Spline discrete differential forms. Application to Maxwell' s equations.

Aurore Back, Eric Sonnendrücker

Spline discrete differential forms. Application to Maxwell' s equations.

Aurore Back and Eric Sonnendrücker

IRMA, CNRS and Université de Strasbourg*

February 23, 2011

Abstract

We construct a new set of discrete differential forms based on B-splines of arbitrary degree as well as an associated Hodge operator. The theory is first developed in 1D and then extended to multi-dimension using tensor products. We link our discrete differential forms with the theory of chains and cochains. The spline discrete differential forms are then applied to the numerical solution of Maxwell's equations.

Keywords: Discrete differential forms, B-splines, Maxwell, Numerical simulation.

1 Introduction

The equations of physics are mathematical models consisting of geometric objects and relationships between them. There are many methods to discretize equations, but few maintain the physical nature of objects that constitute them. To respect the geometrical nature of physics, it is necessary to change the point of view and use differential geometry, also for the numerical study. In differential geometry and tensor calculus, differential forms are an approach to multivariable calculus that is independent of coordinates. The operators such as divergence, curl or gradient are replaced by the exterior derivative d. The exterior derivative acts on a k-form to produce a (k+1)-form. So the fundamental theorem of calculus, the divergence theorem, Green's theorem, and Stokes' theorem are also well defined in differential geometry and we also have the de Rham cohomology. The first who used this point of view to discretize equations is Alain Bossavit [4]. He uses Whitney elements [5] to discretize differential forms and hence, discretize Maxwell equations in the language of differential geometry. Since then, there have been several articles on the subject because there are many problems such as the discretization of Hodge star operator [2, 3, 12] (an important notion which contains all the metric of our domain), and the interpolation of differential forms [1, 4, 5, 6]. Until now, the basis functions used for interpolation have been Whitney forms. In this paper we propose to define a new class of discrete differential forms using B-splines. This new approach proves to have many advantages. It allows to define high order approximation and higher degree B-splines are computed by recurrence with de Boor algorithm [8] so its easy and efficient to implement them; discrete differential forms verify the same properties as "continuous" differential forms especially they preserve the de Rham diagram. Moreover, it appears that in the Finite Element context our B-spline discrete differential forms are naturally related to the B-spline finite elements appearing

*The authors are also affiliated to INRIA-Nancy-Grand Est, CALVI project-team

in isogeometric analysis [7, 14].

In the sequel, we recall the construction of B-splines and their properties (for this part the reader is refered to the book of de Boor [8]) then we explain how to construct discrete differential forms based on B-splines for uniform and non uniform meshes and for periodic and perfect conductors boundary conditions. Our objects are constructed in detail in the $1D$ case and are extended to the $3D$ case by tensor product using the $1D$ forms. We also adapt the technique of T. Tarhasaari, L. Kettunen and A. Bossavit [3] for discretizing the Hodge star operator to the case of B-splines. Then, in fourth part, we show the link between the p-cochains or differential p-forms and p-chains but with a discrete point of view. It is the discrete version of integration of p-forms. This part proves the coherence of discrete differential forms based on B-splines because we obtain the same link that we can find in the continuous case and we show that the discrete de Rham diagram is also preserved. Finally, we apply this theory to the Maxwell equations and we test it on uniform meshes with periodic conditions and on non uniform meshes with perfect conductors boundary conditions. Moreover, since this point of view provides a geometric formulation of Lagrangian equations, this means no reference coordinate system and so the construction of approximation schemes remains valid in case of continuous deformation, so we apply a change of variables on non uniform mesh with perfect conductors boundary conditions.

2 A short overview of B-splines

B-splines on a non uniform set of knots $x_0 < x_1 < \cdots < x_{N-1} < x_N$ can be defined recursively. Some kind of boundary conditions need also be defined. In particular natural boundary conditions (vanishing second derivative), Hermite boundary condition (given derivative), or periodic boundary conditions can be used. Let us denote by B_i^α the B-spline of degree α with support in the interval $[x_i, x_{i+\alpha+1}]$. Then B_i^α is defined recursively by

$$B_i^0(x) = \left| \begin{array}{ll} 1 & \text{if } x_i \leq x < x_{i+1} \\ 0 & \text{else,} \end{array} \right.$$

and for $\alpha \geq 1$

$$B_i^\alpha(x) = \frac{x - x_i}{x_{i+\alpha} - x_i} B_i^{\alpha-1}(x) + \frac{x_{i+\alpha+1} - x}{x_{i+\alpha+1} - x_{i+1}} B_{i+1}^{\alpha-1}(x). \tag{1}$$

The B-splines verify the following properties:

1. The B-spline B_i^α is a polynomial of degree α between two consecutive knots,
2. The B-spline B_i^α is of class $C^{\alpha-1}$,
3. Partition of unity: for any point x, we have $\sum_i B_i^\alpha(x) = 1$.

We shall also need the recursion formula for the derivatives:

$$B_i^{\alpha\prime}(x) = \alpha \left(\frac{B_i^{\alpha-1}(x)}{x_{i+\alpha} - x_i} - \frac{B_{i+1}^{\alpha-1}(x)}{x_{i+\alpha+1} - x_{i+1}} \right). \tag{2}$$

For details, the reader is refered to the book of de Boor [8]

3 Construction of discrete differential forms based on B-splines

3.1 The 1D case

3.1.1 Uniform periodic mesh

The primal 1D mesh of our periodic domain will be $x_0 < x_1 < \cdots < x_{N-1} < x_N$ and the dual mesh will consist of the points $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$. We shall assume $x_N - x_0$ periodicity, so that all functions will be equal at x_0 and x_N. Then x_N will not be part of the primal mesh and both meshes will have N points.

In the 1D case, we need to define discrete 0-forms and 1-forms on both meshes that will be constructed using basis functions denoted respectively by $w_i^{0,\alpha}$ and $w_i^{1,\alpha}$, for the primal mesh, and $\tilde{w}_{i+1/2}^{0,\alpha}$ and $\tilde{w}_{i+1/2}^{1,\alpha}$ for the dual mesh. Those will be defined using the B-splines of degree α, B_i^α.

Let us start with the discrete 0-form on the primal mesh. We define the basis functions $w_i^{0,\alpha} = B_i^\alpha$ and the space of linear spline 0-forms $\mathcal{S}_0^\alpha$ will be the vector space generated by these basis functions. Any function $C^0 \in \mathcal{S}_0^\alpha$ writes

$$C^0(x) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x),$$

with the c_j^0 defined by the interpolation conditions $C^0(x_i) = \sum_{j=0}^{N-1} c_j^0 B_j^\alpha(x_i)$ for $0 \leq i \leq N-1$ which is a linear system that can be written in matrix form $M_\alpha^0 c^0 = \mathbb{C}^0$, with $\mathbb{C}^0 = (C^0(x_0), \ldots, C^0(x_{N-1}))^T$, $c^0 = (c_0^0, \ldots, c_{N-1}^0)^T$ and M_α^0 the square matrix whose components are $m_{ij}^0 = B_j^\alpha(x_i)$ for $i = 0 \ldots N-1$ and $j = -\alpha + 2 \ldots N - \alpha + 2$ with $B_j^\alpha = B_{N+j}^\alpha$ when $j < 0$.

Lemma 3.1 *On uniform set of nodes, we have*

- *for all odd α, the matrix $(M_{\alpha,ij}^0)_{0 \leq i,j \leq N-1}$ is non singular.*
- *for all even α and number of mesh points N an odd number, the matrix $(M_{\alpha,ij}^0)_{0 \leq i,j \leq N-1}$ is non singular.*

Proof. We denote $B_{i,j}^\alpha = B_i^\alpha(x_j) = M_{\alpha,ij}^0$. We can easily show by induction, with help of the formula (1), that $B_{i,j}^\alpha = 0$ for $j \notin \{i+1, \ldots, i+\alpha\}$, $B_{i,i+1}^\alpha = B_{i,i+\alpha}^\alpha = 1$ and $B_{i+1,j}^\alpha = B_{i,j-1}^\alpha$. So the formula (1) becomes:

$$B_{i,j}^\alpha = \frac{j-i}{\alpha} B_{i,j}^{\alpha-1} + \frac{i+\alpha+1-j}{\alpha} B_{i,j-1}^{\alpha-1}.$$

We notice that M_α^0 is a circulant matrix and since $B_{i,j}^\alpha = 0$ for $j \notin \{i+1, \ldots, i+\alpha\}$, it can be written:

$$M_\alpha^0 = \sum_{j=1}^{\alpha} B_{0,j}^\alpha J_N^j, \tag{3}$$

where $J_N = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix}$ with $N \times N$ its size. We can factorize (3) by $\frac{J_N}{\alpha!}$ and we obtain:

$$\begin{aligned} M_\alpha^0 &= \frac{J_N}{\alpha!} \sum_{j=1}^{\alpha} \left(j(\alpha-1)! B_{0,j}^{\alpha-1} + (\alpha+1-j)(\alpha-1)! B_{0,j-1}^{\alpha-1} \right) J_N^{j-1}, \\ &= \frac{J_N}{\alpha!} E^{\alpha-1}(J_N), \end{aligned}$$

where $E^{\alpha-1}(X) = \sum_{j=1}^{\alpha} \left(j(\alpha-1)! B_{0,j}^{\alpha-1} + (\alpha+1-j)(\alpha-1)! B_{0,j-1}^{\alpha-1} \right) X^{j-1}$ are Eulerian polynomials. Their coefficients are positive and symmetric (symmetric means that if $P(X) = a_n X^n + \ldots + a_0$, $a_{n-k} = a_k$). Moreover, the eigenvalues of J_N are the Nth roots of unity: $\{\omega^0, \omega^1, \ldots\}$ with $\omega = \exp(\frac{2i\pi}{N})$. So, the eigenvalues of M_α^0 are $\{\omega^0 E^{\alpha-1}(\omega^0), \omega^1 E^{\alpha-1}(\omega^1), \ldots\}$. Since the determinant is the product of eigenvalues, if an Eulerian polynomial has a root which is a root of unity, M_α^0 is singular. But we know [13] that Eulerian polynomials have real, negative and distinct roots. So, if -1 is the root of Eulerian polynomials, M_α^0 is singular. We can observe that if α is an odd number, $E^{\alpha-1}(X)$ is a polynomial with even degree and positive, symmetric coefficients. So $E^{\alpha-1}(-1) \neq 0$ if α is an odd number that implies M_α^0 is non singular. If α is an even number we have $E^{\alpha-1}(-1) = 0$ and so -1 must not be a root of J_N. However, $\exp(\frac{2ik\pi}{N}) = -1$ if and only if N is an even number and $k = \frac{N}{2}$. So $E^{\alpha-1}(-1) \neq 0$ if α is an even number and N an odd number that implies M_α^0 is non singular. ∎

In order to define the 1-forms we shall need the notation

$$D_i^\alpha(x) = \frac{\alpha}{x_{i+\alpha} - x_i} B_i^{\alpha-1}(x),$$

for this function linked to the derivative of the B-spline $B_i^{\alpha-1}$. We can now define the basis functions for the discrete 1-forms by

$$w_i^{1,\alpha}(x) = D_i^\alpha(x)\, dx.$$

The space of linear spline 1-forms $\mathcal{S}_1^\alpha$ will be the vector space generated by these basis functions. Any 1-form $C^1 \in \mathcal{S}_1^\alpha$ writes

$$C^1(x) = \sum_{j=0}^{N-1} c_j^1 D_j^\alpha(x)\, dx,$$

the coefficients c_j^1 being defined by the relations

$$\int_{x_i}^{x_{i+1}} C^1(x) = \sum_{j=0}^{N-1} c_j^1 \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\, dx \qquad \text{for } 0 \leq i \leq N-1,$$

this also defines a linear system that can be written in matrix form $M_\alpha^1 c^1 = \mathbb{C}^1$, with $\mathbb{C}^1 = (\int_{x_0}^{x_1} C^1(x), \ldots, \int_{x_{N-1}}^{x_N} C^1(x))^T$, $c^1 = (c_0^1, \ldots, c_{N-1}^1)^T$ and M_α^1 the square matrix whose components are $m_{ij}^1 = \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\, dx$.

