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SPENCER OPERATOR AND MACAULAY INVERSE SYSTEM:

A new approach to control identifiability and other engineering applications

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Fifty years ago D.C. Spencer invented the first order operator now wearing his name in order to bring the formal study of systems of ordinary differential (OD) or partial differential (PD) equations to that of equivalent first order systems. However, despite its importance, the Spencer operator is rarely used in mathematics today and, up to our knowledge, has never been used in engineering applications or in physics.

The purpose of this lecture at the second workshop on Differential Equations by Algebraic Methods (DEAM2, february 9-11, 2011, Linz, Austria) is first to recall briefly its definition, both in the framework of systems of OD/PD equations and in the framework of differential modules. The only notation "D" respects the two standard ones existing in the literature but no confusion can be done from the background.

The remaining of the lecture will consist in a series of theorems dealing with explicit and striking applications. In a rough way, the main goal is to prove that the use of the Spencer operator constitutes the *common secret* of the three following famous books published about at the same time in the beginning of the last century, though they do not seem to have anything in common at first sight as they are successively dealing with elasticity theory, commutative algebra, electromagnetism (EM) and general relativity (GR):

- 1) E. and F. COSSERAT: "Théorie des Corps Déformables", Hermann, Paris, 1909.
- 2) F.S. MACAULAY: "The Algebraic Theory of Modular Systems", Cambridge, 1916.
- 3) H. WEYL: "Space, Time, Matter", Springer, Berlin, 1918 (1922, 1958; Dover, 1952).

More precisely, if K is a differential field containing \mathbb{Q} with n commuting derivations ∂_i for $i = 1, \dots, n$, we denote by k a subfield of constants and introduce m differential indeterminates y^k for $k = 1, \dots, m$ and n commuting formal derivatives d_i with $d_i y_\mu^k = y_{\mu+1_i}^k$ where $\mu = (\mu_1, \dots, \mu_n)$ is a multi-index with $length |\mu| = \mu_1 + \dots + \mu_n$, *class* i if $\mu_1 = \dots = \mu_{i-1} = 0, \mu_i \neq 0$ and $\mu + 1_i = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n)$. We set $y_q = \{y_\mu^k | 1 \leq k \leq m, 0 \leq |\mu| \leq q\}$ with $y_\mu^k = y^k$ when $|\mu| = 0$. We introduce the non-commutative ring of differential operators $D = K[d_1, \dots, d_n] = K[d]$ with $d_i a = a d_i + \partial_i a, \forall a \in K$ in the operator sense and the differential module $Dy = Dy^1 + \dots + Dy^m$. If $\{\Phi^\tau = a_k^{\tau\mu} y_\mu^k\}$ is a finite number of elements in Dy indexed by τ , we may introduce the differential module of equations $I = D\Phi \subset Dy$ and the finitely generated residual differential module $M = Dy/I$.

Let now X be a manifold with local coordinates (x^i) for $i = 1, \dots, n$, tangent bundle $T = T(X)$, cotangent bundle $T^* = T^*(X)$, bundle of r -forms $\wedge^r T^*$ and symmetric tensor bundle $S_q T^*$. If E is a vector bundle over X with local coordinates (x^i, y^k) for $i = 1, \dots, n$ and $k = 1, \dots, m$, we denote by $J_q(E)$ the q -jet bundle of E with local coordinates simply denoted by (x, y_q) and sections $f_q : (x) \rightarrow (x, f^k(x), f_i^k(x), f_{ij}^k(x), \dots)$ transforming like the section $j_q(f) : (x) \rightarrow (x, f^k(x), \partial_i f^k(x), \partial_{ij} f^k(x), \dots)$ when f is an arbitrary section of E . For simplicity, we shall de-

