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► **To cite this version:**

Jean-François Pommaret. SPENCER OPERATOR AND APPLICATIONS: From Continuum Mechanics to Mathematical physics. 2011. hal-00568937v2

HAL Id: hal-00568937

<https://hal.science/hal-00568937v2>

Submitted on 8 Jul 2011 (v2), last revised 12 Oct 2011 (v3)

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SPENCER OPERATOR AND APPLICATIONS: From Continuum Mechanics to Mathematical Physics

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SUMMARY:

Fifty years ago D.C. Spencer invented the first order operator now wearing his name in order to bring in a canonical way the formal study of systems of ordinary differential (OD) or partial differential (PD) equations to that of equivalent first order systems. However, despite its importance, the Spencer operator is rarely used in mathematics today and, up to our knowledge, has never been used in engineering or mathematical physics.

The first and technical purpose of this paper, an extended version of a lecture at the second workshop on Differential Equations by Algebraic Methods (DEAM2, february 9-11, 2011, Linz, Austria), is to recall briefly its definition, both in the framework of systems of OD/PD equations and in the framework of differential modules. The only notation "D" respects the two corresponding standard ones existing in the literature but no confusion can be done from the background.

The second and central purpose is to prove that the use of the Spencer operator constitutes the *common secret* of the three following famous books published about at the same time in the beginning of the last century, though they do not seem to have anything in common at first sight as they are successively dealing with the foundations of elasticity theory, commutative algebra, electromagnetism (EM) and general relativity (GR):

- [C] E. and F. COSSERAT: "Théorie des Corps Déformables", Hermann, Paris, 1909.
- [M] F.S. MACAULAY: "The Algebraic Theory of Modular Systems", Cambridge, 1916.
- [W] H. WEYL: "Space, Time, Matter", Springer, Berlin, 1918 (1922, 1958; Dover, 1952).

Meanwhile we shall point out the striking importance of the second book for studying *identifiability* in control theory. We shall also obtain from the previous results the group theoretical unification of finite elements in engineering sciences (elasticity, heat, electromagnetism) recovering in a purely mathematical way known *field-matter coupling phenomena* (piezoelectricity, photoelasticity, streaming birefringence, viscosity, Righi-Leduc effect, ...).

As a byproduct and though disturbing it may be, the third and perhaps essential purpose is to prove that these unavoidable new methods contradict the mathematical foundations of both gauge theory and general relativity.

Many explicit engineering examples will illustrate this chapter which is deliberately written in a rather self-contained way to be accessible to a large audience, which does not mean that it is elementary in view of the number of new concepts that must be patched together.

More precisely, if K is a differential field containing \mathbb{Q} with n commuting derivations ∂_i for $i = 1, \dots, n$, we denote by k a subfield of constants and introduce m differential indeterminates y^k for $k = 1, \dots, m$ and n commuting formal derivatives d_i with $d_i y_\mu^k = y_{\mu+1_i}^k$ where $\mu = (\mu_1, \dots, \mu_n)$ is a multi-index with *length* $|\mu| = \mu_1 + \dots + \mu_n$, *class* i if $\mu_1 = \dots = \mu_{i-1} = 0, \mu_i \neq 0$ and $\mu + 1_i = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n)$. We set $y_q = \{y_\mu^k | 1 \leq k \leq m, 0 \leq |\mu| \leq q\}$

with $y_\mu^k = y^k$ when $|\mu| = 0$. We introduce the non-commutative *ring of differential operators* $D = K[d_1, \dots, d_n] = K[d]$ with $d_i a = a d_i + \partial_i a, \forall a \in K$ in the operator sense and the *differential module* $Dy = Dy^1 + \dots + Dy^m$. If $\{\Phi^\tau = a_k^{\tau\mu} y_\mu^k\}$ is a finite number of elements in Dy indexed by τ , we may introduce the *differential module of equations* $I = D\Phi \subset Dy$ and the finitely generated *residual differential module* $M = Dy/I$.

