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Infinite Feigenbaum Sequences and Spirals in the Vicinity of the Lagrangian Periodic Solutions

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Abstract

We studied systematically cases of the families of non-symmetric periodic orbits in the planar restricted three-body problem. We took interesting information about the evolution, stability and termination of bifurcating families of various multiplicities. We found that the main families of simple non-symmetric periodic orbits present a similar dynamical structure and bifurcation pattern. As the Jacobi constant changes each branch of the characteristic of a main family spirals around a focal point-terminating point in x - at which the Jacobi constant is $C_\infty=3$ and their periodic orbits terminate at the corotation (at the Lagrangian point L_4 or L_5). As the family approaches asymptotically its termination point infinite changes of stability to instability and vice versa occur along its characteristic. Thus, infinite bifurcation points appear and each one of them produces infinite inverse Feigenbaum sequences. That is, every bifurcating family of a Feigenbaum sequence produces the same phenomenon and so on. Therefore, infinite spiral characteristics appear and each one of them generates infinite new inner spirals and so on. Each member of these infinite sets of the spirals reproduces a basic bifurcation pattern. Therefore, we have in general large unstable regions that generate large chaotic regions near the corotation points L_4 , L_5 , which are unstable. As C varies along the spiral characteristic of every bifurcating family, which approaches its focal point, infinite loops, one inside the other, surrounding the unstable triangular points L_4 or L_5 are formed on their orbits. So, each terminating point corresponds to an asymptotic non-symmetric periodic orbit that spirals into the corotation points L_4 , L_5 with infinite period. This is a new mechanism that produces very large degree of stochasticity. These conclusions help us to comprehend better the motions around the points L_4 and L_5 of Lagrange.

Key words: celestial mechanics, restricted three-body problem, chaos, Lagrange points L_4 , L_5

1. Introduction

In general, the stable periodic orbits are followed by many quasi periodic orbits, while the unstable periodic orbits are followed by large numbers of chaotic orbits. There are some routes to chaos in dissipative and conservative dynamical systems. The well-known Feigenbaum sequence of infinite period doubling bifurcations is a particular route to chaos in dissipative and conservative dynamical systems (Feigenbaum 1978; Couillet and Tresser, 1978). Some other routes for the transition from order to chaos in conservative dynamical systems of 2-degrees of freedom have been found by Contopoulos and Zikides (1980), Contopoulos (1983a,b, 1991), Heggie (1983), Bier and Bountis (1984) and Pinotsis (1988). These scenarios of routes to chaos are mentioned by Contopoulos (1993, 2002). Also, some of the routes to chaos in dissipative and conservative systems we find in an article by Eckmann (1981).

It is well known, that in the planar restricted 3-body problem Feigenbaum sequences of families of simple and double symmetric periodic orbits appear, as the Jacobi constant C varies, with a universal ratio consistent with the value $\delta \approx 8.72$. Bifurcated families have in general large unstable regions and generate large chaotic regions (Contopoulos and Pinotsis 1984, Pinotsis 1986, 1987).

Although there exist many studies of the types of bifurcations for the families of symmetric periodic orbits in various Hamiltonian systems of two degrees of freedom, there are limited explorations of the stability and bifurcations of families of non-symmetric periodic orbits. Thus, our study centers on the changes of stability and bifurcations of the families of non-symmetric periodic orbits, which bifurcate from the original families b , c , l and n .

The stability of motions in the neighborhood of the Lagrangian points L_4 and L_5 has been investigated for some values of the mass ratio μ by a number of investigators (Pinotsis 1988, Contopoulos 1991, Sandor et. al. 2000, Bardin 2002, Henrard and Navarro 2004, Efthymiopoulos 2005, Erdi et. al. 2009). Also, Sicardy (2010) examined the stability of these points for some values of μ beyond the Gascheau's value in the planar restricted three-body problem and found that the periodic orbits follow a Feigenbaum cascade leading to chaos at a value of μ_∞ .