note by the same symbol a vector bundle and its set of local sections. Then both $f_q \in J_q(E)$ and $j_q(f) \in J_q(E)$ are over $f \in E$ and the *Spencer operator* just allows to distinguish them by introducing a kind of "difference" through the operator $D : J_{q+1}(E) \rightarrow T^* \otimes J_q(E) : f_{q+1} \rightarrow j_1(f_q) - f_{q+1}$ with local components $(\partial_i f^k(x) - f_i^k(x), \partial_i f_j^k(x) - f_{ij}^k(x), \dots)$ and more generally $(Df_{q+1})_{\mu,i}^k(x) = \partial_i f_{\mu}^k(x) - f_{\mu+1,i}^k(x)$. In a symbolic way, *when changes of coordinates are not involved*, it is sometimes useful to write down the components of D in the form $d_i = \partial_i - \delta_i$ and the restriction of D to the kernel $S_{q+1}T^* \otimes E$ of the canonical projection $\pi_q^{q+1} : J_{q+1}(E) \rightarrow J_q(E)$ is *minus* the *Spencer map* $\delta = dx^i \wedge \delta_i : S_{q+1}T^* \otimes E \rightarrow T^* \otimes S_q T^* \otimes E$. The kernel of D is made by sections such that $f_{q+1} = j_1(f_q) = j_2(f_{q-1}) = \dots = j_{q+1}(f)$. Finally, if $R_q \subset J_q(E)$ is a *system* of order q on E locally defined by linear equations $\Phi^\tau(x, y_q) \equiv a_k^{\tau\mu}(x)y_\mu^k = 0$ and local coordinates (x, z) for the parametric jets up to order q , the first *prolongation* $R_{q+1} = \rho_1(R_q) = J_1(R_q) \cap J_{q+1}(E) \subset J_1(J_q(E))$ is locally defined by the linear equations $\Phi^\tau(x, y_q) = 0, d_i \Phi^\tau(x, y_{q+1}) \equiv a_k^{\tau\mu}(x)y_{\mu+1,i}^k + \partial_i a_k^{\tau\mu}(x)y_\mu^k = 0$ and has *symbol* $g_{q+1} = R_{q+1} \cap S_{q+1}T^* \otimes E \subset J_{q+1}(E)$. If $f_{q+1} \in R_{q+1}$ is over $f_q \in R_q$, differentiating the identity $a_k^{\tau\mu}(x)f_\mu^k(x) \equiv 0$ with respect to x^i and subtracting the identity $a_k^{\tau\mu}(x)f_{\mu+1,i}^k(x) + \partial_i a_k^{\tau\mu}(x)f_\mu^k(x) \equiv 0$, we obtain the identity $a_k^{\tau\mu}(x)(\partial_i f_\mu^k(x) - f_{\mu+1,i}^k(x)) \equiv 0$ and thus the restriction $D : R_{q+1} \rightarrow T^* \otimes R_q$.

DEFINITION: R_q is said to be *formally integrable* when the restriction $\pi_q^{q+1} : R_{q+1} \rightarrow R_q$ is an epimorphism $\forall r \geq 0$ or, equivalently, when all the equations of order $q+r$ are obtained by r prolongations only $\forall r \geq 0$. In that case, $R_{q+1} \subset J_1(R_q)$ is an equivalent formally integrable first order system on R_q , called the *Spencer form*.

In actual practice, instead of having a linear differential operator $\mathcal{D} : E \xrightarrow{j_q} J_q(E) \xrightarrow{\Phi} J_q(E)/R_q = F$ of order q , we have now the *first Spencer operator* $D_1 : C_0 = R_q \xrightarrow{j_1} J_1(R_q) \rightarrow J_1(R_q)/R_{q+1} \simeq T^* \otimes R_q / \delta(g_{q+1}) = C_1$ of order one induced by $D : R_{q+1} \rightarrow T^* \otimes R_q$. More generally, introducing the exterior derivative $d : \wedge^r T^* \rightarrow \wedge^{r+1} T^*$ and the *Spencer bundles* $C_r = \wedge^r T^* \otimes R_q / \delta(\wedge^{r-1} T^* \otimes g_{q+1})$, the $(r+1)$ -*Spencer operator* $D_{r+1} : C_r \rightarrow C_{r+1}$ in the *second Spencer sequence* is induced by $D : \wedge^r T^* \otimes R_{q+1} \rightarrow \wedge^{r+1} T^* \otimes R_q : \alpha \otimes \xi_{q+1} \rightarrow d\alpha \otimes \xi_q + (-1)^r \alpha \wedge D\xi_{q+1}$ in the *first Spencer sequence*.

