial D, M_{n}(\mathbb{C})\right) \rightarrow L^{\infty}\left(\partial D, M_{n}(\mathbb{C})\right)$ and $\|v\|_{L^{\infty}(D)}=\max _{1 \leq i, j \leq n}\left\|v_{i, j}\right\|_{L^{\infty}(D)}$ (likewise for $\|v\|_{C^{2}(\bar{D})}$ ) for a matrixvalued potential $v$.

This is the first global stability result for the multi-channel $(n \geq 2)$ Gel'fand-Calderón inverse problem in two dimension. In addition, Theorem 1.1 is new also for the scalar case, as the estimate obtained in 11] is weaker.

Instability estimates complementing the stability estimates of [1], 10], [11] and of the present work are given in (8], [7].

The proof of Theorem 1.1 is based on results obtained in [11], [12], which takes inspiration mostly from [3] and [1]. In particular, for $z_{0} \in D$ we use the existence and uniqueness of a family of solution $\psi_{z_{0}}(z, \lambda)$ of equation (1.1) where in particular $\psi_{z_{0}} \rightarrow e^{\lambda\left(z-z_{0}\right)^{2}} I$, for $\lambda \rightarrow \infty$ (where $I$ is the identity matrix). Then, using an appropriate matrix-valued version of Alessandrini's identity along with stationary phase techniques, we obtain the result. Note that this matrix-valued identity is one of the new results of this paper.

A generalization of Theorem 1.1 in the case where we do not assume that $\left.v_{j}\right|_{\partial D}=0$ and $\left.\frac{\partial}{\partial \nu} v_{j}\right|_{\partial D}=0$ for $j=1,2$, is given in section 5 .

This work was fulfilled in the framework of researches under the direction of R. G. Novikov.

## 2. Preliminaries

In this section we introduce and give details about the above-mentioned family of solutions of equation (1.1), which will be used throughout all the paper.

We identify $\mathbb{R}^{2}$ with $\mathbb{C}$ and use the coordinates $z=x_{1}+i x_{2}, \bar{z}=x_{1}-i x_{2}$ where $\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$. Let us define the function spaces $C_{\bar{z}}^{1}(\bar{D})=\left\{u: u, \frac{\partial u}{\partial \bar{z}} \in\right.$ $\left.C\left(\bar{D}, M_{n}(\mathbb{C})\right)\right\}$ with the norm $\|u\|_{C_{\bar{z}}^{1}(\bar{D})}=\max \left(\|u\|_{C(\bar{D})},\left\|\frac{\partial u}{\partial \bar{z}}\right\|_{C(\bar{D})}\right)$, where $\|u\|_{C(\bar{D})}=\sup _{z \in \bar{D}}|u|$ and $|u|=\max _{1 \leq i, j \leq n}\left|u_{i, j}\right| ;$ we define also $C_{z}^{1}(\bar{D})=$ $\left\{u: u, \frac{\partial u}{\partial z} \in C\left(\bar{D}, M_{n}(\mathbb{C})\right)\right\}$ with an analogous norm. Following (11], 12], we consider the functions:

$$
\begin{align*}
G_{z_{0}}(z, \zeta, \lambda) & =e^{\lambda\left(z-z_{0}\right)^{2}} g_{z_{0}}(z, \zeta, \lambda) e^{-\lambda\left(\zeta-z_{0}\right)^{2}}  \tag{2.1}\\
g_{z_{0}}(z, \zeta, \lambda) & =\frac{e^{\lambda\left(\zeta-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{\zeta}-\bar{z}_{0}\right)^{2}}}{4 \pi^{2}} \int_{D} \frac{e^{-\lambda\left(\eta-z_{0}\right)^{2}+\bar{\lambda}\left(\bar{\eta}-\bar{z}_{0}\right)^{2}}}{(z-\eta)(\bar{\eta}-\bar{\zeta})} d \operatorname{Re} \eta d \operatorname{Im} \eta  \tag{2.2}\\
\psi_{z_{0}}(z, \lambda) & =e^{\lambda\left(z-z_{0}\right)^{2}} \mu_{z_{0}}(z, \lambda)  \tag{2.3}\\
\mu_{z_{0}}(z, \lambda) & =I+\int_{D} g_{z_{0}}(z, \zeta, \lambda) v(\zeta) \mu_{z_{0}}(\zeta, \lambda) d \operatorname{Re} \zeta d \operatorname{Im} \zeta \tag{2.4}
\end{align*}
$$

$$
\begin{equation*}
h_{z_{0}}(\lambda)=\int_{D} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} v(z) \mu_{z_{0}}(z, \lambda) d \operatorname{Re} z d \operatorname{Im} z \tag{2.5}
\end{equation*}
$$

where $z, z_{0}, \zeta \in D$ and $\lambda \in \mathbb{C}$ and $I$ is the identity matrix. In addition, equation (2.4) at fixed $z_{0}$ and $\lambda$, is considered as a linear integral equation for $\mu_{z_{0}}(\cdot, \lambda) \in C_{\bar{z}}^{1}(\bar{D})$. The functions $G_{z_{0}}(z, \zeta, \lambda), g_{z_{0}}(z, \zeta, \lambda), \psi_{z_{0}}(z, \lambda), \mu_{z_{0}}(z, \lambda)$ defined above, satisfy the following equations (see [11], 12]):

$$
\begin{align*}
& 4 \frac{\partial^{2}}{\partial z \partial \bar{z}} G_{z_{0}}(z, \zeta, \lambda)=\delta(z-\zeta)  \tag{2.6}\\
& 4 \frac{\partial^{2}}{\partial \zeta \partial \bar{\zeta}} G_{z_{0}}(z, \zeta, \lambda)=\delta(\zeta-z)  \tag{2.7}\\
& 4\left(\frac{\partial}{\partial z}+2 \lambda\left(z-z_{0}\right)\right) \frac{\partial}{\partial \bar{z}} g_{z_{0}}(z, \zeta, \lambda)=\delta(z-\zeta)  \tag{2.8}\\
& 4 \frac{\partial}{\partial \bar{\zeta}}\left(\frac{\partial}{\partial \zeta}-2 \lambda\left(\zeta-z_{0}\right)\right) g_{z_{0}}(z, \zeta, \lambda)=\delta(\zeta-z)  \tag{2.9}\\
&-4 \frac{\partial^{2}}{\partial z \partial \bar{z}} \psi_{z_{0}}(z, \lambda)+v(z) \psi_{z_{0}}(z, \lambda)=0  \tag{2.10}\\
&-4\left(\frac{\partial}{\partial z}+2 \lambda\left(z-z_{0}\right)\right) \frac{\partial}{\partial \bar{z}} \mu_{z_{0}}(z, \lambda)+v(z) \mu_{z_{0}}(z, \lambda)=0 \tag{2.11}
\end{align*}
$$

where $z, z_{0}, \zeta \in D, \lambda \in \mathbb{C}, \delta$ is the Dirac's delta. (In addition, it is assumed that (2.4) is uniquely solvable for $\mu_{z_{0}}(\cdot, \lambda) \in C_{\bar{z}}^{1}(\bar{D})$ at fixed $z_{0}$ and $\lambda$.)

