A Hamilton-Jacobi approach to junction problems and application to traffic flows

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Dedicated to J.-B. Hiriart-Urruty

Abstract

This paper is concerned with the study of a model case of first order Hamilton-Jacobi equations posed on a "junction", that is to say the union of a finite number of half-lines with a unique common point. The main result is a comparison principle. We also prove existence and stability of solutions. The two challenging difficulties are the singular geometry of the domain and the discontinuity of the Hamiltonian. As far as discontinuous Hamiltonians are concerned, these results seem to be new. They are applied to the study of some models arising in traffic flows. The techniques developed in the present article provide new powerful tools for the analysis of such problems.

Keywords. Hamilton-Jacobi equations, discontinuous Hamiltonians, viscosity solutions, optimal control, traffic problems, junctions

Mathematical Subject Classication. 35F21, 35D40, 35Q93, 35R05, 35B51

1 Introduction

In this paper we are interested in Hamilton-Jacobi (HJ) equations posed on a one dimensional domain containing one single singularity. This is a special case of a more general setting where HJ equations are posed in domains that are unions of submanifolds whose dimensions are different [12]. An intermediate setting is the study of HJ equations on networks [1]. We will restrict ourselves to a very simple network: the union of a finite numbers of half-lines of the plane with a single common point. Such a domain is referred to as a *junction* and the common point is called the *junction point*. Our motivation comes from traffic flows. For this reason, it is natural to impose different dynamics on each *branch* of the junction. Consequently, the resulting Hamiltonian is by nature discontinuous at the junction point. Together with the singularity of the domain, this is the major technical difficulty to overcome. The analysis relies on the complete study of some *minimal action* (or *metric*) related to the optimal control interpretation of the equation [42, 22]. We prove in particular that this minimal action is semi-concave by computing it.

We first present the problem and the main results in details. Then we recall existing results and compare them with ours.

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1.1 Setting of the problem

In this subsection, the analytical problem is introduced in details. We first define the junction, then the space of functions on the junction and finally the Hamilton-Jacobi equation.

The junction. Let us consider $N \geq 1$ different unit vectors $e_i \in \mathbb{R}^2$ for $i = 1, ..., N$. We define the branches

$$J_i = [0, +\infty) \cdot e_i, \quad J_i^* = J_i \setminus \{0\}, \quad i = 1, ..., N$$

and the *junction* (see Figure 1)

$$J = \bigcup_{i=1,...,N} J_i.$$

The origin $x = 0$ is called the *junction point*. It is also useful to write $J^* = J \setminus \{0\}$. For a time

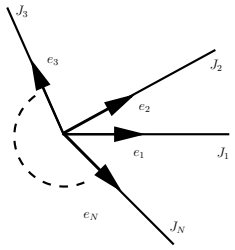

Figure 1: A junction

$T > 0$, we also define

$$J_T = (0, T) \times J.$$

The reader can remark that we chose to embed the junction in a two-dimensional Euclidian space. But we could also have considered an abstract junction, or we could have embedded it for instance in a higher dimensional Euclidian space. We made such a choice for the sake of clarity.

Space of functions. For a function $u \colon J_T \to \mathbb{R}$, we denote by u^i the restriction of u to $(0, T) \times J_i$. Then we define the natural space of functions on the junction

$$C^1_*(J_T) = \left\{ u \in C(J_T), \quad u^i \in C^1((0, T) \times J_i) \quad \text{for} \quad i = 1, ..., N \right\}.$$

In particular for $u \in C^1_*(J_T)$ and $x = x_i e_i$ with $x_i \geq 0$, we define

$$u_t(t, x) = \frac{\partial u^i}{\partial t}(t, x_i e_i) \quad \text{and} \quad u^i_x(t, x) = \frac{\partial u^i}{\partial x_i}(t, x_i e_i).$$

Then we set

$$\begin{cases} u_x(t, x) = u^i_x(t, x) & \text{if} \quad x \neq 0, \\ u_x(t, 0) = (u^j_x(t, 0))_{j=1,...,N} & \text{if} \quad x = 0. \end{cases}$$

HJ equation on the junction. We are interested in continuous functions $u \colon [0, T) \times J \to \mathbb{R}$ which are *viscosity solutions* (see Definition 3.2) on J_T of

$$u_t + H(x, u_x) = 0 \tag{1.1}$$

submitted to an initial condition

$$u(0, x) = u_0(x), \quad x \in J. \tag{1.2}$$

Because of the optimal control problem we have in mind (see Subsection 1.1 below), we restrict ourselves to the simplest case of discontinuous Hamiltonians; precisely, we consider

$$H(x,p) = \begin{cases} H_i(p) & \text{for} \quad p \in \mathbb{R} & \text{if} \quad x \in J_i^* \\ \max_{i=1,...,N} H_i^-(p_i) & \text{for} \quad p = (p_1,...,p_N) \in \mathbb{R}^N & \text{if} \quad x = 0 \end{cases}$$

where H_i are convex functions whose Legendre-Fenchel transform is denoted L_i. We recall that

$$H_i(p) = L_i^*(p) = \sup_{q \in \mathbb{R}} (pq - L_i(q))$$

and

$$H_i = L_i^*. \tag{1.3}$$

We also consider

$$H_i^-(p_i) = \sup_{q \le 0} (p_i q - L_i(q)) \tag{1.4}$$

Therefore equation (1.1) can be rewritten as follows

$$\begin{cases} u_t^i + H_i(u_x^i) = 0 & \text{on} \quad (0,T) \times J_i^* & \text{for} \quad i = 1,...,N, \\ u_t + \max_{i=1,...,N} H_i^-(u_x^i) = 0 & \text{on} \quad (0,T) \times \{0\}. \end{cases} \tag{1.5}$$

The optimal control framework. In this paragraph, we give an optimal control interpretation [36, 8, 5] of the Hamilton-Jacobi equation. We define the set of admissible controls at a point $x \in J$ by

$$A(x) = \begin{cases} \mathbb{R} e_{i_0} & \text{if} \quad x \in J_{i_0}^*, \\ \bigcup_{i=1,...,N} \mathbb{R}^+ e_i & \text{if} \quad x = 0. \end{cases}$$

For $(s,y),(t,x) \in [0,T] \times J$ with $s \le t$ (the case $s = t$ being trivial and forcing $y = x$), we define the set of admissible trajectories from (s,y) to (t,x) by

$$\mathcal{A}(s,y;t,x) = \left\{ X \in W^{1,1}([s,t];\mathbb{R}^2) : \left| \begin{array}{lll} X(\tau) \in J & \text{for all} & \tau \in (s,t), \\ \dot{X}(\tau) \in A(X(\tau)) & \text{for a.e.} & \tau \in (s,t) \\ X(s) = y & \text{and} & X(t) = x \end{array} \right. \right\}. \tag{1.6}$$

For $P = pe_i \in A(x)$ with $p \in \mathbb{R}$, we define the Lagrangian on the junction

$$L(x,P) = \begin{cases} L_i(p) & \text{if} \quad x \in J_i^* \\ L_0(p) & \text{if} \quad x = 0 \end{cases} \tag{1.7}$$

with

$$L_0(p) = \min_{j=1,...,N} L_j(p).$$

The reader can be surprised by the definition of $L(x,P)$ for $x = 0$. In fact, if one considers only trajectories that do not stay at the junction point, then the ones staying at 0 are approximated by those staying very close to 0 on a branch $i \in I_0$ and moving "slowly" ($\dot{X} \simeq 0$).

1.2 Main results

We make the following assumptions:
(A0) The initial data u_0 is Lipschitz continuous.
(A1) There exists a constant $\gamma > 0$, and for all $i = 1,...,N$, there exists $C^2(\mathbb{R})$ functions L_i satisfying $L_i'' \ge \gamma > 0$, such that (1.3) and (1.4) hold.

Theorem 1.1 (Existence and uniqueness)**.** *Assume (A0)-(A1) and let $T > 0$. Then there exists a unique viscosity solution u of (1.1)-(1.2) on J_T in the sense of Definition 3.2, satisfying for some constant $C_T > 0$*

$$|u(t,x) - u_0(x)| \le C_T \quad \textit{for all} \quad (t,x) \in J_T.$$

Moreover the function u is Lipschitz continuous with respect to (t,x) on J_T.

On one hand, we will see below that the existence of a solution can be obtained with Perron's method under weaker assumptions than (A1) (see Theorem A.2). On the other hand, we are able to get uniqueness of the solution only under assumption (A1) and this is a consequence of the following result:

Theorem 1.2 (Comparison principle). *Assume (A0)-(A1). Let $T > 0$ and let u (resp. v) be a subsolution (resp. a supersolution) of (1.1)-(1.2) on J_T in the sense of Definition 3.2. We also assume that there exists a constant $C_T > 0$ such that for all $(t, x) \in J_T$*

$$u(t,x) \geq -C_T(1+|x|) \quad (\textit{resp.} \quad v(t,x) \leq C_T(1+|x|)).$$

Then we have $u \leq v$ on J_T.

In order to prove this strong uniqueness result, we will use in an essential way the *value function* associated to the optimal control problem described in Subsection 1.1: for $t \geq 0$,

$$u_{\rm oc}(t,x) = \inf_{y\in J,\ X\in\mathcal{A}(0,y;t,x)} \left\{ u_0(y) + \int_0^t L(X(\tau),\dot{X}(\tau))d\tau \right\} \tag{1.8}$$

where L is defined in (1.7) and $\mathcal{A}(0,y;t,x)$ is defined in (1.6).

Theorem 1.3 (Optimal control representation of the solution). *Assume (A0)-(A1) and let $T > 0$. The unique solution given by Theorem 1.1 is $u = u_{\rm oc}$ with $u_{\rm oc}$ given in (1.8). Moreover, we have the following Hopf-Lax representation formula*

$$u_{\rm oc}(t,x) = \inf_{y\in J} \{u_0(y) + \mathcal{D}(0,y;t,x)\} \tag{1.9}$$

with

$$\mathcal{D}(0,y;t,x) = \inf_{X\in\mathcal{A}(0,y;t,x)} \left\{ \int_0^t L(X(\tau),\dot{X}(\tau))d\tau \right\}.$$

The comparison principle is obtained by combining

- a super-optimality principle for surpersolutions v, which implies $v \geq u_{\rm oc}$;
- a direct comparison result with subsolutions u, which gives $u_{\rm oc} \geq u$.

We finally have the following result which shed light on the role of the junction condition (see the second line of (1.5)).

Theorem 1.4 (Comparison with continuous solutions outside the junction point). *Assume (A0)-(A1) and let $T > 0$. Let $u \in C([0,T) \times J)$ be such that $u(0,\cdot) = u_0$ and for each $i \in \{1,...,N\}$, the restriction u^i of u to $(0,T) \times J_i$ is a classical viscosity solution of*

$$u^i_t + H_i(u^i_x) = 0 \quad \textit{on} \quad (0,T) \times J_i^*.$$

Then u is a subsolution of (1.1)-(1.2) on J_T in the sense of Definition 3.2, and $u \leq u_{\rm oc}$.

An immediate consequence of Theorem 1.4 is the fact that the optimal control solution $u_{\rm oc}$ is the maximal continuous function which is a viscosity solution on each open branch.

We apply in Section 2 our HJ approach to describe traffic flows on a junction. In particular, we recover the well-known junction conditions of Lebacque (see [32]) or equivalently those for the Riemann solver at the junction as in the book of Garavelo and Piccoli [27].

1.3 Comments

We already mentioned that the main difficulties we have to overcome in order to get our main results are on one hand the singular geometry of the domain and on the other hand the discontinuity of the Hamiltonian.

Discontinuity. There is an important literature concerning Hamilton-Jacobi equations with discontinuous coefficients; see for instance [9, 50, 7, 44, 38, 18, 10, 11, 47, 49, 16, 13, 48, 20, 19]. There are also many results concerning scalar conservation laws with discontinuous flux functions; see for instance [41, 4, 14] and references therein. But there are very few results (if any) building bridges between these two kinds of results.

Networks. As it is explained in [14] for instance, the study of traffic flows on networks is an important source of problems for scalar conservation laws with discontinuous coefficients [27, 21]. The study of Hamilton-Jacobi equations on networks is more recent; the reader is referred to [1, 2, 40, 15] and references therein. In [1], a comparison principle is proved adapting the double variable technique to this new framework, assuming among other things that all the Lagrangian $L_i(p)$ are minimal for $p = 0$ with the same minimal value $L_1(0) = ... = L_N(0)$.

The optimal control interpretation. As explained above, the comparison principle is proved by using in an essential way the optimal control interpretation of the Hamilton-Jacobi equation. The use of representation formulas and/or optimality principles is classical in the study of Hamilton-Jacobi equations [37, 45, 46, 28, 29]. More specifically, it is also known that a "metric" interpretation of the Hamilton-Jacobi equation is fruitful [42]. Such an interpretation plays a central role in the weak KAM theory [22].

As far as our problem is concerned, we are not able to adapt the classical viscosity solution approach to get uniqueness (doubling variables techniques). On the contrary, we get it by proving e.g. a super-optimality principle (see the discussion above) and by using representation formulas in the spirit of the works cited above.

We would like next to be a bit more precise. The technical core of the paper lies in Theorem 4.2. This result implies that the function

$$\mathcal{D}(s, y; t, x) = (t - s)\mathcal{D}_0\left(\frac{y}{t-s}, \frac{x}{t-s}\right)$$

is semi-concave with respect to (t, x) and (s, y) and, if there are at least two branches ($N \geq 2$), that $\mathcal{D}$ satisfies

$$\begin{cases} \mathcal{D}_t + H(x, \mathcal{D}_x) & = 0, \\ -\mathcal{D}_s + H(y, -\mathcal{D}_y) & = 0 \end{cases}$$

(in a weak sense made precise in the statement of Theorem 4.2). In the case where the Lagrangians coincide at the junction point ($L_1(0) = ... = L_N(0)$), it turns out that the restriction $\mathcal{D}_0^{ji}(y, x)$ of $\mathcal{D}_0$ to $J_j \times J_i$ belong to $C^1(J_j \times J_i)$ and is convex. A more general case is considered in this paper: Lagrangians can differ at the junction point and in this case, the functions $\mathcal{D}_0^{ji}$ are not convex nor C^1 anymore for some (i, j).

Viability and state constraints. We would like to make a final comment: in view of the geometric framework we chose, we see that the problem under analysis has to do with state constraints problems [43, 31, 17, 30]; indeed, the trajectories of the dynamical system are constrainted to stay on the junction. Similarly, our problem is related to the viability theory [3, 23, 24, 25, 26]. However, we did not use this approach/these techniques here.

Generalization and open problems. Eventually, we briefly mention natural generalizations of our results and some important open problems. First of all, it is natural to let Hamiltonians H_i depend on the space variable x. It is also natural to consider general networks by considering several junction points. We believe these generalizations can be achieved but this could be very technical. Dealing with non-convex and/or non-coercive Hamiltonians is quite challenging and some intermediate conditions should probably be imposed. For instance, the controllability of the underlying dynamical system ensures that we can work with continuous viscosity solutions.

Organization of the article

Section 2 is devoted to the application of our results to some traffic flow problems. In particular, the HJ equation is derived and the junction condition is interpreted. In Section 3, the definition of (viscosity) solutions is made precise. In Section 4, the first important properties of optimal trajectories are given. Section 5 is devoted to the proof of the main results of the paper. In particular, the comparison principle is proved by proving a super-optimality principle and by comparing subsolutions with the solution given by the optimal control interpretation of the equation. Section 6 is devoted to the proof of the technical core of the paper, namely the existence of test functions for the minimal action associated with the optimal control interpretation.

Notation

Distance and coordinates in the junction. We denote by d the geodesic distance defined on J by

$$d(x,y) = \begin{cases} |x-y| & \text{if} \quad x,y \quad \text{belong to the same branch } J_i \text{ for some } i, \\ |x|+|y| & \text{if} \quad x,y \quad \text{do not belong to the same branch.} \end{cases}$$

For $x \in J$, $B(x,r)$ denotes the (open) ball centered at x of radius r. We also consider balls $B((t,x),r)$ centered at $(t,x) \in (0,+\infty) \times J$ of radius $r > 0$. For $x \in J$, let us define the index $i(x)$ of the branch where x lies. Precisely we set:

$$i(x) = \begin{cases} i_0 & \text{if} \quad x \in J^*_{i_0}, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Up to reordering the indices, we assume that there exists an index $k_0 \in \{1,...,N\}$ such that

$$L_0(0) = L_1(0) = \cdots = L_{k_0}(0) < L_{k_0+1}(0) \leq \cdots \leq L_N(0). \tag{1.10}$$

We also set

$$I_0 = \{1,..,k_0\} \quad \text{and} \quad I_N = \{1,...,N\}.$$

Functions defined in J^2. For a function φ defined on $J \times J$, we call φ^{ij} its restriction to $J_i \times J_j$. Then we define the space

$$C^1_*(J^2) = \left\{ \varphi \in C(J^2), \quad \varphi^{ij} \in C^1(J_i \times J_j) \quad \text{for all} \quad i,j \in I_N \right\}.$$

We also call for $x = x_i e_i$ with $x_i \geq 0$ and $y = y_j e_j$ with $y_j \geq 0$

$$\partial^i_x \varphi(x,y) = \frac{\partial}{\partial x_i} \varphi^{ij}(x_i e_i, y) \quad \text{and} \quad \partial^j_y \varphi(x,y) = \frac{\partial}{\partial y_j} \varphi^{ij}(x, y_j e_j)$$

and

$$\partial_x \varphi(x,y) = \begin{cases} \partial^i_x \varphi(x,y) & \text{if} \quad x \in J^*_i, \\ \left(\partial^i_x \varphi(x,y)\right)_{i=1,...,N} & \text{if} \quad x = 0 \end{cases}$$

and similarly

$$\partial_y \varphi(x,y) = \begin{cases} \partial^j_y \varphi(x,y) & \text{if} \quad y \in J^*_j, \\ \left(\partial^j_y \varphi(x,y)\right)_{j=1,...,N} & \text{if} \quad y = 0. \end{cases}$$

We also set

$$\begin{cases} x\partial_x \varphi(x,y) = x_i \partial^i_x \varphi(x,y) & \text{if} \quad x \in J_i, \\ y\partial_y \varphi(x,y) = y_j \partial^j_y \varphi(x,y) & \text{if} \quad y \in J_j. \end{cases}$$

2 Application to the modeling of traffic flows

In this section we present the derivation of the Hamilton-Jacobi formulation of traffic on a junction. We also discuss the meaning of our junction condition in this framework and relate it to known results.