Let now X be a manifold with local coordinates (x^i) for $i = 1, \dots, n$, tangent bundle $T = T(X)$, cotangent bundle $T^* = T^*(X)$, bundle of r -forms $\wedge^r T^*$ and symmetric tensor bundle $S_q T^*$. If E is a vector bundle over X with local coordinates (x^i, y^k) for $i = 1, \dots, n$ and $k = 1, \dots, m$, we denote by $J_q(E)$ the q -jet bundle of E with local coordinates simply denoted by (x, y_q) and *sections* $f_q : (x) \rightarrow (x, f^k(x), f_i^k(x), f_{ij}^k(x), \dots)$ transforming like the section $j_q(f) : (x) \rightarrow (x, f^k(x), \partial_i f^k(x), \partial_{ij} f^k(x), \dots)$ when f is an arbitrary section of E . For simplicity, we shall denote by the same symbol a vector bundle and its set of local sections. Then both $f_q \in J_q(E)$ and $j_q(f) \in J_q(E)$ are over $f \in E$ and the *Spencer operator* just allows to distinguish them by introducing a kind of "difference" through the operator $D : J_{q+1}(E) \rightarrow T^* \otimes J_q(E) : f_{q+1} \rightarrow j_1(f_q) - f_{q+1}$ with local components $(\partial_i f^k(x) - f_i^k(x), \partial_i f_j^k(x) - f_{ij}^k(x), \dots)$ and more generally $(Df_{q+1})_{\mu,i}^k(x) = \partial_i f_\mu^k(x) - f_{\mu+1_i}^k(x)$. In a symbolic way, *when changes of coordinates are not involved*, it is sometimes useful to write down the components of D in the form $d_i = \partial_i - \delta_i$ and the restriction of D to the kernel $S_{q+1} T^* \otimes E$ of the canonical projection $\pi_q^{q+1} : J_{q+1}(E) \rightarrow J_q(E)$ is *minus* the *Spencer map* $\delta = dx^i \wedge \delta_i : S_{q+1} T^* \otimes E \rightarrow T^* \otimes S_q T^* \otimes E$. The kernel of D is made by sections such that $f_{q+1} = j_1(f_q) = j_2(f_{q-1}) = \dots = j_{q+1}(f)$. Finally, if $R_q \subset J_q(E)$ is a *system* of order q on E locally defined by linear equations $\Phi^\tau(x, y_q) \equiv a_k^{\tau\mu}(x) y_\mu^k = 0$ and local coordinates (x, z) for the parametric jets up to order q , the r -prolongation $R_{q+r} = \rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ is locally defined when $r = 1$ by the linear equations $\Phi^\tau(x, y_q) = 0, d_i \Phi^\tau(x, y_{q+1}) \equiv a_k^{\tau\mu}(x) y_{\mu+1_i}^k + \partial_i a_k^{\tau\mu}(x) y_\mu^k = 0$ and has *symbol* $g_{q+r} = R_{q+r} \cap S_{q+r} T^* \otimes E \subset J_{q+r}(E)$. If $f_{q+1} \in R_{q+1}$ is over $f_q \in R_q$, differentiating the identity $a_k^{\tau\mu}(x) f_\mu^k(x) \equiv 0$ with respect to x^i and subtracting the identity $a_k^{\tau\mu}(x) f_{\mu+1_i}^k(x) + \partial_i a_k^{\tau\mu}(x) f_\mu^k(x) \equiv 0$, we obtain the identity $a_k^{\tau\mu}(x) (\partial_i f_\mu^k(x) - f_{\mu+1_i}^k(x)) \equiv 0$ and thus the restriction $D : R_{q+1} \rightarrow T^* \otimes R_q$ ([2],[4],[5],[10]).

DEFINITION: R_q is said to be *formally integrable* when the restriction $\pi_q^{q+1} : R_{q+1} \rightarrow R_q$ is an epimorphism $\forall r \geq 0$ or, equivalently, when all the equations of order $q+r$ are obtained by r prolongations only $\forall r \geq 0$. In that case, $R_{q+1} \subset J_1(R_q)$ is a canonical equivalent formally integrable first order system on R_q with no zero order equations, called the *Spencer form*.