DEFINITION: R_q is said to be *involutive* when it is formally integrable and all the sequences $\dots \xrightarrow{\delta} \wedge^s T^* \otimes g_{q+r} \xrightarrow{\delta} \dots$ are exact $\forall 0 \leq s \leq n, \forall r \geq 0$. Equivalently, using a linear change of local coordinates if necessary in order to have δ -regular coordinates, we may successively solve the maximum number $\beta_q^n = m - \alpha, \beta_q^{n-1}, \dots, \beta_q^1$ of equations with respect to the jet coordinates of class $n, n-1, \dots, 1$ and R_q is involutive if R_{q+1} is obtained by only prolonging the β_q^i equations of class i with respect to d_1, \dots, d_i for $i = 1, \dots, n$. In that case one can exhibit the *Hilbert polynomial* $\dim(R_{q+r})$ in r with leading term $(\alpha/n!)r^n$.

We obtain the following theorem generalizing to PD control systems the well known first order Kalman form of OD control systems where the derivatives of the input do not appear:

THEOREM 1: When R_q is involutive, its Spencer form is involutive and can be modified to a *reduced Spencer form* in such a way that $\beta = \dim(R_q) - \alpha$ equations can be solved with respect to the jet coordinates z_n^1, \dots, z_n^β while $z_n^{\beta+1}, \dots, z_n^{\beta+\alpha}$ do not appear. In this case $z^{\beta+1}, \dots, z^{\beta+\alpha}$ do not appear in the other equations.

In the algebraic framework already considered, *only two possible formal constructions can be obtained from M* , namely $\text{hom}_D(M, D)$ and $M^* = \text{hom}_K(M, K)$.

THEOREM 2: $\text{hom}_D(M, D)$ is a *right* differential module that can be *converted* to a *left* differential module by introducing the *right* differential module structure of $\wedge^n T^*$. As a differential geometric counterpart, we get the *formal adjoint* $\text{ad}(\mathcal{D}) : \wedge^n T^* \otimes F^* \rightarrow \wedge^n T^* \otimes E^*$ where E^* is obtained from E by inverting the local transition matrices, the simplest example being T^* .

REMARK: Such a result explains why dual objects in physics and engineering are no longer

tensors but tensor *densities*.

The filtration $D_0 = K \subseteq D_1 = K \oplus T \subseteq \dots \subseteq D_q \subseteq \dots \subseteq D$ of D induces a filtration/inductive limit $0 \subseteq M_0 \subseteq M_1 \subseteq \dots \subseteq M_q \subseteq \dots \subseteq M$ and provides by duality *over* K the projective limit $M^* = R \rightarrow \dots \rightarrow R_q \rightarrow \dots \rightarrow R_1 \rightarrow R_0 \rightarrow 0$ of formally integrable systems. As D is generated by K and $T = D_1/D_0$, we can define for any $f \in M^*$:

$$(af)(m) = af(m) = f(am), (\xi f)(m) = \xi f(m) - f(\xi m), \forall a \in K, \forall \xi = a^i d_i \in T, \forall m \in M$$

and check $d_i a = ad_i + \partial_i a$, $\xi \eta - \eta \xi = [\xi, \eta]$ in the operator sense by introducing the standard bracket of vector fields on T . Finally we get $(d_i f)_\mu^k = (d_i f)(y_\mu^k) = \partial_i f_\mu^k - f_{\mu+1_i}^k$ in a coherent way.

THEOREM 3: $R = M^*$ has a structure of differential module induced by the Spencer operator.

REMARK: When $m = 1$ and $D = k[d]$ is a commutative ring isomorphic to the polynomial ring $A = k[\chi]$ for the indeterminates χ_1, \dots, χ_n , this result *exactly* describes the *inverse system* of Macaulay with $-d_i = \delta_i$ ([2], §59,60).