2.1 Primitive of the dentities of cars

We consider a junction (represented on Figure 2) with $m \geq 1$ incoming roads (labeled by the

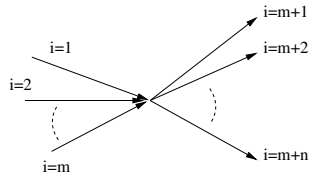

Figure 2: A traffic junction

index $i = 1, ..., m$) and $n \geq 1$ outgoing roads (labeled by $j = m+1, ..., m+n = N$). This means that the cars move on the incoming roads in the direction of the junction and then have to choose to go on one of the n outgoing roads. We assume that the proportion of cars coming from the branch $i = 1, ..., m$ is a fixed number $\gamma_i > 0$ (which may be not realistic for $m \geq 2$), and that the proportion of cars going on each branch $j \in \{m+1, ..., m+n\}$ is also a fixed number $\gamma_j > 0$. We also assume the obvious relations (for conservation of cars)

$$\sum_{i=1,...,m} \gamma_i = 1 \quad \text{and} \quad \sum_{j=m+1,...,m+n} \gamma_j = 1$$

We denote by $\rho^k(t, X) \geq 0$ the car density at time t and at the position X on the branch k. In particular, we assume that the traffic is described on each branch k by a flux function $f^k \colon \mathbb{R} \to \mathbb{R}$. We assume

$$\text{each function } f^k \text{ is concave and has a unique maximum value at } \rho = \rho_c^k > 0 \tag{2.1}$$

The typical example of such flux function is given by the LWR model (Lighthill, Whitham [35] and Richards [39]), with

$$f(\rho) = \rho v(\rho) \quad \text{with the velocity} \quad v(\rho) = V_{\max}(1 - \rho/\rho_{\max}) \tag{2.2}$$

where $V_{\max}$ and $\rho_{\max}$ are respectively the maximal velocity and the maximal car density in the model). In this model the critical car density ρ_c where f is maximal, is equal to $\frac{1}{2}\rho_{\max}$.

We assume that the car densities are solution of non linear transport equations:

$$\begin{cases} \rho_t^i + (f^i(\rho^i))_X = 0, & X < 0, \qquad \text{for} \quad i = 1, ..., m \\ \rho_t^j + (f^j(\rho^j))_X = 0, & X > 0, \qquad \text{for} \quad j = m+1, ..., m+n \end{cases} \tag{2.3}$$

where we assume that the junction point is located at the origin $X = 0$.

We do not precise yet the junction condition at $X = 0$, and we now proceed formally to deduce the Hamilton-Jacobi model of such a junction. For a function g to determine, let us consider the functions

$$\begin{cases} U^i(t, X) = g(t) + \frac{1}{\gamma^i} \int_0^X \rho^i(t, Y)\, dY, & X < 0, \qquad \text{for} \quad i = 1, ..., m, \\ U^j(t, X) = g(t) + \frac{1}{\gamma^j} \int_0^X \rho^j(t, Y)\, dY, & X > 0, \qquad \text{for} \quad j = m+1, ..., m+n. \end{cases} \tag{2.4}$$

Then we can compute formally for $j = m+1, ..., m+n$

$$\begin{aligned} U_t^j &= g'(t) + \tfrac{1}{\gamma^j} \textstyle\int_0^X \rho_t^j(t, Y)\, dY \\ &= g'(t) - \tfrac{1}{\gamma^j} \textstyle\int_0^X (f^j(\rho^j(t, Y)))_X\, dY \\ &= -\tfrac{1}{\gamma^j} f^j(\rho^j(t, X)) + g'(t) + \tfrac{1}{\gamma^j} f^j(\rho^j(t, 0^+)) \end{aligned}$$

This shows that for $j = m+1, ..., m+n$

$$U_t^j + \frac{1}{\gamma^j} f^j(\gamma_j U_X^j) = h^j(t) \tag{2.5}$$

where

$$h^j(t) := g'(t) + \frac{1}{\gamma^j} f^j(\rho^j(t, 0^+)).$$

Remark that we can show similarly that (2.5) is still true for the index j replaced by $i = 1, ..., m$ with

$$h^i(t) = g'(t) + \frac{1}{\gamma^i} f^i(\rho^i(t, 0^-)).$$

In particular, this shows (at least when the quantities in (2.5) are well defined) that we can choose g such that the total flux $-g'(t)$ is given by

$$-g'(t) = \sum_{i=1,...,m} f^i(\rho^i(t, 0^-)) \tag{2.6}$$

and then we have

$$h^k(t) \equiv 0 \quad \text{for} \quad i = 1, ..., m+n$$

if and only if

$$\begin{cases} f^i(\rho^i(t, 0-)) = \gamma^i(-g'(t)) & \text{for} \quad i = 1, ..., m \\ f^j(\rho^j(t, 0^+)) = \gamma^j(-g'(t)) & \text{for} \quad j = m+1, ..., m+n \end{cases} \tag{2.7}$$

which is exactly the expected condition which says that the proportion of incoming cars going in the junction from the branch i is γ^i and the proportion of cars getting out of the junction which choose to go on the branch j is γ^j.

Let us notice that if we choose the initial condition $g(0) = 0$, then we deduce from (2.6) that we have with $l = 1, ..., m+n$

$$-U^l(t, 0) = -g(t) = \int_0^t \sum_{i=1,...,m} f^i(\rho^i(\tau, 0^-))\ d\tau$$

which shows that $-U^l(t, 0)$ *can be interpreted as the total quantity of cars passing through the junction point* $X = 0$ *during the time interval* $[0, t)$. As a consequence, the quantity $-U_t^l(t, 0)$ can also be interpreted as the instantaneous flux of cars passing through the junction point.

We now give a further interpretation of the problem in the special case $m = 1$. In the special case $m = 1$, imagine for a moment, that we come back to a discrete description of the traffic, where each car of label k has a position $x_k(t)$ with the ordering $x_k(t) < x_{k+1}(t) < 0$. We can be interested in the label k of the car $x_k(t) \leq 0$ which is the closest to the junction point $X = 0$. Let us call it $K(t)$. We can normalize the initial data such that $K(0) = 0$. Then the quantity of cars that have passed through the junction point $X = 0$ during the time interval $[0, t)$ is equal to $-K(t)$, which is the exact discrete analogue of the continuous quantity $-U^1(t, 0)$.

On the other hand the number of cars between the positions $a = x_A(t)$ and $b = x_B(t)$ is obviously equal to $B - A$, and its continuous analogue on the branch $i = m = 1$ with $\gamma^1 = 1$, is $\int_a^b \rho^1(t, X)\ dX = U^1(t, b) - U^1(t, a)$. This shows that $U^1(t, X)$ can be interpreted as the exact continuous analogue of the discrete labeling of the cars moving in the traffic.

This interpretation is also meaningful on the "exit" branches, that is to say for $j \in \{m+1, ..., m+n\}$. Indeed, for such j's, $U^j(t, X)$ is the continuous analogue of the discrete label of the car that have decided to choose the branch j and which is at time t close to the position $X > 0$.

2.2 Getting the Hamilton-Jacobi equations

We now set

$$\begin{cases} u^i(t,X) = -U^i(t,-X), & X>0, \quad \text{for} \quad i=1,...,m \\ u^j(t,X) = -U^j(t,X), & X>0, \quad \text{for} \quad j=m+1,...,m+n \end{cases} \tag{2.8}$$

and we define the convex Hamiltonians

$$\begin{cases} H_i(p) = -\frac{1}{\gamma^i} f^i(\gamma^i p) & \text{for} \quad i=1,...,m \\ H_j(p) = -\frac{1}{\gamma^j} f^j(-\gamma^j p) & \text{for} \quad j=m+1,...,m+n \end{cases} \tag{2.9}$$

Then we deduce from (2.5) that we have

$$u_t^k + H_k(u_X^k) = 0, \quad X>0, \quad \text{for} \quad k=1,...,m+n \tag{2.10}$$

with equality of the functions at the origin, i.e.

$$u^k(t,0) = u(t,0) \quad \text{for any} \quad k \in \{1,...,m+n\}\,.$$

Notice that for the choice $V_{\max} = 1 = \rho_{\max}$ in (2.2), we get with $f^k(\rho) = f(\rho) = \rho(1-\rho)$ for all $k \in I_N$, that

$$\begin{cases} L_{\text{ref}}(q) = \frac{1}{4}(1+q)^2 & \\ L_i(q) = \frac{1}{\gamma^i} L_{\text{ref}}(q) & \text{for} \quad i=1,...,m \\ L_j(q) = \frac{1}{\gamma^j} L_{\text{ref}}(-q) & \text{for} \quad j=m+1,...,m+n \end{cases}$$

In particular this shows that the $L_k(0)$ are not all the same, even in the simplest case.

2.3 The junction condition and its interpretation

A junction condition is still needed so that the solution of (2.10) be uniquely defined. Indeed, at first glance, one may think that $u_t(t,0)$ is equal to $-g'(t)$ which is given by (2.6) (where we have assumed (2.7)). The point is that this condition can not be satisfied for every time. One way to be convinced oneself of that fact is to consider the case $m=n=1$ with $f^1 = f^2 = f$. Then, we look at solutions u of the Hamilton-Jacobi equation in $\mathbb{R}$ with the artificial junction. We can simply associate with it the classical conservation law on the whole real line. We can then consider a single shock moving with constant velocity for the conservation law. When this shock will pass through the jonction point (let us say at time t_0), this will mean that $u_t(t_0,\cdot)$ is discontinuous in space at the junction point. In particular the formal computations of Subsection 2.1 are no longer valid at that time t_0, even if they are valid for $t \neq t_0$. For a general problem, one may expect that our formal computations are only valid for almost every time (even if it is not clear for us).

In view of Theorem 1.4, if we restrict our attention to continuous solutions u, then we will have $u \leq u_{\text{oc}}$ where u_{oc} is the solution associated to the optimal control problem. This shows in particular that we have

$$u(t,0) \leq u_{\text{oc}}(t,0)$$

which means (in view of (2.8) and the interpretation of $-U^l$ given in subsection 2.1) that we have a universal bound on the total amount of cars passing through the junction point during the time interval $[0,t)$. If we assume moreover that this amount of cars is maximal, then we can choose (and indeed have to choose) $u = u_{\text{oc}}$ and the natural junction condition is then

$$u_t(t,0) + \max_{k=1,...,N} H_k^-(u_X^k(t,0^+)) = 0 \tag{2.11}$$

with

$$H_k^-(p) = \sup_{q \leq 0} (pq - L_k(q)) \quad \text{and} \quad L_k(p) = \sup_{q \in \mathbb{R}} (pq - H_k(q))\,.$$

Using our assumption (2.1) on the functions f^k, let us define for $k = 1, ..., N$ the Demand functions

$$f_D^k(p) = \begin{cases} f^k(p) & \text{for} \quad p \leq \rho_c^k \\ f^k(\rho_c^k) & \text{for} \quad p \geq \rho_c^k \end{cases}$$

and the Supply functions

$$f_S^k(p) = \begin{cases} f^k(\rho_c^k) & \text{for} \quad p \leq \rho_c^k \\ f^k(p) & \text{for} \quad p \geq \rho_c^k. \end{cases}$$

From assumption (2.1) on the functions f^k, we deduce that

$$\begin{cases} H_i^-(p) = -\frac{1}{\gamma^i} f_D^i(\gamma^i p), & \text{for} \quad i = 1, ..., m \\ H_j^-(p) = -\frac{1}{\gamma^j} f_S^j(-\gamma^j p), & \text{for} \quad j = m+1, ..., m+n = N. \end{cases}$$

Condition (2.11) means that

$$\begin{aligned} -U_t^1(t,0) = u_t(t,0) &= \min_{k=1,...,N} -H_k^-(u_X^k(t,0^+)) \\ &= \min\left(\min_{i=1,...,m} \frac{1}{\gamma^i} f_D^i(\rho^i(t,0^-)), \min_{j=m+1,...,m+n} \frac{1}{\gamma^j} f_S^j(\rho^j(t,0^+)) \right). \end{aligned} \quad (2.12)$$

Notice that from (2.7), it is natural to compare

$$\frac{1}{\gamma^i} f^i(\rho^i(t,0^-)) \quad \text{and} \quad \frac{1}{\gamma^j} f^j(\rho^j(t,0^+)).$$

Then condition (2.12) is nothing else that the Demand and Supply condition of Lebacque, which claims that the passing flux is equal to the mimimum between the Demand and the Supply, as it is defined in [33] (at least in the case $m = 1$).

In the special case $m = 1$, it is explained in [34] that this condition (2.12) is also equivalent to the condition defining the Riemmann solver at the junction point in the book of Garavello and Piccoli [27]. Let us notice that this condition is also related to the Bardos, Le Roux, Nédelec [6] boundary condition.

3 Viscosity solutions

In this section, we consider a weaker assumption than (A1). We introduce the following assumption:

(A1') For each $i \in I_N$,

- the function $H_i \colon \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{|p| \to +\infty} H_i(p) = +\infty$;
- there exists $p_0^i \in \mathbb{R}$ such that H_i is non-increasing on $(-\infty, p_0^i]$ and non-decreasing on $[p_0^i, +\infty)$;

When (A1') holds true, the function H_i^- is defined by $H_i^-(p) = \inf_{q \leq 0} H_i(p+q)$. We now make the following useful remark whose proof is left to the reader.

Lemma 3.1. *Assumption (A1) implies Assumption (A1').*

Next we give equivalent definitions of viscosity solutions for (1.1). We give a first definition where the jonction condition is satisfied in "the classical sense"; we then prove that it is equivalent to impose it in "the generalized sense". It is essential if one expects solutions to be stable.

We give a first definition of viscosity solutions for (1.1) in terms of test functions by imposing the junction condition in the classical sense. We recall the definition of the upper and lower semi-continuous envelopes u^* and u_* of a function $u \colon [0,T) \times J$:

$$u^*(t,x) = \limsup_{(s,y) \to (t,x)} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{(s,y) \to (t,x)} u(s,y).$$

Definition 3.2 (Viscosity solutions)**.** *A function* $u\colon [0,T)\times J\to\mathbb{R}$ *is a* subsolution (resp. supersolution) of (1.1) on J_T *if it is upper semi-continuous (resp. lower semi-continuous) and if for any* $\phi\in C^1_*(J_T)$ *such that* $u\le\phi$ *in* $B(P,r)$ *for some* $P=(t,x)\in J_T$, $r>0$ *and such that* $u=\phi$ *at* $P\in J_T$, *we have*

$$\phi_t(t,x)+H(x,\phi_x(t,x))\le 0 \quad \textit{(resp. } \ge 0\textit{)},$$

that is to say

- *if* $x\in J_i^*$, *then*
$$\phi_t(t,x)+H_i(\phi_x(t,x))\le 0 \quad \textit{(resp. } \ge 0\textit{)};$$
- *if* $x=0$, *then*
$$\phi_t(t,0)+\max_{i\in I_N} H_i^-(\phi_x^i(t,0))\le 0 \quad \textit{(resp. } \ge 0\textit{)}. \tag{3.1}$$

A function $u\colon [0,T)\times J\to\mathbb{R}$ *is a* subsolution (resp. supersolution) of (1.1)-(1.2) on J_T *if it is a subsolution (resp. supersolution) of* (1.1) *on* J_T *and moreover satisfies* $u(0,\cdot)\le u_0$ *(resp.* $u(0,\cdot)\ge u_0$*).*

A function $u\colon [0,T)\times J\to\mathbb{R}$ *is a* (viscosity) solution of (1.1) (resp. (1.1)-(1.2)) on J_T *if* u^* *is a subsolution and* u_* *is a supersolution of* (1.1) *(resp.* (1.1)-(1.2)*) on* J_T.

As mentioned above, the following proposition is important in order to get discontinuous stability results for the viscosity solutions of Definition 3.2.