We say that the functions $G_{z_{0}}, g_{z_{0}}, \psi_{z_{0}}, \mu_{z_{0}}, h_{z_{0}}$ are the Bukhgeim-type analogues of the Faddeev functions (see [12]).

Now we state some fundamental lemmata. Let

$$
\begin{equation*}
g_{z_{0}, \lambda} u(z)=\int_{D} g_{z_{0}}(z, \zeta, \lambda) u(\zeta) d \operatorname{Re} \zeta d \operatorname{Im} \zeta, z \in \bar{D}, z_{0}, \lambda \in \mathbb{C} \tag{2.12}
\end{equation*}
$$

where $g_{z_{0}}(z, \zeta, \lambda)$ is defined by (2.2) and $u$ is a test function.
Lemma 2.1 (11]). Let $g_{z_{0}, \lambda} u$ be defined by (2.12). Then, for $z_{0}, \lambda \in \mathbb{C}$, the following estimates hold:

$$
\begin{align*}
g_{z_{0}, \lambda} u \in C_{\bar{z}}^{1}(\bar{D}), \quad \text { for } u \in C(\bar{D})  \tag{2.13}\\
\left\|g_{z_{0}, \lambda} u\right\|_{C^{1}(\bar{D})} \leq c_{1}(D, \lambda)\|u\|_{C(\bar{D})}, \quad \text { for } u \in C(\bar{D})  \tag{2.14}\\
\left\|g_{z_{0}, \lambda} u\right\|_{C_{\bar{z}}^{1}(\bar{D})} \leq \frac{c_{2}(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C_{\bar{z}}^{1}(\bar{D})}, \quad \text { for } u \in C_{\bar{z}}^{1}(\bar{D}),|\lambda| \geq 1 \tag{2.15}
\end{align*}
$$

Given a potential $v \in C_{\bar{z}}^{1}(\bar{D})$ we define the operator $g_{z_{0}, \lambda} v$ simply as $\left(g_{z_{0}, \lambda} v\right) u(z)=g_{z_{0}, \lambda} w(z), w=v u$, for a test function $u$. If $u \in C_{\bar{z}}^{1}(\bar{D})$, by Lemma 2.1 we have that $g_{z_{0}, \lambda} v: C_{\bar{z}}^{1}(\bar{D}) \rightarrow C_{\bar{z}}^{1}(\bar{D})$,

$$
\begin{equation*}
\left\|g_{z_{0}, \lambda} v\right\|_{C_{\bar{z}}^{1}(\bar{D})}^{o p} \leq 2 n\left\|g_{z_{0}, \lambda}\right\|_{C_{\bar{z}}^{1}(\bar{D})}^{o p}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \tag{2.16}
\end{equation*}
$$

where $\|\cdot\|_{C_{\bar{z}}^{1}(\bar{D})}^{o p}$ denotes the operator norm in $C_{\bar{z}}^{1}(\bar{D}), z_{0}, \lambda \in \mathbb{C}$. In addition, $\left\|g_{z_{0}, \lambda}\right\|_{C_{\bar{z}}^{1}(\bar{D})}^{o p}$ is estimated in Lemma 2.1. Inequality (2.16) and Lemma 2.1 imply existence and uniqueness of $\mu_{z_{0}}(z, \lambda)$ (and thus also $\psi_{z_{0}}(z, \lambda)$ ) for $|\lambda|>\rho(D, K, n)$, where $\|v\|_{C_{\bar{z}}^{1}(\bar{D})}<K$.

Let

$$
\begin{aligned}
\mu_{z_{0}}^{(k)}(z, \lambda) & =\sum_{j=0}^{k}\left(g_{z_{0}, \lambda} v\right)^{j} I \\
h_{z_{0}}^{(k)}(\lambda) & =\int_{D} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} v(z) \mu_{z_{0}}^{(k)}(z, \lambda) d \operatorname{Re} z d \operatorname{Im} z,
\end{aligned}
$$

where $z, z_{0} \in D, \lambda \in \mathbb{C}, k \in \mathbb{N} \cup\{0\}$.
Lemma $2.2(11)$. For $v \in C_{\bar{z}}^{1}(\bar{D})$ such that $\left.v\right|_{\partial D}=0$ the following formula holds:

$$
\begin{equation*}
v\left(z_{0}\right)=\frac{2}{\pi} \lim _{\lambda \rightarrow \infty}|\lambda| h_{z_{0}}^{(0)}(\lambda), z_{0} \in D \tag{2.17}
\end{equation*}
$$

In addition, if $v \in C^{2}(\bar{D}),\left.v\right|_{\partial D}=0$ and $\left.\frac{\partial v}{\partial \nu}\right|_{\partial D}=0$ then

$$
\begin{equation*}
\left|v\left(z_{0}\right)-\frac{2}{\pi}\right| \lambda\left|h_{z_{0}}^{(0)}(\lambda)\right| \leq c_{3}(D, n) \frac{\log (3|\lambda|)}{|\lambda|}\|v\|_{C^{2}(\bar{D})} \tag{2.18}
\end{equation*}
$$

for $z_{0} \in D, \lambda \in \mathbb{C},|\lambda| \geq 1$.
Let

$$
W_{z_{0}}(\lambda)=\int_{D} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} w(z) d \operatorname{Re} z d \operatorname{Im} z
$$

where $z_{0} \in \bar{D}, \lambda \in \mathbb{C}$ and $w$ is some $M_{n}(\mathbb{C})$-valued function on $\bar{D}$. (One can see that $W_{z_{0}}=h_{z_{0}}^{(0)}$ for $w=v$.)