DEFINITION: R_q is said to be *involutive* when it is formally integrable and all the sequences $\dots \xrightarrow{\delta} \wedge^s T^* \otimes g_{q+r} \xrightarrow{\delta} \dots$ are exact $\forall 0 \leq s \leq n, \forall r \geq 0$. Equivalently, using a linear change of local coordinates if necessary, we may *successively* solve the maximum number $\beta_q^n, \beta_q^{n-1}, \dots, \beta_q^1$ of equations with respect to the jet coordinates of class $n, n-1, \dots, 1$ and introduce the *characters* $\alpha_q^i = m \frac{(q+n-i-1)!}{(q-1)!((n-i)!)} - \beta_q^i$ for $i = 1, \dots, n$ with $\alpha_q^n = \alpha$. Then R_q is involutive if R_{q+1} is obtained by only prolonging the β_q^i equations of class i with respect to d_1, \dots, d_i for $i = 1, \dots, n$. In that case one can exhibit the *Hilbert polynomial* $\dim(R_{q+r})$ in r with leading term $(\alpha/n!)r^n$ when $\alpha \neq 0$.

We obtain the following theorem generalizing for PD control systems the well known first order Kalman form of OD control systems where the derivatives of the input do not appear ([5]):

THEOREM 1: When R_q is involutive, its Spencer form is involutive and can be modified to a *reduced Spencer form* in such a way that $\beta = \dim(R_q) - \alpha$ equations can be solved with respect to the jet coordinates z_n^1, \dots, z_n^β while $z_n^{\beta+1}, \dots, z_n^{\beta+\alpha}$ do not appear. In this case $z^{\beta+1}, \dots, z^{\beta+\alpha}$ do not appear in the other equations.

When R_q is involutive, the linear differential operator $\mathcal{D} : E \xrightarrow{j_q} J_q(E) \xrightarrow{\Phi} J_q(E)/R_q = F_0$ of order q with space of solutions $\Theta \subset E$ is said to be *involutive* and one has the canonical *Janet sequence* ([4], p 144):

$$0 \longrightarrow \Theta \longrightarrow T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \dots \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0$$

where each other operator is first order involutive and generates the *compatibility conditions* (CC)

The filtration $D_0 = K \subseteq D_1 = K \oplus T \subseteq \dots \subseteq D_q \subseteq \dots \subseteq D$ of D induces a filtration/inductive limit $0 \subseteq M_0 \subseteq M_1 \subseteq \dots \subseteq M_q \subseteq \dots \subseteq M$ and provides by duality *over* K the projective limit $M^* = R \rightarrow \dots \rightarrow R_q \rightarrow \dots \rightarrow R_1 \rightarrow R_0 \rightarrow 0$ of formally integrable systems. As D is generated by K and $T = D_1/D_0$, we can define for any $f \in M^*$:

$$(af)(m) = af(m) = f(am), (\xi f)(m) = \xi f(m) - f(\xi m), \forall a \in K, \forall \xi = a^i d_i \in T, \forall m \in M$$

and check $d_i a = ad_i + \partial_i a$, $\xi \eta - \eta \xi = [\xi, \eta]$ in the operator sense by introducing the standard bracket of vector fields on T . Finally we get $(d_i f)_\mu^k = (d_i f)(y_\mu^k) = \partial_i f_\mu^k - f_{\mu+1_i}^k$ in a coherent way.

THEOREM 3: $R = M^*$ has a structure of differential module induced by the Spencer operator.

REMARK: When $m = 1$ and $D = k[d]$ is a commutative ring isomorphic to the polynomial ring $A = k[\chi]$ for the indeterminates χ_1, \dots, χ_n , this result *exactly* describes the *inverse system* of Macaulay with $-d_i = \delta_i$ ([M], §59,60).

DEFINITION: A *simple* module is a module having no other proper submodule than 0. A *semi-simple* module is a direct sum of simple modules. When A is a commutative integral domain and M a finitely generated module over A , the *socle* of M is the largest semi-simple submodule of M , that is $soc(M) = \bigoplus soc_{\mathfrak{m}}(M)$ where $soc_{\mathfrak{m}}(M)$ is the direct sum of all the *isotypical* simple submodules of M isomorphic to A/\mathfrak{m} for $\mathfrak{m} \in max(A)$ the set of maximal proper ideals of A . The *radical* of a module is the intersection of all its maximum *proper* submodules. The quotient of a module by its radical is called the *top* and is a semi-simple module ([1]).