Proposition 3.3 (Equivalence with relaxed junction conditions)**.** *Assume (A1'). A function* $u\colon J_T\to\mathbb{R}$ *is a subsolution (resp. supersolution) of* (1.1) *on* J_T *if and only if for any function* $\phi\in C^1_*(J_T)$ *such that* $u\le\phi$ *in* J_T *and* $u=\phi$ *at* $(t,x)\in J_T$,

- *if* $x\in J_i^*$, *then*
$$\phi_t(t,x)+H_i(\phi_x(t,x))\le 0 \quad \textit{(resp. } \ge 0\textit{)}$$
- *if* $x=0$, *then either there exists* $i\in I_N$ *such that*
$$\phi_t(t,0)+H_i(\phi_x(t,0))\le 0 \quad \textit{(resp. } \ge 0\textit{)}$$
or (3.1) *holds true.*

Proof of Proposition 3.3. We classically reduce to the case where the ball $B(P,r)$ is replaced with J_T.

The "if" part is clear. Let us prove the "only if" one. We distinguish the subsolution case and the supersolution one. We start with supersolutions since it is slightly easier.

Case 1: supersolution case. We consider a test function $\phi\in C^1_*(J_T)$ such that $u\ge\phi$ in J_T and $u=\phi$ at (t_0,x_0). There is nothing to prove if $x_0\neq 0$ so we assume $x_0=0$. We have to prove that $\phi_t(t_0,0)+\sup_{i\in I_N} H_i^-(\phi_x^i(t_0,0))\ge 0$. We argue by contradiction and we assume that

$$\phi_t(t_0,0)+\sup_{i\in I_N} H_i^-(\phi_x^i(t_0,0))<0. \tag{3.2}$$

Then it is easy to see that there exists a function $\tilde\phi\in C^1_*(J_T)$ such that $\phi\ge\tilde\phi$ with equality at the point $(t_0,0)$ and such that

$$\tilde\phi_x^i(t_0,0)=\min(\phi_x^i(t_0,0),p_0^i) \quad \text{and} \quad \tilde\phi_t(t_0,0)=\phi_t(t_0,0). \tag{3.3}$$

Notice that

$$H_i^-(\tilde\phi_x^i(t_0,0))\le H_i(\tilde\phi_x^i(t_0,0))\le H_i^-(\phi_x^i(t_0,0)). \tag{3.4}$$

The first inequality is straightforward. To check the second inequality, we have to distinguish two cases. Either we have $\tilde\phi_x^i(t_0,0)<\phi_x^i(t_0,0)$, and then $\tilde\phi_x^i(t_0,0)=p_0^i$ and we use the fact that the

minimum of H_i^- is $H_i(p_0^i)$. Or $\tilde{\phi}_x^i(t_0,0) = \phi_x^i(t_0,0)$ and then this common value belongs to the interval $(-\infty, p_0^i]$ on which we have $H_i = H_i^-$.

Since $u \geq \tilde{\phi}$ in J_T and $u = \tilde{\phi}$ at $(t_0,0)$, we conclude that either

$$\tilde{\phi}_t(t_0,0) + \sup_{i\in I_N} H_i^-(\tilde{\phi}_x^i(t_0,0)) \geq 0$$

or there exists $i \in I_N$ such that

$$\tilde{\phi}_t(t_0,0) + H_i(\tilde{\phi}_x^i(t_0,0)) \geq 0.$$

In view of (3.3) and (3.4), we obtain a contradiction with (3.2).

Case 2: subsolution case. We consider a function $\phi \in C^1(J_T)$ such that $u \leq \phi$ in J_T and $u = \phi$ at (t_0, x_0). There is nothing to prove if $x_0 \neq 0$ and we thus assume $x_0 = 0$. We have to prove that $\phi_t(t_0,0) + \sup_{i\in I_N} H_i^-(\phi_x^i(t_0,0)) \leq 0$. We argue by contradiction and we assume that

$$\phi_t(t_0,0) + \sup_{i\in I_N} H_i^-(\phi_x^i(t_0,0)) > 0. \tag{3.5}$$

In order to construct a test function $\tilde{\phi}$, we first consider $\bar{I}_1 \subset I_N$ the set of j's such that

$$H_j^-(\phi_x^j(t_0,0)) < \sup_{i\in I_N} H_i^-(\phi_x^i(t_0,0)).$$

Since H_j is coercive, there exists $q^j \geq p_0^j$ such that $H_j(q^j) = \sup_{i\in I_N} H_i^-(\phi_x^i(t_0,0))$.

We next consider a test function $\tilde{\phi} \in C_*^1(J_T)$ such that $\phi \leq \tilde{\phi}$ with equality at $(t_0,0)$ and such that

$$\tilde{\phi}_x^i(t_0,0) = \begin{cases} \max(\phi_x^i(t_0,0), q^i) & \text{if } i \in \bar{I}_1, \\ \phi_x^i(t_0,0) & \text{if not,} \end{cases} \qquad \text{and} \qquad \tilde{\phi}_t(t_0,0) = \phi_t(t_0,0). \tag{3.6}$$

Notice that for all $j \in I_N$,

$$H_j(\tilde{\phi}_x^j(t_0,0)) \geq \sup_{i\in I_N} H_i^-(\tilde{\phi}_x^i(t_0,0)) = \sup_{i\in I_N} H_i^-(\phi_x^i(t_0,0)) \tag{3.7}$$

where for the inequality, we have in particular used the fact that H_j is non-decreasing on $[p_0^j, +\infty)$.

Since $u \leq \tilde{\phi}$ in J_T and $u = \tilde{\phi}$ at $(t_0,0)$, we conclude that either

$$\tilde{\phi}_t(t_0,0) + \sup_{i\in I_N} H_i^-(\tilde{\phi}_x^i(t_0,0)) \leq 0$$

or there exists $j \in I_N$ such that

$$\tilde{\phi}_t(t_0,0) + H_j(\tilde{\phi}_x^j(t_0,0)) \leq 0.$$

In view of (3.6) and (3.7), we obtain a contradiction with (3.5). This ends the proof of the Proposition. □

We now prove Theorem 1.4.

Proof of Theorem 1.4. Let us consider a function $\phi \in C_*^1(J_T)$ such that $u \leq \phi$ with equality at $(t_0,0)$ with $t_0 \in (0,T)$. Modifying ϕ if necessary, we can always assume that the supremum of $u - \phi$ is strict (and reached at $(t_0,0)$). For $\eta = (\eta_1, ..., \eta_N) \in (\mathbb{R}^+)^N$, we set

$$M_\eta = \sup_{(t,x=x_je_j)\in J_T} \left(u(t,x) - \phi(t,x) - \frac{\eta_j}{|x_j|} \right).$$

Because u is continuous at $(t_0,0)$, we get for $\eta \in (\mathbb{R}_*^+)^N$ that

$$\left\{ \begin{array}{l} M_\eta \to M_0 = 0 \\ (t^\eta, x^\eta) \to (t_0,0) \end{array} \right| \quad \text{as soon as one of the component } \eta_{i_0} \to 0. \tag{3.8}$$

where $(t^\eta, x^\eta) \in J_T$ is a point where the suppremum in M_η is reached.

Moreover given the components $\eta_j > 0$ for $j \in I_N \setminus \{i_0\}$, we can use (3.8) in order to find $\eta_{i_0} > 0$ small enough to ensure that $x^\eta \in J^*_{i_0}$. Then we have in particular the following subsolution viscosity inequality at that point (t^η, x^η):

$$\phi_t + H_{i_0}\left(\phi_x - \frac{\eta_{i_0}}{|x^\eta|^2}\right) \leq 0.$$

Therefore passing to the limit $\eta_{i_0} \to 0$, we get

$$\phi_t + H^-_{i_0}(\phi^{i_0}_x) \leq 0 \quad \text{at} \quad (t_0, 0).$$

Because this is true for any $i_0 \in I_N$, we finally get the subsolution viscosity inequality at the junction:

$$\phi_t + \max_{i \in I_N} H^-_i(\phi^i_x) \leq 0 \quad \text{at} \quad (t_0, 0).$$

Now the fact that $u \leq u_{\text{oc}}$ follows from the comparison principle. This ends the proof of the Theorem. □

4 The minimal action

We already mentioned that the optimal control solution of the Hamilton-Jacobi equation defined by (1.8) plays a central role in our analysis. We remark that for $x \in J$ and $t > 0$,

$$u_{\text{oc}}(t, x) = \inf_{y \in J} \{u_0(y) + \mathcal{D}(0, y; t, x)\} \tag{4.1}$$

where

$$\mathcal{D}(0, y; t, x) = \min_{X \in \mathcal{A}(0,y;t,x)} \int_0^t L(X(\tau), \dot{X}(\tau)) d\tau.$$

More generally, keeping in mind the weak KAM theory, we define the so-called *minimal action* $\mathcal{D} : \{(s, y, t, x) \in ([0, \infty) \times J)^2, s < t\} \to \mathbb{R}$ by

$$\mathcal{D}(s, y; t, x) = \inf_{X \in \mathcal{A}(s,y;t,x)} \int_s^t L(X(\tau), \dot{X}(\tau)) d\tau. \tag{4.2}$$

It is convenient to extend $\mathcal{D}$ to $\{s = t\}$. We do so by setting

$$\mathcal{D}(t, y, t, x) = \begin{cases} 0 & \text{if } y = x, \\ +\infty & \text{if } y \neq x. \end{cases}$$

Remark 4.1 (Dynamic Programming Principle). Under assumptions (A0)-(A1), it is possible (and easy) to prove the following Dynamic Programming Principle: for all $x \in J$ and $s \in [0, t]$,

$$u_{\text{oc}}(t, x) = \inf_{y \in J} \{u_{\text{oc}}(s, y) + \mathcal{D}(s, y; t, x)\}.$$

Notice that a super-optimality principle will be proved in Proposition 5.1.

Before stating the main result of this section, we

The following result can be considered as the core of our analysis. The most important part of the following theorem is the fact that the minimal action is semi-concave with respect to (t, x) (resp. (s, y)).

Theorem 4.2 (Key inequalities for $\mathcal{D}$)**.** *$\mathcal{D}$ is finite, continuous in $\{(s, y; t, x) \colon 0 < s < t, x, y \in J\}$ and lower semi-continuous in $\{(s, y; t, x) \colon 0 < s \leq t, x, y \in J\}$. Moreover, for all (s_0, y_0) and $(t_0, x_0) \in (0, T) \times J$, $s_0 < t_0$, there exist two functions $\phi, \psi \in C^1_*(J_T)$ and $r > 0$ such that*

- $\phi \geq \mathcal{D}(s_0, y_0; \cdot, \cdot)$ *on a ball* $B(P_0, r)$ *with equality at* $P_0 = (t_0, x_0)$ *and*

$$\phi_t + H(x_0, \phi_x) \geq 0 \quad \text{at} \quad (t_0, x_0). \tag{4.3}$$

- $\psi \geq \mathcal{D}(\cdot, \cdot; t_0, x_0)$ *on a ball* $B(Q_0, r)$ *with equality at* $Q_0 = (s_0, y_0)$ *and*

$$\begin{cases} -\psi_s + H(y_0, -\psi_y) \leq 0 \quad \text{at} \quad (s_0, y_0) & \text{if} \quad N \geq 2, \\ -\psi_s + H_1(-\psi_y) \leq 0 \quad \text{at} \quad (s_0, y_0) & \text{if} \quad N = 1. \end{cases} \tag{4.4}$$

Moreover, for all $R > 0$*, there exists a constant* $C_R > 0$ *such that we have*

$$d(y_0, x_0) \leq R \quad \Longrightarrow \quad |\phi_x(t_0, x_0)| + |\psi_y(s_0, y_0)| \leq C_R. \tag{4.5}$$

Remark 4.3. As we shall see when proving this result, we can even require equalities instead of inequalities in (4.3) and (4.4).

Since the proof of Theorem 4.2 is lengthy and technical, we postpone it until Section 6. When proving the main results of our paper in the next section, we also need the following lower bound on $\mathcal{D}$. We remark that this bound ensures in particular that it is finite.

Lemma 4.4. *Assume (A1). Then*

$$\mathcal{D}(s, y; t, x) \geq \frac{\gamma}{4(t-s)} d(y, x)^2 - C_0(t - s)$$

where $C_0 \colon = \max(0, -L_0(0) + \frac{\gamma_0^2}{\gamma})$*,* γ *appears in (A1),* $\gamma_0 = \max_{i \in I_N} |L_i'(0)|$ *and* $L_0(0)$ *is chosen as in (1.10). Moreover,*

$$\mathcal{D}(s, x; t, x) \leq L_0(0)(t - s).$$

In particular, if $(t_n, x_n) \to (t, x)$*, then* $\mathcal{D}(t_n, x_n; t, x_n) \to 0$ *as* $n \to \infty$*.*

Proof of Lemma 4.4. We only prove the first inequality since the remaining of the statement is elementary. We have

$$L_i(p) \geq \frac{\gamma}{2} p^2 + L_i'(0) p + L_i(0) \geq \frac{\gamma}{2} p^2 - \gamma_0 |p| + L_0(0) \geq \frac{\gamma}{4} p^2 + L_0(0) - \frac{\gamma_0^2}{\gamma}.$$

This shows that

$$L_i(p) \geq \frac{\gamma}{4} p^2 - C_0. \tag{4.6}$$

Thus we can write for $X(\cdot) \in \mathcal{A}(s, y; t, x)$,

$$\int_s^t L(X(\tau), \dot{X}(\tau))\, d\tau \geq -C_0(t - s) + \frac{\gamma}{4} \int_s^t (\dot{X}(\tau))^2\, d\tau.$$

Then Jensen's inequality allows us to conclude. □

5 Proofs of the main results

In this section, we investigate the uniqueness of the solution of (1.1)-(1.2). In particular, we will show that the solution constructed by Perron's method coincide with the function u_{oc} coming from the associated optimal control problem.

5.1 Supersolutions and super-optimality

In this subsection, we will show that a supersolution satisfies a super-optimality principle. For the sake of clarity, we first give a formal argument to understand this claim. We consider the auxiliary function, for $s \leq t$,

$$U_{t,x}(s) = \inf_{y \in J} \{u(s,y) + \mathcal{D}(s,y;t,x)\} \tag{5.1}$$

and we are going to explain formally that it is non-decreasing with respect to s as soon as u is a supersolution of (1.1). We call this property a super-optimality principle. Notice that this is strongly related to the fact that the quantity $U_{t,x}(s)$ is constant in s if u is equal to the optimal control solution $u_{\rm oc}$.

Assume that the infimum defining U is attained for some $\bar{y} \in J$. Then we write

$$\begin{aligned} U'_{t,x}(s) &= \partial_s u(s,\bar{y}) + \partial_s \mathcal{D}(s,\bar{y};t,x) \\ \partial_x u(s,\bar{y}) &= -\partial_y \mathcal{D}(s,\bar{y};t,x). \end{aligned}$$

Moreover assuming $\mathcal{D}$ to be smooth (which is not the case), we formally get from (4.4) the fact that $\partial_s \mathcal{D}(s,\bar{y};t,x) \geq H(\bar{y}, -\partial_y \mathcal{D}(\bar{s},\bar{y};t,x))$ (at least in the case $N \geq 2$). Hence

$$U'_{t,x}(s) \geq \partial_s u(s,\bar{y}) + H(\bar{y}, \partial_x u(s,\bar{y})) \geq 0.$$

We thus conclude that $U_{t,x}$ is non-decreasing if u is a supersolution of (1.1). We now give a precise statement and a rigourous proof.

Proposition 5.1 (Super-optimality of supersolutions)**.** *Assume (A1). Let $u\colon [0,T) \times J \to \mathbb{R}$ be a supersolution of (1.1) on J_T such that there exists $\sigma > 0$ such that for all $(t,x) \in J_T$,*

$$u(t,x) \geq -\sigma(1 + d(x,0)) \tag{5.2}$$

Then for all $(t,x) \in J_T$ and $s \in (0,t]$,

$$u(t,x) \geq \inf_{y \in J} \{u(s,y) + \mathcal{D}(s,y;t,x)\} \tag{5.3}$$

Assume moreover (A0) and that u is a supersolution of (1.1)-(1.2) on J_T. Then we have $u \geq u_{\rm oc}$ on $[0,T) \times J$.

Proof of Proposition 5.1. The proof proceeds in several steps.

Step 1: preliminary. Notice first that from (6.7), we get

$$u(s,y) + \mathcal{D}(s,y;t,x) \geq \frac{\gamma}{4(t-s)} d(y,x)^2 - C_0(t-s) - \sigma(1+|y|).$$

Using the lower semi-continuity of $\mathcal{D}$, we see that the infimum in y of this function is then reached for bounded y's. Moreover by lower semi-continuity of the map $(s,y;t,x) \mapsto u(s,y) + \mathcal{D}(s,y;t,x)$, we deduce in particular that the map $(s;t,x) \mapsto U_{t,x}(s)$ (and then also $s \mapsto U_{t,x}(s)$) is lower semi-continuous.