Lemma 3.2 *Under the conditions of the previous lemma, the matrix M_α^1 is non singular.*

Proof. The basis functions $\mathbf{w}_i^{1,\alpha}$ have been chosen using the recursion formula for the spline derivatives (2) and verify

$$\mathbf{w}_i^{1,\alpha}(x) - \mathbf{w}_{i+1}^{1,\alpha}(x) = B_i^{\alpha\prime}(x)\, dx.$$

Using this relation and the fact that $B_i^\alpha(x)$ vanishes for $x \notin [x_i, x_{i+\alpha}]$, it follows easily by recurrence on the degree α that

$$\int_{x_{i+\nu}}^{x_{i+\nu+1}} \mathbf{w}_i^{1,\alpha}(x) = \sum_{k=0}^{\nu} B_{i+k}^\alpha(x_{i+\nu+1}) - \sum_{k=0}^{\nu-1} B_{i+k}^\alpha(x_{i+\nu}).$$

In the case of uniform meshes, we have the property $B_{i+k}^\alpha(x_{i+\nu+1}) = B_{i+k+\nu+1}^\alpha(x_i)$. So, we deduce that $(M_\alpha^1)_{i,j} = \int_{x_i}^{x_{i+1}} D_j^\alpha(x)\, dx = B_{j+1}^\alpha(x_{i+1})$. ∎

Denoting by $B^{\alpha}_{j+1/2}$ the splines whose knots are based on the dual mesh, the discrete 0-forms and 1-forms on the dual mesh are defined in the same way by

$$\tilde{C}^0(x) = \sum_{j=0}^{N-1} \tilde{c}^0_{j+1/2} B^{\alpha}_{j+1/2}(x),$$

with the $\tilde{c}^0_{j+1/2}$ defined by the interpolation conditions $\tilde{C}^0(x_{i+1/2}) = \sum_{j=0}^{N-1} \tilde{c}^0_{j+1/2} B^{\alpha}_{j+1/2}(x_{i+1/2})$ for $0 \leq i \leq N-1$ which is a linear system that can be written in matrix form $\tilde{M}^0_{\alpha}\tilde{c}^0 = \tilde{\mathbb{C}}^0$, with $\tilde{\mathbb{C}}^0 = (\tilde{C}^0(x_{1/2}), \ldots, \tilde{C}^0(x_{N-1/2}))^T$, $\tilde{c}^0 = (\tilde{c}^0_{1/2}, \ldots, \tilde{c}^0_{N-1/2})^T$ and $\tilde{M}^0_{\alpha}$ the square matrix whose components are $\tilde{m}^0_{ij} = B^{\alpha}_{j+1/2}(x_{i+1/2})$. We have that $\tilde{M}^0_{\alpha}$ meet the conditions of the first lemma and so the square matrix is non singular.

A discrete 1-form on the dual mesh is defined by

$$\tilde{C}^1(x) = \sum_{j=0}^{N-1} \tilde{c}^1_{j+1/2} D^{\alpha}_{j+1/2}(x)\, dx,$$

the coefficients $\tilde{c}^1_{j+1/2}$ being defined by the relations

$$\int_{x_{i+1/2}}^{x_{i+3/2}} \tilde{C}^1(x) = \sum_{j=0}^{N-1} \tilde{c}^1_{j+1/2} \int_{x_{i+1/2}}^{x_{i+3/2}} D^{\alpha}_{j+1/2}(x)\, dx \qquad \text{for } 0 \leq i \leq N-1,$$

this also defines a linear system that can be written in matrix form $\tilde{M}^1_{\alpha}\tilde{c}^1 = \tilde{\mathbb{C}}^1$, with $\tilde{\mathbb{C}}^1 = (\int_{x_{1/2}}^{x_{3/2}} \tilde{C}^1(x), \ldots, \int_{x_{N-1/2}}^{x_{N+1/2}} \tilde{C}^1(x))^T$, $\tilde{c}^1 = (\tilde{c}^1_{1/2}, \ldots, \tilde{c}^1_{N-1/2})^T$ and $\tilde{M}^1_{\alpha}$ the square matrix whose components are $\tilde{m}^1_{ij} = \int_{x_{i+1/2}}^{x_{i+3/2}} D^{\alpha}_{j+1/2}(x)\, dx$. For the same reason as M^1, $\tilde{M}^1_{\alpha}$ is non singular. Note that due to the periodicity hypothesis $\int_{x_{N-1/2}}^{x_{N+1/2}} \tilde{C}^1(x) = \int_{x_{N-1/2}}^{x_N} \tilde{C}^1(x) + \int_{x_0}^{x_{1/2}} \tilde{C}^1(x)$.

The discrete Hodge operator: Having defined discrete 0-forms and 1-forms on both grids, we can now define in a natural way the discrete Hodge operators [2, 3], mapping primal 0-forms to dual 1-forms, primal 1-forms to dual 0-forms and the other way round.

As discrete differential forms are defined by their coefficients in the appropriate basis, the discrete Hodge operator should map those coefficients to those on the image basis. Let us start with the discrete Hodge mapping primal 0-forms to dual 1-forms. Given a discrete 0-form on the primal mesh

$$C^0(x) = \sum_{j=0}^{N-1} c^0_j B^{\alpha}_j(x),$$

we can apply the continuous Hodge operator to it, as $\star 1 = dx$, we get

$$\star C^0(x) = \sum_{j=0}^{N-1} c^0_j B^{\alpha}_j(x)\, dx.$$

Now, as B^{α}_j are not splines on the dual mesh, this does not define a discrete differential form on the dual mesh. We need an additional projection step. Denoting by πC^0 the projection of $\star C^0$ on the space of discrete differential forms of the same order on the dual mesh, we can write

$$\pi C^0(x) = \sum_{j=0}^{N-1} \tilde{c}^1_{j+1/2} D^{\alpha}_{j+1/2}(x)\, dx,$$

with

$$\int_{x_{i+1/2}}^{x_{i+3/2}} \star C^0(x) = \sum_{j=0}^{N-1} \tilde{c}^1_{j+1/2} \int_{x_{i+1/2}}^{x_{i+3/2}} D^\alpha_{j+1/2}(x)\,dx \qquad \text{for } 0 \le i \le N-1.$$

Now defining $\tilde{S}^1$ the matrix whose i,j coefficient is $\int_{x_{i+1/2}}^{x_{i+3/2}} B^\alpha_j(x)\,dx$, this relation becomes in matrix form

$$\tilde{S}^1 c^0 = \tilde{M}^1_\alpha \tilde{c}^1,$$

so that the discrete Hodge operator mapping c^0 to $\tilde{c}^1$ is

$$(\tilde{M}^1_\alpha)^{-1}\tilde{S}^1 \quad \text{with } \tilde{S}^1_{i,j} = \int_{x_{i+1/2}}^{x_{i+3/2}} B^\alpha_j(x)\,dx.$$

In order to define the Hodge operator mapping discrete 1-forms on the primal grid to discrete 0-forms on the dual grid, we apply the continuous Hodge operator ($\star dx = 1$) to a discrete 1-form on the primal grid

$$\star C^1(x) = \sum_{j=0}^{N-1} c^1_j D^\alpha_j(x).$$

Its projection on the space of discrete 0-forms on the dual grid is defined by the point values $\star C^1(x_{i+1/2})$. Hence in the same way as before the discrete Hodge in this case is defined by

$$(\tilde{M}^0_\alpha)^{-1}\tilde{S}^0 \quad \text{with } \tilde{S}^0_{i,j} = D^\alpha_j(x_{i+1/2}).$$

The Hodge operators mapping from the dual grid to the primal grid are naturally defined in the same way by

$$(M^1_\alpha)^{-1}S^1 \quad \text{with } S^1_{i,j} = \int_{x_i}^{x_{i+1}} B^\alpha_{j+1/2}(x)\,dx,$$

$$(M^0_\alpha)^{-1}S^0 \quad \text{with } S^0_{i,j} = D^\alpha_{j+1/2}(x_i).$$

Let us finally explicit the different matrices involved in the Hodge operators for the case of uniform periodic linear and cubic splines. In the case of a uniform mesh, due to the recurrence relation on spline derivatives we have

$$D^\alpha_j(x) - D^\alpha_{j+1}(x) = {B^\alpha_{j+1}}'(x).$$

Integrating between i an $i+1$ and using that $B^\alpha_j(x) = B^\alpha_0(x - x_j)$ yields

$$\int_{x_i}^{x_{i+1}} (D^\alpha_0(x - x_j) - D^\alpha_0(x - x_j - \Delta x))\,dx = B^\alpha_0((i-j)\Delta x) - B^\alpha_0((i-j-1)\Delta x).$$

So that

$$\int_{x_i}^{x_{i+1}} D^\alpha_j(x)\,dx = B^\alpha_{j+1}(x_{i+1}).$$

From this it follows that on a uniform grid $M^0_\alpha = M^1_\alpha = \tilde{M}^0_\alpha = \tilde{M}^1_\alpha$ are all the usual degree α periodic spline interpolation matrix.

For linear splines ($\alpha = 1$) the matrices $M^0_1 = M^1_1 = \tilde{M}^0_1 = \tilde{M}^1_1 = \mathbb{I}$ are all the identity matrix. For cubic splines ($\alpha = 3$) these matrices are the circulant matrices with $2/3$ on the diagonal and $1/6$ on the upper and lower diagonal.

Let us now come to the Hodge matrices. Due to their expressions the matrices are also constant circulant matrices with $S^0_\alpha = \tilde{S}^0_\alpha$ and $S^1_\alpha = \tilde{S}^1_\alpha$. In the case $\alpha = 1$, $D^\alpha_i(x) = \frac{1}{\Delta x}$ for $x_{i-1} \leq x \leq x_i$ and 0 elsewhere. Hence

$$S^0_1 = \tilde{S}^0_1 = \frac{1}{\Delta x}\mathbb{I}.$$

And a simple computation yields that $S^1_1 = \tilde{S}^1_1$ are circulant matrices with three diagonals that read

$$S^1_1 = \tilde{S}^1_1 = \Delta x \,\mathrm{circ}[\frac{1}{8}, \frac{3}{4}, \frac{1}{8}],$$

where

$$\mathrm{circ}[\frac{1}{8}, \frac{3}{4}, \frac{1}{8}] = \begin{pmatrix} \frac{1}{8} & 0 & \cdots & \frac{1}{8} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{8} & 0 & \cdots & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{pmatrix}.$$

Notice that S^0_1 and $\tilde{S}^1_1$ are not exactly the inverse of each other as is the case for their continuous counterparts and the same for S^1_1 and $\tilde{S}^0_1$, but for example $\tilde{S}^1_1$ and $(S^0_1)^{-1}$ can be used as approximations for the discrete Hogde operator.

In the case of cubic splines we have, the following circulant matrices

$$S^0_3 = \tilde{S}^0_3 = \frac{1}{\Delta x}\,\mathrm{circ}[\frac{1}{8}, \frac{3}{4}, \frac{1}{8}],$$

$$S^1_3 = \tilde{S}^1_3 = \Delta x \,\mathrm{circ}[\frac{1}{384}, \frac{19}{96}, \frac{115}{192}, \frac{19}{96}, \frac{1}{384}].$$

3.1.2 Non uniform mesh with perfect conductors boundary conditions

The primal $1D$ mesh of our domain will be a non uniform set of knots $x_0 < x_1 < \cdots < x_{N-1} < x_N$. With perfect conductors boundary conditions, using a degree α taller than one for our interpolation, we have to contend a problem. We have $N + \alpha$ splines functions and $N + 1$ knots in the primal mesh. So we must add knots for interpolation on primal mesh. As α is odd, we add the middle of $\frac{(\alpha-1)}{2}$ first primal cells and $\frac{(\alpha-1)}{2}$ last primal cells. For example, if α equal 3, we must add 2 knots, so we take the middle of $[x_0, x_1]$ and $[x_{N-1}, x_N]$.
The primal mesh will be

$$\{x_0, \cdots, x_N\} \cup \left\{ \frac{(x_{i+1} + x_i)}{2} | i \in \{0, \cdots, \frac{(\alpha-1)}{2} - 1\} \right\} \cup \left\{ \frac{(x_{i-1} + x_i)}{2} | i \in \{N - \frac{(\alpha-1)}{2} + 1, \cdots, N\} \right\}.$$

Denoting by $n_0 < n_1 < \cdots < n_{N+\alpha-1}$ the knots of our primal mesh in increasing order. The dual mesh will consist of the middle points of the primal mesh with the extremal points x_0 and x_N, i.e. $n_0 = n_{-1/2} < n_{1/2} < \cdots < n_{N+\alpha-3/2} < n_{N+\alpha-1/2} = n_{N+\alpha-1}$. To conclude we have $N+\alpha$ knots for primal meshes and $N + \alpha + 1$ knots for dual meshes. With this construction, we can interpolate primal and dual 0 and 1-forms.
Let us start with the discrete 0-form on the primal mesh. Any function $C^0 \in \mathcal{S}^\alpha_0$ writes

$$C^0(x) = \sum_{j=-\alpha}^{N-1} c^0_j B^\alpha_j(x),$$

with the c^0_j defined by the interpolation conditions $C^0(n_i) = \sum_{j=-\alpha}^{N-1} c^0_j B^\alpha_j(n_i)$ for $i \in \{0, \cdots N + \alpha - 1\}$ on the primal mesh which is a linear system that can be written in matrix form $M^0_\alpha c^0 = \mathbb{C}^0$, with

$\mathbb{C}^0 = \left(C^0(n_0), \ldots, C^0(n_{N+\alpha-1})\right)^T$, $c^0 = \left(c^0_{-\alpha}, \ldots, c^0_{N-1}\right)^T$ and M^0_α the square matrix with size $N+\alpha$ whose components are $m^0_{i,j+\alpha} = B^\alpha_j(n_i)$ for $j = -\alpha, \cdots, N-1$ and $i = 0, \cdots N+\alpha-1$.