DEFINITION: When A is a commutative integral domain and M a finitely generated module over A , the *socle* of M is $\text{soc}(M) = \bigoplus \text{soc}_{\mathfrak{m}}(M)$ where $\text{soc}_{\mathfrak{m}}(M)$ is the direct sum of all the *isotypical* simple submodules of M isomorphic to A/\mathfrak{m} for $\mathfrak{m} \in \text{ass}(M) \cap \text{max}(A)$. The *radical* of a module is the intersection of all its maximum *proper* submodules. The quotient of a module by its radical is called the *top*.

The secret of Macaulay is expressed by the next theorem:

THEOREM 4: Instead of using the socle of M over A , one may use duality over k in order to deal with the short exact sequence $0 \rightarrow \text{rad}(R) \rightarrow R \rightarrow \text{top}(R) \rightarrow 0$ where $\text{top}(R)$ is the dual of $\text{soc}(M)$.

However, Nakayama's lemma cannot be used in general unless R is finitely generated over k and thus over D . The main idea of Macaulay has been to overcome this difficulty by dealing only with *unmixed* ideals when $m = 1$. As a generalization, one can state:

DEFINITION: One has the *purity filtration* $0 = t_n(M) \subseteq \dots \subseteq t_0(M) = t(M) \subseteq M$ where the dimension of the *characteristic variety* of Dm is $< n - r$ when $m \in t_r(M)$ and M is said to be *r-pure* if $t_r(M) = 0, t_{r-1}(M) = M$. A 0-pure module is a torsion-free module.

In actual practice, using an involutive Spencer form and δ -regular coordinates, let us define a differential module N_r by the first order involutive system made up by the equations of class 1+ class 2 + ... + class $(n - r)$, obtaining therefore epimorphisms $N_{r+1} \rightarrow N_r \rightarrow 0$ and $N_r \rightarrow M \rightarrow 0, \forall 0 \leq r \leq n$ with $N_0 = M$. One can prove that the image of the induced morphism $t(N_r) \rightarrow t(M)$ is $t_r(M)$ with $t_{r+1}(M) \subseteq t_r(M)$.

THEOREM 5: The sequence $0 \rightarrow M \rightarrow \bigoplus_{\mathfrak{p} \in \text{ass}(M)} M_{\mathfrak{p}}$ is exact. Moreover the images of all the localizing morphisms $M \rightarrow M_{\mathfrak{p}}$ are primary modules if and only if M is pure, that is $\text{ass}(M)$ only contains equidimensional minimum primes. Moreover this primary embedding corresponds to a primary decomposition of I and leads to decompose R into a sum of subsystems.

Theorem 1 and a partial localization providing the exact sequence $0 \rightarrow M \rightarrow k(\chi_1, \dots, \chi_{n-r}) \otimes M$ when M is r -pure, also discovered by Macaulay ([2], §77, 82), lead to the following key result for studying the *identifiability* of OD/PD control systems.

THEOREM 6: When M is n -pure the monomorphism of the preceding theorem becomes an isomorphism (*chinese remainder* theorem) and the minimum number of generators of R is equal to the maximum number of isotypical components that can be found among the various components of $\text{soc}(M)$ or $\text{top}(R)$, that is $\text{max}_{\mathfrak{m} \in \text{ass}(M)} \{ \dim_{A/\mathfrak{m}} \text{soc}_{\mathfrak{m}}(M) \}$.

EXAMPLE: When $n = 1, m = 2, k = \mathbb{R}$ and a is a constant parameter, the OD system $y_{xx}^1 - ay^1 = 0, y_x^2 = 0$ needs two generators when $a = 0$ but only one generator when $a \neq 0$, namely $\{ch(x), 1\}$ when $a = 1$. Setting $z = y^1 - y^2$ when $a \neq 0$ brings an isomorphic module defined by the single OD equation $z_{xxx} - az_x = 0$ for the only z .