Step 2: the map $\mathbf{s \mapsto U_{t,x}(s)}$ is non-decreasing. We are going to prove that for $s \in (0,t)$, $U'_{t,x}(s) \geq 0$ in the viscosity sense. We consider a test function φ touching $U_{t,x}$ from below at $\bar{s} \in (0,t)$. There exists $\bar{y}$ such that

$$U_{t,x}(\bar{s}) = u(\bar{s},\bar{y}) + \mathcal{D}(\bar{s},\bar{y};t,x).$$

We deduce from the definition of $U_{t,x}$ that

$$\varphi(s) - \mathcal{D}(s,y;t,x) - [\varphi(\bar{s}) - \mathcal{D}(\bar{s},\bar{y};t,x)] \leq u(s,y) - u(\bar{s},\bar{y}).$$

By Theorem 4.2, there exists a test function ψ such that $\psi \geq \mathcal{D}(\cdot,\cdot;t,x)$ on a ball $B(\bar{Q},r)$ with equality at $\bar{Q} = (\bar{s},\bar{y})$. Hence, we can rewrite the previous inequality by replacing $\mathcal{D}$ with ψ. We

then obtain that $(s,y) \mapsto \varphi(s) - \psi(s,y)$ is a test function touching u at $(\bar{s},\bar{y})$ from below. Since u is a super-solution of (1.1), we have in the cases $N \geq 2$ or $N = 1$ and $\bar{y} \neq 0$

$$\varphi'(\bar{s}) \geq \psi_s(\bar{s},\bar{y}) - H(\bar{y}, -\partial_y\psi(\bar{s},\bar{y})) \geq 0$$

and in the case $N = 1$ and $\bar{y} = 0$

$$\varphi'(\bar{s}) \geq \psi_s(\bar{s},\bar{y}) - H_1^-(-\partial_y\psi(\bar{s},\bar{y})) \geq \psi_s(\bar{s},\bar{y}) - H_1(-\partial_y\psi(\bar{s},\bar{y})) \geq 0$$

where we used the properties of the function ψ given by Theorem 4.2.

Step 3: conclusion. Let us define for $(t,x) \in J_T$ the following kind of lower semi-continuous envelope (for the past in time)

$$\underline{u}(t,x) = \liminf\{u(t_n,x_n) : (t_n,x_n) \to (t,x), t_n < t\}.$$

Let us notice that we have

$$\underline{u}_* = u_* = u \quad \text{on} \quad J_T. \tag{5.4}$$

Given a point $(t,x) \in J_T$, let us consider a sequence $(t_n,x_n) \to (t,x)$ such that

$$\underline{u}(t,x) = \lim_{n\to+\infty} u(t_n,x_n).$$

Using Lemma 4.4, we have for any $s < t_n < t$

$$U_{t,x_n}(s) \leq U_{t,x_n}(t_n) \leq u(t_n,x_n) + \mathcal{D}(t_n,x_n,t,x_n) \to \underline{u}(t,x).$$

Therefore from the lower semicontinuity of U, we get

$$U_{t,x}(s) \leq \underline{u}(t,x).$$

Again from the lower semi-continuity of the map $(t,x) \mapsto U_{t,x}(s)$, we get passing to the lower semi-continuous envelopes in (t,x):

$$U_{t,x}(s) \leq \underline{u}_*(t,x) = u(t,x)$$

where we have used (5.4). This shows (5.3) for $0 < s < t$. This is still true for $s = t$ by definition of $\mathcal{D}$. The proof is now complete. □

5.2 Comparison with subsolutions

Proposition 5.2 (Comparison with subsolutions)**.** *Let $u\colon J_T \to \mathbb{R}$ be a subsolution of (1.1)-(1.2) on J_T, such that there exists $\sigma > 0$ such that for all $(t,x) \in J_T$,*

$$u(t,x) \leq \sigma(1 + d(x,0)). \tag{5.5}$$

Then we have $u \leq u_{\rm oc}$ on J_T.

In order to prove Proposition 5.2, we first state and prove two lemmas.

Lemma 5.3. *Assume (A0)-(A1). Then the function $u_{\rm oc}$ defined in (1.8) satisfies*

$$|u_{\rm oc}(t,x) - u_0(x)| \leq Ct.$$

Proof of Lemma 5.3. We first get a bound from below. Using (6.7), we deduce (denoting by L_{u_0} the Lipschitz constant for u_0):

$$\begin{aligned} u_0(y) + \mathcal{D}(s,y;t,x) &\geq u_0(x) + \tfrac{\gamma}{4t}(d(y,x))^2 - C_0 t - L_{u_0} d(y,x) \\ &\geq u_0(x) - C_2 t \end{aligned}$$

with

$$-C_2 = \inf_{a\in[0,+\infty)} \left\{ \frac{\gamma}{4}a^2 - C_0 - L_{u_0}a \right\} > -\infty.$$

This implies that

$$u_{\rm oc}(x) \geq u_0(x) - C_2 t.$$

We next get a bound from above. We have

$$u_{\rm oc}(x) \leq u_0(x) + \mathcal{D}(0,x;t,x) \leq u_0(x) + Mt$$

with

$$M = \sup_{i\in I_N} L_i(0).$$

This ends the proof of the lemma. □

Lemma 5.4. *Assume (A0)-(A1). Let $u : [0,T)\times J \to \mathbb{R}$ be a subsolution of (1.1)-(1.2) on J_T, satisfying (5.5). Then there exists a constant $C > 0$ such that*

$$u(t,x) \leq u_0(x) + Ct \quad \textit{for all} \quad (t,x) \in J_T. \tag{5.6}$$

Proof of Lemma 5.4. Using the Lipschitz regularity of u_0, we can easily consider a smooth approximation u_0^η of u_0 such that $u_0^\eta \geq u_0$ and $|u_0^\eta - u_0|_{L^\infty(J)} \to 0$ as $\eta \to 0$. Then consider the following supremum for $\eta, \alpha > 0$

$$N_{\eta,\alpha} = \sup_{(t,x)\in[0,T)\times J} \{u(t,x) - u_0^\eta(x) - Ct - \alpha d(x,0)^2 - \frac{\eta}{T-t}\}.$$

We claim that $N_{\eta,\alpha} \leq 0$ for some C large enough independent on $\eta, \alpha > 0$ small enough. The lemma will be obtained by letting α and η go to 0. We argue by contradiction and assume that $N_{\eta,\alpha} > 0$. Thanks to (5.5), the supremum $N_{\eta,\alpha}$ is attained for some $(t,x) \in [0,T)\times J$. If $t = 0$, we have $N_{\eta,\alpha} \leq 0$. Therefore $t > 0$ and we can use the fact that u is a subsolution to obtain for $x = x_i e_i$

$$\frac{\eta}{T^2} + C - \max_{j\in I_N} L_j(0) \leq \frac{\eta}{T^2} + C + H(x, \partial_x u_0^\eta(x) + 2\alpha x_i) \leq 0$$

where we have used assumption (A1) to estimate H from below. Notice that we have also made use of a slight abuse of notation in the case $x = 0$. Choosing $C = \max_{j\in I_N} |L_j(0)|$ allows us to conclude to a contradiction. This ends the proof of Lemma 5.4. □

We now turn to the proof of Proposition 5.2.

Proof of Proposition 5.2. The proof proceeds in several steps.

Step 1: preliminaries. Let us consider

$$M = \sup_{(t,x)\in[0,T)\times J} \{u(t,x) - u_{\rm oc}(t,x)\}.$$

From Lemmas 5.3 and 5.4, we deduce that we have $M \leq 2CT < +\infty$. We want to prove that $M \leq 0$.

To this end, we perform the usual corrections considering the following supremum for $\eta, \alpha > 0$

$$M_{\eta,\alpha} = \sup_{(t,x)\in[0,T)\times J} \left\{ u(t,x) - u_{\rm oc}(t,x) - \alpha d(x,0)^2 - \frac{\eta}{T-t} \right\}.$$

As it is proved classically, we also have that $M_{\eta,\alpha} \to M_{\eta,0}$ as $\alpha \to 0$ where

$$M_{\eta,0} = \sup_{(t,x)\in[0,T)\times J} \left\{ u(t,x) - u_{\rm oc}(t,x) - \frac{\eta}{T-t} \right\}.$$

We argue by contradiction by assuming that $M > 0$ and then $M_{\eta,0} \geq M/2 > 0$ for $\eta > 0$ small enough and fixed for the rest of the proof.

Step 2: reduction to $\bar{\mathbf{t}} > \mathbf{0}$. Notice that the supremum $M_{\eta,\alpha}$ is achieved for points $(\bar{t}, \bar{x}) \in [0,T) \times J$. Using again Lemmas 5.3 and 5.4, we also deduce that

$$M/2 < M_{\eta,0} \leq M_{\eta,\alpha} + o_\alpha(1) \leq 2C\bar{t}$$

and hence $\bar{t} \geq \frac{M}{4C} > 0$ for α small enough.

Step 3: a priori bounds. Using the argument of Step 1 of the proof of Proposition 5.1, we see that there exists $\bar{y} \in J$ such that

$$u_{\rm oc}(\bar{t}, \bar{x}) = u_0(\bar{y}) + \mathcal{D}(0, \bar{y}; \bar{t}, \bar{x}).$$

Therefore we can rewrite $M_{\eta,\alpha}$ as

$$M_{\eta,\alpha} = \sup_{0 \leq t < T, x, y \in J} \{u(t,x) - u_0(y) - \mathcal{D}(0,y;t,x) - \alpha d(x,0)^2 - \frac{\eta}{T-t}\}.$$

and the supremum is achieved for $(\bar{t}, \bar{x}, \bar{y}) \in (0,T) \times J^2$. Notice that this supremum looks like the classical one for proving the comparison principle for viscosity solutions, with the usual penalization term $(y-x)^2/\varepsilon$ replaced here by the function $\mathcal{D}(0,y;t,x)$.

In view of the bound (6.7) from below on $\mathcal{D}$ and (5.6), we derive from $M_{\eta,\alpha} > 0$ that

$$\frac{\eta}{T-\bar{t}} + \alpha d(\bar{x},0)^2 + \frac{\gamma}{4\bar{t}} d(\bar{y},\bar{x})^2 \leq C_0\bar{t} + C\bar{t} + L_{u_0} d(\bar{y},\bar{x})$$

where L_{u_0} denotes the Lipschitz constant of u_0. We conclude that there exists C_T such that

$$\alpha d(\bar{x},0)^2 \leq C_T \quad \text{and} \quad d(\bar{y},\bar{x}) \leq C_T \tag{5.7}$$

where C_T depends on T, C_0,C, L_{u_0} and γ.

Step 4: getting the viscosity inequality. Since $\bar{t} > 0$, we have in particular that

$$u(t,x) - \left(\mathcal{D}(0,\bar{y};t,x) + \alpha d(x,0)^2 + \frac{\eta}{T-t}\right) \leq u(\bar{t},\bar{x}) - \left(\mathcal{D}(0,\bar{y};\bar{t},\bar{x}) + \alpha d(\bar{x},0)^2 + \frac{\eta}{T-\bar{t}}\right).$$

By Theorem 4.2, there exists a test function ϕ such that $\phi \geq \mathcal{D}(0,\bar{y};\cdot,\cdot)$ on a ball $B(\bar{P},r)$ with equality at $\bar{P} = (\bar{t},\bar{x})$. Hence, we can rewrite the previous inequality by replacing $\mathcal{D}$ with ϕ. We then obtain that $(t,x) \mapsto \phi(t,x) + \alpha d(x,0)^2 + \frac{\eta}{T-t}$ touches u from above at $(\bar{t},\bar{x})$ with $\bar{t} > 0$. We use next that u is a subsolution of (1.1) and get for $\bar{x} = \bar{x}_i e_i$

$$\frac{\eta}{T^2} + \phi_t(\bar{t},\bar{x}) + H(\bar{x}, \phi_x(\bar{t},\bar{x}) + 2\alpha\bar{x}_i) \leq 0$$

where we have made use of a slight abuse of notation in the case $\bar{x} = 0$. On the other hand, we have

$$\phi_t(\bar{t},\bar{x}) + H(\bar{x}, \phi_x(\bar{t},\bar{x})) \geq 0$$

therefore

$$\frac{\eta}{T^2} + H(\bar{x}, \phi_x(\bar{t},\bar{x}) + 2\alpha\bar{x}_i) - H(\bar{x}, \phi_x(\bar{t},\bar{x})) \leq 0.$$

On the one hand, from (5.7), we have $0 \leq \alpha x_i \leq \sqrt{\alpha C_T}$. On the other hand, we can use (5.7) and (4.5) in order to conclude that

$$|\phi_x(\bar{t},\bar{x})| \leq \bar{C}$$

for some constant $\bar{C}$ which does not depend on α. We can now use the fact that the Hamiltonians are locally Lipschitz continuous in order to get the desired contradiction for α small enough. This ends the proof of the Proposition. □

5.3 Proof of the main results

In this subsection, we prove the main results announced in the introduction.

Proof of Theorem 1.2. We simply apply Propositions 5.1 and 5.2 and get $u \leq u_{\rm oc} \leq v$ which implies the result. □

In order to prove Theorem 1.1, we should first prove that solutions are Lipschitz continuous.

Lemma 5.5. *Assume (A0)-(A1). Let u be a solution of* (1.1)-(1.2) *on J_T. Then u is Lipschitz continuous with respect to (t,x) on J_T.*

Proof of Lemma 5.5. We first recall (see Lemma 3.1) that (A1) implies (A1'). We know that the solution $u = u^*$ given by Theorem A.2 satisfies for some constant $C > 0$ and all $(t,x) \in J_T$,

$$|u(t,x) - u_0(x)| \leq Ct.$$

From the comparison principle (Theorem 1.2), we deduce that $u = u^* \leq u_*$ and then the solution $u = u^* = u_*$ is continuous.

For $h_0 > 0$ small (with $h_0 < T$), we now consider $h \in (0, h_0)$ and

$$v(t,x) = u(t+h,x) - \sup_{x \in J}(u(h,x) - u_0(x)).$$

This new function satisfies in particular $v(0,x) \leq u_0(x)$. Therefore v is a subsolution of (1.1)-(1.2) on J_{T-h_0}. We thus conclude from the comparison principle that $v(t,x) \leq u(t,x)$, which implies

$$u(t+h,x) \leq u(t,x) + Ch \quad \text{for all} \quad (t,x) \in J_{T-h_0}.$$

Arguing similarly, we can prove that $u(t+h,x) \geq u(t,x) - Ch$. Because h_0 can be chosen arbitrarily small, we conclude that u is Lipschitz continuous with respect to time on the whole J_T.

Since u is a viscosity solution of (1.1), it satisfies in particular (in the viscosity sense) for each $i \in I_N$:

$$H_i(u_x) \leq C \quad \text{on} \quad (0,T) \times J_i^*.$$

This implies that there exists a constant $\tilde{C}$ such that (in the viscosity sense)

$$|u_x| \leq \tilde{C} \quad \text{on} \quad (0,T) \times J^*.$$

This implies that u is Lipschitz continuous with respect to the space variable. This ends the proof of the lemma. □

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. The uniqueness of the solution follows from Theorem 1.2. The existence is obtained thanks to the optimal control interpretation ($u_{\rm oc}$ is a solution). The Lipschitz regularity was proved in Lemma 5.5 above. The proof of Theorem 1.1 is now complete. □

Proof of Theorem 1.3. The fact that the solution is equal to $u_{\rm oc}$ follows from Propositions 5.1 and 5.2. The representation formula (1.9) follows from (4.1). □

6 A complete study of the minimal action

6.1 Reduction of the study

We start this section with the following remark: the analysis can be reduced to the case $(s,t) = (0,1)$. Precisely, using the fact that the Hamiltonian does not depend on time and is positively homogeneous with respect to the state, the reader can check that a change of variables in time yields the following

Lemma 6.1. *For all $x, y \in J$ and $s < t$, we have*

$$\mathcal{D}(s, y; t, x) = (t-s)\mathcal{D}\left(0, \frac{y}{t-s}; 1, \frac{x}{t-s}\right). \tag{6.1}$$

This is the reason why we consider the reduced minimal action $\mathcal{D}_0 \colon J^2 \to \mathbb{R}$ defined by

$$\mathcal{D}_0(y, x) = \mathcal{D}(0, y; 1, x).$$

Thanks to the previous observation, it is enough to prove the following theorem in order to get Theorem 4.2.

Theorem 6.2 (Key equalities for $\mathcal{D}_0$)**.** *Let us assume (A1). Then for all $y, x \in J$, the $\mathcal{D}_0(y, x)$ is finite. It is continuous in J^2 and for all $y, x \in J$, there exists a function $\varphi_0 \in C^1_*(J^2)$ such that $\varphi_0 \geq \mathcal{D}_0$ on J^2, $\varphi_0(y, x) = \mathcal{D}_0(y, x)$ and we have*

$$\text{if } x \neq 0 \colon (\varphi_0 - x\partial_x\varphi_0 - y\partial_y\varphi_0)(y, x) + H(x, \partial_x\varphi_0(y, x)) = 0 \tag{6.2}$$

$$\text{if } x = 0 \colon (\varphi_0 - x\partial_x\varphi_0 - y\partial_y\varphi_0)(y, 0) + \sup_{i \in I_N} H_i^-(\partial_x^i\varphi_0(y, 0)) = 0 \tag{6.3}$$

and if $y \neq 0$,

$$(\varphi_0 - x\partial_x\varphi_0 - y\partial_y\varphi_0)(y, x) + H(y, -\partial_y\varphi_0(y, x)) = 0 \tag{6.4}$$

and if $y = 0$,

$$\begin{cases} (\varphi_0 - x\partial_x\varphi_0 - y\partial_y\varphi_0)(0, x) + \sup_{j \in I_N} H_j^-(-\partial_y^j\varphi_0(0, x)) = 0 & \text{if } N \geq 2, \\ (\varphi_0 - x\partial_x\varphi_0 - y\partial_y\varphi_0)(0, x) + H_1(-\partial_y^j\varphi_0(0, x)) = 0 & \text{if } N = 1. \end{cases} \tag{6.5}$$

Moreover, for all $R > 0$, there exists $C_R > 0$ such that for all $x, y \in J$,

$$d(y, x) \leq R \Rightarrow |\partial_x\varphi_0(y, x)| + |\partial_y\varphi_0(y, x)| \leq C_R. \tag{6.6}$$

Remark 6.3. If $I_0 = I_N$, then we have $\mathcal{D}_0 \in C^1_*(J^2)$. This good case corresponds to the case where all the $L_i(0)$'s are equal.