Lemma $2.3(11)$. For $w \in C_{\bar{z}}^{1}(\bar{D})$ the following estimate holds:

$$
\begin{equation*}
\left|W_{z_{0}}(\lambda)\right| \leq c_{4}(D) \frac{\log (3|\lambda|)}{|\lambda|}\|w\|_{C_{\bar{z}}^{1}(\bar{D})}, \quad z_{0} \in \bar{D},|\lambda| \geq 1 \tag{2.19}
\end{equation*}
$$

Lemma 2.4 ([12]). For $v \in C_{\bar{z}}^{1}(\bar{D})$ and for $\left\|g_{z_{0}, \lambda} v\right\|_{C_{\bar{z}}^{1}(\bar{D})}^{o p} \leq \delta<1$ we have that

$$
\begin{align*}
& \left\|\mu_{z_{0}}(\cdot, \lambda)-\mu_{z_{0}}^{(k)}(\cdot, \lambda)\right\|_{C_{\bar{z}}^{1}(\bar{D})} \leq \frac{\delta^{k+1}}{1-\delta}  \tag{2.20}\\
& \left|h_{z_{0}}(\lambda)-h_{z_{0}}^{(k)}(\lambda)\right| \leq c_{5}(D, n) \frac{\log (3|\lambda|)}{|\lambda|} \frac{\delta^{k+1}}{1-\delta}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \tag{2.21}
\end{align*}
$$

where $z_{0} \in D, \lambda \in \mathbb{C},|\lambda| \geq 1, k \in \mathbb{N} \cup\{0\}$.
The proofs of Lemmata 2.1-2.4 can be found in the references given.
We will also need the following two new lemmata.

Lemma 2.5. Let $g_{z_{0}, \lambda} u$ be defined by (2.12), where $u \in C_{\bar{z}}^{1}(\bar{D}), z_{0}, \lambda \in \mathbb{C}$.
Then the following estimate holds:

$$
\begin{equation*}
\left\|g_{z_{0}, \lambda} u\right\|_{C(\bar{D})} \leq c_{6}(D) \frac{\log (3|\lambda|)}{|\lambda|}\|u\|_{C_{\bar{z}}^{1}(\bar{D})},|\lambda| \geq 1 \tag{2.22}
\end{equation*}
$$

Lemma 2.6. The expression

$$
\begin{equation*}
W(u, v)(\lambda)=\int_{D} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} u(z)\left(g_{z_{0}, \lambda} v\right)(z) d \operatorname{Re} z d \operatorname{Im} z \tag{2.23}
\end{equation*}
$$

defined for $u, v \in C_{\overline{\bar{z}}}^{1}(\bar{D})$ with $\|u\|_{C_{\bar{z}}^{1}(\bar{D})},\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \leq N_{1}, \lambda \in \mathbb{C}, z_{0} \in D$, satisfies the estimate

$$
\begin{equation*}
|W(u, v)(\lambda)| \leq c_{7}\left(D, N_{1}, n\right) \frac{(\log (3|\lambda|))^{2}}{|\lambda|^{1+3 / 4}}, \quad|\lambda| \geq 1 \tag{2.24}
\end{equation*}
$$

The proofs of Lemmata 2.5, 2.6 are given in section 0.

## 3. Proof of Theorem 1.1

We begin with a technical lemma, which will be useful to generalise Alessandrini's identity.

Lemma 3.1. Let $v \in C^{1}\left(\bar{D}, M_{n}(\mathbb{C})\right)$ be a matrix-valued potential which satisfies condition ( $\overline{1.4}$ ) (i.e. 0 is not a Dirichlet eigeinvalue for the operator $-\Delta+v$ in $D$ ). Then ${ }^{t} v$, the transpose of $v$, also satisfies condition (1.4).

The proof of Lemma 3.1 is given in section 4.
We can now state and prove a matrix-valued version of Alessandrini's identity (see [1] for the scalar case).

Lemma 3.2. Let $v_{1}, v_{2} \in C^{1}\left(\bar{D}, M_{n}(\mathbb{C})\right)$ be two matrix-valued potentials which satisfy (1.4), $\Phi_{1}, \Phi_{2}$ their associated Dirichlet-to-Neumann operators, respectively, and $u_{1}, u_{2} \in C^{2}\left(\bar{D}, M_{n}(\mathbb{C})\right)$ matrix-valued functions such that

$$
\left(-\Delta+v_{1}\right) u_{1}=0, \quad\left(-\Delta+{ }^{t} v_{2}\right) u_{2}=0 \quad \text { on } D
$$

where ${ }^{t} A$ stand for the transpose of $A$. Then we have the identity

$$
\begin{equation*}
\int_{\partial D}^{t} u_{2}(z)\left(\Phi_{2}-\Phi_{1}\right) u_{1}(z)|d z|=\int_{D}{ }^{t} u_{2}(z)\left(v_{2}(z)-v_{1}(z)\right) u_{1}(z) d \operatorname{Re} z d \operatorname{Im} z \tag{3.1}
\end{equation*}
$$

Proof. If $v \in C^{1}\left(\bar{D}, M_{n}(\mathbb{C})\right)$ is any matrix-valued potential (which satisfies (1.4)) and $f_{1}, f_{2} \in C^{1}\left(\partial D, M_{n}(\mathbb{C})\right)$ then we have

$$
\begin{equation*}
\int_{\partial D}{ }^{t} f_{2} \Phi f_{1}|d z|=\int_{\partial D}{ }^{t}\left({ }^{t} f_{1} \Phi^{*} f_{2}\right)|d z|, \tag{3.2}
\end{equation*}
$$

where $\Phi$ and $\Phi^{*}$ are the Dirichlet-to-Neumann operators associated to $v$ and ${ }^{t} v$, respectively (these operators are well-defined thanks to Lemma 3.1).

Indeed, it is sufficient to extend $f_{1}$ and $f_{2}$ in $D$ as the solutions of the Dirichlet problems $(-\Delta+v) \tilde{f}_{1}=0,\left(-\Delta+{ }^{t} v\right) \tilde{f}_{2}=0$ on $D$ and $\left.\tilde{f}_{j}\right|_{\partial D}=f_{j}$, for $j=1,2$, so that one obtains

$$
\begin{aligned}
\int_{\partial D} & \left({ }^{t} f_{2} \Phi f_{1}-{ }^{t}\left({ }^{t} f_{1} \Phi^{*} f_{2}\right)\right)|d z| \\
& =\int_{\partial D}\left({ }^{t} f_{2} \frac{\partial \tilde{f}_{1}}{\partial \nu}-{ }^{t}\left(\frac{\partial \tilde{f}_{2}}{\partial \nu}\right) f_{1}\right)|d z| \\
& =\int_{D}\left({ }^{t} \tilde{f}_{2} \Delta \tilde{f}_{1}-{ }^{t}\left(\Delta \tilde{f}_{2}\right) \tilde{f}_{1}\right) d \operatorname{Re} z d \operatorname{Im} z \\
& =\int_{D}\left({ }^{t} \tilde{f}_{2} v \tilde{f}_{1}-{ }^{t}\left({ }^{t} v \tilde{f}_{2}\right) \tilde{f}_{1}\right) d \operatorname{Re} z d \operatorname{Im} z=0
\end{aligned}
$$