Lemma 3.3 *The matrix M^0_α is non singular.*

Proof. M^0_α meets the conditions of the Schoenberg-Whitney Theorem [8] and so the square matrix is non singular. ∎

The space of linear spline 1-forms $\mathcal{S}^\alpha_1$ will be the vector space generated by these basis functions. Any 1-form $C^1 \in \mathcal{S}^\alpha_1$ writes

$$C^1(x) = \sum_{j=-\alpha+1}^{N-1} c^1_j D^\alpha_j(x)\, dx,$$

the coefficients c^1_j being defined by the relations

$$\int_{n_i}^{n_{i+1}} C^1(x) = \sum_{j=-\alpha+1}^{N-1} c^1_j \int_{n_i}^{n_{i+1}} D^\alpha_j(x)\, dx \qquad \text{for } 0 \le i \le N+\alpha-2,$$

this also defines a linear system that can be written in matrix form $M^1_\alpha c^1 = \mathbb{C}^1$, with $\mathbb{C}^1 = \left(\int_{n_0}^{n_1} C^1(x), \ldots, \int_{n_{N+\alpha-2}}^{n_{N+\alpha-1}} C^1(x)\right)^T$, $c^1 = \left(c^1_{-\alpha+1}, \ldots, c^1_{N-1}\right)^T$ and M^1_α the square matrix with size $N+\alpha-1$ whose components are $m^1_{i,j+\alpha-3} = \int_{n_i}^{n_{i+1}} D^\alpha_j(x)\, dx$ for $j = -\alpha+1, \cdots, N-1$ and $i = 0, \cdots N+\alpha-2$.

Lemma 3.4 *The matrix M^1_α is non singular.*

Proof. Using the relation $\mathbf{w}^{1,\alpha}_i(x) - \mathbf{w}^{1,\alpha}_{i+1}(x) = B^{\alpha\prime}_i(x)\, dx$ and the fact that $B^\alpha_i(x)$ vanishes for $x \notin [x_i, x_{i+\alpha}]$, it follows easily by recurrence on the degree α that

$$\int_{n_i}^{n_{i+1}} \mathbf{w}^{1,\alpha}_i(x) = \sum_{k=j}^{N-1} B^\alpha_k(n_{i+1}) - \sum_{k=j}^{N-1} B^\alpha_k(n_i) =: A^\alpha_j(n_{i+1}) - A^\alpha_j(n_i).$$

We can observe that M^1_α is the principal minor (1,1) of the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} M^0_\alpha \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} = \begin{pmatrix} A^\alpha_{-\alpha}(n_0) & A^\alpha_{-\alpha+1}(n_0) & \cdots & A^\alpha_{N-1}(n_0) \\ \int_{n_0}^{n_1} D_{-\alpha} & & & \\ \int_{n_1}^{n_2} D_{-\alpha} & & & \\ \vdots & & M^1_\alpha & \\ \int_{n_{N+\alpha-2}}^{n_{N+\alpha-1}} D_{-\alpha} & & & \end{pmatrix}.$$

The matrix $\mathcal{A}$ on the left hand side is non singular because it's the product of non singular matrices. Since for all i, $n_i \geqslant x_0$ and the support of $D_{-\alpha}$ is $]x_{-\alpha}, x_0[$, we have $\int_{n_i}^{n_{i+1}} D_{-\alpha} = 0$. Furthermore, $A^\alpha_{-\alpha}(n_0) \neq 0$ because $B_i(x) \geqslant 0$ and $B_{-\alpha}(n_0) \neq 0$. So, $0 \neq det(\mathcal{A}) = A^\alpha_{-\alpha}(n_0) det(M^1_\alpha)$ that implies $det(M^1_\alpha) \neq 0$. ∎

The discrete 0-forms and 1-forms on the dual mesh are defined in the same way by

$$\tilde{C}^0(x) = \sum_{j=-\alpha-1}^{N-1} \tilde{c}^0_{j+1/2} B^\alpha_{j+1/2}(x),$$

with the $\tilde{c}^0_{j+1/2}$ defined by the interpolation conditions

$$\tilde{C}^0(n_{i-1/2}) = \sum_{j=-\alpha-1}^{N-1} \tilde{c}^0_{j+1/2} B^\alpha_{j+1/2}(n_{i-1/2}) \qquad \text{for } i \in \{0, \cdots, N+\alpha\}$$

on the dual mesh which is a linear system that can be written in matrix form $\tilde{M}^0_\alpha \tilde{c}^0 = \tilde{\mathbb{C}}^0$, with $\tilde{\mathbb{C}}^0 = \left(\tilde{C}^0(n_{-1/2}), \tilde{C}^0(n_{1/2}), \cdots, \tilde{C}^0(n_{N+\alpha-1/2})\right)^T$, $\tilde{c}^0 = \left(\tilde{c}^0_{-\alpha+3/2}, \ldots, \tilde{c}^0_{N+3/2}\right)^T$ and $\tilde{M}^0_\alpha$ the square matrix with size $N+\alpha+1$ whose components are $\tilde{m}^0_{i,j+\alpha-1} = B^\alpha_{j+1/2}(n_{i-1/2})$ and is nonsingular with help Schoenberg-Whitney Theorem.

A discrete 1-form on the dual mesh is defined by

$$\tilde{C}^1(x) = \sum_{j=-\alpha}^{N-1} \tilde{c}^1_{j+1/2} D^\alpha_{j+1/2}(x)\, dx,$$

the coefficients $\tilde{c}^1_{j+1/2}$ being defined by the relations

$$\int_{n_{i-1/2}}^{n_{i+1/2}} \tilde{C}^1(x) = \sum_{j=-\alpha}^{N-1} \tilde{c}^1_{j+1/2} \int_{n_{i-1/2}}^{n_{i+1/2}} D^\alpha_{j+1/2}(x)\, dx \qquad \text{for } 0 \le i \le N+\alpha-1,$$

this also defines a linear system that can be written in matrix form $\tilde{M}^1_\alpha \tilde{c}^1 = \tilde{\mathbb{C}}^1$, with $\tilde{\mathbb{C}}^1 = \left(\int_{n_{-1/2}}^{n_{1/2}} \tilde{C}^1(x), \cdots, \int_{n_{N+\alpha-3/2}}^{n_{N+\alpha-1/2}} \tilde{C}^1(x)\right)^T$, $\tilde{c}^1 = \left(\tilde{c}^1_{-\alpha+3/2}, \ldots, \tilde{c}^1_{N+3/2}\right)^T$ and $\tilde{M}^1_\alpha$ the square matrix with size $N+\alpha$ whose components are $\tilde{m}^1_{i,j+\alpha-2} = \int_{n_{i-1/2}}^{n_{i+1/2}} D^\alpha_{j+1/2}(x)\, dx$ and is nonsingular thanks to the previous lemma.

The discrete Hodge operator: Let us start with the discrete Hodge [2, 3] mapping primal 0-forms to dual 1-forms. Given a discrete 0-form on the primal mesh

$$C^0(x) = \sum_{j=-\alpha}^{N-1} c^0_j B^\alpha_j(x),$$

and so

$$\star C^0(x) = \sum_{j=-\alpha}^{N-1} c^0_j B^\alpha_j(x)\, dx.$$

Denoting by πC^0 the projection of $\star C^0$ on the space of discrete differential forms of the same order on the dual mesh, we can write

$$\pi C^0(x) = \sum_{j=-\alpha}^{N-1} \tilde{c}^1_{j+1/2} D^\alpha_{j+1/2}(x)\, dx,$$

with

$$\int_{n_{i-1/2}}^{n_{i+1/2}} \star C^0(x) = \sum_{j=-\alpha}^{N-1} \tilde{c}^1_{j+1/2} \int_{n_{i-1/2}}^{n_{i+1/2}} D^\alpha_{j+1/2}(x)\, dx \qquad \text{for } 0 \le i \le N+\alpha-1,$$

where $n_{-1/2} = n_0$ and $n_{N+\alpha-1/2} = n_{N+\alpha-1}$. Now defining $\tilde{S}^1$ the square matrix with a size $N+\alpha$ whose i,j coefficient is $\int_{n_{i-1/2}}^{n_{i+1/2}} B^\alpha_j(x)\, dx$, this relation becomes in matrix form

$$\tilde{S}^1 c^0 = \tilde{M}^1_\alpha \tilde{c}^1,$$

so that the discrete Hodge operator mapping c^0 to $\tilde{c}^1$ is

$$(\tilde{M}^1_\alpha)^{-1}\tilde{S}^1 \quad \text{with } \tilde{S}^1_{i,j} = \int_{n_{i-1/2}}^{n_{i+1/2}} B^\alpha_j(x)\,dx.$$

In order to define the Hodge operator mapping discrete 1-forms in the primal grid to discrete 0-forms on the dual grid, we apply the continuous Hodge operator to a discrete 1-form on the primal grid

$$\star C^1(x) = \sum_{j=-\alpha+1}^{N-1} c^1_j D^\alpha_j(x).$$

Its projection on the space of discrete 0-forms on the dual grid is defined by the point values $\star C^1(n_{i-1/2})$ for $i \in \{0, \cdots, N+\alpha\}$. Hence in the same way as before the discrete Hodge in this case is defined by

$$(\tilde{M}^0_\alpha)^{-1}\tilde{S}^0 \quad \text{with } \tilde{S}^0_{i,j} = D^\alpha_j(n_{i-1/2}).$$

We can notice that this matrix is not square because its size is $(N+\alpha+1)\times(N+\alpha-1)$.

The Hodge operators mapping from the dual grid to the primal grid are naturally defined in the same way by

$$(M^1_\alpha)^{-1}S^1 \quad \text{with } S^1_{i,j} = \int_{n_i}^{n_{i+1}} B^\alpha_{j+1/2}(x)\,dx.$$

This matrix is not square, its size is $(N+\alpha-1)\times(N+\alpha+1)$, but the matrix S^0 is square, with a size $(N+\alpha)\times(N+\alpha)$:

$$(M^0_\alpha)^{-1}S^0 \quad \text{with } S^0_{i,j} = D^\alpha_{j+1/2}(n_i).$$

3.2 The 3D case

We are now going to define the 3D discrete differential forms on a periodic cartesian grid, which will be needed for Maxwell's equations, by tensor product using the 1D form. This procedure can be generalized in a natural way to any number of dimensions.

The set of 3D discrete differential forms will be defined as the span of the following basis functions:

- The basis functions for the 0-forms are

$$^0w^\alpha_{i,j,k}(x,y,z) = B^\alpha_i(x)B^\alpha_j(y)B^\alpha_k(z).$$

- The basis functions for the 1-forms are

$$\begin{aligned}
^1\mathbf{w}^{\alpha,x}_{i,j,k}(x,y,z) &= D^\alpha_i(x)B^\alpha_j(y)B^\alpha_k(z)\,dx,\\
^1\mathbf{w}^{\alpha,y}_{i,j,k}(x,y,z) &= B^\alpha_i(x)D^\alpha_j(y)B^\alpha_k(z)\,dy,\\
^1\mathbf{w}^{\alpha,z}i,j,k(x,y,z) &= B^\alpha_i(x)B^\alpha_j(y)D^\alpha_k(z)\,dz.
\end{aligned}$$

- The basis functions for the 2-forms are

$$\begin{aligned}
^2\mathbf{w}^{\alpha,x}_{i,j,k}(x,y,z) &= B^\alpha_i(x)D^\alpha_j(y)D^\alpha_k(z)\,dy\wedge dz,\\
^2\mathbf{w}^{\alpha,y}_{i,j,k}(x,y,z) &= D^\alpha_i(x)B^\alpha_j(y)D^\alpha_k(z)\,dz\wedge dx,\\
^2\mathbf{w}^{\alpha,z}_{i,j,k}(x,y,z) &= D^\alpha_i(x)D^\alpha_j(y)B^\alpha_k(z)\,dx\wedge dy.
\end{aligned}$$

- The basis functions for the 3-forms are

$$^3w^{\alpha}_{i,j,k}(x,y) = D^{\alpha}_i(x)D^{\alpha}_j(y)D^{\alpha}_k(z)\,dx \wedge dy \wedge dz.$$

This construction will yield the same basis functions as in [7] and [14] where vector calculus is used.

4 Link with the theory of chains and cochains

We now point out the link of our discrete differential forms with the theory of chains and cochains [4, 6, 9, 15].

4.1 Notion of chain

Hypercubes [16, 17]: Let us denote by Ω a domain and by T its boundary. We pave Ω with squares and we denote by m this mesh. We call 0-cubes, the nodes of our mesh, 1-cubes, the edges, 2-cubes, the squares and 3-cubes, the cubes (4-cubes, the tesseract, 5-cubes for the penteract, ..). We define an orientation for all of elements of m and denote by $H_{\alpha,p}(m)$ the set of p-cubes (we will see that in the case of a non periodic domain the number of p-cubes depends on the degree of B-splines α). Now, we define the boundary operator ∂ that maps a p-cube to a $(p-1)$-cube. For example: let e_i a vector who has for origin x_i and for extremity point x_{i+1}, so the boundary of e_i is $\partial e_i = x_{i+1} - x_i$. In fact, we can define the boundary operator ∂^p, mapping a p-cube to a $(p-1)$-cube, by a sparse matrix containing only $-1, 1$ or 0 with size $|H_{\alpha,p-1}(m)| \times |H_{\alpha,p}(m)|$. These matrices are called the transpose of incidence matrices and verify $\partial^p\,\partial^{p+1} = 0$.