We can interpret Lemma 4.4 as follows.

Lemma 6.4. *Assume (A1). Then*

$$\mathcal{D}_0(y; x) \geq \frac{\gamma}{4} d(y, x)^2 - C_0 \tag{6.7}$$

where constants are made precise in Lemma 4.4.

6.2 Piecewise linear trajectories

We are going to see that the infimum defining the minimal action can be computed among piecewise linear trajectories, and more precisely among trajectories that are linear as long as they do not reach the junction point. This is a consequence of the fact that the Hamiltonians do not depend on x and are convex (through Jensen's inequality).

In order to state a precise statement, we first introduce that optimal curves are of two types: either they reach the junction point, or they stay in a branch and are straight lines. This is the reason why we introduce first the action associated with straight line trajectories

$$\mathcal{D}_{\text{straight}}(y, x) = \begin{cases} L_i\left(\dfrac{x_i - y_i}{}\right) & \text{if} \quad (y, x) \in J_i^2 \setminus \{(0, 0)\}, \\ L_0(0) & \text{if} \quad y = 0 = x, \\ +\infty & \text{otherwise} \end{cases}$$

and the action associated with piecewise linear trajectories passing through the junction point

$$\mathcal{D}_{\text{junction}}(y, x) = \inf_{0 \leq \tau_1 \leq \tau_2 \leq 1} \{\mathcal{E}_1(\tau_1, y) + \mathcal{E}_2(\tau_2, x)\}$$

where

$$\mathcal{E}_1(\tau_1, y) = \begin{cases} \tau_1 L_j\left(-\frac{y_j}{\tau_1}\right) - \tau_1 L_0(0) & \text{for } y = y_j e_j \neq 0, \tau_1 \neq 0 \\ 0 & \text{for } y = 0 \\ +\infty & \text{for } \tau_1 = 0, y \neq 0 \end{cases}$$

and

$$\mathcal{E}_2(\tau_2, x) = \begin{cases} (1-\tau_2) L_i\left(\frac{x_i}{1-\tau_2}\right) + \tau_2 L_0(0), & \text{for } x = x_i e_i \neq 0, \tau_2 \neq 1 \\ L_0(0) & \text{for } x = 0 \\ +\infty & \text{for } \tau_2 = 1, x \neq 0. \end{cases}$$

Remark 6.5. By defining the $\mathcal{E}_i$'s in such a way, we treat the degenerate cases: $x = 0$ or $y = 0$. Indeed, τ_1 (resp. τ_2) measures how long it takes to the trajectory to reach the junction point (resp. the final point x) from the starting point y (resp. the junction point).

The following facts will be used several times.

Lemma 6.6. *The function $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) is continuous in $(0,1] \times J^*$ (resp. in $[0,1) \times J^*$).*

Lemma 6.7. *The function $\mathcal{E}_i$, $i = 1, 2$ are lower semi-continuous in $[0,1] \times J$.*

Proof. Consider the function defined for $(\tau, y) \in [0,1] \times J$ by

$$g(\tau, y) = \begin{cases} \tau L_i(-\frac{y_i}{\tau}) & \text{if } y = y_i e_i \neq 0, \tau \neq 0 \\ \tau L_0(0) & \text{if } y = 0 \\ +\infty & \text{if } y \neq 0, \tau = 0. \end{cases}$$

From the inequality for $\tau > 0$ (consequence of (4.6)):

$$g(\tau, y) \geq \frac{\gamma}{4} \frac{|y|^2}{\tau} - C_0 \tau,$$

we deduce that g is lower semi-continuous. Consequently, the map $\mathcal{E}_1$ is lower semi-continuous. We proceed similarly for $\mathcal{E}_2$. □

We first show the main lemma of this subsection.

Lemma 6.8. *The infimum defining the reduced minimal action $\mathcal{D}_0$ can be computed among piecewise linear trajectories; more precisely, for all $x, y \in J$,*

$$\mathcal{D}_0(y, x) = \min\left(\mathcal{D}_{\text{straight}}(y, x), \mathcal{D}_{\text{junction}}(y, x)\right).$$

Proof. We write with obvious notation $\mathcal{D}_0(y, x) = \inf_{X \in \mathcal{A}_0(y,x)} \mathcal{E}(X)$. In order to prove the lemma, it is enough to consider a curve $X \in \mathcal{A}_0(y, x)$ and prove that

$$\mathcal{E}(X) \geq \min(\mathcal{D}_{\text{straight}}(y, x), \mathcal{D}_{\text{junction}}(y, x)).$$

To do so, we first remark that the uniform convexity of L_i implies that for all $p_0 \in \mathbb{R}$, we have

$$L_i(p) \geq L_i(p_0) + L_i'(p_0)(p - p_0) + \frac{\gamma}{2}(p - p_0)^2. \tag{6.8}$$

We now consider an admissible trajectory $X \colon [0,1] \to J$ and we treat different cases.

Case A: $X((t_1, t_2)) \subset J_i^*$. We assume that a curve X stays in one of the branch J_i^* on the time interval (t_1, t_2) with $t_1 < t_2$. In such a case, we consider the curve $\tilde{X}$ with same end points $X(t_1)$ and $X(t_2)$ in J_i but linear. If $p_0 \in \mathbb{R}$ is such that $p_0 e_i = \dot{\tilde{X}}(\tau)$ for $\tau \in (t_1, t_2)$ and $p e_i = \dot{X}(\tau)$, we deduce from (6.8) that

$$\int_{t_1}^{t_2} L(X(\tau), \dot{X}(\tau)) d\tau \geq \int_{t_1}^{t_2} L(\tilde{X}(\tau), \dot{\tilde{X}}(\tau)) d\tau + \frac{\gamma}{2} \int_{t_1}^{t_2} |\dot{X}(\tau) - \dot{\tilde{X}}(\tau)|^2 d\tau. \tag{6.9}$$

Case B: $X([t_1,t_2]) \subset J_i$ with $X(t_1) = X(t_2) = 0$. In that case, let us set $\tilde{X}(\tau) = 0$ for $\tau \in [t_1,t_2]$. Using (6.8) with $p_0 = 0$ and the definition of L_0 as a minimum of the L_j's (see (1.10)), we get that

$$L_i(p) \geq L_0(0) + L_i'(0)p + \frac{\gamma}{2}p^2$$

from what we deduce that (6.9) still holds true.

Case C: the general case. By assumption, we have $X \in \mathcal{A}_0(y;x) \subset C([0,1])$. We then distinguish two cases. Either $0 \notin X([0,1])$, and then we define $\tilde{X}$ as in Case A. In this case, (6.9) implies that

$$\mathcal{E}(X) \geq \mathcal{D}_{\text{straight}}(y,x).$$

Or $0 \in X([0,1])$, and then we call $[\tau_1,\tau_2] \subset [s,t]$ the largest interval such that $X(\tau_1) = 0 = X(\tau_2)$, and define $\tilde{X}$ as follows: it is linear between 0 and τ_1, and reaches 0 at τ_1; it stays at 0 in (τ_1,τ_2); then it is linear in $(\tau_2,1)$ and reaches x at $t = 1$. Using again the continuity of X, we can find a decomposition of $[\tau_1,\tau_2]$ as a disjoint union of intervals $\mathcal{I}_k$ (with an at most countable union)

$$[\tau_1,\tau_2] = \bigcup_k \mathcal{I}_k$$

such that for each k, $X(\mathcal{I}_k) \subset J_{i_k}$ for some $i_k \in I_N$ and $X = 0$ on $\partial\mathcal{I}_k$. Using Case A or Case B on each segment $\overline{\mathcal{I}}_k$, we deduce that

$$\mathcal{E}(X) \geq \mathcal{D}_{\text{junction}}(y,x).$$

□

6.3 Continuity of the (reduced) minimal action

Lemma 6.9 (Continuity of $\mathcal{D}_{\text{junction}}$)**.** *The function $\mathcal{D}_{\text{junction}}$ is continuous in J^2.*

Proof. We first prove that $\mathcal{D}_{\text{junction}}$ is lower semi-continuous. We know from Lemma 6.7 that the function

$$G(\tau_1,\tau_2;y;x) = \mathcal{E}_1(\tau_1,y) + \mathcal{E}_2(\tau_2,x)$$

is lower semi-continuous for $y,x \in J$ and $0 \leq \tau_1 \leq \tau_2 \leq 1$. Therefore the function

$$\mathcal{D}_{\text{junction}}(y;x) = \inf_{0\leq\tau_1\leq\tau_2\leq 1} G(\tau_1,\tau_2;y,x)$$

is also lower semi-continuous (since the infimum is taken over a compact set). Besides, the infimum is in fact a minimum.

We now prove that $\mathcal{D}_{\text{junction}}$ is upper semi-continuous at any point (y,x). Consider first $(\tau_1,\tau_2) \in [0,1]^2$ such that

$$\mathcal{D}_{\text{junction}}(y,x) = \mathcal{E}_1(\tau_1,y) + \mathcal{E}_2(\tau_2,x).$$

Given any sequence $(y^k,x^k) \to (y,x)$, we want to show that

$$\mathcal{D}_{\text{junction}}(y^k,x^k) \leq \mathcal{D}_{\text{junction}}(y,x) + o_k(1) \tag{6.10}$$

We use

$$\mathcal{D}_{\text{junction}}(y^k,x^k) \leq \mathcal{E}_1(\tau_1^k,y^k) + \mathcal{E}_2(\tau_2^k,x^k)$$

with an appropriate choice of (τ_1^k,τ_2^k).

Case 1: $y \in J_j^*$, $x \in J_i^*$. In this case, we choose $(\tau_1^k, \tau_2^k) = (\tau_1, \tau_2) \in (0,1)^2$ and we use Lemma 6.6 in order to get

$$\mathcal{E}_1(\tau_1^k, y^k) \to \mathcal{E}_1(\tau_1, y)$$

and

$$\mathcal{E}_2(\tau_2^k, x^k) \to \mathcal{E}_2(\tau_2, x)$$

Hence we conclude that (6.10) holds true.

Case 2: $y = 0$, $x \in J_i^*$. We choose $(\tau_1^k, \tau_2^k) = (|y^k|, \max(\tau_2, |y^k|)) \in [0,1)^2$. We still have $\tau_2^k \to \tau_2$ and we can use Lemma 6.6 in order to get

$$\mathcal{E}_2(\tau_2^k, x^k) \to \mathcal{E}_2(\tau_2, x).$$

We also have (if $y^k \in J_j$)

$$\mathcal{E}_1(\tau_1^k, y^k) \leq |y^k| L_j \left(-\frac{y_j^k}{|y^k|} \right) - |y^k| L_0(0) \to 0 = \mathcal{E}_1(\tau_1, 0). \tag{6.11}$$

Hence we conclude that (6.10) holds true.

Case 3: $y \in J_j^*$, $x = 0$. We choose $(\tau_1^k, \tau_2^k) = (\min(\tau_1, 1 - |x^k|), 1 - |x^k|) \in (0,1]^2$ We still have $\tau_1^k \to \tau_1$ and then

$$\mathcal{E}_1(\tau_1^k, y^k) \to \mathcal{E}_1(\tau_1, y)$$

(since $\mathcal{E}_1$ is continuous in $(0,1] \times J^*$). We also have (if $x^k \in J_i$)

$$\mathcal{E}_2(\tau_2^k, x^k) \leq |x^k| L_i \left(\frac{x_i^k}{|x^k|} \right) + (1 - |x^k|) L_0(0) \to L_0(0) = \mathcal{E}_2(\tau_2, 0) \tag{6.12}$$

Hence we conclude that (6.10) holds true.

Case 4: $y = 0$, $x = 0$. We choose $(\tau_1^k, \tau_2^k) = (|y^k|, 1 - |x^k|) \in [0,1) \times (0,1]$. We deduce (6.10) from (6.11) and (6.12). □

Lemma 6.10. *The function $\mathcal{D}_0$ is continuous in J^2.*

Proof. Since $\mathcal{D}_{\text{straight}}$ is lower semi-continuous, we can use Lemmas 6.8 and 6.9 in order to conclude that $\mathcal{D}_0$ is lower semi-continuous.

Consider $(y, x) \in \partial(J_i \times J_i) \setminus \{(0,0)\}$. Then either $x = 0$ or $y = 0$. Moreover for $y = y_i e_i$ and $x = x_i e_i$,

$$\mathcal{D}_{\text{junction}}(y, x) \leq \left\{ \begin{array}{ll} \mathcal{E}_1(1, y) + \mathcal{E}_2(1, x) & \text{if } x_i = 0 \\ \mathcal{E}_1(0, y) + \mathcal{E}_2(0, x) & \text{if } y_i = 0 \end{array} \right\} \leq L_i (x_i - y_i).$$

Therefore for each $i \in I_N$, we have for $(y, x) \in \partial(J_i \times J_i)$,

$$\mathcal{D}_{\text{junction}}(y, x) \leq \mathcal{D}_{\text{straight}}(y, x)$$

Therefore we have with $y = y_i e_i$, $x = x_i e_i$

$$\mathcal{D}_0(y, x) = \begin{cases} \min(\mathcal{D}_{\text{junction}}(y, x), L_i(x_i - y_i)) & \text{if } (y, x) \in J_i \times J_i \\ \mathcal{D}_{\text{junction}}(y, x) & \text{if } (y, x) \in \partial(J_i \times J_i) \\ \mathcal{D}_{\text{junction}}(y, x) & \text{otherwise.} \end{cases} \tag{6.13}$$

This implies that $\mathcal{D}_0$ is continuous in J^2. □

6.4 Study of $\mathcal{D}_{\text{junction}}$

In view of (6.13), we see that the study of $\mathcal{D}_0$ can now be reduced to the study of $\mathcal{D}_{\text{junction}}$. The function $\mathcal{D}_{\text{junction}}$ is defined as a minimum over a triangle $\{(\tau_1, \tau_2) \in [0,1]^2 : \tau_1 \le \tau_2\}$. We will see below that $\mathcal{D}_{\text{junction}}$ is defined implicitely when the constraint $\tau_1 \le \tau_2$ is active ($\mathcal{D}_{\text{implicit}}$) or defined explicitely if not ($\mathcal{D}_{\text{linear}}$). In other words, it will be linear "as long as" trajectories stay some time ($\tau_2 - \tau_1 > 0$) at the junction point.

We first define for $(y, x) \in J^2$,

$$\mathcal{D}_{\text{implicit}}(y, x) = \inf_{0 \le \tau \le 1} \{\mathcal{E}_1(\tau, y) + \mathcal{E}_2(\tau, x)\}. \tag{6.14}$$

The continuity of $\mathcal{D}_{\text{implicit}}$ will be used later on.

Lemma 6.11 (Continuity of $\mathcal{D}^{ji}_{\text{implicit}}$)**.** *The restrictions $\mathcal{D}^{ji}_{\text{implicit}}$ of $\mathcal{D}_{\text{implicit}}$ are continuous in $(J_j \times J_i) \setminus \{(0,0)\}$ and continuous at $(0,0)$ if $j \in I_0$ or $i \in I_0$.*

Proof of Lemma 6.11. From Lemma 6.7, we deduce that $\mathcal{D}^{ji}_{\text{implicit}}$ is lower semi-continuous on $J_j \times J_i$. We now show that $\mathcal{D}^{ji}_{\text{implicit}}$ is upper semi-continuous at any point $(y, x) \in (J_j \times J_i) \setminus \{(0,0)\}$ and also at $(0,0)$ if $j \in I_0$ or $i \in I_0$. We first consider $\tau \in [0,1]$ such that

$$\mathcal{D}^{ji}_{\text{implicit}}(y, x) = \mathcal{E}_1(\tau, y) + \mathcal{E}_2(\tau, x) \quad \text{with} \quad 0 \le \tau \le 1.$$

For any sequence $(y^k, x^k) \to (y, x)$ with $(y^k, x^k) \in J_j \times J_i$, we want to show that

$$\mathcal{D}^{ji}_{\text{implicit}}(y^k, x^k) \le \mathcal{D}^{ji}_{\text{implicit}}(y, x) + o_k(1) \tag{6.15}$$

Arguing as in Lemma 6.9, we use

$$\mathcal{D}^{ji}_{\text{implicit}}(y^k, x^k) \le \mathcal{E}_1(\tau^k, y^k) + \mathcal{E}_2(\tau^k, x^k)$$

and we choose τ^k as follows

$$\begin{array}{ll}
\text{if } y \in J_j^*, x \in J_i^* : & \tau^k = \tau \in (0,1), \\
\text{if } y = 0, x \in J_i^* : & \tau^k = |y_k| \in [0,1), \\
\text{if } y \in J_j^*, x = 0 : & \tau^k = 1 - |x^k| \in (0,1], \\
\text{if } x = 0, j \in I_0 : & \tau^k = 1 - |x^k| \in (0,1], \\
\text{if } y = 0, x = 0, i \in I_0 : & \tau^k = |y^k| \in [0,1).
\end{array}$$

This ends the proof of the lemma. □

We next define for $(y, x) \in J^j \times J^i$

$$\mathcal{D}^{ji}_{\text{linear}}(y, x) = -L_j'(\xi_j^-)y + L_i'(\xi_i^+)x + L_0(0) \tag{6.16}$$

where $\xi_l^\pm$ are defined thanks to the following function (for $l \in I_N$)

$$K_l(\xi) = L_l(\xi) - \xi L_l'(\xi) - L_0(0).$$

Precisely, $\xi_l^\pm = (K_l^\pm)^{-1}(0) \neq 0$ when $l \notin I_0$ (see Lemma 6.17 below). We will see that K_l plays an important role in the analysis of $\mathcal{D}_{\text{junction}}$. In particular, it allows us to define, when $i \notin I_0$ and $j \notin I_0$, the following convex subset (triangle) of $J_j \times J_i$:

$$\Delta^{ji} = \left\{ (y, x) \in J_j \times J_i, \quad \frac{x}{\xi_i^+} - \frac{y}{\xi_j^-} < 1 \right\}$$

It is convenient to set $\Delta^{ji} = \emptyset$ if $i \in I_0$ or $j \in I_0$. We next state a series of lemmas before proving them.