We made a preliminary study for the non-symmetric short period orbits (SPO) of the family b (Pinotsis 1988) and we found infinite spiral characteristics around infinite focal points. Also, spiral characteristics of families of periodic solutions have been found in the case of a satellite oscillations (Bruno, A.D. and Varin, V.P., 1997; Bruno, A.D., 2002). In this paper we make a systematic study of the families of non-symmetric periodic orbits that bifurcate at the points b_1 , c_1 , l_2 , $n_{2,4}$ (Hénon 1965) and z_1 (Pinotsis 1986), in the planar circular restricted 3-body problem and in the case of equal masses of the primaries. We found interesting information about the evolution and termination of the bifurcating families of various multiplicities of non-symmetric periodic orbits. The relevant topological characteristics change not only quantitatively but also qualitatively. We found a new mechanism that introduces a large degree of stochasticity, which could not be predicted a priori. Thus, we found a new route for the transition from order to chaos. This work required very large computing time because as the Jacobi constant C varies infinite successive stable and unstable segments along the characteristic of each bifurcating family appear, while each bifurcation point generates infinite inverse Feigenbaum sequences. So, each bifurcating family forms infinite spirals and each one of them generates infinite new inner spirals and so on. Therefore, we have in general large unstable regions that generate large chaotic regions near the corotation points L_4 , L_5 and so these points are unstable.

2. The method

For the planar restricted 3-body problem the equations of motion are (Szebehely 1967)

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y} \quad (1)$$

The potential function Ω is given by

$$\Omega = \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}((1-\mu)r_1^2 + \mu r_2^2) \quad (2)$$

and $r_1^2=(x-\mu)^2+y^2$, $r_2^2=(x-\mu+1)^2+y^2$

The Jacobian integral of equations (1) is

$$C = 2\Omega - (\dot{x}^2 + \dot{y}^2) \quad (3)$$

where C is the so-called Jacobian constant. We study non-symmetric periodic solutions with respect to the x -axis (the axis joining the two primaries), in the physical plane (x,y) , for the Copenhagen case ($\mu=0.5$). If we vary the Jacobi constant C , the values of x , \dot{x} and \dot{y} vary continuously and the characteristic of the family is a curve in the 3-dimensional space (C, x, \dot{x}) , since the value of \dot{y} can be calculated from the Jacobi integral. For the numerical calculations, in double precision, we used the recurrent power series method with variable step in time.

In order to find non-symmetric periodic orbits we applied essentially a Newton method. A similar method has been applied already in 3-dimensional Hamiltonian systems (Magenat 1982). In order to calculate a periodic orbit approximate, initial conditions $(x_0, y_0=0, \dot{x}_0, \dot{y}_0)$ are given. Each orbit intersects again the $y=0$ axis with \dot{y} in the same direction as \dot{y}_0 and at the point x, \dot{y}, \dot{x} . If we consider x, \dot{x} as functions f_i ($i=1, 2$) of initial values of x_0, \dot{x}_0 (we can find \dot{y}_0 from the Jacobi constant) then,

$$x=f_1(x_0, \dot{x}_0), \quad \dot{x}=f_2(x_0, \dot{x}_0) \quad (4)$$

In order to find a periodic orbit we give a small change of the initial conditions, $\Delta x_0, \Delta \dot{x}_0$ which implies a change of the final point $\Delta x, \Delta \dot{x}$. In order for the orbit to be periodic we must have,

$$\begin{aligned} x_0+\Delta x_0 &= f_1(x_0+\Delta x_0, \dot{x}_0+\Delta \dot{x}_0) \\ \dot{x}_0+\Delta \dot{x}_0 &= f_2(x_0+\Delta x_0, \dot{x}_0+\Delta \dot{x}_0) \end{aligned} \quad (5)$$

Neglecting terms of order higher than one in Taylor's expansion, in matrix form equations (5) can be written

$$\begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} + \begin{pmatrix} \Delta x_0 \\ \Delta \dot{x}_0 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \Delta x_0 \\ \Delta \dot{x}_0 \end{pmatrix} \quad (6)$$

or

$$\begin{pmatrix} a_1 - 1 & a_2 \\ a_3 & a_4 - 1 \end{pmatrix} \begin{pmatrix} \Delta x_0 \\ \Delta \dot{x}_0 \end{pmatrix} = \begin{pmatrix} x_0 - x \\ \dot{x}_0 - \dot{x} \end{pmatrix} \quad (7)$$

where $\frac{\partial f_1}{\partial x_0}=a_1$, $\frac{\partial f_1}{\partial \dot{x}_0}=a_2$, $\frac{\partial f_2}{\partial x_0}=a_3$, $\frac{\partial f_2}{\partial \dot{x}_0}=a_4$