Notion of Chains and discrete manifolds: A p-chain $\mathbf{c}_p$ is a linear combination of all p-cubes on m i.e.:

$$\mathbf{c}_p = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s}\, h^{p,s},$$

where $c_{p,s} \in \mathbb{R}$. We will denote the set of all p-chains as $C_{\alpha,p}(m)$. We define that a p-dimensional manifold is discretized by a p-chain and the boundary operator acts on p-chains by linearity as:

$$\partial \mathbf{c}_p = \partial \left(\sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s}\, h^{p,s} \right) = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s}\, \partial\, h^{p,s},$$

and when we collect the s-th $(p-1)$-cube we obtain:

$$\partial \mathbf{c}_p = \sum_{h^{p-1,s} \in H_{\alpha,p-1}(m)} (\partial^p c_p)_s\, h^{p-1,s},$$

where $c_p = (c_{p,s})_s$ is a column vector containing the coefficients of a p-chain. And so, applying the boundary operator on a p-chain is equivalent to applying the operator ∂^p on coefficients of a p-chain. Furthermore, since all p-chains are defined by their $|H_{\alpha,p}(m)|$ coefficients $c_{p,s}$, we can find a bijection mapping $C_{\alpha,p}(m)$ to $\mathbb{R}^{|H_{\alpha,p}(m)|}$. But, before, we will define the mapping that determines the coefficients $c_{p,s}$.
A p-chain $\mathbf{c}_p$ represents a discrete p-manifold, so for all basis functions $^p\mathbf{w}^{\alpha}_i \in W^{\alpha,p}(m)$ we can compute the integral of $^p\mathbf{w}^{\alpha}_i$ over $\mathbf{c}_p$. So, coefficient $c_{p,s}$ is defined by the relations, for all $i \in |H_{\alpha,p}(m)|$:

$$\int_{\mathbf{c}_p} {}^p\mathbf{w}^{\alpha}_i = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s} \int_{h^{p,s}} {}^p\mathbf{w}^{\alpha}_i.$$

This yields a linear system. For example, in one dimension, we must solve

- for a 0-chain
$$\mathbb{C}_0 = (M^0_\alpha)^t\, c_0,$$
where $\mathbb{C}_0 = (B^\alpha_0(\mathbf{c}_0), \ldots, B^\alpha_{|H_{\alpha,0}(m)|-1}(\mathbf{c}_0))^t$, $c_0 = (c_{0,0}, \ldots, c_{0,|H_{\alpha,0}(m)|-1})^t$ and M^0_α is the square matrix we constructed for discrete differential 0-forms.

- for a 1-chain
$$\mathbb{C}_1 = (M^1_\alpha)^t\, c_1,$$
where $\mathbb{C}_1 = (\int_{\mathbf{c}_1} D^\alpha_0(x)dx, \ldots, \int_{\mathbf{c}_1} D^\alpha_{|H_{\alpha,1}(m)|-1}(x)dx)^t$, $c_1 = (c_{1,0}, \ldots, c_{1,|H_{\alpha,1}(m)|-1})^t$ and M^1_α is the square matrix we constructed for discrete differential 1-forms.

In the same way, in two or three dimension, we can remark that we also must solve linear systems involving tensor products of matrices $(M^1_\alpha)^t$ and $(M^0_\alpha)^t$. More generally, for all dimensions, we denote by $\mathbb{M}^p_\alpha$ the square matrix that we use for finding the coefficients of p-chain $\mathbf{c}_p$ i.e. for solving the linear system $\mathbb{C}_p = (\mathbb{M}^p_\alpha)^t c_p$. Now, we can define $\mathcal{P}^t_\alpha$ mapping $C_{\alpha,p}(m)$ to $\mathbb{R}^{|H_{\alpha,p}(m)|}$ and $\mathcal{R}^t_\alpha$ mapping $\mathbb{R}^{|H_{\alpha,p}(m)|}$ to $C_{\alpha,p}(m)$ such that

$$\begin{aligned}\mathcal{P}^t_\alpha :\quad & C_{\alpha,p}(m) \to \mathbb{R}^{|H_{\alpha,p}(m)|}\\ & \mathbf{c}_p \mapsto \{c_p \mid \mathbb{C}_p = (\mathbb{M}^p_\alpha)^t\, c_p\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^t_\alpha :\quad & \mathbb{R}^{|H_{\alpha,p}(m)|} \to C_{\alpha,p}(m)\\ & c_p = (c_{p,s})_s \mapsto \sum_{h^{p,s}\in H_{\alpha,p}(m)} c_{p,s}\, h^{p,s}.\end{aligned}$$

Now we can show that we have a de Rham complex:

$$\begin{array}{ccccccc} C_{\alpha,3}(m) & \xrightarrow{\partial} & C_{\alpha,2}(m) & \xrightarrow{\partial} & C_{\alpha,1}(m) & \xrightarrow{\partial} & C_{\alpha,0}(m)\\ \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha & & \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha & & \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha & & \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha\\ \mathbb{R}^{|H_{\alpha,3}(m)|} & \xrightarrow{\partial^3} & \mathbb{R}^{|H_{\alpha,2}(m)|} & \xrightarrow{\partial^2} & \mathbb{R}^{|H_{\alpha,1}(m)|} & \xrightarrow{\partial^1} & \mathbb{R}^{|H_{\alpha,0}(m)|}\end{array}$$

with $\partial^2\partial^3 = 0$ and $\partial^1\partial^2 = 0$.

Lemma 4.1 *This diagram is commutative.*

Proof. Remember a property of B-splines $B^\alpha_j(x_{i+1}) - B^\alpha_j(x_i) = \int_{x_i}^{x_{i+1}} D_j(x)\,dx - \int_{x_i}^{x_{i+1}} D_{j+1}(x)\,dx$, this becomes in matrix from $(\mathbb{M}^0_\alpha)^t\,\partial^1 = \partial^1\,(\mathbb{M}^1_\alpha)^t$ and we have also $(\mathbb{M}^1_\alpha)^t\,\partial^2 = \partial^2\,(\mathbb{M}^2_\alpha)^t$ and $(\mathbb{M}^2_\alpha)^t\,\partial^3 = \partial^3\,(\mathbb{M}^3_\alpha)^t$. Then, remenber also that applying the boundary operator on a p-chain its equivalent to applying the operator ∂^p on the coefficients of the p-chain. So, for a 1-chain $\mathbf{c}_1$, $\partial\,\mathbf{c}_1 = \sum_{h^{0,s}\in H_{\alpha,0}(m)} (\partial^1 c_1)_s\, h^{0,s}$ and so $\mathcal{P}^t_\alpha\,\partial\,\mathbf{c}_1 = ((\mathbb{M}^0_\alpha)^t)^{-1}\,\partial^1\,c_1$ and $\partial^1\,\mathcal{P}^t_\alpha\,\mathbf{c}_1 = \partial^1\,((\mathbb{M}^1_\alpha)t)^{-1}\,c_1$. We can deduce that $\partial^1\,(\mathcal{P}_\alpha)^t\,\mathbf{c}_1 = \mathcal{P}^t_\alpha\,\partial\,\mathbf{c}_1$. Proceeding in the same way, we also obtain that $\partial^2\,\mathcal{P}^t_\alpha\,\mathbf{c}_2 = \mathcal{P}^t_\alpha\,\partial\,\mathbf{c}_2$ and $\partial^3\,\mathcal{P}^t_\alpha\,\mathbf{c}_3 = \mathcal{P}^t_\alpha\,\partial\,\mathbf{c}_3$. ∎

In the dual mesh, denoting by $H^\star_{\alpha,p}(m)$ the p-cell, dual of $(n-p)$-cubes and $C^\star_{\alpha,p}(m)$ the set of p-chain in the dual mesh. Similarly, we obtain a de Rham complex, also commutative:

$$\begin{array}{ccccccc} C^\star_{\alpha,3}(m) & \xrightarrow{\partial} & C^\star_{\alpha,2}(m) & \xrightarrow{\partial} & C^\star_{\alpha,1}(m) & \xrightarrow{\partial} & C^\star_{\alpha,0}(m)\\ \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha & & \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha & & \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha & & \mathcal{P}^t_\alpha\uparrow\downarrow\mathcal{R}^t_\alpha\\ \mathbb{R}^{|H_{\alpha,0}(m)|} & \xrightarrow{(\partial^1)^t} & \mathbb{R}^{|H_{\alpha,1}(m)|} & \xrightarrow{(\partial^2)^t} & \mathbb{R}^{|H_{2,\alpha}(m)|} & \xrightarrow{(\partial^3)^t} & \mathbb{R}^{|H_{\alpha,3}(m)|}\end{array}$$

4.2 Notion of cochain

Cochains: A p-cochain $\mathbf{w}^p$ is the dual of a p-chain. That is to say $\mathbf{w}^p$ is a linear mapping that takes p-chains to $\mathbb{R}$:

$$\begin{aligned} \mathbf{w}^p : \quad & C_{\alpha,p}(m) \to \mathbb{R} \\ & \mathbf{c}_p \mapsto \mathbf{w}^p(\mathbf{c}_p) \end{aligned}$$

We denote the set of p-cochains or p-forms by $W^{\alpha,p}(m)$. By duality, this space has a finite dimension $|H_{\alpha,p}(m)|$. A p-cochain $\mathbf{w}^p$ operates on a p-chain $\mathbf{c}_p$ and returns a linear combination of the values of the cochain on each p-cube.

Link with differentials forms: p-cochains are discrete analogs of differential forms. So, a discrete differential p-form, $\mathbf{w}^p$, is a linear mapping that takes p-chains to $\mathbb{R}$. We have seen, during the construction of discrete differential p-forms, or p-cochains, that they can be written as:

$$\mathbf{w}^p = \sum_{{}^p\mathbf{w}_s^\alpha \in W^{\alpha,p}(m)} w^{p,s}\, {}^p\mathbf{w}_s^\alpha,$$

where the coefficients $w^{p,s}$ are defined as the solution of the linear system:

$$\mathbb{W}^p = \mathbb{M}_\alpha^p\, w^p.$$

Now, we can define the non degenerate bilinear form:

$$\begin{aligned} \langle \quad , \quad \rangle : \quad & W^{\alpha,p}(m) \times C_{\alpha,p}(m) \to \mathbb{R} \\ & (\mathbf{w}^p, \mathbf{c}_p) \longmapsto \langle \mathbf{w}^p, \mathbf{c}_p \rangle = \int_{\mathbf{c}_p} \mathbf{w}^p. \end{aligned}$$

Let $\mathbf{c}_p$ a p-chain and $\mathbf{w}^p$ a differential p-form, we obtain by linearity:

$$\int_{\mathbf{c}_p} \mathbf{w}^p = \sum_{h^{p,s} \in H_{\alpha,p}(m)} c_{p,s} \int_{h^{p,s}} \mathbf{w}^p.$$

So, the bilinear mapping acts on the spline coefficients and on the coefficients of p-chains as:

$$\langle \mathbf{w}^p, \mathbf{c}_p \rangle = c_p^t\, \mathbb{M}_\alpha^p\, w^p,$$

where c_p and w^p are column vectors of coefficients of the p-chain $\mathbf{c}_p$ and the p-cochain $\mathbf{w}^p$ respectively.

Exterior derivative, Stokes theorem and de Rham complex: The exterior derivative maps a $(p-1)$-form to p-form. For example, in the $3D$ case and $p=1$:

$$\begin{aligned}
d\mathbf{w}^0 &= d\left(\sum w^{i,j,k} B_i^\alpha(x) B_j^\alpha(y) B_k^\alpha(z)\right) \\
&= \sum w^{i,j,k} d\left(B_i^\alpha(x) B_j^\alpha(y) B_k^\alpha(z)\right) \\
&= \sum w^{i,j,k} (D_i^\alpha(x) - D_{i+1}^\alpha(x)) B_j^\alpha(y) B_k^\alpha(z)\, dx \\
&\qquad + w^{i,j,k} (D_j^\alpha(y) - D_{j+1}^\alpha(y)) B_i^\alpha(x) B_k^\alpha(z)\, dy \\
&\qquad + w^{i,j,k} (D_k^\alpha(z) - D_{k+1}^\alpha(z)) B_i^\alpha(x) B_j^\alpha(y)\, dz \\
&= \sum (w^{i,j,k} - w^{i-1,j,k}) D_i^\alpha(x) B_j^\alpha(y) B_k^\alpha(z)\, dx \\
&\qquad + (w^{i,j,k} - w^{i,j-1,k}) B_i^\alpha(x) D_j^\alpha(y) B_k^\alpha(z)\, dy \\
&\qquad + (w^{i,j,k} - w^{i,j,k-1}) B_i^\alpha(x) B_j^\alpha(y) D_k^\alpha(z)\, dz \\
&= \sum (w^{i,j,k} - w^{i-1,j,k})\, {}^1\mathbf{w}_{i,j,k}^{\alpha,x}(x,y,z) \\
&\qquad + (w^{i,j,k} - w^{i,j-1,k})\, {}^1\mathbf{w}_{i,j,k}^{\alpha,y}(x,y,z) \\
&\qquad + (w^{i,j,k} - w^{i,j,k-1})\, {}^1\mathbf{w}_{i,j,k}^{\alpha,z}(x,y,z).
\end{aligned}$$