where for the second equality we used the following matrix-valued version of the classical scalar Green's formula:

$$
\begin{equation*}
\int_{\partial D}\left({ }^{t}\left(\frac{\partial f}{\partial \nu}\right) g-{ }^{t} f \frac{\partial g}{\partial \nu}\right)|d z|=\int_{D}\left({ }^{t}(\Delta f) g-{ }^{t} f \Delta g\right) d \operatorname{Re} z d \operatorname{Im} z \tag{3.3}
\end{equation*}
$$

for any $f, g \in C^{2}\left(D, M_{n}(\mathbb{C})\right) \cap C^{1}\left(\bar{D}, M_{n}(\mathbb{C})\right)$.
Identities (3.2) and (3.3) imply

$$
\begin{aligned}
\int_{\partial D}{ }^{t} & u_{2}(z)\left(\Phi_{2}-\Phi_{1}\right) u_{1}(z)|d z| \\
& =\int_{\partial D}\left({ }^{t}\left({ }^{t} u_{1}(z) \Phi_{2}^{*} u_{2}(z)\right)-{ }^{t} u_{2}(z) \Phi_{1} u_{1}(z)\right)|d z| \\
& =\int_{\partial D}\left({ }^{t}\left(\frac{\partial u_{2}(z)}{\partial \nu}\right) u_{1}(z)-{ }^{t} u_{2}(z) \frac{\partial u_{1}(z)}{\partial \nu}\right)|d z| \\
& =\int_{D}\left({ }^{t}\left(\Delta u_{2}(z)\right) u_{1}(z)-{ }^{t} u_{2}(z) \Delta u_{1}(z)\right) d \operatorname{Re} z d \operatorname{Im} z \\
& =\int_{D}\left({ }^{t}\left({ }^{t} v_{2}(z) u_{2}(z)\right) u_{1}(z)-{ }^{t} u_{2}(z) v_{1}(z) u_{1}(z)\right) d \operatorname{Re} z d \operatorname{Im} z \\
& =\int_{D}{ }^{t} u_{2}(z)\left(v_{2}(z)-v_{1}(z)\right) u_{1}(z) d \operatorname{Re} z d \operatorname{Im} z
\end{aligned}
$$

Now let $\bar{\mu}_{z_{0}}$ denote the complex conjugated of $\mu_{z_{0}}$ (the solution of (2.4)) for a $M_{n}(\mathbb{R})$-valued potential $v$ and, more generally, the solution of (2.4) with $g_{z_{0}}(z, \zeta, \lambda)$ replaced by $\overline{g_{z_{0}}(z, \zeta, \lambda)}$ for a $M_{n}(\mathbb{C})$-valued potential $v$. In order to make use of (3.1) we define

$$
\begin{aligned}
& u_{1}(z)=\psi_{1, z_{0}}(z, \lambda)=e^{\lambda\left(z-z_{0}\right)^{2}} \mu_{1}(z, \lambda) \\
& u_{2}(z)=\bar{\psi}_{2, z_{0}}(z,-\lambda)=e^{-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} \bar{\mu}_{2}(z,-\lambda)
\end{aligned}
$$

for $z_{0} \in D, \lambda \in C,|\lambda|>\rho(\rho$ is mentioned in section 2 ), where we called for simplicity $\mu_{1}=\mu_{1, z_{0}}, \mu_{2}=\mu_{2, z_{0}}$ and $\mu_{1, z_{0}}, \mu_{2, z_{0}}$ are the solutions of (2.4) with $v$ replaced by $v_{1},{ }^{t} v_{2}$, respectively.

Equation (3.1), with the above-defined $u_{1}, u_{2}$, now reads

$$
\begin{array}{r}
\int_{\partial D} \int_{\partial D} e^{-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2} t} \bar{\mu}_{2}(z,-\lambda)\left(\Phi_{2}-\Phi_{1}\right)(z, \zeta) e^{\lambda\left(\zeta-z_{0}\right)^{2}} \mu_{1}(\zeta, \lambda)|d \zeta||d z|  \tag{3.4}\\
=\int_{D} e_{\lambda, z_{0}}(z)^{t} \bar{\mu}_{2}(z,-\lambda)\left(v_{2}-v_{1}\right)(z) \mu_{1}(z, \lambda) d \operatorname{Re} z d \operatorname{Im} z
\end{array}
$$

with $e_{\lambda, z_{0}}(z)=e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}}$ and $\left(\Phi_{2}-\Phi_{1}\right)(z, \zeta)$ is the Schwartz kernel of the operator $\Phi_{2}-\Phi_{1}$.

The right side $I(\lambda)$ of (3.4) can be written as the sum of four integrals, namely

$$
\begin{aligned}
& I_{1}(\lambda)=\int_{D} e_{\lambda, z_{0}}(z)\left(v_{2}-v_{1}\right)(z) d \operatorname{Re} z d \operatorname{Im} z \\
& I_{2}(\lambda)=\int_{D} e_{\lambda, z_{0}}(z)^{t}\left(\bar{\mu}_{2}-I\right)\left(v_{2}-v_{1}\right)(z)\left(\mu_{1}-I\right) d \operatorname{Re} z d \operatorname{Im} z \\
& I_{3}(\lambda)=\int_{D} e_{\lambda, z_{0}}(z)^{t}\left(\bar{\mu}_{2}-I\right)\left(v_{2}-v_{1}\right)(z) d \operatorname{Re} z d \operatorname{Im} z \\
& I_{4}(\lambda)=\int_{D} e_{\lambda, z_{0}}(z)\left(v_{2}-v_{1}\right)(z)\left(\mu_{1}-I\right) d \operatorname{Re} z d \operatorname{Im} z
\end{aligned}
$$

for $z_{0} \in D$.
The first term, $I_{1}$, can be estimated using Lemma 2.2 as follows:

$$
\begin{equation*}
\left|\frac{2}{\pi}\right| \lambda\left|I_{1}-\left(v_{2}\left(z_{0}\right)-v_{1}\left(z_{0}\right)\right)\right| \leq c_{3}(D, n) \frac{\log (3|\lambda|)}{|\lambda|}\left\|v_{2}-v_{1}\right\|_{C^{2}(\bar{D})} \tag{3.5}
\end{equation*}
$$

for $|\lambda| \geq 1$. The other terms, $I_{2}, I_{3}, I_{4}$, satisfy, by Lemmata 2.1 and 2.4,