Let us now consider the *conformal Killing system* $\hat{R}_1 \subset J_1(T)$:

$$\omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r \partial_r \omega_{ij} = A(x)\omega_{ij} \Rightarrow n\xi_{ij}^k - \delta_i^k \xi_{rj}^r - \delta_j^k \xi_{ri}^r + \omega_{ij}\omega^{ks}\xi_{rs}^r \Rightarrow \xi_{ijr}^k = 0, \forall n \geq 3$$

obtained by eliminating the arbitrary function $A(x)$, where ω is the Euclidean metric when $n = 2$ (plane) or $n = 3$ (space) and the Minkowskian metric when $n = 4$ (space-time). The brothers Cosserat were only dealing with the *Killing subsystem* $R_1 \subset \hat{R}_1$:

$$\omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r \partial_r \omega_{ij} = 0$$

that is with $\{\xi^k, \xi_i^k \mid \xi_r^r = 0, \xi_{ij}^k = 0\}$ when $A(x) = 0$ while, *in a somehow complementary way*, Weyl was mainly dealing with $\{\xi_r^r, \xi_{ri}^r\}$. Accordingly, one has:

THEOREM 7: The *Cosserat equations* ([1], p 137 for $n = 3$, p 167 for $n = 4$):

$$\partial_r \sigma^{ir} = f^i \quad , \quad \partial_r \mu^{ij,r} + \sigma^{ij} - \sigma^{ji} = m^{ij}$$

are *exactly* described by the formal adjoint of the first Spencer operator $D_1 : R_1 \rightarrow T^* \otimes R_1$. Introducing $\phi^{r,ij} = -\phi^{r,ji}$ and $\psi^{rs,ij} = -\psi^{rs,ji} = -\psi^{sr,ij}$, they can be *parametrized* by the formal adjoint of the second Spencer operator $D_2 : T^* \otimes R_1 \rightarrow \wedge^2 T^* \otimes R_1$:

$$\sigma^{ij} = \partial_r \phi^{i,jr} \quad , \quad \mu^{ij,r} = \partial_s \psi^{ij,rs} + \phi^{j,ir} - \phi^{i,jr}$$

EXAMPLE: When $n = 2$, lowering the indices by means of the constant metric ω , we just need to look for the factors of ξ_1, ξ_2 and $\xi_{1,2}$ in the integration by part of the sum:

$$\sigma^{11}(\partial_1 \xi_1 - \xi_{1,1}) + \sigma^{12}(\partial_2 \xi_1 - \xi_{1,2}) + \sigma^{21}(\partial_1 \xi_2 - \xi_{2,1}) + \sigma^{22}(\partial_2 \xi_2 - \xi_{2,2}) + \mu^{12,r}(\partial_r \xi_{1,2} - \xi_{1,2r})$$

THEOREM 8: The *Weyl equations* ([3], §35) are *exactly* described by the formal adjoint of the first Spencer operator $D_1 : \hat{R}_2 \rightarrow T^* \otimes \hat{R}_2$ when $n = 4$ and can be parametrized by the formal adjoint of the second Spencer operator $D_2 : T^* \otimes \hat{R}_2 \rightarrow T^* \otimes \hat{R}_2$. In particular, among the components of the Spencer operator, one has $\partial_i \xi_{rj}^r - \xi_{ijr}^r = \partial_i \xi_{rj}^r$ and thus the components $\partial_i \xi_{rj}^r - \partial_j \xi_{ri}^r = F_{ij}$ of the EM *field* with EM *potential* $\xi_{ri}^r = A_i$ coming from the second order jets (*elations*). It follows that D_2 *projects onto* the first set of Maxwell equations described by the exterior derivative $d : \wedge^2 T^* \rightarrow \wedge^3 T^*$ while, by duality, the second set of Maxwell equations thus *appears among* the Weyl equations which *project onto* the Cosserat equations because of the inclusion $R_1 \simeq R_2 \subset \hat{R}_2$.

REMARK: Though striking it may look like, *there is no conceptual difference between the Cosserat and Maxwell equations on space-time*. As a byproduct, separating space from time, there is no conceptual difference between the Lamé constants (mass per unit volume) of elasticity and the magnetic (dielectric) constants of EM appearing in the respective wave speeds. This result perfectly agrees with *piezoelectricity* (quadratic Lagrangian in strain and electric fields $A^{ijk}\epsilon_{ij}E_k \Rightarrow \sigma^{ij} = A^{ijk}E_k$) and *photoelasticity* (cubic Lagrangian $B^{ijkl}\epsilon_{ij}E_kE_l \Rightarrow D^l = (B^{ijkl}\epsilon_{ij})E_k \Rightarrow$ refraction index $n(\epsilon)$) which are field-matter coupling phenomena, but contradicts *gauge theory*.