We remember that we have an equivalence between applying the boundary operator on a p-chain and applying ∂^p on coefficients of a p-chain for $p = 1, 2, 3$ respectively. Here, we have the dual property. We can see that applying the exterior derivative on a differential p-form is equivalent to applying the incidence matrix $(\partial^p)^t$ on the spline coefficients.
Now, construct a diagram. Denoting by $F^p(\Omega)$ the set of differential p-forms on Ω (not discrete). We can pass from p-form to a $(p+1)$-form by exterior derivative and the sequence

$$F^0(\Omega) \xrightarrow{d} F^1(\Omega) \xrightarrow{d} F^2(\Omega) \xrightarrow{d} F^3(\Omega)\ ,$$

is exact when Ω is star-shaped [10] and we have, in the same way, for discrete differential p-forms,

$$W^{\alpha,0}(m) \xrightarrow{d} W^{\alpha,1}(m) \xrightarrow{d} W^{\alpha,2}(m) \xrightarrow{d} W^{\alpha,3}(m)\ ,$$

is exact when m is the mesh of a star-shaped Ω. Seeing that $W^{\alpha,p}(m)$ was constructed as a finite dimensional subspace of $F^p(\Omega)$, we can project the set of differential p-forms on $W^{p,\alpha}(m)$. We obtain the de Rham complex:

$$\begin{array}{ccccccc}
F^0 & \xrightarrow{d} & F^1 & \xrightarrow{d} & F^2 & \xrightarrow{d} & F^3 \\
\downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} \\
W^{\alpha,0}(m) & \xrightarrow{d} & W^{\alpha,1}(m) & \xrightarrow{d} & W^{\alpha,2}(m) & \xrightarrow{d} & W^{\alpha,3}(m) \\
{\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} & & {\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} & & {\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} & & {\scriptstyle \mathcal{P}_\alpha}\uparrow\downarrow{\scriptstyle \mathcal{R}_\alpha} \\
\mathbb{R}^{|H_{\alpha,0}(m)|} & \xrightarrow{(\partial^1)^t} & \mathbb{R}^{|H_{\alpha,1}(m)|} & \xrightarrow{(\partial^2)^t} & \mathbb{R}^{|H_{\alpha,2}(m)|} & \xrightarrow{(\partial^3)^t} & \mathbb{R}^{|H_{\alpha,3}(m)|}
\end{array}$$

where $proj$ is a projection and mapping $\mathcal{P}_\alpha$ and $\mathcal{R}_\alpha$ are the dual mapping of $\mathcal{P}_\alpha^t$ and $\mathcal{R}_\alpha^t$ defined by :

$$\begin{aligned}
\mathcal{P}_\alpha :\quad & W^{\alpha,p}(m) \to \mathbb{R}^{|H_{\alpha,p}(m)|} \\
& \mathbf{w}^p \mapsto \{w^p | \mathbb{W}^p = \mathbb{M}_\alpha^p\, w^p\}
\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_\alpha : \quad & \mathbb{R}^{|H_{\alpha,p}(m)|} \to W^{\alpha,p}(m) \\ & w^p \mapsto \sum_{{}^p\mathbf{w}_s^\alpha \in W^{\alpha,p}(m)} w^{p,s}\ {}^p\mathbf{w}_s^\alpha.\end{aligned}$$

Lemma 4.2 *This diagram is commutative.*

Proof. First, let us show that the part involving discrete differential p-forms and spline coefficients, a commutative diagram.
With help of a property of splines, we have seen that $(\mathbb{M}_\alpha^0)^t\,\partial^1 = \partial^1\,(\mathbb{M}_\alpha^1)^t$, $(\mathbb{M}_\alpha^1)^t\,\partial^2 = \partial^2\,(\mathbb{M}_\alpha^2)^t$ and $(\mathbb{M}_\alpha^2)^t\,\partial^3 = \partial^3\,(\mathbb{M}_\alpha^3)^t$. We take the tranpose and we have $(\partial^1)^t\,\mathbb{M}_\alpha^0 = \mathbb{M}_\alpha^1\,(\partial^1)^t$, $(\partial^2)^t\,\mathbb{M}_\alpha^1 = \mathbb{M}_\alpha^2\,(\partial^2)^t$ and $(\partial^3)^t\,\mathbb{M}_\alpha^2 = \mathbb{M}_\alpha^3\,(\partial^3)^t$. Furthermore, applying the exterior derivative on a differential p-form is equivalent to applying the incidence matrix on the spline coefficients. We have $\mathcal{P}_\alpha\,d\,\mathbf{w}^0 = (\mathbb{M}_\alpha^1)^{-1}\,(\partial^1)^t\,w^0$ and $(\partial^1)^t\,\mathcal{P}_\alpha\,\mathbf{w}^0 = (\partial^1)^t\,(\mathbb{M}_\alpha^0)^{-1}\,w^0$ and so $(\partial^1)^t\,\mathcal{P}_\alpha\,\mathbf{w}^0 = \mathcal{P}_\alpha\,d\,\mathbf{w}^0$. In the same way, we obtain $(\partial^2)^t\,\mathcal{P}_\alpha\,\mathbf{w}^1 = \mathcal{P}_\alpha\,d\,\mathbf{w}^1$, $(\partial^3)^t\,\mathcal{P}_\alpha\,\mathbf{w}^2 = \mathcal{P}_\alpha\,d\,\mathbf{w}^2$.

Secondly, let us show that the part concerning differential p-forms and discrete differential p-forms is a commutative diagram.
On the one hand, let $\omega^{p-1} \in F^{p-1}(\Omega)$ a differential $(p-1)$-form. The exterior derivative of a $(p-1)$-form $d\omega^{p-1}$ is p-form. So, with help of the Leibniz property , we obtain that

$$proj\ d\,\omega^{p-1} = \sum_{{}^p\mathbf{w}_s^\alpha \in W^{\alpha,p}(m)} \left((\mathbb{M}_\alpha^p)^{-1}\ (\partial^p)^t\mathbb{W}^{p-1}\right)_s\ {}^p\mathbf{w}_s^\alpha.$$

On other hand, with help of exterior derivative on splines coefficients

$$d\ proj\ \omega^{p-1} = \sum_{{}^p\mathbf{w}_s^\alpha \in W^{\alpha,p}(m)} \left((\partial^p)^t\ \left(\mathbb{M}_\alpha^{p-1}\right)^{-1}\ \mathbb{W}^{p-1}\right)_s\ {}^p\mathbf{w}_s^\alpha.$$

And so, showing that $d\ proj\ \omega^{p-1} = proj\ d\,\omega^{p-1}$ is equivalent to showing that $(\partial^p)^t\ \left(\mathbb{M}_\alpha^{p-1}\right)^{-1} = (\mathbb{M}_\alpha^p)^{-1}\ (\partial^p)^t$, that we have. ∎

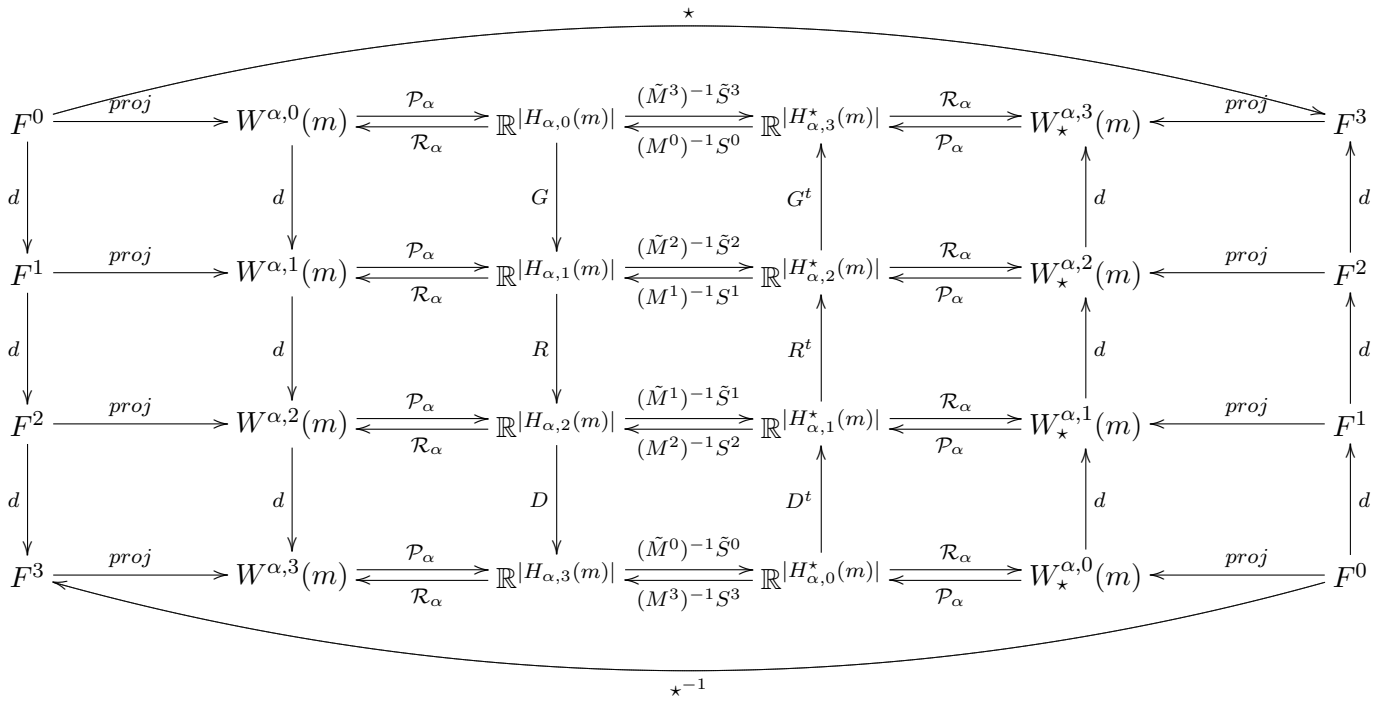
Remark 4.1 *Since splines have a compact support, we can observe that $W^{\alpha,0}(m) \subset H(grad,\Omega)$, $W^{\alpha,1}(m) \subset H(curl,\Omega)$, $W^{\alpha,2}(m) \subset H(div,\Omega)$ and $W^{\alpha,3}(m) \subset L_2(\Omega)$ in the sense of differential forms. That is, if $\mathbf{w}^p \in W^{\alpha,p}(m)$, $\mathbf{w}^p \wedge \star\mathbf{w}^p \in L_2(\Omega)$ and $d\mathbf{w}^p \wedge \star d\mathbf{w}^p \in L_2(\Omega)$.*

Also, for p-cochains on the dual mesh $W_\star^{\alpha,p}$, we have similarly a commutative diagram:

$$\begin{array}{ccccccc}
F^0 & \xrightarrow{d} & F^1 & \xrightarrow{d} & F^2 & \xrightarrow{d} & F^3 \\
\downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} \\
W_\star^{\alpha,0}(m) & \xrightarrow{d} & W_\star^{\alpha,1}(m) & \xrightarrow{d} & W_\star^{\alpha,2}(m) & \xrightarrow{d} & W_\star^{\alpha,3}(m) \\
\uparrow{\scriptstyle \mathcal{P}_\alpha}\downarrow{\scriptstyle \mathcal{R}_\alpha} & & \uparrow{\scriptstyle \mathcal{P}_\alpha}\downarrow{\scriptstyle \mathcal{R}_\alpha} & & \uparrow{\scriptstyle \mathcal{P}_\alpha}\downarrow{\scriptstyle \mathcal{R}_\alpha} & & \uparrow{\scriptstyle \mathcal{P}_\alpha}\downarrow{\scriptstyle \mathcal{R}_\alpha} \\
\mathbb{R}^{|H^\star_{\alpha,0}(m)|} & \xrightarrow{\partial^3} & \mathbb{R}^{|H^\star_{\alpha,1}(m)|} & \xrightarrow{\partial^2} & \mathbb{R}^{|H^\star_{\alpha,2}(m)|} & \xrightarrow{\partial^1} & \mathbb{R}^{|H^\star_{\alpha,3}(m)|}
\end{array}$$

4.3 A broader view

In parallel with gradient, curl and divergence, we denote by G, R, D the matrix $(\partial^p)^t$ for $p = 1, 2, 3$ respectively. Furthermore, using Hodge star operator $\star$ mapping a p-form to a $(n-p)$-form, where n is the dimension of space, we have



In the n-dimensional space , we denote by $(\tilde{M}^{n-p})^{-1}\tilde{S}^{n-p}$ maps from a discrete primal p-form to a discrete dual $(n-p)$-form and denote by $(M^{n-p})^{-1}S^{n-p}$ maps from a discrete dual p-form to a discrete primal $(n-p)$-form. In theory, we have $\star\star\omega^p = (-1)^{p(n-p)}\omega^p$ for all differential p-forms. We do not have this property with discrete differential forms but $(M^{n-p})^{-1}S^{n-p}(\tilde{M}^{n-p})^{-1}\tilde{S}^{n-p}$ tends to $(-1)^{p(n-p)}Id$ when we increase the order of spline for interpolation. We remark that it does not depend on the number of mesh points. In the case of periodic domain in 1-dimension, we obtain the following table:

Degree of Spline	$Max\,(\lvert \star\star - Id\rvert)$
$\alpha = 1$	0.25
$\alpha = 3$	0.011031684
$\alpha = 5$	0.001809231

5 Application to the Maxwell equations

5.1 General case

The 3D Maxwell equations can be written in terms of differential forms [4] in the following way

$$-\partial_t\, {}^2\mathbf{D} + d\, {}^1\mathbf{H} = {}^2\mathbf{J}, \tag{4}$$

$$\partial_t\, {}^2\mathbf{B} + d\, {}^1\mathbf{E} = 0, \tag{5}$$

$$d\, {}^2\mathbf{D} = {}^3\rho, \tag{6}$$

$$d\, {}^2\mathbf{B} = 0, \tag{7}$$

where ${}^2\mathbf{D}$, ${}^2\mathbf{B}$, ${}^2\mathbf{J}$ are 2-forms, ${}^1\mathbf{E}$, ${}^1\mathbf{H}$ are 1-forms and ${}^3\rho$ is a 3-form.