$$
\begin{align*}
\left|I_{2}\right| \leq & \left|\int_{D} e_{\lambda, z_{0}}(z)^{t}\left({\overline{g_{z_{0}, \lambda}}}^{t} v_{2}\right)\left(v_{2}-v_{1}\right)(z)\left(g_{z_{0}, \lambda} v_{1}\right) d \operatorname{Re} z d \operatorname{Im} z\right|  \tag{3.6}\\
& +O\left(\frac{\log (3|\lambda|)}{|\lambda|^{2}}\right) c_{8}(D, N, n) \\
\left|I_{3}\right| \leq & \left|\int_{D} e_{\lambda, z_{0}}(z)^{t}\left({\overline{g_{z_{0}, \lambda}}}^{t} v_{2}\right)\left(v_{2}-v_{1}\right)(z) d \operatorname{Re} z d \operatorname{Im} z\right|  \tag{3.7}\\
& +O\left(\frac{\log (3|\lambda|)}{|\lambda|^{2}}\right) c_{9}(D, N, n) \\
\left|I_{4}\right| \leq & \left|\int_{D} e_{\lambda, z_{0}}(z)\left(v_{2}-v_{1}\right)(z)\left(g_{z_{0}, \lambda} v_{1}\right) d \operatorname{Re} z d \operatorname{Im} z\right|  \tag{3.8}\\
& +O\left(\frac{\log (3|\lambda|)}{|\lambda|^{2}}\right) c_{10}(D, N, n)
\end{align*}
$$

where $N$ is the costant in the statement of Theorem 1.1 and $|\lambda|$ is sufficiently large, for example for $\lambda$ such that

$$
\begin{equation*}
2 n \frac{c_{2}(D)}{|\lambda|^{\frac{1}{2}}} \leq \frac{1}{2}, \quad|\lambda| \geq 1 \tag{3.9}
\end{equation*}
$$

Lemmata 2.5, 2.6, applied to (3.6)-(3.8), give us

$$
\begin{align*}
& \left|I_{2}\right| \leq c_{11}(D, N, n) \frac{(\log (3|\lambda|))^{2}}{|\lambda|^{2}}  \tag{3.10}\\
& \left|I_{3}\right| \leq c_{12}(D, N, n) \frac{(\log (3|\lambda|))^{2}}{|\lambda|^{1+3 / 4}}  \tag{3.11}\\
& \left|I_{4}\right| \leq c_{13}(D, N, n) \frac{(\log (3|\lambda|))^{2}}{|\lambda|^{1+3 / 4}} \tag{3.12}
\end{align*}
$$

The left side $J(\lambda)$ of (3.4) can be estimated as follows:

$$
\begin{equation*}
\left|\lambda\left\|J(\lambda) \mid \leq c_{14}(D, n) e^{\left(2 L^{2}+1\right)|\lambda|}\right\| \Phi_{2}-\Phi_{1} \|\right. \tag{3.13}
\end{equation*}
$$

for $\lambda$ which satisfies (3.9), and $L=\max _{z \in \partial D, z_{0} \in D}\left|z-z_{0}\right|$.
Putting together estimates (3.5)-(3.13) we obtain
$\left|v_{2}\left(z_{0}\right)-v_{1}\left(z_{0}\right)\right| \leq c_{15}(D, N, n) \frac{(\log (3|\lambda|))^{2}}{|\lambda|^{3 / 4}}+\frac{2}{\pi} c_{14}(D, n) e^{\left(2 L^{2}+1\right)|\lambda|}\left\|\Phi_{2}-\Phi_{1}\right\|$
for any $z_{0} \in D$. We call $\varepsilon=\left\|\Phi_{2}-\Phi_{1}\right\|$ and impose $|\lambda|=\gamma \log \left(3+\varepsilon^{-1}\right)$, where $0<\gamma<\left(2 L^{2}+1\right)^{-1}$ so that (3.14) reads

$$
\begin{align*}
\left|v_{2}\left(z_{0}\right)-v_{1}\left(z_{0}\right)\right| \leq & c_{15}(D, N, n)\left(\gamma \log \left(3+\varepsilon^{-1}\right)\right)^{-\frac{3}{4}}\left(\log \left(3 \gamma \log \left(3+\varepsilon^{-1}\right)\right)\right)^{2}  \tag{3.15}\\
& +\frac{2}{\pi} c_{14}(D, n)\left(3+\varepsilon^{-1}\right)^{\left(2 L^{2}+1\right) \gamma} \varepsilon
\end{align*}
$$

for every $z_{0} \in D$, with

$$
\begin{equation*}
0<\varepsilon \leq \varepsilon_{1}(D, N, \gamma, n) \tag{3.16}
\end{equation*}
$$

where $\varepsilon_{1}$ is sufficiently small or, more precisely, where (3.16) implies that $|\lambda|=\gamma \log \left(3+\varepsilon^{-1}\right)$ satisfies (3.9).

As $\left(3+\varepsilon^{-1}\right)^{\left(2 L^{2}+1\right) \gamma} \varepsilon \rightarrow 0$ for $\varepsilon \rightarrow 0$ more rapidly then the other term, we obtain that

$$
\begin{equation*}
\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq c_{16}(D, N, \gamma, n) \frac{\left(\log \left(3 \log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)\right)\right)^{2}}{\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)\right)^{\frac{3}{4}}} \tag{3.17}
\end{equation*}
$$

for any $\varepsilon=\left\|\Phi_{2}-\Phi_{1}\right\| \leq \varepsilon_{1}(D, N, \gamma, n)$.
Estimate (3.17) for general $\varepsilon$ (with modified $c_{16}$ ) follows from (3.17) for $\varepsilon \leq \varepsilon_{1}(D, N, \gamma, n)$ and the assumption that $\left\|v_{j}\right\|_{L^{\infty}(D)} \leq N, j=1,2$. This completes the proof of Theorem 1.1.

## 4. Proofs of Lemmata 2.5, 2.6, 3.1.

Proof of Lemma 2.5. We decompose the operator $g_{z_{0}, \lambda}$, defined in (2.12), as the product $\frac{1}{4} T_{z_{0}, \lambda} \bar{T}_{z_{0}, \lambda}$, where

$$
\begin{align*}
& T_{z_{0}, \lambda} u(z)=\frac{1}{\pi} \int_{D} \frac{e^{-\lambda\left(\zeta-z_{0}\right)^{2}+\bar{\lambda}\left(\bar{\zeta}-\bar{z}_{0}\right)^{2}}}{z-\zeta} u(\zeta) d \operatorname{Re} \zeta d \operatorname{Im} \zeta  \tag{4.1}\\
& \bar{T}_{z_{0}, \lambda} u(z)=\frac{1}{\pi} \int_{D} \frac{e^{\lambda\left(\zeta-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{\zeta}-\bar{z}_{0}\right)^{2}}}{\bar{z}-\bar{\zeta}} u(\zeta) d \operatorname{Re} \zeta d \operatorname{Im} \zeta \tag{4.2}
\end{align*}
$$