The "secret" of Macaulay is expressed by the next theorem:

THEOREM 4: Instead of using the socle of M over A , one may use duality over k in order to deal with the short exact sequence $0 \rightarrow rad(R) \rightarrow R \rightarrow top(R) \rightarrow 0$ where $top(R)$ is the dual of $soc(M)$.

However, Nakayama's lemma ([1],[3],[9]) cannot be used in general unless R is finitely generated over k and thus over D . The main idea of Macaulay has been to overcome this difficulty by dealing only with *unmixed* ideals when $m = 1$. As a generalization, one can state ([5]):

DEFINITION: One has the *purity filtration* $0 = t_n(M) \subseteq \dots \subseteq t_0(M) = t(M) \subseteq M$ where *any* involutive system of order p defining Dm is such that $\alpha_p^{n-r} = 0, \dots, \alpha_p^n = 0$ when $m \in t_r(M)$ and M is said to be *r-pure* if $t_r(M) = 0, t_{r-1}(M) = M$. A 0-pure module is a *torsion-free* module that is $t(M) = 0$ with $t(M) = \{m \in M \mid \exists 0 \neq a \in A, am = 0\}$.

EXAMPLE: With $n = 2, q = 2$, let us consider the involutive system $y_{(0,2)} \equiv y_{22} = 0, y_{(1,1)} \equiv y_{12} = 0$. Then $z' = y_1$ satisfies $z'_2 = 0$ while $z'' = y_2$ satisfies $z''_2 = 0, z''_1 = 0$ and we have the filtration $0 = t_2(M) \subset t_1(M) \subset t_0(M) = t(M) = M$ with $z'' \in t_1(M), z' \in t_0(M)$ but $z' \notin t_1(M)$. This classification of observables has never been applied to engineering systems like the ones to be found in magnetohydrodynamics (MHD) because the mathematics involved are not known.

REMARK: A standard result in commutative algebra allows to embed any torsion-free module into a free module ([9]). Such a property provides the possibility to *parametrize* the solution space of the corresponding system of OD/PD equations by a finite number of potential like arbitrary functions. When $n = 1$ this result amounts to Kalman test and the fact that a classical OD control system is controllable if and only if it is parametrizable, a result showing that *controllability is an intrinsic structural property of a control system*, not depending on the choice of inputs and outputs contrary to a well established engineering tradition ([5],[7]).

When M is *r-pure*, the use of Theorem 1 provides the exact sequence $0 \rightarrow M \rightarrow k(\chi_1, \dots, \chi_{n-r}) \otimes M$, also discovered by Macaulay ([M], §77, 82), and one obtains the following key result for studying the *identifiability* of OD/PD control systems (*localization* in [3] and [5], [8]).

THEOREM 5: When M is *n-pure* one may use the *chinese remainder theorem* ([3], p 41) in

order to prove that the minimum number of generators of R is equal to the maximum number of isotypical components that can be found among the various components of $\text{soc}(M)$ or $\text{top}(R)$.

EXAMPLE: When $n = 1, m = 1, q = 3$, using a sub-index x for the derivatives, the general solution of $y_{xxx} - y_x = 0$ is $y = ae^x + be^{-x} + c1$ with a, b, c constants and the derivative of e^x is e^x , the derivative of e^{-x} is $-e^{-x}$ and the derivative of 1 is 0. Hence we could believe that we need a basis $\{1, e^x, e^{-x}\}$ with *three* generators for obtaining all the solutions through derivatives. Also, when $n = 1, m = 2, k = \mathbb{R}$ and a is a constant real parameter, the OD system $y_{xx}^1 - ay^1 = 0, y_x^2 = 0$ needs two generators $\{(x, 0), (0, 1)\}$ when $a = 0$ with the only $\mathfrak{m} = (\chi)$ killing both y_x^1 and y_2 but only one generator when $a \neq 0$, namely $\{(ch(x), 1)\}$ when $a = 1$. In this case, setting $y = y^1 - y^2$ brings an isomorphic module defined by the single OD equation $y_{xxx} - y_x = 0$ for the only y and $(\chi^3 - \chi) = (\chi) \cap (\chi - 1) \cap (\chi + 1)$, a result leading to the *only* generator $\{ch(x) - 1\}$.