For the purpose of numerical validation, let us consider the 2D case, where all functions depend only on the x and y variables. In this case, the Maxwell equations keep their three components, we therefore still need to consider the differential forms in a 3D space but restrict them to coefficients depending only on x and y. For the discretization of Maxwell's equations with our spline discrete differential forms, we can define two dual uniform cartesian grids of $[0,1]^3$. The mesh we shall consider will be a the cartesian product of a 2D mesh with one cell of length one in the z direction. The primal 2D grid is based on the points $x_i = i/\Delta x$, $y_j = j/\Delta y$, with $(i,j) \in [0, N_x] \times [0, N_y]$ and $N_x \Delta x = N_y \Delta y$. In case of periodic boundary conditions in the x direction, the point x_{N_x} corresponds to x_0 and is omitted from the grid. Periodic boundary conditions in the other direction are dealt with in the same manner.

The points of the dual grid are $x_{i+1/2} = (i+1/2)/\Delta x$, $y_{j+1/2} = (j+1/2)/\Delta y$, with $(i,j,k) \in [0, N_x - 1] \times [0, N_y - 1]$.

The basis functions for our spline discrete differential forms in this case will be

- for the 0-forms
$$ {}^0w^\alpha_{i,j}(x,y) = B^\alpha_i(x) B^\alpha_j(y), $$
- for the 1-forms
$$ {}^1\mathbf{w}^{\alpha,x}_{i,j}(x,y) = D^\alpha_i(x) B^\alpha_j(y)\, dx, \quad {}^1\mathbf{w}^{\alpha,y}_{i,j}(x,y) = B^\alpha_i(x) D^\alpha_j(y)\, dy, \quad {}^1\mathbf{w}^{\alpha,z}_{i,j}(x,y) = B^\alpha_i(x) B^\alpha_j(y)\, dz, $$
- for the 2-forms
$$ {}^2\mathbf{w}^{\alpha,x}_{i,j}(x,y) = B^\alpha_i(x) D^\alpha_j(y)\, dy \wedge dz, \quad {}^2\mathbf{w}^{\alpha,y}_{i,j}(x,y) = D^\alpha_i(x) B^\alpha_j(y)\, dz \wedge dx, $$
$$ {}^2\mathbf{w}^{\alpha,z}_{i,j}(x,y) = D^\alpha_i(x) D^\alpha_j(y)\, dx \wedge dy, $$
- for the 3-forms
$$ {}^3w^\alpha_{i,j}(x,y) = D^\alpha_i(x) D^\alpha_j(y)\, dx \wedge dy \wedge dz. $$

The discrete differential forms on the dual mesh are defined in the same way with their indices on the dual mesh.

Let us now introduce approximations of the unknowns of Maxwell's equations as linear combinations of these basis functions for the corresponding p-forms. Let us denote by ${}^1\tilde{\mathbf{w}}^\alpha$ and ${}^2\tilde{\mathbf{w}}^\alpha$, the discrete spline one form and two form basis functions for the splines with knots being the points of the dual mesh. In 2D, Maxwell's equations decouple into two systems, one linking the x and y components of ${}^2\mathbf{D}$ and ${}^1\mathbf{E}$ and the z component of ${}^2\mathbf{B}$ and ${}^1\mathbf{H}$, and the other one linking the x and y components of ${}^2\mathbf{B}$ and ${}^1\mathbf{H}$ and the z component of ${}^2\mathbf{D}$ and ${}^1\mathbf{E}$. We shall only consider the first one here. The other can be dealt with similarly.

Let us now express the relevant components of our electromagnetic field in the appropriate basis of discrete differential forms

$$ {}^2\mathbf{D}^x_h(t,x,y) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t)\, {}^2\tilde{\mathbf{w}}^{\alpha,x}_{i+1/2,j+1/2}(x,y), \qquad {}^1\mathbf{E}^x_h(t,x,y) = \sum_{i,j} e^x_{i,j}(t)\, {}^1\mathbf{w}^{\alpha,x}_{i,j}(x,y), $$
$$ {}^2\mathbf{D}^y_h(t,x,y) = \sum_{i,j} d^y_{i+1/2,j+1/2}(t)\, {}^2\tilde{\mathbf{w}}^{\alpha,y}_{i+1/2,j+1/2}(x,y) \qquad {}^1\mathbf{E}^y_h(t,x,y) = \sum_{i,j} e^y_{i,j}(t)\, {}^1\mathbf{w}^{\alpha,y}_{i,j}(x,y), $$
$$ {}^1\mathbf{H}^z_h(t,x,y) = \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\, {}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(x,y), \qquad {}^2\mathbf{B}_h(t,x,y) = \sum_{i,j} b^z_{i,j}(t)\, {}^2\mathbf{w}^{\alpha,z}_{i,j}(x,y). $$

In order to obtain equations relating the coefficients of this discrete differential forms, we inject these expressions into the Maxwell equations (4)-(7), and take the De Rahm maps for two forms on each facet of the the dual mesh for (4) and (6) and of primal mesh for (5) and (7) .

Let us first compute the exterior derivatives of ${}^1\mathbf{H}_h$ and ${}^1\mathbf{E}_h$:

$$\begin{aligned} d\,{}^1\mathbf{H}_h^z(t,x,y) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t) d\,{}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(x,y) \\ &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t) \left(-{B^\alpha_{i+1/2}}'(x) B^\alpha_{j+1/2}(y)\, dz\wedge dx + B^\alpha_{i+1/2}(x) {B^\alpha_{j+1/2}}'(y)\, dy\wedge dz \right), \end{aligned}$$

using the formula (2) for the derivative of the spline functions and the definition of D^α_i, (3.1.1), we get

$$\begin{aligned} d\,{}^1\mathbf{H}_h^z(t,x,y) &= \sum_{i,j} h^z_{i+1/2,j+1/2}(t) \Big((D^\alpha_{i+3/2}(x) - D^\alpha_{i+1/2}(x)) B^\alpha_{j+1/2}(y)\, dz\wedge dx \\ &\qquad + B^\alpha_{i+1/2}(x)(D^\alpha_{j+1/2} - D^\alpha_{j+3/2})(y)\, dy\wedge dz \Big) \\ &= \sum_{i,j} (h^z_{i-1/2,j+1/2}(t) - h^z_{i+1/2,j+1/2}(t)) D^\alpha_{i+1/2}(x) B^\alpha_{j+1/2}(y)\, dz\wedge dx \\ &\quad + \sum_{i,j} (h^z_{i+1/2,j+1/2}(t) - h^z_{i+1/2,j-1/2}(t)) B^\alpha_{i+1/2}(x) D^\alpha_{j+1/2}(y)\, dy\wedge dz. \end{aligned} \tag{8}$$

In the same way

$$\begin{aligned} d\,{}^1\mathbf{E}_h^x(t,x,y) &= \sum_{i,j} e^x_{i,j}(t) d\,{}^1\tilde{\mathbf{w}}^{\alpha,x}_{i,j}(x,y) \\ &= -\sum_{i,j} e^x_{i,j}(t) D^\alpha_i(x) {B^\alpha_j}'(y)\, dx\wedge dy \\ &= -\sum_{i,j} (e^x_{i,j}(t) - e^x_{i,j-1}(t)) D^\alpha_i(x) D^\alpha_j(y)\, dx\wedge dy, \end{aligned} \tag{9}$$

and

$$\begin{aligned} d\,{}^1\mathbf{E}_h^y(t,x,y) &= \sum_{i,j} e^y_{i,j}(t) d\,{}^1\tilde{\mathbf{w}}^{\alpha,y}_{i,j}(x,y) \\ &= \sum_{i,j} e^y_{i,j}(t) {B^\alpha_i}'(x) D^\alpha_j(y)\, dx\wedge dy \\ &= \sum_{i,j} (e^y_{i,j}(t) - e^y_{i-1,j}(t)) D^\alpha_i(x) D^\alpha_j(y)\, dx\wedge dy, \end{aligned} \tag{10}$$

Ampere's law (4), without current, for the first two components can be written

$$\partial_t\, {}^2\mathbf{D}^x + \partial_t\, {}^2\mathbf{D}^y - d\,{}^1\mathbf{H} = 0.$$

Then using expression (8) and identifying the components on the basis vectors of the discrete differential forms we get the following relation between the spline coefficients

$$\begin{aligned} {d^x_{i+1/2,j+1/2}}'(t) + h^z_{i+1/2,j+1/2}(t) - h^z_{i+1/2,j-1/2}(t) = 0, \\ {d^y_{i+1/2,j+1/2}}'(t) - h^z_{i+1/2,j+1/2}(t) + h^z_{i-1/2,j+1/2}(t) = 0. \end{aligned}$$

On the other hand, Faraday's law (5), for the third component can be written

$$\partial_t \, {}^2\mathbf{B}^z + d \, {}^1\mathbf{E} = 0.$$

This becomes using (9) and (10) and identifying the components on the basis vectors of the discrete differential forms

$$b^z_{i,j}{}'(t) + (e^y_{i,j}(t) - e^y_{i-1,j}(t)) - (e^x_{i,j}(t) - e^x_{i,j-1}(t)) = 0.$$

Discrete Hodge operators: Let us denote by

$$\mathbf{b}^z = ((b^z_{i,j}))_{1\le i\le N_x-1, 1\le j\le N_y-1}, \quad \mathbf{e}^x = ((e^x_{i,j}))_{1\le i\le N_x-1, 1\le j\le N_y-1}, \quad \mathbf{e}^y = ((e^y_{i,j}))_{1\le i\le N_x-1, 1\le j\le N_y-1},$$

$$\mathbf{h}^z = ((h^z_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1}, \quad \mathbf{d}^x = ((d^x_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1},$$

$$\mathbf{d}^y = ((d^y_{i+1/2,j+1/2}))_{1\le i\le N_x-1, 1\le j\le N_y-1},$$

the matrices of spline coefficients on the discrete grids.

We now need to define the discrete Hodge operators mapping $\mathbf{d}^x$ to $\mathbf{e}^x$, $\mathbf{d}^y$ to $\mathbf{e}^y$ and $\mathbf{b}^z$ to $\mathbf{h}^z$. The same procedure as for the 1D case will be used. We have

$${}^2\mathbf{D}^x_h(t,x,y) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(x) D^\alpha_{j+1/2}(y)\, dy \wedge dz,$$

defines a 2-form to which we can apply the continuous Hodge operator, yielding the one form

$$\star \, {}^2\mathbf{D}^x_h(t,x,y) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(x) B^\alpha_{j+1/2}(y)\, dx.$$

We now define the image of ${}^2\mathbf{D}$ by the discrete Hodge operator as the projection of this 1-form onto the primal grid. Denoting ${}^1\mathbf{E}_h = \sum_{i,j} e^x_{i,j}(t) D^\alpha_i(x) B^\alpha_j(y)\, dx$ this image. Then we have for any $(k,l) \in [0, N_x-1] \times [0, N_y-1]$

$$\int_{x_k}^{x_{k+1}} \star \, {}^2\mathbf{D}^x_h(t,x,y_l) = \sum_{i,j} d^x_{i+1/2,j+1/2}(t) \int_{x_k}^{x_{k+1}} B^\alpha_{i+1/2}(x)\, dx \, D^\alpha_{j+1/2}(y_l) = \sum_{i,j} e^x_{i,j}(t) \int_{x_k}^{x_{k+1}} D^\alpha_i(x)\, dx \, B^\alpha_j(y_l).$$