for $z_{0}, \lambda \in \mathbb{C}$. From the proof of 11 , Lemma 3.1] we have the estimate

$$
\begin{equation*}
\left\|\bar{T}_{z_{0}, \lambda} u\right\|_{C(\bar{D})} \leq \frac{\eta_{1}(D)}{|\lambda|^{1 / 2}}\|u\|_{C(\bar{D})}+\eta_{2}(D) \frac{\log (3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial \bar{z}}\right\|_{C(\bar{D})} \tag{4.3}
\end{equation*}
$$

for $u \in C_{\bar{z}}^{1}(\bar{D}), z_{0} \in D,|\lambda| \geq 1$. As the kernel of $T_{z_{0}, \lambda}$ and $\bar{T}_{z_{0}, \lambda}$ are conjugated each other we deduce immediately

$$
\begin{equation*}
\left\|T_{z_{0}, \lambda} u\right\|_{C(\bar{D})} \leq \frac{\eta_{1}(D)}{|\lambda|^{1 / 2}}\|u\|_{C(\bar{D})}+\eta_{2}(D) \frac{\log (3|\lambda|)}{|\lambda|}\left\|\frac{\partial u}{\partial z}\right\|_{C(\bar{D})},|\lambda| \geq 1 \tag{4.4}
\end{equation*}
$$

for $u \in C_{z}^{1}(\bar{D})$. Combining the two estimates we obtain

$$
\begin{aligned}
& \left\|g_{\lambda, z_{0}} u\right\|_{C(\bar{D})}=\frac{1}{4}\left\|T_{z_{0}, \lambda} \bar{T}_{z_{0}, \lambda} u\right\|_{C(\bar{D})} \\
& \quad \leq \frac{1}{4}\left(\eta_{1}(D) \frac{\left\|\bar{T}_{z_{0}, \lambda} u\right\|_{C(\bar{D})}}{|\lambda|^{1 / 2}}+\eta_{2}(D) \frac{\log (3|\lambda|)}{|\lambda|}\left\|\frac{\partial}{\partial z} \bar{T}_{z_{0}, \lambda} u\right\|_{C(\bar{D})}\right) \\
& \quad \leq \eta_{3}(D)\left(\frac{\|u\|_{C(\bar{D})}}{|\lambda|}+\frac{\log (3|\lambda|)}{|\lambda|^{3 / 2}}\left\|\frac{\partial u}{\partial \bar{z}}\right\|_{C(\bar{D})}+\frac{\log (3|\lambda|)}{|\lambda|}\|u\|_{C(\bar{D})}\right) \\
& \quad \leq \eta_{4}(D) \frac{\log (3|\lambda|)}{|\lambda|}\|u\|_{C_{\bar{z}}^{1}(\bar{D})}, \quad|\lambda| \geq 1,
\end{aligned}
$$

where we used the fact that $\left\|\frac{\partial}{\partial z} \bar{T}_{z_{0}, \lambda} u\right\|_{C(D)}=\|u\|_{C(D)}$.
Proof of Lemma 2.9. For $0<\varepsilon \leq 1, z_{0} \in D$, let $B_{z_{0}, \varepsilon}=\left\{z \in \mathbb{C}:\left|z-z_{0}\right| \leq\right.$ $\varepsilon\}$. We write $W(u, v)(\lambda)=W^{1}(\lambda)+W^{2}(\lambda)$, where

$$
\begin{aligned}
W^{1}(\lambda) & =\int_{D \cap B_{z_{0}, \varepsilon}} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} u(z) g_{z_{0}, \lambda} v(z) d \operatorname{Re} z d \operatorname{Im} z \\
W^{2}(\lambda) & =\int_{D \backslash B_{z_{0}, \varepsilon}} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} u(z) g_{z_{0}, \lambda} v(z) d \operatorname{Re} z d \operatorname{Im} z
\end{aligned}
$$

The first term, $W^{1}$, can be estimated as follows:

$$
\begin{equation*}
\left|W^{1}(\lambda)\right| \leq \sigma_{1}(D, n)\|u\|_{C(\bar{D})}\|v\|_{C \bar{z}(\bar{D})} \frac{\varepsilon^{2} \log (3|\lambda|)}{|\lambda|}, \quad|\lambda| \geq 1 \tag{4.5}
\end{equation*}
$$

where we used estimates (2.16) and (2.22).

For the second term, $W^{2}$, we proceed using integration by parts, in order to obtain

$$
\begin{aligned}
W^{2}(\lambda)= & \frac{1}{4 i \bar{\lambda}} \int_{\partial\left(D \backslash B_{z_{0}, \varepsilon}\right)} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} \frac{u(z) g_{z_{0}, \lambda} v(z)}{\bar{z}-\bar{z}_{0}} d z \\
& -\frac{1}{2 \bar{\lambda}} \int_{D \backslash B_{z_{0}, \varepsilon}} e^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} \frac{\partial}{\partial \bar{z}}\left(\frac{u(z) g_{z_{0}, \lambda} v(z)}{\bar{z}-\bar{z}_{0}}\right) d \operatorname{Re} z d \operatorname{Im} z .
\end{aligned}
$$

This imply

$$
\begin{align*}
& \left|W^{2}(\lambda)\right| \leq \frac{1}{4|\lambda|} \int_{\partial\left(D \backslash B_{z_{0}, \varepsilon}\right)} \frac{\left\|u(z) g_{z_{0}, \lambda} v(z)\right\|_{C(\bar{D})}}{}|d z|  \tag{4.6}\\
& \quad+\frac{1}{2|\lambda|}\left|\int_{D \backslash \bar{z}_{0} \mid} e_{z_{0}, \varepsilon}^{\lambda\left(z-z_{0}\right)^{2}-\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} \frac{\partial}{\partial \bar{z}}\left(\frac{u(z) g_{z_{0}, \lambda} v(z)}{\bar{z}-\bar{z}_{0}}\right) d \operatorname{Re} z d \operatorname{Im} z\right|,
\end{align*}
$$

for $\lambda \neq 0$. Again by estimates (2.16) and (2.22) we obtain

$$
\begin{align*}
\left|W^{2}(\lambda)\right| \leq & \sigma_{2}(D, n)\|u\|_{C \bar{z}}^{1}(\bar{D})
\end{aligned}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \frac{\log \left(3 \varepsilon^{-1}\right) \log (3|\lambda|)}{|\lambda|^{2}}, \quad \begin{aligned}
&  \tag{4.7}\\
& \\
& +\frac{1}{8|\lambda|}\left|\int_{D \backslash B_{z_{0}, \varepsilon}} u(z) \frac{\bar{T}_{z_{0}, \lambda} v(z)}{\bar{z}-\bar{z}_{0}} d \operatorname{Re} z d \operatorname{Im} z\right|, \quad|\lambda| \geq 1
\end{align*}
$$

where we used the fact that $\frac{\partial}{\partial \bar{z}} g_{z_{0}, \lambda} v(z)=\frac{1}{4} e^{-\lambda\left(z-z_{0}\right)^{2}+\bar{\lambda}\left(\bar{z}-\bar{z}_{0}\right)^{2}} \bar{T}_{z_{0}, \lambda} v(z)$, with $\bar{T}_{z_{0}, \lambda}$ defined in ( 4.2 ).