Let us now consider the *conformal Killing system* $\hat{R}_1 \subset J_1(T)$:

$$\omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r \partial_r \omega_{ij} = A(x)\omega_{ij} \Rightarrow n\xi_{ij}^k - \delta_i^k \xi_{rj}^r - \delta_j^k \xi_{ri}^r + \omega_{ij}\omega^{ks}\xi_{rs}^r \Rightarrow \xi_{ijr}^k = 0, \forall n \geq 3$$

obtained by eliminating the arbitrary function $A(x)$, where ω is the Euclidean metric when $n = 2$ (plane) or $n = 3$ (space) and the Minkowskian metric when $n = 4$ (space-time). The brothers Cosserat were only dealing with the *Killing subsystem* $R_1 \subset \hat{R}_1$:

$$\omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r \partial_r \omega_{ij} = 0$$

that is with $\{\xi^k, \xi_i^k \mid \xi_r^r = 0, \xi_{ij}^k = 0\} = \{\text{translations, rotations}\}$ when $A(x) = 0$, while, *in a somehow complementary way*, Weyl was mainly dealing with $\{\xi_r^r, \xi_{ri}^r\} = \{\text{dilatation, elations}\}$. Accordingly, one has ([7]):

THEOREM 6: The *Cosserat equations* ([C], p 137 for $n = 3$, p 167 for $n = 4$):

$$\partial_r \sigma^{i,r} = f^i \quad , \quad \partial_r \mu^{ij,r} + \sigma^{i,j} - \sigma^{j,i} = m^{ij}$$

are *exactly* described by the formal adjoint of the first Spencer operator $D_1 : R_1 \rightarrow T^* \otimes R_1$. Introducing $\phi^{r,ij} = -\phi^{r,ji}$ and $\psi^{rs,ij} = -\psi^{rs,ji} = -\psi^{sr,ij}$, they can be *parametrized* by the formal adjoint of the second Spencer operator $D_2 : T^* \otimes R_1 \rightarrow \wedge^2 T^* \otimes R_1$:

$$\sigma^{i,j} = \partial_r \phi^{i,jr} \quad , \quad \mu^{ij,r} = \partial_s \psi^{ij,rs} + \phi^{j,ir} - \phi^{i,jr}$$

EXAMPLE: When $n = 2$, lowering the indices by means of the constant metric ω , we just need to look for the factors of ξ_1, ξ_2 and $\xi_{1,2}$ in the integration by part of the sum:

$$\sigma^{1,1}(\partial_1 \xi_1 - \xi_{1,1}) + \sigma^{1,2}(\partial_2 \xi_1 - \xi_{1,2}) + \sigma^{2,1}(\partial_1 \xi_2 - \xi_{2,1}) + \sigma^{2,2}(\partial_2 \xi_2 - \xi_{2,2}) + \mu^{12,r}(\partial_r \xi_{1,2} - \xi_{1,2r})$$

Finally, setting $\phi^{1,12} = \phi^1, \phi^{2,12} = \phi^2, \psi^{12,12} = \phi^3$, we obtain the nontrivial parametrization $\sigma^{1,1} = \partial_2 \phi^1, \sigma^{1,2} = -\partial_1 \phi^1, \sigma^{2,1} = -\partial_2 \phi^2, \sigma^{2,2} = \partial_1 \phi^2, \mu^{12,1} = \partial_2 \phi^3 + \phi^1, \mu^{12,2} = -\partial_1 \phi^3 - \phi^2$ in a coherent way with the Airy parametrization obtained when $\phi^1 = \partial_2 \phi, \phi^2 = \partial_1 \phi, \phi^3 = -\phi$.