The quantities a_1, a_2, a_3, a_4 can be found by calculating two orbits near the original one (i.e. we need to compute 3 orbits) with deviations $(\Delta x_0, 0)$ and $(0, \Delta \dot{x}_0)$. In order to find approximate initial conditions for different bifurcating families at the bifurcation points we used a method of trial and error. Since we neglected higher order terms in Taylor's expansion the solution of the

system (7) does not give exactly the periodic orbit. Adding the increments $\Delta x_0, \Delta \dot{x}_0$, given by equation (7), to x_0, \dot{x}_0 we obtain better initial conditions for the next iteration. This iterative process has to be repeated a number of times in order to give a periodic orbit with the required accuracy.

The stability of the families of non-symmetric periodic orbits can be studied in general by evaluating the Hénon (1965) stability parameters a, b, c, d or by evaluating the monodromy matrix of the variational equations (Hadjidemetriou 1989). In this work the stability of the families is studied by evaluating the stability parameters which are functions of the initial values x_0, \dot{x}_0 and the Jacobian C (Hénon 1965). For a non-symmetric periodic orbit the stability condition is

$$|A| = \left| \frac{a + d}{2} \right| < 1 \quad (8)$$

3. Infinite sets of spirals and self-similar orbits

We consider the original families of simple symmetric periodic orbits b, l and c . At the previously mentioned bifurcation points -critical points- with initial conditions: **b1** (C_{in}, x_{up}, \dot{x} , for the upper branch and x_l for the lower one) (Table I), **l2** ($C_{in}, x_{up}, \dot{x}, x_l$) (Table IV) and **c1** ($C_{in}, x_{up}, \dot{x}, x_l$) (Table VI) bifurcate new families of simple non-symmetric periodic orbits of the same period with their corresponding symmetric periodic orbits (we call these families *main families fb1, fl2 and fc1* respectively) (Figures 1, 4, 7).

Each main family of the above three cases and each bifurcating families of various multiplicities presents the Feigenbaum scenario of infinite period doubling pitchfork bifurcations and each bifurcation ratio tends to approach the universal number $\delta \approx 8.72$ at a finite value of the Jacobian constant, which we have already found for the families of symmetric periodic orbits (Contopoulos and Pinotsis 1984, Pinotsis 1986, 1987).

Along the characteristic curve of each main family, as the serial number of its bifurcation points increases the distances between two neighbouring characteristics of the bifurcating families become smaller and smaller. So, it is more and more difficult to separate the bifurcating families of different multiplicity. Also, they are strongly unstable and the instability of the families becomes larger and larger from one bifurcating family to the next one. Therefore more accuracy is necessary to study them, that is, a new magnification of the space ($x-C$ diagram) is needed and so on. Thus, it is difficult to follow further some of these families in order to find their termination point because of the very large computing time needed.

As we mention below in these cases we took many interesting information about the evolution and termination of the bifurcating families of various multiplicities of periodic orbits.

Also, we studied the main family of double non-symmetric periodic orbits, fz_1 , which bifurcates at the critical point z_1 (C_{in}, x, \dot{x}) (Table VIII). We found here the Feigenbaum phenomenon. We followed the evolution of this main family of double non-symmetric periodic orbits beyond its first $2 \rightarrow 4$ transition point. The family becomes more and more unstable, as C increases, while the period of the orbits continuously increases. We did not follow further this family because of the very large computing time needed. We stopped the numerical calculations at the point (C_{st}, x, \dot{x}).

As regards the main family of double non-symmetric periodic orbits, fn_2A , which bifurcates at the critical point n_2A , it is unstable and becomes more and more unstable as the Jacobi

constant C varies. So, it does not appear that it has any bifurcation of other families of periodic orbits. In Tables (I-VIII) we give some of the bifurcation points of all these families. The tables "are published as attachments to the online version of this article".