The last term in (4.7) can be estimated independently on $\varepsilon$ by

$$
\begin{equation*}
\sigma_{3}(D, n)\|u\|_{C(\bar{D})}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \frac{\log (3|\lambda|)}{|\lambda|^{1+3 / 4}} \tag{4.8}
\end{equation*}
$$

This is a consequence of (4.3) and of the estimate

$$
\begin{equation*}
\left|\bar{T}_{z_{0}, \lambda} u(z)\right| \leq \frac{\log (3|\lambda|)\left(1+\left|z-z_{0}\right|\right) \tau_{1}(D)}{|\lambda|\left|z-z_{0}\right|^{2}}\|u\|_{C_{\bar{z}}^{1}(\bar{D})},|\lambda| \geq 1 \tag{4.9}
\end{equation*}
$$

for $u \in C_{\bar{z}}^{1}(\bar{D}), z, z_{0} \in D$ (a proof of (4.9) can be found in the proof of 11 , Lemma 3.1]).

Indeed, for $0<\delta \leq \frac{1}{2}$ we have

$$
\begin{aligned}
& \left|\int_{D} u(z) \frac{\bar{T}_{z_{0}, \lambda} v(z)}{\bar{z}-\bar{z}_{0}} d \operatorname{Re} z d \operatorname{Im} z\right| \\
& \quad \leq \int_{B_{z_{0}, \delta \cap D}}|u(z)| \frac{\left|\bar{T}_{z_{0}, \lambda} v(z)\right|}{\left|z-z_{0}\right|} d \operatorname{Re} z d \operatorname{Im} z+\int_{D \backslash B_{z_{0}, \delta}}|u(z)| \frac{\left|\bar{T}_{z_{0}, \lambda} v(z)\right|}{\left|z-z_{0}\right|} d \operatorname{Re} z d \operatorname{Im} z \\
& \quad \leq\|u\|_{C(\bar{D})}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \frac{\tau_{2}(D, n)}{|\lambda|^{1 / 2}} \int_{B_{z_{0}, \delta} \cap D} \frac{d \operatorname{Re} z d \operatorname{Im} z}{\left|z-z_{0}\right|} \\
& \quad+\|u\|_{C(\bar{D})}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \frac{\log (3|\lambda|)}{|\lambda|} \tau_{3}(D, n) \int_{D \backslash B_{z_{0}, \delta}} \frac{d \operatorname{Re} z d \operatorname{Im} z}{\left|z-z_{0}\right|^{3}} \\
& \quad \leq 2 \pi\|u\|_{C(\bar{D})}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \tau_{2}(D, n) \frac{\delta}{|\lambda|^{\frac{1}{2}}}+\|u\|_{C(\bar{D})}\|v\|_{C_{\bar{z}}^{1}(\bar{D})} \tau_{4}(D, n) \frac{\log (3|\lambda|)}{|\lambda| \delta}
\end{aligned}
$$

for $|\lambda| \geq 1$. Putting $\delta=\frac{1}{2}|\lambda|^{-1 / 4}$ in the last inequality gives (4.8).
Finally, defining $\varepsilon=|\lambda|^{-1 / 2}$ in (4.7), (4.5) and using (4.8), we obtain the main estimate (2.24), which thus finishes the proof of Lemma 2.6.

Proof of Lemma 3.1. Take $u \in H^{1}\left(D, M_{n}(\mathbb{C})\right)$ such that $\left(-\Delta+{ }^{t} v\right) u=0$ on $D$ and $\left.u\right|_{\partial D}=0$. We want to prove that $u \equiv 0$ on $D$.

By our hypothesis, for any $f \in C^{1}\left(\partial D, M_{n}(\mathbb{C})\right)$ there exists a unique $\tilde{f} \in H^{1}\left(D, M_{n}(\mathbb{C})\right)$ such that $(-\Delta+v) \tilde{f}=0$ on $D$ and $\left.\tilde{f}\right|_{\partial D}=f$. Thus we have, using Green's formula (3.3),

$$
\begin{gathered}
\int_{\partial D}{ }^{t}\left(\frac{\partial u}{\partial \nu}\right) f|d z|=\int_{D}\left({ }^{t}(\Delta u) \tilde{f}-{ }^{t} u \Delta \tilde{f}\right) d \operatorname{Re} z d \operatorname{Im} z \\
=\int_{D}\left({ }^{t}\left({ }^{t} v u\right) \tilde{f}-{ }^{t} u v \tilde{f}\right) d \operatorname{Re} z d \operatorname{Im} z=0
\end{gathered}
$$

which yields $\left.\frac{\partial u}{\partial \nu}\right|_{\partial D}=0$. Now consider the following straightforward generalization of Green's formula (3.3),
$\int_{\partial D}\left({ }^{t}\left(\frac{\partial f}{\partial \nu}\right) g-{ }^{t} f \frac{\partial g}{\partial \nu}\right)|d z|=\int_{D}{ }^{t}\left(\left(\Delta-{ }^{t} v\right) f\right) g-{ }^{t} f((\Delta-v) g) d \operatorname{Re} z d \operatorname{Im} z$,
which holds (weakly) for any $f, g \in H^{1}\left(D, M_{n}(\mathbb{C})\right.$ ). If we put $f=u$ we obtain

$$
\begin{equation*}
\int_{D}{ }^{t} u(-\Delta+v) g d \operatorname{Re} z d \operatorname{Im} z=0 \tag{4.11}
\end{equation*}
$$

for any $g \in H^{1}\left(D, M_{n}(\mathbb{C})\right)$. By Fredholm alternative (see [5, Sec. 6.2]), for each $h \in L^{2}\left(D, M_{n}(\mathbb{C})\right)$ there exists a unique $g \in H_{0}^{1}\left(D, M_{n}(\mathbb{C})\right)=\{g \in$ $\left.H^{1}\left(D, M_{n}(\mathbb{C})\right):\left.g\right|_{\partial D}=0\right\}$ such that $(-\Delta+v) g=h$ : this yields $u \equiv 0$ on $D$. Thus Lemma 3.1 is proved.