Recalling that $\int_{x_k}^{x_{k+1}} B^\alpha_i(x)\, dx$ is the term at position (k,i) of matrix M^1_α, that $B^\alpha_j(y_l)$ is the term at position (l,j) of matrix M^0_α and denoting by S^1 the matrix whose (k,i) component is $\int_{x_k}^{x_{k+1}} B^\alpha_{i+1/2}(x)\, dx$ and by S^0 the matrix whose (l,j) component is $D^\alpha_{j+1/2}(y_l)$, the above relation can be written in matrix form

$$S^1 \mathbf{d}^x (S^0)^T = M^1_\alpha \mathbf{e}^x (M^0_\alpha)^T.$$

This defines the discrete Hodge operator mapping $\mathbf{d}$ to $\mathbf{e}$. Note that on a uniform grid all these matrices are symmetric so that the transpose can be omitted. For the y component, the D^α_i and B^α_j are interchanged so that we get

$$S^0 \mathbf{d}^y (S^1)^T = M^0_\alpha \mathbf{e}^y (M^1_\alpha)^T.$$

And the same computation gives us the discrete Hodge operator for the z component

$$S^1 \mathbf{b}^z (S^1)^T = M^0_\alpha \mathbf{h}^z (M^0_\alpha)^T.$$

5.2 Change of variables in two dimensions

Let us now consider a computational domain defined by a mapping f from a square. The change of variables $f : (r,s) \to (f_1(r,s), f_2(r,s)) = (x,y)$ is a diffeomorphism. Let us denote J the jacobian matrix of f

$$J(r,s) = \begin{pmatrix} \frac{\partial f_1}{\partial r}(r,s) & \frac{\partial f_1}{\partial s}(r,s) \\ \frac{\partial f_2}{\partial r}(r,s) & \frac{\partial f_2}{\partial s}(r,s) \end{pmatrix},$$

and $|J|$ its determinant. By definition we also have that the jacobian matrix of the inverse of f at point $(f_1(r,s), f_2(r,s))$ is the inverse of the jacobian matrix of f,

$$J^{-1}(r,s) = \begin{pmatrix} \frac{\partial f_1^{-1}}{\partial x}(f_1(r,s), f_2(r,s)) & \frac{\partial f_1^{-1}}{\partial y}(f_1(r,s), f_2(r,s)) \\ \frac{\partial f_2^{-1}}{\partial x}(f_1(r,s), f_2(r,s)) & \frac{\partial f_2^{-1}}{\partial y}(f_1(r,s), f_2(r,s)) \end{pmatrix} = \frac{1}{|J|} \begin{pmatrix} \frac{\partial f_2}{\partial s}(r,s) & -\frac{\partial f_1}{\partial s}(r,s) \\ -\frac{\partial f_2}{\partial r}(r,s) & \frac{\partial f_1}{\partial r}(r,s) \end{pmatrix}.$$

We discretize the cartesian domain parametrized by (r,s) and we will work only on this domain. The primal 2D mesh of our domain will be $r_0 < r_1 < \cdots < r_{Nr-1} < r_{Nr}$ in the r-direction and $s_0 < s_1 < \cdots < s_{Ns-1} < s_{Ns}$ in the s-direction. In $2D$, Maxwell's equations use the following differential forms

$$\begin{aligned} {}^1\mathbf{H}(x,y) &= h_z(x,y)dz, \\ {}^2\mathbf{B}(x,y) &= b_z(x,y)dx \wedge dy, \\ {}^1\mathbf{E}(x,y) &= e_x(x,y)dx + e_y(x,y)dy, \\ {}^2\mathbf{D}(x,y) &= d_x(x,y)dy \wedge dz + d_y(x,y)dz \wedge dx. \end{aligned}$$

When we do the change of coordinates, we obtain

$$\begin{aligned} f^\star {}^1\mathbf{H}(r,s) &= h_z(f_1(r,s), f_2(r,s))dz, \\ f^\star {}^2\mathbf{B}(r,s) &= b_z(f_1(r,s), f_2(r,s))|J|dr \wedge ds, \\ f^\star {}^1\mathbf{E}(r,s) &= (e_x(f_1(r,s), f_2(r,s))\frac{\partial f_1}{\partial r}(r,s) + e_y(f_1(r,s), f_2(r,s))\frac{\partial f_2}{\partial r}(r,s))dr \\ &\quad +(e_x(f_1(r,s), f_2(r,s))\frac{\partial f_1}{\partial s}(r,s) + e_y(f_1(r,s), f_2(r,s))\frac{\partial f_2}{\partial s}(r,s))ds, \\ f^\star {}^2\mathbf{D}(r,s) &= (d_x(f_1(r,s), f_2(r,s))\frac{\partial f_2}{\partial r}(r,s) - d_y(f_1(r,s), f_2(r,s))\frac{\partial f_1}{\partial r}(r,s))dr \wedge dz \\ &\quad +(d_x(f_1(r,s), f_2(r,s))\frac{\partial f_2}{\partial s}(r,s) - d_y(f_1(r,s), f_2(r,s))\frac{\partial f_1}{\partial s}(r,s))ds \wedge dz. \end{aligned}$$

Let us now express differentials forms in the new coordinates in the appropriate basis of discrete differential forms

$$f^\star {}^2\mathbf{D}_h^r(t,r,s) = \sum_{i,j} d^r_{i+1/2,j+1/2}(t)\, {}^2\tilde{\mathbf{w}}^{\alpha,r}_{i+1/2,j+1/2}(r,s), \quad f^\star {}^1\mathbf{E}_h^r(t,r,s) = \sum_{i,j} e^r_{i,j}(t)\, {}^1\mathbf{w}^{\alpha,r}_{i,j}(r,s),$$

$$f^\star {}^2\mathbf{D}_h^s(t,r,s) = \sum_{i,j} d^s_{i+1/2,j+1/2}(t)\, {}^2\tilde{\mathbf{w}}^{\alpha,s}_{i+1/2,j+1/2}(r,s) \quad f^\star {}^1\mathbf{E}_h^s(t,r,s) = \sum_{i,j} e^s_{i,j}(t)\, {}^1\mathbf{w}^{\alpha,s}_{i,j}(r,s),$$

$$f^\star {}^1\mathbf{H}_h^z(t,r,s) = \sum_{i,j} h^z_{i+1/2,j+1/2}(t)\, {}^1\tilde{\mathbf{w}}^{\alpha,z}_{i+1/2,j+1/2}(r,s), \quad f^\star {}^2\mathbf{B}_h(t,r,s) = \sum_{i,j} b^z_{i,j}(t)\, {}^2\mathbf{w}^{\alpha,z}_{i,j}(r,s).$$

Coefficients associated to splines are calculated with help of degree of freedom on the cartesian grid (r,s,z).

The exterior derivative d does not depend on the coordinate system, we have $f^\star d = df^\star$. This proprety can be verified on discrete differential forms with help of the discrete exterior derivative acting on spline coefficients. So Ampere's law (4), without current, for the first two components can be written

$$\partial_t f^{\star\,2}\mathbf{D}^r + \partial_t f^{\star\,2}\mathbf{D}^s - df^{\star\,1}\mathbf{H} = 0.$$

On the other hand, Faraday's law (5), without current, for the third component can be written

$$\partial_t f^{\star\,2}\mathbf{B}^z + df^{\star\,1}\mathbf{E} = 0.$$

Discrete Hodge operators: We must be careful since $f^\star$ does not commute with the Hodge operator, so we must define discrete Hodge operators in the new coordinate system such that the following equalities are true:

$$\begin{aligned} f^{\star\,2}\mathbf{B} &= f^\star\left(\star\,^1\mathbf{H}\right), \\ f^{\star\,1}\mathbf{E} &= f^\star\left(\star\,^2\mathbf{D}\right). \end{aligned}$$

We know $f^{\star\,1}\mathbf{H}, f^{\star\,2}\mathbf{D}$ and we want calculate the spline coeficients for $f^{\star\,2}\mathbf{B}, f^{\star\,1}\mathbf{E}$. For this, we apply the pullback $f^{-1\star}$ on $f^{\star\,1}\mathbf{H}, f^{\star\,2}\mathbf{D}$ to come back in the old variables (x, y, z) to have ${}^1\mathbf{H}$, ${}^2\mathbf{D}$ then we apply the Hodge Star operator followed by the pullback $f^\star$. After simplication, we obtain

$$\begin{aligned} f^\star\left(\star\,^1\mathbf{H}\right) &= \sum_{i,j} d^z_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s) |J(r,s)| dr \wedge ds, \\ (f^\star(\star\,^2\mathbf{D}_h))^r(t,r,s) &= \sum_{i,j} \Bigg(d^r_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(r) D^\alpha_{j+1/2}(s) \frac{(J^t J(r,s))_{1,1}}{|J(r,s)|} \\ &\qquad + d^s_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s) \frac{(J^t J(r,s))_{2,1}}{|J(r,s)|} \Bigg) dr, \\ (f^\star(\star\,^2\mathbf{D}_h))^s(t,r,s) &= \sum_{i,j} \Bigg(d^r_{i+1/2,j+1/2}(t) B^\alpha_{i+1/2}(r) D^\alpha_{j+1/2}(s) \frac{(J^t J(r,s))_{1,2}}{|J(r,s)|} \\ &\qquad + d^s_{i+1/2,j+1/2}(t) D^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s) \frac{(J^t J(r,s))_{2,2}}{|J(r,s)|} \Bigg) ds, \end{aligned}$$

where $(J^t J(r,s))_{i,j}$ is the (i,j) coefficient of the matrix $J^t J(r,s)$. We now define the image of ${}^1\mathbf{H}$ by $f^\star\star$ as the projection of this 2-form, denoting by $f^{\star\,2}\mathbf{B}_h(t,r,s) = \sum_{i,j} b^z_{i,j}(t)\,{}^2\mathbf{w}^{\alpha,z}_{i,j}(r,s)$, onto primal grid. Then we have for any $(k,l) \in [0, N_r - 1] \times [0, N_s - 1]$

$$\begin{aligned} \int_{r_k}^{r_{k+1}} \int_{s_l}^{s_{l+1}} f^\star(\star\,^1\mathbf{H}_h)(t,r,s) &= \sum_{i,j} d^z_{i+1/2,j+1/2}(t) \int_{r_k}^{r_{k+1}} \int_{s_l}^{s_{l+1}} B^\alpha_{i+1/2}(r) B^\alpha_{j+1/2}(s) |J(r,s)| dr \wedge ds \\ &= \int_{r_k}^{r_{k+1}} \int_{s_l}^{s_{l+1}} f^{\star\,2}\mathbf{B}_h(t,r,s) = \sum_{i,j} b^z_{i,j}(t) \int_{r_k}^{r_{k+1}} \int_{s_l}^{s_{l+1}} {}^2\mathbf{w}^{\alpha,z}_{i,j}(r,s). \end{aligned}$$

Also, we denote by $f^{\star\,1}\mathbf{E}_h = f^{\star\,1}\mathbf{E}_h^r(t,r,s)+f^{\star\,1}\mathbf{E}_h^s(t,r,s) = \sum_{i,j} e_{i,j}^r(t)\,{}^1\mathbf{w}_{i,j}^{\alpha,r}(r,s)+\sum_{i,j} e_{i,j}^s(t)\,{}^1\mathbf{w}_{i,j}^{\alpha,s}(r,s)$ the projection of ${}^2\mathbf{D}$ and we have

$$\begin{aligned}
\int_{r_k}^{r_{k+1}} (f^\star \star {}^2\mathbf{D}_h)^r(t,r,s_l) &= \sum_{i,j} d_{i+1/2,j+1/2}^r(t) D_{j+1/2}^\alpha(s_l) \int_{r_k}^{r_{k+1}} \left(B_{i+1/2}^\alpha(r) \frac{(J^tJ)_{1,1}(r,s_l)}{|J(r,s_l)|} \right) dr \\
&\quad + d_{i+1/2,j+1/2}^s(t) B_{j+1/2}^\alpha(s_l) \int_{r_k}^{r_{k+1}} \left(D_{i+1/2}^\alpha(r) \frac{(J^tJ)_{2,1}(r,s_l)}{|J(r,s_l)|} \right) dr \\
&= \int_{r_k}^{r_{k+1}} f^{\star\,1}\mathbf{E}_h^r(t,r,s_l) = \sum_{i,j} e_{i,j}^r(t) \int_{r_k}^{r_{k+1}} {}^1\mathbf{w}_{i,j}^{\alpha,r}(r,s_l), \\
\int_{s_l}^{s_{l+1}} (f^\star(\star\,{}^2\mathbf{D}_h))^s(t,r_k,s) &= \sum_{i,j} d_{i+1/2,j+1/2}^r(t) B_{i+1/2}^\alpha(r_k) \int_{s_l}^{s_{l+1}} \left(D_{j+1/2}^\alpha(s) \frac{(J^tJ)_{1,2}(r_k,s)}{|J(r_k,s)|} \right) ds \\
&\quad + d_{i+1/2,j+1/2}^s(t) D_{i+1/2}^\alpha(r_k) \int_{s_l}^{s_{l+1}} \left(B_{j+1/2}^\alpha(s) \frac{(J^tJ)_{2,2}(r_k,s)}{|J(r_k,s)|} \right) ds \\
&= \int_{s_l}^{s_{l+1}} f^{\star\,1}\mathbf{E}_h^s(t,r_k,s) = \sum_{i,j} e_{i,j}^s(t) \int_{s_l}^{s_{l+1}} {}^1\mathbf{w}_{i,j}^{\alpha,s}(r_k,s).
\end{aligned}$$

We proceed in the same way if we must solve $f^{\star\,1}\mathbf{H} = f^\star\left(\star\,{}^2\mathbf{B}\right)$ knowing $f^{\star\,2}\mathbf{B}$. We apply the pullback $f^{-1\star}$ on $f^{\star\,2}\mathbf{B}$ to come back in the old variables (x,y,z) to have ${}^2\mathbf{B}$ then we apply the Hodge Star operator followed by the pullback $f^\star$. After simplification, we obtain

$$f^\star\left(\star\,{}^2\mathbf{B}\right) = \sum_{i,j} b_{i,j}^z(t) D_i^\alpha(r) D_j^\alpha(s) \frac{1}{|J(r,s)|} dz.$$