EXAMPLE: The free movement of a body in a constant static gravitational field \vec{g} is described by $\frac{d\vec{x}}{dt} - \vec{v} = 0, \frac{d\vec{v}}{dt} - \vec{g} = 0, \frac{\partial \vec{q}}{\partial x^i} - 0 = 0$ where the "speed" is considered as a Lorentz rotation, that is as a first jet. Hence an *accelerometer* merely helps measuring the part of the Spencer operator dealing with second order jets (*equivalence principle*).

In order to justify the last remark, let G be a Lie group with identity e and parameters a acting on X through the group action $X \times G \rightarrow X : (x, a) \rightarrow y = f(x, a)$ and (local) infinitesimal generators θ_τ satisfying $[\theta_\rho, \theta_\sigma] = c_{\rho\sigma}^\tau \theta_\tau$ for $\rho, \sigma, \tau = 1, \dots, \dim(G)$. We may prolong the graph of this action by differentiating q times the action law in order to eliminate the parameters in the

following commutative and exact diagram where \mathcal{R}_q is a Lie groupoid with *source* projection α_q and *target* projection β_q when q is large enough:

$$\begin{array}{ccccccc}
0 \rightarrow & X \times G & \longrightarrow & & \mathcal{R}_q & \longrightarrow & 0 \\
& \parallel & & & \alpha_q \swarrow & & \searrow \beta_q \\
& X \times G & \rightarrow & X & \times & X &
\end{array}$$

The link between the various sections of the trivial principal bundle on the left (*gauging procedure*) and the various corresponding sections of the Lie groupoid on the right with respect to the source projection is expressed by the next commutative and exact diagram:

$$\begin{array}{ccccccc}
0 \rightarrow & X & \times & G & = & \mathcal{R}_q & \rightarrow 0 \\
a = cst \uparrow\downarrow a(x) & & & & = & j_q(f) \uparrow\downarrow f_q & \\
& X & & & = & X &
\end{array}$$

Introducing the Lie algebra $\mathcal{G} = T_e(G)$ and the corresponding Lie algebroid $R_q \subset J_q(T)$, we obtain the following commutative and exact diagram:

$$\begin{array}{ccccccc}
0 \rightarrow & X & \times & \mathcal{G} & = & R_q & \rightarrow 0 \\
\lambda = cst \uparrow\downarrow \lambda(x) & & & & = & j_q(\xi) \uparrow\downarrow \xi_q & \\
& X & & & = & X &
\end{array}$$

where the upper isomorphism is described by $\lambda^\tau(x) \rightarrow \xi_\mu^k(x) = \lambda^\tau(x) \partial_\mu \theta_\tau^k(x)$ for q large enough. The unusual Lie algebroid structure on $X \times \mathcal{G}$ is described by the formula: $([\lambda, \lambda'])^\tau = c_{\rho\sigma}^\tau \lambda^\rho \lambda'^\sigma + (\lambda^\rho \theta_\rho) \cdot \lambda'^\tau - (\lambda'^\sigma \theta_\sigma) \cdot \lambda^\tau$ which is induced by the ordinary bracket $[\xi, \xi']$ on T and thus depends on the action. Applying the Spencer operator, we finally obtain $\partial_i \xi_\mu^k(x) - \xi_{\mu+1_i}^k(x) = \partial_i \lambda^\tau(x) \partial_\mu \theta_\tau^k(x)$.

CONCLUSION:

In gauge theory, the structure of EM is coming from the unitary group $U(1)$, the unit circle in the complex plane, which is *not* acting on space-time while we have explained the structure of EM from that of the conformal group of space-time, with a shift by one step in the interpretation of the (second) Spencer sequence involved because the "fields" are now sections of C_1 parametrized by D_1 and thus killed by D_2 . Accordingly, we may say:

" TO ACT OR NOT TO ACT, THAT IS THE QUESTION "

and hope future will fast give an answer !.

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