THEOREM 7: The *Weyl equations* ([W], §35) are *exactly* described by the formal adjoint of the first Spencer operator $D_1 : \hat{R}_2 \rightarrow T^* \otimes \hat{R}_2$ when $n = 4$ and can be parametrized by the formal adjoint of the second Spencer operator $D_2 : T^* \otimes \hat{R}_2 \rightarrow T^* \otimes \hat{R}_2$. In particular, among the components of the first Spencer operator, one has $\partial_i \xi_{rj}^r - \xi_{ijr}^r = \partial_i \xi_{rj}^r$ and thus the components $\partial_i \xi_{rj}^r - \partial_j \xi_{ri}^r = F_{ij}$ of the EM *field* with EM *potential* $\xi_{ri}^r = A_i$ coming from the second order jets (*elations*). It follows that D_1 projects onto $d : T^* \rightarrow \wedge^2 T^*$ and thus D_2 *projects onto* the first set of Maxwell equations described by $d : \wedge^2 T^* \rightarrow \wedge^3 T^*$. Indeed, the Spencer sequence projects onto the Poincaré sequence *with a shift by +1 in the degree of the exterior forms involved* because both sequences are made with first order involutive operators and the comment after diagram (1) can thus be used. By duality, the second set of Maxwell equations thus *appears among* the Weyl equations which *project onto* the Cosserat equations because of the inclusion $R_1 \simeq R_2 \subset \hat{R}_2$.

REMARK: When $n = 4$, the *Poincaré group* (10 parameters) is a subgroup of the *conformal group* (15 parameters) which is not a maximal subgroup because it is a subgroup of the *Weyl group* (11 parameters) obtained by adding the only dilatation with infinitesimal generators $x^i \partial_i$. However, the *optical group* is another subgroup with 10 parameters which is maximal and the same procedure may be applied to all these subgroups in order to study coupling phenomena.

REMARK: Though striking it may look like, *there is no conceptual difference between the Cosserat and Maxwell equations on space-time*. As a byproduct, separating space from time, there is no conceptual difference between the Lamé constants (mass per unit volume) of elasticity and the magnetic (dielectric) constants of EM appearing in the respective wave speeds. For example, the speed of longitudinal free vibrations of a thin elastic bar with Young modulus E and mass per unit volume ρ is $v = \sqrt{\frac{E}{\rho}}$ while the speed of light in a medium with magnetic constant μ and dielectric constant ϵ is $v = \sqrt{\frac{1/\mu}{\epsilon}}$. This result perfectly agrees with *piezoelectricity* (quadratic Lagrangian in strain and electric fields $A^{ijk} \epsilon_{ij} E_k \Rightarrow \sigma^{ij} = A^{ijk} E_k$) and *photoelasticity* (cubic Lagrangian $B^{ijkl} \epsilon_{ij} E_k E_l \Rightarrow D^l = (B^{ijkl} \epsilon_{ij}) E_k \Rightarrow$ refraction index $n(\epsilon)$) which are field-matter coupling phenomena ([6]), but contradicts *gauge theory* ([2],[4]).

In order to justify the last remark, let G be a Lie group with identity e and parameters a acting on X through the group action $X \times G \rightarrow X : (x, a) \rightarrow y = f(x, a)$ and (local) infinitesimal generators θ_τ satisfying $[\theta_\rho, \theta_\sigma] = c_{\rho\sigma}^\tau \theta_\tau$ for $\rho, \sigma, \tau = 1, \dots, \dim(G)$. We may prolong the *graph* of this action by differentiating q times the action law in order to eliminate the parameters in the following commutative and exact diagram where \mathcal{R}_q is a Lie groupoid with local coordinates (x, y_q) , *source* projection $\alpha_q : (x, y_q) \rightarrow (x)$ and *target* projection $\beta_q : (x, y_q) \rightarrow (y)$ when q is large enough:

$$\begin{array}{ccccccc} 0 & \rightarrow & X \times G & \longrightarrow & \mathcal{R}_q & \longrightarrow & 0 \\ & & \parallel & & \alpha_q \swarrow & & \searrow \beta_q \\ & & X \times G & \rightarrow & X & \times & X \end{array}$$

The link between the various sections of the trivial principal bundle on the left (*gauging procedure*) and the various corresponding sections of the Lie groupoid on the right with respect to the source projection is expressed by the next commutative and exact diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & X & \times & G & = & \mathcal{R}_q & \rightarrow & 0 \\ & & a = cst \uparrow \downarrow a(x) & & & & j_q(f) \uparrow \downarrow f_q & & \\ & & X & & = & & X & & \end{array}$$