3.1 Evolution and stability of the main family $fb1$

I. Simple periodic orbits

The orbits of the main family $fb1$ are the short period orbits (SPO). The second branch of its characteristic is outside the Figure 1. As the Jacobi constant increases the stability parameter decreases and at the $1 \rightarrow 2$ point (C_1, x_1, \dot{x}) (Tables I, II, III) (first bifurcation point) bifurcates a new stable family of double non-symmetric periodic orbits (we call this family $fbd1$) (Table II). This new family, as C increases, it becomes critically unstable at the $2 \rightarrow 4$ transition point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table III). Here we have the Feigenbaum scenario. The characteristic of each bifurcating family has two branches which extend towards larger values of C (Figure 1).

Looking for possible inverse Feigenbaum sequences, we followed the evolution of the main family $fb1$. At first, as C increases the characteristic of the family extends to the right and downwards (the absolute value of x decreases) in the x - C diagram. At the $1 \rightarrow 2$ bifurcation point (C_2, x_2, \dot{x}) (Tables I, III) -second bifurcation point-a new stable family of non-symmetric double periodic orbits- we call this family $fbd2$ - with two branches towards smaller values of the Jacobi constant appears. As C decreases the new stable family $fbd2$ becomes unstable and so on. Here the Feigenbaum scenario appears.

As C increases beyond the second bifurcation point, the characteristic of the main family continues to the right and downwards while the family is stable until the first turning point $(C_{t1}, x_{t1}, \dot{x})$ and the family has a maximum C . Following the evolution of the main family beyond the first turning point the characteristic of the family has an initial part with C decreasing for decreasing absolute value of x . As C decreases further the characteristic turns upwards in x . At the point $(C_{t2}, x_{t2}, \dot{x})$, second turning point, the family becomes critically stable again and the characteristic has a minimum C . After the second turning point, for C increasing the absolute value of x increases and the family has a small stable segment until the third $1 \rightarrow 2$ critical point (C_3, x_3, \dot{x}) at which it bifurcates a new stable family of non-symmetric double periodic orbits with two branches towards larger values of C . This family loses its stability at the $2 \rightarrow 4$ bifurcation point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table III) and so on. A cascade of Feigenbaum bifurcations is generated here.

After the third bifurcation point, for C increasing, the main family becomes more and more unstable but after a while the family becomes stable again at the fourth $1 \rightarrow 2$ bifurcation point (C_4, x_4, \dot{x}) . At this point it bifurcates a new stable family of non-symmetric double periodic orbits with two branches towards smaller values of C . At this point the Feigenbaum scenario appears.

After the fourth bifurcation point the main family is stable until the third turning point $(C_{t3}, x_{t3}, \dot{x})$ at which it has a maximum C and so on (Figure 1). The same phenomenon occurs in the other (lower) branch of the main family. We stopped the numerical calculations at the point C_{st}, x_{st}, \dot{x} (Table I) after the fifth bifurcation point $(C_{5,}, x_{5,}, \dot{x})$ at which we found the Feigenbaum scenario.

As C changes along the spiral characteristic, which approaches asymptotically its focal point-terminating point, infinite changes from stability to instability and vice versa appear, while each bifurcation point generates infinite inverse Feigenbaum sequences. That is, each

branch of a Feigenbaum sequence presents the same phenomenon and so on. Therefore there exist infinite spirals and each one of them produces infinite new inner spirals and so on. After infinite successive small stable and large unstable segments the characteristic spirals inwards around its terminating point. Each branch of the main family spirals into its focal point $C_\infty = 3$, $x_{up} \approx -1.624989$ (or $x_l \approx -0.684635$ for the lower branch). The values of x (for each branch of a family) were calculated as the average of the extreme –closest to the focal point– values of the corresponding final spiral, which was found numerically. Contopoulos (1991) investigated periodic orbits of a galactic type potential and explained how the spirals are generated as soon as the Lagrangian points L_4, L_5 become unstable.