We denote $f^{\star\,1}\mathbf{H}_h^z(t,r,s) = \sum_{i,j} h_{i+1/2,j+1/2}^z(t)\,{}^1\tilde{\mathbf{w}}_{i+1/2,j+1/2}^{\alpha,z}(r,s)$ the projection onto the dual grid and so we obtain

$$\begin{aligned}
\int_0^1 f^\star \star {}^2\mathbf{B}(t,r_{i+1/2},s_{j+1/2}) &= \sum_{i,j} b_{i,j}^z(t) \frac{D_i^\alpha(r_{i+1/2}) D_j^\alpha(s_{j+1/2})}{|J(r_{i+1/2},s_{j+1/2})|} \\
= f^{\star\,1}\mathbf{H}_h^z(t,r_{i+1/2},s_{j+1/2}) &= \sum_{i,j} h_{i+1/2,j+1/2}^z(t)\,{}^1\tilde{\mathbf{w}}_{i+1/2,j+1/2}^{\alpha,z}(r_{i+1/2},s_{j+1/2}).
\end{aligned}$$

6 Numerical results

6.1 Test case in $2D$ with periodic boundary conditions

In 2 dimension with periodic boundary condition, let us consider the following solution of Maxwell's equations: the electric field, a 1-form, is given by ${}^1\mathbf{E} = -k_y \sin(k_x\,x + k_y\,y - \omega\,t)dx - k_x \sin(k_x\,x + k_y\,y - \omega\,t)dy$, and the magnetic field, a 2- form, is given by ${}^2\mathbf{B} = \omega\,\cos(k_x\,x + k_y\,y - \omega\,t)dx\,dy$. Then, with help of the Hodge star operator we obtain the formula for the electric displacement field: ${}^1\mathbf{D} = -k_y \sin(k_x\,x + k_y\,y - \omega\,t)dy - k_x \sin(k_x\,x + k_y\,y - \omega\,t)dx$ and for the magnetizing field: ${}^0\mathbf{H} = \omega\,\cos(k_x\,x + k_y\,y - \omega\,t)$. Constants are given by $k_x = \frac{2\,\pi}{L_x}, k_y = \frac{2\,\pi}{L_y}$, where L_x and L_y are the length of our domain in x and y respectively and $\omega = \sqrt{k_x^2 + k_y^2}$. We test our code with $L_x = L_y = 1$ and with a time scheme of order 4 and we observe that the order of our scheme is given by the spline order.

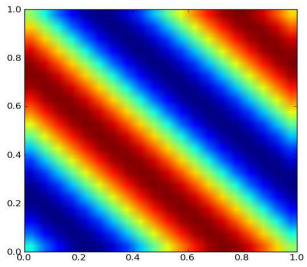


Figure 1: The magnetizing field H in $2D$ with periodic boundary conditions.

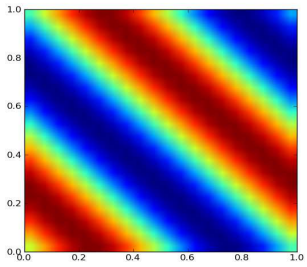


Figure 2: The first component of the electric displacement field Dx in $2D$ with periodic boundary conditions.

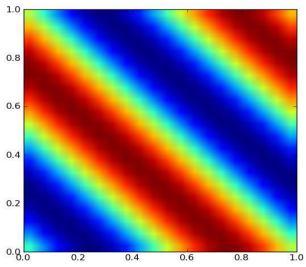


Figure 3: The second component of the electric displacement field Dy in $2D$ with periodic boundary conditions.

spline degree $\alpha = 1$		
Number of points	L^2 errors for D_x	conv. order in D_x
10	0.963466687612	
20	0.244961611114	1.97567910882
40	0.0617111800439	1.98895188936
80	0.015439432949	1.99891211506

spline degree $\alpha = 3$		
Number of points	L^2 errors for D_x	conv. order in D_x
10	0.0346451065418	
20	0.00221920090229	3.96453940841
40	0.000139584011258	3.99083467768
80	8.73723793614e-06	3.99781260726

6.2 Test case in $1D$ with perfect electric conductor boundary conditions

In one dimension with perfect electric conductor boundary conditions, we have a solution where the electric field, a 0-form, is given by ${}^0\mathbf{E} = \frac{k}{\omega}\sin(k\,x)\,\cos(\omega\,t)$, and the magnetic field, a 1- form, has the form, ${}^1\mathbf{B} = -\cos(k_x\,x)\,\sin(\omega\,t)dx$. Then, with help of the Hodge star operator we obtain the formula for the electric displacement field: ${}^1\mathbf{D} = \frac{k}{\omega}\sin(k\,x)\,\cos(\omega\,t)dx$ and for the magnetizing field : ${}^0\mathbf{H} = -\cos(k_x\,x)\,\sin(\omega\,t)$. Constants are given by $k = \frac{2\pi}{L}$, where L are the length of our domain in x and $\omega = |k_x|$. We test our code with $L_x = 1$ and with a time scheme of order 4 and we observe that the order of our scheme is given by the spline order.

spline degree $\alpha = 1$		
Number of points	L^2 errors for H	conv. order in H
10	0.14405968937	
20	0.0365895785828	1.97716199381
40	0.0091869581337	1.99377364042
80	0.00229392316474	2.00177018523

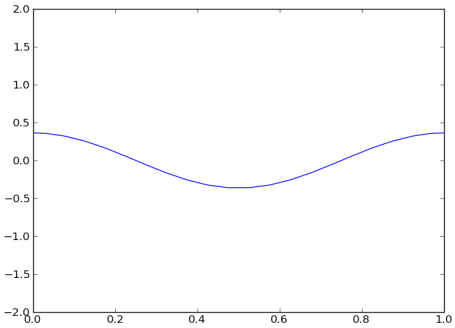


Figure 4: The magnetizing field H in $1D$ with perfect electric conductor boundary conditions.

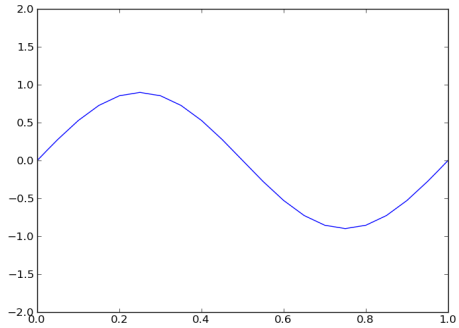


Figure 5: The electric field D in $1D$ with perfect electric conductor boundary conditions.

spline degree$\alpha = 3$		
Number of points	L^2 errors for H	conv. order in H
10	0.00291782099873	
20	0.000191704646607	3.92793416901
40	1.22331803519e-05	3.97001387997
80	7.61847256487e-07	4.00515393133

We are going up to time $T = 2$. These convergence order are obtain with a time step $\Delta t = \Delta x/0.5$ when we use B-spline of degree 1 and $\Delta t = \Delta x/3.5$ when we use B-spline of degree 3.

6.3 Test case in $2D$ with a change of variables and with perfect electric conductor boundary conditions

We consider a physical domain which is a ring with perfect electric conductor boundary conditions. We have an exact solution where the electric field, a 1-form, is given by

$$^1\mathbf{E} = -\cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r)d\theta + \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J'_{2\,k}(\omega\,r)dr,$$

and the magnetic field, a 2- form, has the form,

$$^2\mathbf{B} = \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r)dr\,d\theta,$$

where $J_{2\,k}$ is the Bessel's function of the first kind with order $2\,k$ and $J'_{2\,k}$ her derivative. With the help of the Hodge star operator, we obtain the formula for electric displacement field:

$$^1\mathbf{D} = -\cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r)dr + \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J'_{2\,k}(\omega\,r)d\theta,$$

and for magnetizing field:

$$^0\mathbf{H} = \cos(2\,k\,\theta)\ (A\,\cos(\omega\,t) + B\,\sin(\omega\,t))\ J_{2\,k}(\omega\,r).$$

The intervals on which we have boundary conditions on perfect conductors are $(r, \theta) \in [\gamma_1, \gamma_2] \times [0, \frac{\pi}{2}]$ where γ_1, γ_2 are the first and second zero of Bessel's function of the first kind with order $2k$.
In this case the change of variables is nothing other than change of coordinates into polar coordinates.

We test our code with a time scheme of order 4 and we observe that the order of our scheme is given by spline's order as expected.

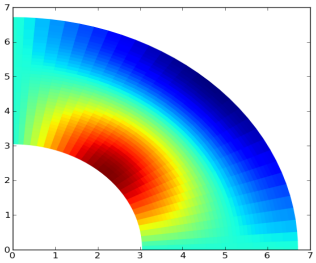


Figure 6: The first component of the electric displacement field Dx in physic domain with boundary conditions on perfect conductors.

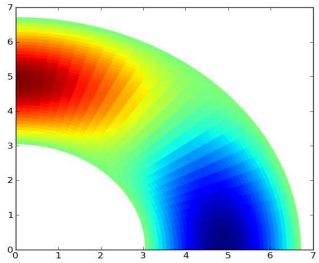


Figure 7: The second component of the electric displacement field Dy in physic domain with boundary conditions on perfect conductors.

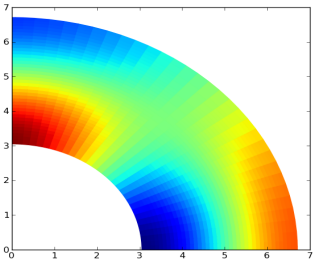


Figure 8: The magnetizing field H in physic domain with boundary conditions on perfect conductors.

spline degree $\alpha = 1$		
Number of points	Errors L^2 for H	conv. order in H
10	0.00513786236267	
20	0.00238054129513	1.67627824073
40	0.000947723465878	1.81682625586
80	0.00035447325496	1.85997897801

spline degree $\alpha = 3$		
Number of points	Errors L^2 for H	conv. order in H
10	0.000245195436118	
20	1.91520985487e-05	3.67835774002
40	1.4651702633e-06	3.7083622499
80	9.27738247413e-08	3.9812066957

We are going up to time $T = 1$ for order 1 and $T = 2$ for order 3. These convergence orders are obtained with a time step $\Delta t = \min(\Delta_x, \Delta_y)/0.5$ when we use B-splines of degree 1 and $\Delta t = \min(\Delta_x, \Delta_y)/3.5$ when we use B-splines of degree 3.

References

[1] R. Hiptmair, Finite elements in computational electromagnetism. Acta Numerica 11 (2002), 237–339.

[2] R. Hiptmair, Discrete Hodge operators. Numer. Math. 90 (2001), 265–289.

[3] T. Tarhasaari, L. Kettunen and A. Bossavit. Some realizations of a discrete Hodge: A reinterpretation of finite element techniques, IEEE Trans. Magnetics 35 (1999), 1494–1497.

[4] A. Bossavit, Computational electromagnetism, Academic Press (Boston), 1998.

[5] A. Bossavit, Generating Whitney Forms of Polynomial Degree One and High, IEEE Trans. on Magnetics (2002), 341–344.

[6] F. Rapetti and A. Bossavit, Whitney forms of higher degree, SIAM J. Numer. Anal. 47 no. 3 (2009), 2369–2386.

[7] A. Buffa and G. Sangalli and R. Vazquez, Isogeometric analysis in electromagnetics: B-splines approximation, Comput. Methods Appl. Mech. Engrg. 199 (2010), no. 17-20, 1143–1152

[8] C. de Boor, A practical guide to splines, Revised edition. Applied Mathematical Sciences, 27. Springer-Verlag, New York, 2001.

[9] M. Desbrun, E. Kanso and Y. Tong , Discrete differential forms for computational modeling, Oberwolfach Semin. (2008), 287–324.

[10] M. Desbrun, M. Leok and J.E. Marsden, Discrete Poincaré lemma, Appl. Numer. Math. 53 no. 2 (2005), 231–248.

[11] Douglas N. Arnold, Richard S. Falk, Ragnar Winther, Finite element exterior calculus, homological techniques, and applications. Acta Numerica 15 (2006), 1–155.

[12] B. He and F. L. Teixeira, Geometric finite element discretization of Maxwell equations in primal and dual spaces, Physics Letters. A 349, no. 1-4 (2006), 1–14.

[13] S.R. Weller, W. Moran, B. Ninness and A.D. Pollington, Sampling zeros and the Euler-Frobenius polynomials, IEEE Trans. Autom. Control 46 no. 2 (2001), 340–343.

[14] A. Ratnani and E. Sonnendrücker, Arbitrary High-Order Spline Finite Element Solver for the Time Domain Maxwell equations, `http://hal.archives-ouvertes.fr/hal-00507758/fr, 2010`.

[15] V.I. Arnold, Mathematical methods of classical mechanics, Graduate Texts in Mathematics, Springer-Verlag, New York, 1976.

[16] J.P. Bowen, Hypercubes, Practical Computing, 5 no. 4 (1982), 97–99.

[17] H. S. M. Coxeter, The Beauty of Geometry: Twelve Essays, Dover Publications 1999.