Introducing the Lie algebra $\mathcal{G} = T_e(G)$ and the Lie algebroid $R_q \subset J_q(T)$, namely the linearization of \mathcal{R}_q at the q -jet of the identity $y = x$, we get the commutative and exact diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & X & \times & \mathcal{G} & = & R_q & \rightarrow & 0 \\ & & \lambda = cst \uparrow \downarrow \lambda(x) & & & & j_q(\xi) \uparrow \downarrow \xi_q & & \\ & & X & & = & & X & & \end{array}$$

where the upper isomorphism is described by $\lambda^\tau(x) \rightarrow \xi_\mu^k(x) = \lambda^\tau(x) \partial_\mu \theta_\tau^k(x)$ for q large enough. The unusual Lie algebroid structure on $X \times \mathcal{G}$ is described by the formula: $([\lambda, \lambda'])^\tau = c_{\rho\sigma}^\tau \lambda^\rho \lambda'^\sigma + (\lambda^\rho \theta_\rho) \cdot \lambda'^\tau - (\lambda'^\sigma \theta_\sigma) \cdot \lambda^\tau$ which is induced by the ordinary bracket $[\xi, \xi']$ on T and thus depends on the action. Applying the Spencer operator, we finally obtain $\partial_i \xi_\mu^k(x) - \xi_{\mu+1, i}^k(x) = \partial_i \lambda^\tau(x) \partial_\mu \theta_\tau^k(x)$ and the isomorphic *gauge sequence* no longer depending on the action:

$$\wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^2 T^* \otimes \mathcal{G} \xrightarrow{d} \dots \xrightarrow{d} \wedge^n T^* \otimes \mathcal{G} \longrightarrow 0$$

EXAMPLE: Even in the elementary situation of the group of projective transformations of the real line $y = (ax + b)/(cx + d)$ with $n = 1$, $\dim(G) = 3$, the computation is not so easy. It leads to $q = 3$ with \mathcal{R}_3 defined by the nonlinear system $(y_{xxx}/y_x) - \frac{3}{2}(y_{xx}/y_x)^2 = 0$, R_3 defined by $\xi_{xxx} = 0$

and the three infinitesimal generators $\theta_1 = \frac{\partial}{\partial x}, \theta_2 = x \frac{\partial}{\partial x}, \theta_3 = x^2 \frac{\partial}{\partial x}$.

It finally remains to study GR within this framework, as it is only "added" by Weyl in an independent way and, for simplicity, we shall restrict to the linearized aspect. First of all, it becomes clear from diagram (1) that the mathematical foundation of GR is based on a confusion between the operator \mathcal{D}_1 (*classical curvature*) in the Janet sequence where \mathcal{D} is the Killing operator brought to involution and the operator D_2 (*gauge curvature*) in the corresponding Spencer sequence. It must also be noticed that, according to the same diagram, the bigger is the underlying group, the bigger are the Spencer bundles while, on the contrary, the smaller are the Janet bundles depending on the invariants of the group action (deformation tensor in classical elasticity is a good example). Precisely, as already noticed in Theorem 7, if $G \subset \hat{G}$, the Spencer sequence for G is *contained into* the Spencer sequence for \hat{G} while the Janet sequence for G *projects onto* the Janet sequence for \hat{G} , the best picture for understanding such a phenomenon is that of two children sitting on the ends of a beam and playing at see-saw.

Such a confusion is also combined with another one well described in ([11]) by the chinese saying "To put Chang's cap on Li's head", namely to relate the Ricci tensor (usually obtained from the Riemann tensor by contraction of indices) to the energy-momentum tensor (space-time stress), without taking into account the previous confusion relating the gauge curvature to *rotations* only while the (Cosserat) stress has only to do with *translations*.