## 5. An extension of Theorem 1.1

As an extension of Theorem 1.1 for the case when we do not assume that $\left.v_{j}\right|_{\partial D} \equiv 0,\left.\frac{\partial}{\partial \nu} v_{j}\right|_{\partial D} \equiv 0, j=1,2$, we give the following result.

Proposition 5.1. Let $D \subset \mathbb{R}^{2}$ be an open bounded domain with $C^{2}$ boundary, let $v_{1}, v_{2} \in C^{2}\left(\bar{D}, M_{n}(\mathbb{C})\right)$ be two matrix-valued potentials which satisfy (1.4), with $\left\|v_{j}\right\|_{C^{2}(\bar{D})} \leq N$ for $j=1,2$, and $\Phi_{1}, \Phi_{2}$ the corresponding Dirichlet-to-Neumann operators. Then, for any $0<\alpha<\frac{1}{5}$, there exists a constant $C=C(D, N, n, \alpha)$ such that

$$
\begin{equation*}
\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq C\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}^{-1}\right)\right)^{-\alpha} \tag{5.1}
\end{equation*}
$$

where $\|A\|_{1}$ is the norm for an operator $A: L^{\infty}\left(\partial D, M_{n}(\mathbb{C})\right) \rightarrow L^{\infty}\left(\partial D, M_{n}(\mathbb{C})\right)$, with kernel $A(x, y)$, defined as $\|A\|_{1}=\sup _{x, y \in \partial D}|A(x, y)|\left(\log \left(3+|x-y|^{-1}\right)\right)^{-1}$ and $|A(x, y)|=\max _{1 \leq i, j \leq n}\left|A_{i, j}(x, y)\right|$.

The only properties of $\left\|\|_{1}\right.$ we will use are the following:
i) $\|A\|_{L^{\infty}(\partial D) \rightarrow L^{\infty}(\partial D)} \leq \operatorname{const}(D, n)\|A\|_{1}$;
ii) In a similar way as in formula (4.9) of [9] one can deduce

$$
\|v\|_{L^{\infty}(\partial D)} \leq \operatorname{const}(n)\left\|\Phi_{v}-\Phi_{0}\right\|_{1}
$$

for a matrix-valued potential $v, \Phi_{v}$ its associated Dirichlet-to-Neumann operator and $\Phi_{0}$ the Dirichlet-to-Neumann operator of the 0 potential.

We recall a lemma from [11], which generalize Lemma 2.2 to the case of potentials without boundary conditions. We define $(\partial D)_{\delta}=\{z \in \mathbb{C}$ : $\operatorname{dist}(z, \partial D)<\delta\}$.

Lemma 5.2. For $v \in C^{2}(\bar{D})$ we have that

$$
\begin{align*}
\left|v\left(z_{0}\right)-\frac{2}{\pi}\right| \lambda\left|h_{z_{0}}^{(0)}(\lambda)\right| \leq & \kappa_{1}(D, n) \delta^{-4} \frac{\log (3|\lambda|)}{|\lambda|}\|v\|_{C^{2}(\bar{D})}  \tag{5.2}\\
& +\kappa_{2}(D, n) \log \left(3+\delta^{-1}\right)\|v\|_{C(\partial D)}
\end{align*}
$$

for $z_{0} \in D \backslash(\partial D)_{\delta}, 0<\delta<1, \lambda \in \mathbb{C},|\lambda| \geq 1$.

The proof of Lemma 5.2 for the scalar case can be found in 11]: the generalization to the matrix-valued case is straightforward.

Proof of Proposition 5.1. Fix $0<\alpha<\frac{1}{5}$ and $0<\delta<1$. We have the following chain of inequalities

$$
\begin{aligned}
& \| v_{2}= \\
& =v_{1} \|_{L^{\infty}(D)}=\max \left(\left\|v_{2}-v_{1}\right\|_{L^{\infty}\left(D \cap(\partial D)_{\delta}\right)},\left\|v_{2}-v_{1}\right\|_{L^{\infty}\left(D \backslash(\partial D)_{\delta)}\right)}\right. \\
& \leq \\
& \quad C_{1} \max \left(2 N \delta+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}, \frac{\log \left(3 \log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)\right)}{\delta^{4} \log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)}\right. \\
& \left.\quad+\log \left(3+\frac{1}{\delta}\right)\left\|\Phi_{2}-\Phi_{1}\right\|_{1}+\frac{\left(\log \left(3 \log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)\right)^{2}\right.}{\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|^{-1}\right)\right)^{\frac{3}{4}}}\right) \\
& \leq \\
& \quad C_{2} \max \left(2 N \delta+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}, \frac{1}{\delta^{4}}\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}^{-1}\right)\right)^{-5 \alpha}\right. \\
& \\
& \left.\quad+\log \left(3+\frac{1}{\delta}\right)\left\|\Phi_{2}-\Phi_{1}\right\|_{1}+\frac{\left(\log \left(3 \log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}^{-1}\right)\right)^{2}\right.}{\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}^{-1}\right)\right)^{\frac{3}{4}}}\right)
\end{aligned}
$$

where we followed the scheme of the proof of Theorem 1.1 with the following modifications: we made use of Lemma 5.2 instead of Lemma 2.2 and we also used i)-ii); note that $C_{1}=C_{1}(D, N, n)$ and $C_{2}=C_{2}(D, N, n, \alpha)$.

Putting $\delta=\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}^{-1}\right)\right)^{-\alpha}$ we obtain the desired inequality

$$
\begin{equation*}
\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq C_{3}\left(\log \left(3+\left\|\Phi_{2}-\Phi_{1}\right\|_{1}^{-1}\right)\right)^{-\alpha} \tag{5.3}
\end{equation*}
$$

with $C_{3}=C_{3}(D, N, n, \alpha),\left\|\Phi_{2}-\Phi_{1}\right\|_{1}=\varepsilon \leq \varepsilon_{1}(D, N, n, \alpha)$ with $\varepsilon_{1}$ sufficiently small or, more precisely when $\delta_{1}=\left(\log \left(3+\varepsilon_{1}^{-1}\right)\right)^{-\alpha}$ satisfies:

$$
\delta_{1}<1, \quad \varepsilon_{1} \leq 2 N \delta_{1}, \quad \log \left(3+\frac{1}{\delta_{1}}\right) \varepsilon_{1} \leq \delta_{1}
$$

Estimate (5.3) for general $\varepsilon$ (with modified $C_{3}$ ) follows from (5.3) for $\varepsilon \leq \varepsilon_{1}(D, N, n, \alpha)$ and the assumption that $\left\|v_{j}\right\|_{L^{\infty}(\bar{D})} \leq N$ for $j=1,2$. This completes the proof of Proposition 5.1.

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