Lemma 6.12 (Link between $\mathcal{D}_{\text{junction}}, \mathcal{D}_{\text{linear}}, \mathcal{D}_{\text{implicit}}$)**.**

$$\mathcal{D}^{ji}_{\text{junction}}(y,x) = \begin{cases} \mathcal{D}^{ji}_{\text{linear}}(y,x) & \text{if} \quad (y,x) \in \Delta^{ji} \\ \mathcal{D}^{ji}_{\text{implicit}}(y,x) & \text{if} \quad (y,x) \in (J_j \times J_i) \setminus \Delta^{ji}. \end{cases} \tag{6.17}$$

Lemma 6.13 (The equations in the interior)**.** *The functions $\mathcal{D}^{ji}_{\text{junction}}$, $\mathcal{D}^{ji}_{\text{linear}}$ and $\mathcal{D}^{ji}_{\text{implicit}}$ are convex and C^1 in $J_j^* \times J_i^*$ and, if $\tilde{\mathcal{D}}$ is one of them, it satisfies for $(y,x) \in J_j^* \times J_i^*$*

$$\begin{cases} \tilde{\mathcal{D}}(y,x) - x\partial_x\tilde{\mathcal{D}}(y,x) - y\partial_y\tilde{\mathcal{D}}(y,x) + H_i(\partial_x\tilde{\mathcal{D}}(y,x)) = 0, \\ \tilde{\mathcal{D}}(y,x) - x\partial_x\tilde{\mathcal{D}}(y,x) - y\partial_y\tilde{\mathcal{D}}(y,x) + H_j(-\partial_y\tilde{\mathcal{D}}(y,x)) = 0. \end{cases} \tag{6.18}$$

Lemma 6.14 (Study of $\mathcal{D}_{\text{implicit}}$)**.** *For $(y,x) \in J_j^* \times J_i^*$, there exists a unique $\tau = T(y,x) \in (0,1)$ such that*

$$\mathcal{D}^{ji}_{\text{implicit}}(y,x) = \mathcal{E}_1(\tau, y) + \mathcal{E}_2(\tau, x);$$

Moreover,

$$\begin{cases} \partial_x\mathcal{D}^{ji}_{\text{implicit}}(y,x) = L_i'(\xi_x) & \text{with} \quad \xi_x = \frac{x}{1-T(y,x)}, \\ \partial_y\mathcal{D}^{ji}_{\text{implicit}}(y,x) = -L_j'(\xi_y) & \text{with} \quad \xi_y = -\frac{y}{T(y,x)}. \end{cases}$$

Lemma 6.15 (Study of T)**.** *For $(y,x) \in (J_j \times J_i) \setminus \{(0,0)\}$, there is a unique $\tau = T(y,x) \in [0,1]$ such that*

$$\mathcal{D}^{ji}_{\text{implicit}}(y,x) = \mathcal{E}_1(\tau, y) + \mathcal{E}_2(\tau, x).$$

Moreover $T \in C(J_j \times J_i \setminus \{(0,0)\})$ and

$$T(y,x) = \begin{cases} \max\left(0, 1 - \frac{x}{\xi_i^+}\right) & \text{if} \quad (y,x) \in (\{0\} \times J_i^*) \setminus \Delta^{ji}, \\ \min\left(1, -\frac{y}{\xi_j^-}\right) & \text{if} \quad (y,x) \in (J_j^* \times \{0\}) \setminus \Delta^{ji}. \end{cases}$$

Lemma 6.16 ($\mathcal{D}^{ji}_{\text{junction}}$ at the boundary)**.** *Then we have $\mathcal{D}^{ji}_{\text{junction}} \in C^1(J_j \times J_i)$ with*

$$\begin{cases} \partial_x\mathcal{D}^{ji}_{\text{junction}}(y,x) = L_i'(\xi_x) \\ \partial_y\mathcal{D}^{ji}_{\text{junction}}(y,x) = -L_j'(\xi_y) \end{cases} \tag{6.19}$$

where $\xi_y \le 0 \le \xi_x$ satisfy

$$(\xi_x, \xi_y) = \begin{cases} (\max(x, \xi_i^+), (K_j^-)^{-1}(K_i(\xi_x))) & \text{if } (y,x) \in (\{0\} \times J_i) \setminus \Delta^{ji} \\ (\xi_i^+, \xi_j^-) & \text{if } (y,x) \in (\{0\} \times J_i) \cap \Delta^{ji} \\ ((K_i^+)^{-1}(K_j(\xi_y)), -\max(y, -\xi_j^-)) & \text{if } (y,x) \in (J_j \times \{0\}) \setminus \Delta^{ji} \\ (\xi_i^+, \xi_j^-) & \text{if } (y,x) \in (J_j \times \{0\}) \cap \Delta^{ji}. \end{cases} \tag{6.20}$$

Moreover we have

$$\begin{cases} \mathcal{D}^{ji}_{\text{junction}}(0,x) = \frac{x}{\xi_x}(L_i(\xi_x) - L_0(0)) + L_0(0) & \text{for} \quad x \in J_i^* \\ \mathcal{D}^{ji}_{\text{junction}}(y,0) = -\frac{y}{\xi_y}(L_j(\xi_y) - L_0(0)) + L_0(0) & \text{for} \quad y \in J_j^* \end{cases} \tag{6.21}$$

and

$$\begin{aligned} &\mathcal{D}^{ji}_{\text{junction}}(x,y) - x\partial_x\mathcal{D}^{ji}_{\text{junction}}(x,y) - y\partial_y\mathcal{D}^{ji}_{\text{junction}}(x,y) \\ &\qquad = \begin{cases} L_0(0) + K_i(\max(x, \xi_i^+)) & \text{if} \quad (y,x) \in \{0\} \times J_i, \\ L_0(0) + K_j(-\max(y, -\xi_j^-)) & \text{if} \quad (y,x) \in J_j \times \{0\}. \end{cases} \end{aligned} \tag{6.22}$$

Before proving these lemmas, the reader can check the following useful properties of the function K_l that will be used in their proofs.

Lemma 6.17 (Properties of K_l)**.** *Assume (A1). Then for any $l \in I_N$, we have*

$$\begin{aligned} K_l'(\xi) &\geq \gamma|\xi| \quad \text{for} \quad \xi \in (-\infty, 0), \\ K_l'(\xi) &\leq -\gamma|\xi| \quad \text{for} \quad \xi \in (0, +\infty). \end{aligned}$$

We define $(K_l^-)^{-1}$ as the inverse of the function K_l restricted to $(-\infty, 0]$, and $(K_l^+)^{-1}$ as the inverse of the function K_l restricted to $[0, +\infty)$. We set

$$\xi_l^{\pm} = (K_l^{\pm})^{-1}(0).$$

Then we have

$$\begin{aligned} \pm\xi_l^{\pm} &= 0 \quad \text{if} \quad l \in I_0, \\ \pm\xi_l^{\pm} &> 0 \quad \text{if} \quad l \in I_N \setminus I_0. \end{aligned}$$

Moreover we have

$$K_l(\xi) = -H_l(L_l'(\xi)) - L_0(0). \tag{6.23}$$

Proof of Lemmas 6.12-6.14. The proof proceeds in several steps.

Step 1: first study of $\mathcal{D}^{ji}_{\text{junction}}$. Let us define

$$G(\tau_1, \tau_2, y, x) = \mathcal{E}_1(\tau_1, y) + \mathcal{E}_2(\tau_2, x).$$

For $\tau_1, \tau_2 \in (0, 1)$, and setting

$$\xi_y = -\frac{y}{\tau_1}, \quad \xi_x = \frac{x}{1 - \tau_2} \tag{6.24}$$

and $V_y = (\xi_y, 0, 1, 0)$ and $V_x = (0, \xi_x, 0, 1)$, we compute

$$D^2 G(\tau_1, \tau_2, y, x) = \frac{L_j''(\xi_y)}{\tau_1} V_y^T V_y + \frac{L_i''(\xi_x)}{1 - \tau_2} V_x^T V_x \geq 0.$$

Therefore G is in particular convex on $(0, 1) \times (0, 1) \times J_j^* \times J_i^*$. Because G is in particular lower semi-continuous on $[0, 1] \times [0, 1] \times J_j^* \times J_i^*$, and

$$G(0, \tau_2, y, x) = +\infty = G(\tau_1, 1, y, x) \quad \text{for } (y, x) \in J_j^* \times J_i^*, \tag{6.25}$$

we deduce that

$$\mathcal{D}^{ji}_{\text{junction}}(y, x) = \inf_{0 < \tau_1 \leq \tau_2 < 1} G(\tau_1, \tau_2, y, x) \quad \text{for} \quad (y, x) \in J_j^* \times J_i^*.$$

This implies that $\mathcal{D}^{ji}_{\text{junction}}$ is also convex in $J_j^* \times J_i^*$. Notice that in particular

$$D^2_{\tau_1 \tau_1} G(\tau_1, \tau_2, y, x) = \frac{y^2}{\tau_1^3} L_j''(\xi_y) > 0$$

and

$$D^2_{\tau_2 \tau_2} G(\tau_1, \tau_2, y, x) = \frac{x^2}{(1 - \tau_2)^3} L_i''(\xi_x) > 0.$$

The map $(\tau_1, \tau_2) \mapsto G(\tau_1, \tau_2, y, x)$ is then strictly convex on the convex set

$$\left\{(\tau_1, \tau_2) \in (0, 1)^2, \quad \tau_1 \leq \tau_2\right\}.$$

Therefore using again (6.25) and the lower semi-continuity of G, we deduce that it has a unique minimum that we denote by (τ_1, τ_2) satisfying $0 < \tau_1 \leq \tau_2 < 1$.

Step 2: study of $\mathcal{D}^{ji}_{\text{implicit}}$. Let us consider the following function

$$e(\tau, y, x) = G(\tau, \tau, y, x).$$

For $\tau \in (0,1)$, setting

$$\xi_y = -\frac{y}{\tau}, \quad \xi_x = \frac{x}{1-\tau}$$

and proceeding similarly as in Step 1, we can deduce that

$$\mathcal{D}^{ji}_{\text{implicit}}(y,x) = \inf_{\tau\in(0,1)} e(\tau, y, x) \quad \text{for} \quad (y,x) \in J_j^* \times J_i^*$$

and that $\mathcal{D}^{ji}_{\text{implicit}}$ is also convex on $J_j^* \times J_i^*$. We can also deduce that the map $\tau \mapsto e(\tau, y, x)$ is strictly convex on $(0,1)$ for $(y,x) \in J_j^* \times J_i^*$ and that it has a unique minimum that we denote by $\tau \in (0,1)$ such that

$$\mathcal{D}^{ji}_{\text{implicit}}(y,x) = e(\tau, y, x).$$

Using the derivative with respect to τ, we see that τ is characterized by the equation

$$F = 0 \quad \text{with} \quad F(\tau, y, x) := K_j(-\frac{y}{\tau}) - K_i(\frac{x}{1-\tau}). \tag{6.26}$$

Moreover

$$\partial_\tau F(\tau, y, x) = D^2_{\tau\tau} e(\tau, y, x) > 0.$$

Using the regularity C^2 of L_l given in assumption (A1), we see that the unique solution $\tau = T(y,x)$ of $F(\tau, y, x) = 0$ is continuously differentiable with respect to (y,x). Therefore we deduce that $\mathcal{D}^{ji}_{\text{implicit}} \in C^1(J_j^* \times J_i^*)$.

We have

$$\mathcal{D}^{ji}_{\text{implicit}}(y,x) = \mathcal{E}_1(T(y,x), y) + \mathcal{E}_2(T(y,x), x), \tag{6.27}$$

$$\partial_y \mathcal{D}^{ji}_{\text{implicit}}(y,x) = (\partial_y \mathcal{E}_1)(T(y,x), y) = -L_j'(\xi_y), \tag{6.28}$$

$$\partial_x \mathcal{D}^{ji}_{\text{implicit}}(y,x) = (\partial_x \mathcal{E}_2)(T(y,x), x) = L_i'(\xi_x). \tag{6.29}$$

Writing τ for $T(y,x)$, and using the optimality condition (6.26), we get

$$\begin{aligned}
&(\mathcal{D}^{ji}_{\text{implicit}} - x\partial_x \mathcal{D}^{ji}_{\text{implicit}} - y\partial_y \mathcal{D}^{ji}_{\text{implicit}})(y,x) \\
&\quad = \tau K_j(-\frac{y}{\tau}) + (1-\tau) K_i(\frac{x}{1-\tau}) + L_0(0) \\
&\quad = K_j(-\frac{y}{\tau}) + L_0(0) = -H_j(L_j'(-\frac{y}{\tau})) \\
&\quad = -H_j(-\partial_y \mathcal{D}^{ji}_{\text{implicit}}(y,x)) \\
&\quad = K_i(\frac{x}{1-\tau}) + L_0(0) = -H_i(L_i'(\frac{x}{1-\tau})) \\
&\quad = -H_i(\partial_x \mathcal{D}^{ji}_{\text{implicit}}(y,x))
\end{aligned}$$

where we have used (6.23) in the second and in the fourth line. Hence $\mathcal{D}^{ji}_{\text{implicit}}$ satisfies (6.18) on $J_j^* \times J_i^*$.

Step 3: further study of $\mathcal{D}^{ji}_{\text{junction}}$. We concluded at the end of Step 1 that for $(y,x) \in J_j^* \times J_i^*$ we have

$$\mathcal{D}^{ji}_{\text{junction}}(y,x) = \mathcal{E}_1(\tau_1, y) + \mathcal{E}_2(\tau_2, x)$$

with $0 < \tau_1 \le \tau_2 < 1$. Then we can distinguish two cases.

Case 1: $\tau_1 < \tau_2$. In that case this implies that

$$\partial_{\tau_1} \mathcal{E}_1(\tau_1, y) = 0, \quad \partial_{\tau_2} \mathcal{E}_2(\tau_2, x) = 0$$

which can be written as

$$K_j(\xi_y) = 0, \quad K_i(\xi_x) = 0 \tag{6.30}$$

with $\xi_y \leq 0 \leq \xi_x$ defined in (6.24).

Using Lemma 6.17, we conclude that (6.30) holds true if and only if $K_j(0) > 0$ and $K_i(0) > 0$; i.e. $j, i \in I_N \setminus I_0$. In this case we have $\xi_y = \xi_j^-$ and $\xi_x = \xi_i^+$ and then

$$\tau_1 = -\frac{y}{\xi_j^-}, \quad \tau_2 = 1 - \frac{x}{\xi_i^+}. \tag{6.31}$$

Moreover, we have in this case $\mathcal{D}^{ji}_{\text{junction}}(y,x) = \mathcal{D}^{ji}_{\text{linear}}(y,x)$.

Using Legendre-Fenchel's equality together with $K_j(\xi_j^-) = 0$ and $K_i(\xi_i^+) = 0$, we have

$$\mathcal{D}^{ji}_{\text{linear}}(y,x) - y\partial_y \mathcal{D}^{ji}_{\text{linear}}(y,x) - x\partial_x \mathcal{D}^{ji}_{\text{linear}}(y,x) = L_0(0), \tag{6.32}$$

and

$$\begin{aligned} H_i(\partial_x \mathcal{D}^{ji}_{\text{linear}}(y,x)) = H_i(L_i'(\xi_i^+)) = \xi_i^+ L_i'(\xi_i^+) - L_i(\xi_i^+) = -L_0(0), \\ H_j(-\partial_y \mathcal{D}^{ji}_{\text{linear}}(y,x)) = H_j(L_j'(\xi_j^-)) = \xi_j^- L_j'(\xi_j^-) - L_j(\xi_j^-) = -L_0(0). \end{aligned}$$

Hence $\mathcal{D}^{ji}_{\text{linear}}$ satisfies (6.18) on $J_j^* \times J_i^*$.