In order to escape from this dilemma, let us denote by $B^2(g_q), Z^2(g_q)$ and $H^2(g_q) = Z^2(g_q)/B^2(g_q)$ the coboundary (image of the left δ), cocycle (kernel of the right δ) and cohomology bundles of the δ -sequence $T^* \otimes g_{q+1} \xrightarrow{\delta} \wedge^2 T^* \otimes g_q \xrightarrow{\delta} \wedge^3 T^* \otimes S_{q-1} T^* \otimes T$. It can be proved that the Riemann tensor is a section of $Riemann = H^2(g_1)$ in the Killing case with $dim(Riemann) = (n^2(n-1)^2/4) - (n^2(n-1)(n-2)/6) = n^2(n^2-1)/12$ while the Weyl tensor is a section of $Weyl = H^2(\hat{g}_1)$ in the conformal Killing case ([4]). It can also be proved that the order of the generating CC of a formally integrable operator of order q is equal to $r+1$ when r is the smallest integer such that $H^2(g_{q+s}) = 0, \forall s \geq r$ ([4]). For the Killing system we have $q=1$ and $r=1$ because $g_2=0$. However, for the conformal Killing system with $q=1$ the situation is much more delicate because $g_3=0$ for $n \geq 3$ but $H^2(\hat{g}_2)=0$ only for $n \geq 4$ ([4], p 435). *This is the reason for which both CC are second order only.* Similarly, when $n=3$, we let the reader prove that the second order systems $y_{33}=0, y_{23}-y_{11}=0, y_{22}=0$ and $y_{33}-y_{11}=0, y_{23}=0, y_{22}-y_{11}=0$ have both three second order generating CC ([8]). The inclusion $g_1 \subset \hat{g}_1$ and the relation $g_2=0$ finally induce the following *crucial* commutative and exact diagram (2) ([4], p 446):

$$\begin{array}{ccccccc}
& & & & & & 0 \\
& & & & & & \downarrow \\
& & & & & & Ricci \\
& & & & & & \downarrow \\
& & & & 0 & \rightarrow & Z^2(g_1) \rightarrow Riemann \rightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & & & JANET \\
& & & 0 & \rightarrow & T^* \otimes \hat{g}_2 & \xrightarrow{\delta} Z^2(\hat{g}_1) \rightarrow Weyl \rightarrow 0 \\
& & & \downarrow & & \downarrow & \downarrow \\
& & & & & & 0 \\
0 \rightarrow & S_2 T^* & \xrightarrow{\delta} & T^* \otimes T^* & \xrightarrow{\delta} & \wedge^2 T^* & \rightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & \\
& & & & & & SPENCER
\end{array}$$

A diagonal chase allows to identify *Ricci* with $S_2 T^*$ *without contracting indices* and provides the splitting of $T^* \otimes T^*$ into $S_2 T^*$ (*gravitation*) and $\wedge^2 T^*$ (*electromagnetism*) in the lower horizontal sequence *obtained by using the Spencer sequence*, solving thus an old conjecture. However, $T^* \otimes T^* \simeq T^* \otimes \hat{g}_2$ has only to do with second order jets (*relations*) and not a word is left from the standard approach to GR. In addition, we obtain the following important theorem explaining for the first time classical results in an intrinsic way:

THEOREM 8: There exist canonical splittings of the various δ -maps appearing in the above diagram which allow to split the vertical short exact sequence on the right *obtained by using the Janet sequence*.

EXAMPLE: The free movement of a body in a constant static gravitational field \vec{g} is described by $\frac{d\vec{x}}{dt} - \vec{v} = 0$, $\frac{d\vec{v}}{dt} - \vec{g} = 0$, $\frac{\partial \vec{g}}{\partial x^i} - 0 = 0$ where the "speed" is considered as a Lorentz rotation, that is as a first jet. Hence an *accelerometer* merely helps measuring the part of the Spencer operator dealing with second order jets (*equivalence principle*).

CONCLUSION:

In gauge theory, the structure of EM is coming from the unitary group $U(1)$, the unit circle in the complex plane, which is *not* acting on space-time, as the *only* possibility to obtain a pure 2-form from $\wedge^2 T^* \otimes \mathcal{G}$ is to have $\dim(G) = 1$. However, we have explained the structure of EM from that of the conformal group of space-time, with a shift by one step in the interpretation of the Spencer sequence involved because the "fields" are now sections of $C_1 \simeq T^* \otimes \mathcal{G}$ parametrized by D_1 and thus killed by D_2 .

In general relativity, we have similarly proved that the standard way of introducing the Ricci tensor was based on a double confusion between the Janet and Spencer sequences described by diagrams (1) and (2). In particular we have explained why the intrinsic structure of this tensor *necessarily* depends on the difference existing between the Weyl group and the conformal group which is coming from second order jets, relating for the first time on equal footing EM and GR to the Spencer δ -cohomology of various symbols.

Accordingly, we may say:

" TO ACT OR NOT TO ACT, THAT IS THE QUESTION "

and hope future will fast give an answer !.

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