Finally we deduce from (6.31) that the condition: $0 < \tau_1 < \tau_2 < 1$ is equivalent to $(y,x) \in \Delta^{ji} \cap (J^*)^2$ and then by continuity of $\mathcal{D}^{ji}_{\text{junction}}$ and $\mathcal{D}^{ji}_{\text{linear}}$, we get

$$\mathcal{D}^{ji}_{\text{junction}} = \mathcal{D}^{ji}_{\text{linear}} \quad \text{on} \quad \Delta^{ji}.$$

Case 2: $\tau_1 = \tau_2$**.** If for $(y,x) \in J_j^* \times J_i^*$ we have

$$\mathcal{D}^{ji}_{\text{junction}}(y,x) = \mathcal{E}_1(\tau_1, y) + \mathcal{E}_2(\tau_2, x)$$

with $\tau_1 = \tau_2$, then we have seen that $(y,x) \in (J_j^* \times J_i^*) \setminus \Delta^{ji}$ and $\mathcal{D}^{ji}_{\text{junction}}(y,x) = \mathcal{D}^{ji}_{\text{implicit}}(y,x)$. From Lemma 6.11, we also have that $\mathcal{D}^{ji}_{\text{implicit}} \in C(J_j \times J_i)$ if $j \in I_0$ or $i \in I_0$ and in that case $\Delta^{ji} = \emptyset$. On the other hand, we have $\mathcal{D}^{ji}_{\text{implicit}} \in C((J_j \times J_i) \setminus \{(0,0)\})$ if $j, i \in I_N \setminus I_0$ with $\{(0,0)\} \in \Delta^{ij}$ in that case. Therefore in all cases we have

$$\mathcal{D}^{ji}_{\text{implicit}} \in C((J_j \times J_i) \setminus \Delta^{ji}).$$

Now from the continuity of $\mathcal{D}_{\text{junction}}$, we deduce that

$$\mathcal{D}^{ji}_{\text{junction}} = \mathcal{D}^{ji}_{\text{implicit}} \quad \text{on} \quad (J_j \times J_i) \setminus \Delta^{ji}.$$

Step 4: on the boundary $(\partial\Delta^{ji}) \cap (J^*)^2$**.** We already know that $\mathcal{D}_{\text{junction}}$ is continuous, therefore if $j, i \in I_N \setminus I_0$:

$$\mathcal{D}^{ji}_{\text{linear}} = \mathcal{D}^{ji}_{\text{implicit}} \quad \text{on} \quad \left\{ (y,x) \in J_j \times J_i, \quad \frac{x}{\xi_i^+} - \frac{y}{\xi_j^-} = 1 \right\}$$

On the other hand, recall that $(y,x) \in J_j^* \times J_i^*$, the real $\tau \in (0,1)$ is characterized by (6.26), i.e.

$$K_j(\xi_y) = K_i(\xi_x) \quad \text{with} \quad \xi_y = -\frac{y}{\tau}, \quad \xi_x = \frac{x}{1-\tau}. \tag{6.33}$$

Notice that if we choose

$$\tau = -\frac{y}{\xi_j^-}$$

we deduce from $\frac{x}{\xi_i^+} - \frac{y}{\xi_j^-} = 1$ that

$$\xi_y = \xi_j^- \quad \text{and} \quad \xi_x = \xi_i^+ \tag{6.34}$$

which are obvious solutions of (6.33). Therefore we conclude that this is the solution. Using (6.28)-(6.29) and the expression of $\mathcal{D}^{ji}_{\text{linear}}$, (6.34) implies the equality of the gradients of $\mathcal{D}^{ji}_{\text{linear}}$ and $\mathcal{D}^{ji}_{\text{implicit}}$ on the boundary $(\partial\Delta^{ji}) \cap (J^*)^2$. Finally this shows that $\mathcal{D}^{ji}_{\text{junction}} \in C^1(J_j^* \times J_i^*)$. This ends the proof of the lemmas. □

Proof of Lemma 6.15. The proof proceeds in several steps.

Continuity of T. We set for $(\tau, y, x) \in [0,1] \times J_j \times J_i$

$$e(\tau, y, x) = \mathcal{E}_1(\tau, y) + \mathcal{E}_2(\tau, x).$$

From Proposition 6.12, we already know that there exists a unique $\tau \in [0,1]$ such that

$$\mathcal{D}^{ji}_{\text{implicit}}(y, x) = e(\tau, y, x) \quad \text{if} \quad (y, x) \in J_j^* \times J_i^*.$$

On the other hand, we have

$$e(\tau, y, x) = \begin{cases} (1-\tau) L_i\left(\frac{x}{1-\tau}\right) + \tau L_0(0) & \text{if } (y,x) \in \{0\} \times J_i^* \quad \text{(case 1)}, \\ \tau L_j\left(-\frac{y}{\tau}\right) + (1-\tau) L_0(0) & \text{if } (y,x) \in J_j^* \times \{0\} \quad \text{(case 2)}. \end{cases} \tag{6.35}$$

Notice that in Cases 1 and 2, there is a unique $\tau \in [0,1]$ such that

$$\mathcal{D}^{ji}_{\text{implicit}}(y, x) = e(\tau, y, x) \tag{6.36}$$

and $\tau \in [0,1)$ in case 1, $\tau \in (0,1]$ in case 2. Then the continuity of $\tau = T(y,x)$ in $(J_j \times J_i) \setminus \{(0,0)\}$ follows from the lower semi-continuity of e on $[0,1] \times J_j \times J_i$ and the uniqueness of τ such that (6.36) holds.

Computation of T. We distinguish cases.

Case 1: $(y,x) \in (\{0\} \times J_i^*) \setminus \Delta^{ji}$. Notice that we have

$$\partial_\tau e(\tau, 0, x) = -K_i(\xi_x) \quad \text{with} \quad \xi_x = \frac{x}{1-\tau}.$$

If $x \geq \xi_i^+$, then $\partial_\tau e(\tau, 0, x) \geq 0$ and $T(0,x) = 0$.

If $x < \xi_i^+$, then $\xi_x = \xi_i^+$ is a solution of $\partial_\tau e(\tau, 0, x) = -K_i(\xi_x) = 0$ and $T(0,x) = 1 - \frac{x}{\xi_i^+}$.

Case 2: $(y,x) \in (J_j^* \times \{0\}) \setminus \Delta^{ji}$. Notice that we have

$$\partial_\tau e(\tau, y, 0) = K_j(\xi_y) \quad \text{with} \quad \xi_y = -\frac{y}{\tau}.$$

If $y \geq -\xi_j^-$, then $\partial_\tau e(\tau, y, 0) \leq 0$ and $T(y,0) = 1$.

If $y < -\xi_j^-$, then $\xi_y = \xi_j^-$ is a solution of $\partial_\tau e(\tau, y, 0) = K_j(\xi_y) = 0$ and $T(y,0) = -\frac{y}{\xi_j^-}$. This ends the proof of the lemma. □

Proof of Lemma 6.16. The proof proceeds in several steps.

Step 1: continuity. From Proposition 6.12, we already know that $\mathcal{D}^{ji}_{\text{junction}} \in C^1((J_j^* \times J_i^*) \cup \Delta^{ji})$ and (6.19) holds true with

$$\begin{cases} \xi_x = \frac{x}{1-\tau}, & \xi_y = -\frac{y}{\tau} & \text{if} \quad (y,x) \in (J_j^* \times J_i^*) \setminus \Delta^{ji} \\ \xi_x = \xi_i^+, & \xi_y = \xi_j^- & \text{if} \quad (y,x) \in \Delta^{ji} \end{cases}$$

where $\tau = T(y,x)$ in the first line. Therefore, in order to prove that $\mathcal{D}^{ji}_{\text{junction}} \in C^1(J_j \times J_i)$, it is sufficient to prove that if $(y,x) \in (\partial(J_j \times J_i)) \setminus \Delta^{ji} = (\{0\} \times J_i) \times (J_j \times \{0\})) \setminus \Delta^{ji}$, and if $(y^k, x^k) \in (J_j^* \times J_i^*) \setminus \Delta^{ji}$ is a sequence of points such that $(y^k, x^k) \to (y,x)$, then we have with $\tau^k = T(y^k, x^k)$

$$-\frac{y^k}{\tau^k} \to \xi_y \quad \text{and} \quad \frac{x^k}{1-\tau^k} \to \xi_x \tag{6.37}$$

where (ξ_y, ξ_x) is given by (6.20). Let us recall that τ^k is characterized by the equation

$$K_j\left(-\frac{y^k}{\tau^k}\right) = K_i\left(\frac{x^k}{1-\tau^k}\right) \tag{6.38}$$

We will assume (up to extract a subsequence) that $\tau^k \to \tau_0$ for some limit $\tau_0 \in [0,1]$. Because we have $|x^k|^2 + |y^k|^2 \le R^2$, it is easy to deduce from (6.38), that there exists a constant C_R such that

$$\left|-\frac{y^k}{\tau^k}\right| + \left|\frac{x^k}{1-\tau^k}\right| \le C_R \tag{6.39}$$

This can be proved by contradiction, distinguishing the cases $\tau_0 = 0$, $\tau_0 = 1$ and $\tau_0 \in (0,1)$. Up to extract a subsequence, we can then pass to the limit in (6.38) and get

$$K_j(\xi_y) = K_i(\xi_x) \quad \text{with} \quad \xi_y \le 0 \le \xi_x \tag{6.40}$$

In the following cases, we now identify one of the two quantities ξ_y or ξ_x, the other one being determined by (6.40).

Case 1: $(y,x) \in (\{0\} \times J_i^*) \setminus \Delta^{ji}$**.** From Lemma 6.15, we know that $\tau_0 = \max\left(0, 1 - \frac{x}{\xi_i^+}\right)$, and then

$$\xi_x = \max(x, \xi_i^+), \quad \xi_y = (K_j^-)^{-1}(K_i(\xi_x))$$

and from (6.35), we get

$$\mathcal{D}^{ji}_{\text{junction}}(0,x) = \frac{x}{\xi_x}\left(L_i(\xi_x) - L_0(0)\right) + L_0(0) \tag{6.41}$$

Case 2: $(y,x) \in (J_j^* \times \{0\}) \setminus \Delta^{ji}$**.** From Lemma 6.15, we know that $\tau_0 = \min\left(1, -\frac{y}{\xi_j^-}\right)$, and then

$$-\xi_y = \max(y, -\xi_j^-), \quad \xi_x = (K_i^+)^{-1}(K_j(\xi_y))$$

and from (6.35), we get

$$\mathcal{D}^{ji}_{\text{junction}}(y,0) = -\frac{y}{\xi_y}\left(L_j(\xi_y) - L_0(0)\right) + L_0(0) \tag{6.42}$$

Case 3: $(y,x) \in \{(0,0)\} \setminus \Delta^{ji}$**.** This case only occurs if $j \in I_0$ or $i \in I_0$. Moreover at least one of the two quantities $-\frac{y^k}{\tau^k}$ and $\frac{x^k}{1-\tau^k}$ tends to zero.

If $\xi_y = 0$, then $K_i(\xi_x) = K_j(0)$ and hence

$$\xi_y = 0 \quad \Longrightarrow \quad L_i(0) \ge L_j(0) = L_0(0)$$

If $\xi_x = 0$, then $K_j(\xi_y) = K_i(0)$ and hence

$$\xi_x = 0 \quad \Longrightarrow \quad L_j(0) \ge L_i(0) = L_0(0)$$

This implies that

$$\begin{cases} \xi_x = \xi_i^+ = 0, & \xi_y = \xi_j^- < 0, & \text{if} \quad L_i(0) = L_0(0) < L_j(0), \\ \xi_x = \xi_i^+ > 0, & \xi_y = \xi_j^- = 0, & \text{if} \quad L_i(0) > L_j(0) = L_0(0), \\ \xi_x = \xi_i^+ = 0, & \xi_y = \xi_j^- = 0, & \text{if} \quad L_i(0) = L_j(0) = L_0(0). \end{cases}$$

By the uniqueness of the limit, this finally shows that $\mathcal{D}^{ji}_{\text{junction}} \in C^1(J_j \times J_i)$ and (6.20) holds.

Step 2: checking (6.35) and (6.22). From (6.41) and (6.42), we deduce (6.35) on $((J_j^* \times \{0\}) \cup (\{0\} \times J_i^*)) \setminus \Delta^{ji}$. From $\mathcal{D}^{ji}_{\text{junction}} = \mathcal{D}^{ji}_{\text{linear}}$ on Δ^{ji}, we deduce that (6.35) is also true on $((J_j^* \times \{0\}) \cup (\{0\} \times J_i^*)) \cap \Delta^{ji}$.

Then (6.22) follows from a simple computation for $(y, x) \neq (0, 0)$. This is still true for $(y, x) = 0$, because $\mathcal{D}^{ji}_{\text{junction}}$ is C^1. This ends the proof of the lemma. □

6.5 Study of $\mathcal{D}_{\text{straight}}$

The following lemma will be used below. Since it is elementary, its proof is omitted.

Lemma 6.18 (Properties of $\mathcal{D}^{ji}_{\text{straight}}$)**.** *For $j = i \in I_N$, we have for $(y, x) \in J_j \times J_i$ with $(y, x) \neq (0, 0)$ if $j = i \in I_N \setminus I_0$:*

$$\begin{aligned}\mathcal{D}^{ji}_{\text{straight}}(y,x) - x\partial_x \mathcal{D}^{ji}_{\text{straight}}(y,x) - y\partial_y \mathcal{D}^{ji}_{\text{straight}}(y,x) &= L_0(0) + K_i(x-y) \\ &= -H_i(\partial_x \mathcal{D}^{ji}_{\text{straight}}(y,x)) \\ &\quad - H_j(-\partial_y \mathcal{D}^{ji}_{\text{straight}}(y,x))\end{aligned}$$

and

$$\partial_x \mathcal{D}^{ji}_{\text{straight}}(y,x) = L_i'(x-y), \quad \partial_y \mathcal{D}^{ji}_{\text{straight}}(y,x) = -L_j'(x-y).$$

6.6 Proof of Theorem 6.2

We are now in position to prove Theorem 6.2. We prove several lemmas successively.

Lemma 6.19 (Properties of $\mathcal{D}_0^{ji}$)**.** *For $(y, x) \in J_j \times J_i$, we have*

$$\mathcal{D}_0^{ji}(y,x) = \begin{cases} L_i(x-y) & \text{if} \quad i = j \in I_0, \\ \mathcal{D}^{ji}_{\text{junction}}(y,x) & \text{if} \quad i \neq j, \\ \min(\mathcal{D}^{ji}_{\text{junction}}(y,x), L_i(x-y)) & \text{if} \quad i = j \in I_N \setminus I_0. \end{cases}$$

In particular $\mathcal{D}_0^{ji} \in C^1(J_j \times J_i)$ in the first two cases.

Lemma 6.20 (Singularities of the gradient of $\mathcal{D}_0$)**.** *In the case $i = j \in I_N \setminus I_0$, we have*

$$\mathcal{D}_0^{ji}(y,x) = \begin{cases} \mathcal{D}^{ji}_{\text{linear}} & \text{in a neighborhood of } (\partial(J_j \times J_i)) \cap \Delta^{ji}, \\ L_i(x-y) & \text{in a neighborhood of } (\partial(J_j \times J_i)) \setminus \overline{\Delta^{ji}}; \end{cases} \tag{6.43}$$

moreover, in this case there exists a curve Γ^{ji} such that $\mathcal{D}_0^{ji} \in C^1((J_j \times J_i) \setminus (\Gamma^{ji} \cup \{Y_j, X_i\}))$. This curve connects $Y_j = (-\xi_j^-, 0)$ and $X_i = (0, \xi_i^+)$ and is contained in $(J_j^* \times J_i^*) \cap \Delta^{ji}$

The results of these two lemmas are illustrated in Figures 3 and 4.

Proof of Lemma 6.19. We only have to treat the case $i = j$. The convexity of L_i implies that for $\tau \in (0, 1)$:

$$e(\tau, y, x) = \tau L_i\left(-\frac{y}{\tau}\right) + (1-\tau) L_i\left(\frac{x}{1-\tau}\right) \geq L_i(x-y).$$

Therefore for $(y, x) \in J_j^* \times J_i^*$ with $j = i$, we have

$$\mathcal{D}^{ji}_{\text{implicit}}(y,x) = \inf_{0<\tau<1} e(\tau, y, x) \geq L_i(x-y)$$

When $i = j \in I_0$, we have $\mathcal{D}^{ji}_{\text{junction}}(y,x) = \mathcal{D}^{ji}_{\text{implicit}}(y,x)$, and then

$$\mathcal{D}^{ji}_{\text{junction}}(y,x) \geq L_i(x-y) = \mathcal{D}^{ji}_{\text{straight}}(y,x) = \mathcal{D}_0^{ji}(y,x)$$

for $(y, x) \in J_j^* \times J_i^*$ and then also for $(y, x) \in J_j \times J_i$, by continuity of the functions. □

Proof of Lemma 6.20. We first prove (6.43) and then describe the curve $\Gamma_{j,i}$.

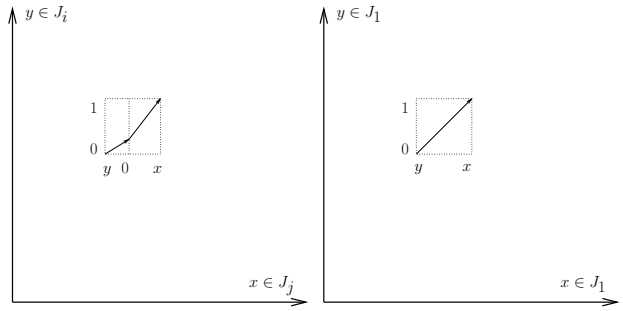

Figure 3: $i \in I_0$ or $j \in I_0$: trajectories never stay at the junction point

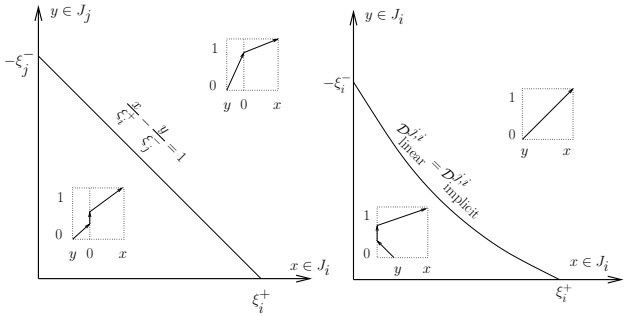

Figure 4: $i, j \in I_N \setminus I_0$: trajectories do stay at the junction point if $(y, x) \sim (0, 0)$

Proof of (6.43). Combining (6.13) and (6.17), we obtain

$$\begin{aligned}\mathcal{D}_0^{ji}(y,x) &= \min(\mathcal{D}^{ji}_{\text{junction}}(y,x), \mathcal{D}^{ji}_{\text{straight}}(y,x)) \\ &= \begin{cases} \mathcal{D}^{ji}_{\text{straight}}(y,x) = L_i(x-y) & \text{for } (y,x) \in (J_j \times J_i) \setminus \Delta^{ji}, \\ \min(\mathcal{D}^{ji}_{\text{linear}}(y,x), \mathcal{D}^{ji}_{\text{straight}}(y,x)) & \text{for } (y,x) \in \Delta^{ji}. \end{cases} \end{aligned} \tag{6.44}$$

On the other hand, we have (a striclty convex function being above its tangent) for $x \neq \xi_i^+$ and $y \neq -\xi_j^-$

$$\begin{aligned} L_i(x) &> L_i(\xi_i^+) + (x - \xi_i^+)L_i'(\xi_i^+) = xL_i'(\xi_i^+) + L_0(0) = \mathcal{D}^{ji}_{\text{linear}}(0,x) \\ L_j(-y) &> L_j(\xi_j^-) + (-y - \xi_j^-)L_j'(\xi_j^-) = -yL_j'(\xi_j^-) + L_0(0) = \mathcal{D}^{ji}_{\text{linear}}(y,0). \end{aligned}$$

This shows that

$$\mathcal{D}^{ji}_{\text{straight}} > \mathcal{D}^{ji}_{\text{linear}} \quad \text{on} \quad (\partial(J_j \times J_i)) \cap \Delta^{ji}. \tag{6.45}$$

We see that (6.44) and (6.45) imply (6.43).

Description of $\left\{\mathcal{D}^{ji}_{\text{linear}} = \mathcal{D}^{ji}_{\text{straight}}\right\} \cap \Delta^{ji}$. Notice that

$$\begin{cases} \mathcal{D}^{ji}_{\text{linear}}(0, \xi_i^+) = \xi_i^+ L_i'(\xi_i^+) + L_0(0) = L_i(\xi_i^+) = \mathcal{D}^{ji}_{\text{straight}}(0, \xi_i^+), \\ \mathcal{D}^{ji}_{\text{linear}}(-\xi_j^-, 0) = \xi_j^- L_j'(\xi_j^-) + L_0(0) = L_j(\xi_j^-) = \mathcal{D}^{ji}_{\text{straight}}(-\xi_j^-, 0). \end{cases}$$

This means that the functions $\mathcal{D}^{ji}_{\text{linear}}$ and $\mathcal{D}^{ji}_{\text{straight}}$ coincide at the two points $X_i = (0, \xi_i^+)$ and $Y_j = (-\xi_j^-, 0)$. Therefore we have

$$\mathcal{D}^{ji}_{\text{straight}} < \mathcal{D}^{ji}_{\text{linear}} \quad \text{on the open interval} \quad]X_i, Y_j[$$

because $\mathcal{D}^{ji}_{\text{linear}}$ is linear and $\mathcal{D}^{ji}_{\text{straight}}$ is strictly convex as a function of $y-x$.
The function $(y,x)\mapsto \mathcal{D}^{ji}_{\text{straight}}(y,x)-\mathcal{D}^{ji}_{\text{linear}}(y,x)$ being convex because $\mathcal{D}^{ji}_{\text{linear}}(y,x)$ is linear, we can consider the convex set

$$K^{ji}=\left\{(y,x)\in J_j\times J_i,\quad \mathcal{D}^{ji}_{\text{straight}}(y,x)\le \mathcal{D}^{ji}_{\text{linear}}(y,x)\right\}.$$

Then for $i=j\in I_N\setminus I_0$, the set

$$\Gamma^{ji}=\left\{(y,x)\in \Delta^{ji},\quad \mathcal{D}^{ji}_{\text{linear}}(y,x)=\mathcal{D}^{ji}_{\text{straight}}(y,x)\right\}$$

is contained in the boundary of the convex set K^{ji}. More precisely, we have

$$\Gamma^{ji}=((\partial K^{ji})\cap\Delta^{ji})\subset J_j^*\times J_i^*$$

which shows that Γ^{ji} is a curve and

$$\overline{\Gamma^{ji}}=\Gamma^{ji}\cup\{X_i,Y_j\}.$$

□

Lemma 6.21 (The equations for $\mathcal{D}_0$)**.** *For all i,j, and x,y where $\mathcal{D}^{ji}_0$ is C^1:*

$$(\mathcal{D}^{ji}_0-x\partial_x\mathcal{D}^{ji}_0-y\partial_y\mathcal{D}^{ji}_0)(y,x)$$
$$=-H_i((\partial_x\mathcal{D}^{ji}_0)(y,x))=-H_j((-\partial_y\mathcal{D}^{ji}_0)(y,x)). \quad (6.46)$$

Moreover for all $x\in J_i$ (with $x\neq\xi_i^+$ if $j=i\in I_N\setminus I_0$)

$$(\mathcal{D}^{ji}_0-x\partial_x\mathcal{D}^{ji}_0-y\partial_y\mathcal{D}^{ji}_0)(0,x)=L_0(0)+K_i(\max(x,\xi_i^+)) \quad (6.47)$$

and for all $y\in J_j$ (with $y\neq-\xi_j^-$ if $j=i\in I_N\setminus I_0$)

$$(\mathcal{D}^{ji}_0-x\partial_x\mathcal{D}^{ji}_0-y\partial_y\mathcal{D}^{ji}_0)(y,0)=L_0(0)+K_j(-\max(y,-\xi_j^-)). \quad (6.48)$$

We also have

$$\partial_x\mathcal{D}^{ji}_0(y,x)=L_i'(\xi_x),\quad \partial_y\mathcal{D}^{ji}_0(y,x)=-L_j'(\xi_y) \quad (6.49)$$

for all $(y,x)\in\partial(J_j\times J_i)$ except for $i=j\in I_N\setminus I_0$ for which we exclude points $(y,x)\in\{Y_j,X_i\}$.
Moreover for $j=i\in I_0$, we have

$$\xi_x=\xi_y=x-y\quad \textit{for all}\quad (y,x)\in\partial(J_j\times J_i) \quad (6.50)$$

and $j=i\in I_N\setminus I_0$, we have

$$\begin{cases}\xi_y=x-y, & \xi_x=x-y\\ \xi_y=\xi_j^-, & \xi_x=\xi_i^+\end{cases}\qquad \textit{for}\quad (y,x)\in(\partial(J_j\times J_i))\cap\overline{\Delta^{ji}} \quad (6.51)$$

Proof. Using Proposition 6.12 for $\mathcal{D}^{ji}_{\text{junction}}$, Lemma 6.18 for $\mathcal{D}^{ji}_{\text{straight}}$, and (6.32) for $\mathcal{D}^{ji}_{\text{linear}}$ and the property (6.43), we get

$$(\mathcal{D}^{ji}_0-x\partial_x\mathcal{D}^{ji}_0-y\partial_y\mathcal{D}^{ji}_0)(0,x)=\begin{cases}L_0(0)+K_i(\max(x,\xi_i^+)) & \text{if}\quad i\neq j\\ L_0(0)+K_i(x) & \text{if}\quad i=j\in I_0\\ \left\{\begin{array}{lll}L_0(0)+K_i(x) & \text{if} & x>\xi_i^+\\ L_0(0) & \text{if} & x<\xi_i^+\end{array}\right. & \text{if}\quad i=j\in I_N\setminus I_0\end{cases}$$

which implies (6.47). Similarly we get

$$(\mathcal{D}^{ji}_0-x\partial_x\mathcal{D}^{ji}_0-y\partial_y\mathcal{D}^{ji}_0)(y,0)=\begin{cases}L_0(0)+K_j(-\max(y,-\xi_j^-)) & \text{if}\quad i\neq j\\ L_0(0)+K_j(-y) & \text{if}\quad i=j\in I_0\\ \left\{\begin{array}{lll}L_0(0)+K_j(-y) & \text{if} & y>-\xi_j^-\\ L_0(0) & \text{if} & y<-\xi_j^-\end{array}\right. & \text{if}\quad i=j\in I_N\setminus I_0\end{cases}$$

which implies (6.48). Relations (6.46) and (6.49) follow both from Proposition 6.12 and Lemma 6.18. Finally (6.50) and (6.51) follows from the previous results. This ends the proof of the lemma. □

We now can check the equations satisfied by $\mathcal{D}_0$ at the boundary.

Lemma 6.22 (Boundary properties of $\mathcal{D}_0$)**.** *At any point $(y,x) \in \{0\}\times J_i$ with $x \neq \xi_i^+$ if $i \in I_N\backslash I_0$, we have for any $j \in I_N$*

$$(\mathcal{D}_0^{ji} - x\partial_x\mathcal{D}_0^{ji} - y\partial_y\mathcal{D}_0^{ji})(y,x) = \begin{cases} -\max_{k\in I_N} H_k^-(-\partial_y\mathcal{D}_0^{ki}(y,x)) & \text{if} \quad N \geq 2, \\ -H_1(-\partial_y\mathcal{D}_0^{ki}(y,x)) & \text{if} \quad N = 1. \end{cases} \tag{6.52}$$

Lemma 6.23 (Boundary properties of $\mathcal{D}_0$ (continued))**.** *At any point $(y,x) \in J_j \times \{0\}$ with $y \neq -\xi_j^-$ if $j \in I_N \setminus I_0$, we have for any $i \in I_N$*

$$(\mathcal{D}_0^{ji} - x\partial_x\mathcal{D}_0^{ji} - y\partial_y\mathcal{D}_0^{ji})(y,x) = -\max_{k\in I_N} H_k^-(\partial_x\mathcal{D}_0^{jk}(y,x)). \tag{6.53}$$

Proof of Lemma 6.22. We first remark the general fact that

$$H_k(L_k'(\xi)) = H_k^-(L_k'(\xi)) \quad \text{if} \quad \xi \leq 0.$$

On the one hand, from Lemma 6.19, we have for points $(y,x) \in \{0\} \times J_i$ where $\mathcal{D}_0^{ki}$ is C^1

$$-(\mathcal{D}_0^{ki} - x\partial_x\mathcal{D}_0^{ki} - y\partial_y\mathcal{D}_0^{ki})(y,x) = H_k(-\partial_y\mathcal{D}_0^{ki}(y,x)) \geq H_k^-(-\partial_y\mathcal{D}_0^{ki}(y,x))$$

and this commom quantity is independent on k. Therefore to conclude to (6.52) in the case $N \geq 2$, it is enough to show that there exists at least an index k such that

$$H_k(-\partial_y\mathcal{D}_0^{ki}(y,x)) = H_k^-(-\partial_y\mathcal{D}_0^{ki}(y,x)). \tag{6.54}$$

Case 1: $N \geq 2$ and $k \neq i$. Then we have $\xi_y \leq 0$ and then

$$H_k(-\partial_y\mathcal{D}_0^{ki}(y,x)) = H_k(L_k'(\xi_y)) = H_k^-(L_k'(\xi_y)) = H_k^-(-\partial_y\mathcal{D}_0^{ki}(y,x)).$$

Therefore (6.54) holds true for $k \neq i$.

Case 2: $N = 1$ and $k = i = 1 \in I_0$. Then we have

$$\mathcal{D}_0(y,x) = \mathcal{D}_0^{11}(y,x) = L_1(x-y)$$

and by Lemma 6.18, we have for

$$(\mathcal{D}_0^{11} - x\partial_x\mathcal{D}_0^{11} - y\partial_y\mathcal{D}_0^{11})(y,x) = -H_1(-\partial_y\mathcal{D}_0^{11}(y,x))$$

which is in particular true for $y = 0$. This shows (6.52) in the case $N = 1$. □

Proof of Lemma 6.23. From Lemma 6.19, we have for points $(y,x) \in J_j \times \{0\}$ where $\mathcal{D}_0^{jk}$ is C^1

$$-(\mathcal{D}_0^{jk} - x\partial_x\mathcal{D}_0^{jk} - y\partial_y\mathcal{D}_0^{jk})(y,x) = H_k(\partial_x\mathcal{D}_0^{jk}(y,x)) \geq H_k^-(\partial_x\mathcal{D}_0^{jk}(y,x))$$

and this commom quantity is independent on k. Therefore to conclude to (6.53), it is enough to show that there exists at least an index k such that

$$H_k(\partial_x\mathcal{D}_0^{jk}(y,x)) = H_k^-(\partial_x\mathcal{D}_0^{jk}(y,x)). \tag{6.55}$$

Case 1: $j \in I_0$. Then from Lemma 6.19, we have with $k = j$

$$\partial_x\mathcal{D}_0^{jk}(y,x) = L_k'(\xi_x) \quad \text{with} \quad \xi_x = x - y \leq 0. \tag{6.56}$$

Therefore (6.53) holds true for $k = j$.

Case 2: $j \in I_N \setminus I_0$. We distinguish subcases.

SUBCASE 2.1: $y > -\xi_j^-$. From Lemma 6.19, we still have (6.56) with $k = j$, which again implies (6.53) for $k = j$.

SUBCASE 2.2: $y < -\xi_j^-$. Then we choose an index $k \in I_0$, and Lemma 6.19 implies that

$$\partial_x \mathcal{D}_0^{jk}(y,x) = L_k'(\xi_x) \quad \text{with} \quad \xi_x = \xi_k^+ = 0$$

which again implies (6.53) for such $k \in I_0$. This ends the proof of the lemma. □

We can now prove Theorem 6.2.

Proof of Theorem 6.2. From Lemma 6.19, we know that $\mathcal{D}_0$ has the regularity C_*^1 except on certain curves $\Gamma^{ji} \cup \{Y_j, X_i\}$ for $j = i \in I_N \setminus I_0$. So if (y,x) is a point of local C_*^1 regularity of $\mathcal{D}_0$, then we simply set

$$\varphi_0 = \mathcal{D}_0 \quad \text{locally around} \quad (y,x).$$

If (y,x) is a point where $\mathcal{D}_0$ is not C_*^1, then we have $\mathcal{D}_0(y,x) = \mathcal{D}_{\text{junction}}(y,x)$, and we can simply set

$$\varphi_0 = \mathcal{D}_{\text{junction}} \quad \text{on} \quad J^2.$$

The required equalities follow from Lemmas 6.19, 6.22, 6.23. Estimate (6.6) follows from the fact that $\mathcal{D}_0$ is the mimimum of $\mathcal{D}_{\text{junction}} \in C_*^1(J^2)$ and of functions in $C^1(J_i^2)$ for some i. This ends the proof of the theorem. □

A Appendix: Stability and Perron's method

This section contains classical results from viscosity solutions, whose statements are adapted to the equation studied in the present paper.

1.1 Stability results

In view of Proposition 3.3, the following stability results are classical in the viscosity solution framework. See for instance [8].

Proposition A.1 (Stability)**.** *Assume (A1') and let $T > 0$.*

- *Consider a family of subsolutions (resp. supersolutions) $(u_\alpha)_{\alpha \in A}$ of (1.1) on J_T such that the u.s.c. (resp. l.s.c.) envelope u of*

$$\sup_{\alpha \in A} u_\alpha \qquad (\textit{resp.} \ \inf_{\alpha \in A} u_\alpha)$$

  is finite everywhere. Then u is a subsolution (resp. supersolution) of (1.1) on J_T.

- *Consider a family of subsolutions (resp. supersolutions) $(u_\varepsilon)_{\varepsilon \in (0,1)}$ of (1.1) on J_T such that the upper (resp. lower) relaxed semi-limit u is finite everywhere. Then u is a subsolution (resp. supersolution) of (1.1) on J_T.*

1.2 Perron's method

In this subsection, we state the existence of a solution of (1.1)-(1.2) which can be constructed by using Perron's method. This method is the classical way to get existence in a viscosity solution framework.

Theorem A.2 (Existence)**.** *Assume (A0)-(A1') and let $T > 0$. Then there exists an upper semi-continuous function $u\colon [0,T) \times J \to \mathbb{R}$ which is a viscosity solution of (1.1)-(1.2) on J_T and satisfies*

$$|u(t,x) - u_0(x)| \le Ct \quad \textit{for} \quad (t,x) \in [0,T) \times J$$

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