Figure 2 shows a set of some simple periodic orbits. We found that near the first turning point a cusp appears close to the Lagrangian point L_4 , which turns to loop in the second turning point. As C varies, the size of the loop of the orbit grows and encloses the triangular libration point L_4 . As C varies further, a new inner cusp appears which evolves to a new inner loop, near the fourth turning point, surrounding the triangular point L_4 and so on. As we approach the terminating point infinite loops, one inside the other, emanate, surrounding the Lagrangian point L_4 . Thus, an infinite set of self-similar orbits is constructed, each member of this set reproducing a basic pattern. The orbits spiral around the triangular point L_4 . So, the terminating point corresponds to an asymptotic non-symmetric periodic orbit spiraling into the point L_4 , which has the same Jacobi constant as the Lagrangian point L_4 ($C_\infty = 3$). In this limiting point we have an infinite self-similar orbit with infinite period and so the termination principle $T \rightarrow \infty$ is satisfied (Szebehely 1967).

II. Families of double periodic orbits fbd1 and fbd2

We followed the evolution of the family *fbd1* beyond the $2 \rightarrow 4$ transition point (first bifurcation point of the family *fbd1*) (Tables II, III and Figure 3a). As C increases, the two branches of the characteristic of the family *fbd1* extend to the right in the x - C diagram. The upper branch initially goes upwards, with decreasing period and then turns downwards while the period continuously increases. The lower branch goes downwards. These branches are outside and far from the characteristic of the main family (Figure 1). As C increases further the family becomes critically stable again at the $2 \rightarrow 4$ bifurcation point, second bifurcation point, ($C_2, x_l, \dot{x}, x_{up}$) (Table II) a new stable family of quadruple non-symmetric periodic orbits with two branches towards smaller C is generated and so on. Here the Feigenbaum scenario appears.

After its second bifurcation point, the family *fbd1* has a stable segment and at its first turning point ($C_{t1}, x_l, \dot{x}, x_{up}$) the family has a maximum C . After this point C decreases with the absolute value of x decreasing for the upper branch and increasing for the lower one (Figure 1). At the second turning point ($C_{t2}, x_l, \dot{x}, x_{up}$) the family has a minimum C . After this point the family has a stable segment until the third $2 \rightarrow 4$ bifurcation point ($C_3, x_l, \dot{x}, x_{up}$) and so on. At this point a new stable family with two branches towards larger values of C emanates, giving birth to a Feigenbaum sequence. We stopped at the point ($C_{st}, x_l, \dot{x}, x_{up}$) (Table II, Figure 3c) which is after the fourth $2 \rightarrow 4$ bifurcation point.

In this case a similar bifurcation pattern as in the case of the main family of simple periodic orbits *fb1* appears. The two branches of the family *fbd1*, which pass outside the spirals of the main family, spiral around their focal points with $C_\infty = 3$, $x_{up} \approx -1.6558$ and $x_l \approx -1.5038$. As the spiral characteristic approaches asymptotically its focal point infinite changes from stability to instability and vice versa appear while from each bifurcation point infinite inverse Feigenbaum sequences are generated. Each branch has infinite successive small stable and large unstable

segments. Every bifurcating family of a Feigenbaum sequence produces the same phenomenon and so on. Therefore, infinite spiral characteristics appear and each one of them generates infinite new inner spirals and so on. Each member of these infinite sets of the spirals reproduces a basic bifurcation pattern. Therefore, we have in general large unstable regions that generate large chaotic regions near the corotation point L_4 . We have here self-similar orbits, which grow loops, as C varies. These loops surround the Langrangian point L_4 . The terminating point corresponds to an infinite self-similar asymptotic orbit with $C_\infty=3$, while its period is infinite.

We also followed the evolution of the family *fbd2* (Table III, Figure 1). As C decreases the two branches of the characteristic curve of the family extend to the left in the x - C diagram, while its period continuously increases. This family becomes critically unstable at the $2 \rightarrow 4$ bifurcation point $(C_{2 \rightarrow 4}, x_l, \dot{x}, x_{up})$ (Figure 3d). We have here the Feigenbaum sequences (Figure 1).

As C decreases, the family *fbd2* becomes more and more unstable and its two branches continue further to the left and upwards in the x - C diagram. The further study of this family is very difficult because of the very large degree of stochasticity and so the very large computing time needed to extend it to its terminating point. We stopped the numerical calculations at the point $(C_{st}, x_l, \dot{x}, x_{up})$ (Table III, Figure 3e).

The two branches of this family are close to the characteristic of the main family *fb1* and inside of the two branches of the previous family *fbd1* (Figure 1). Also, the shapes and the initial values of x of the orbits (b) and (e) of Figure 3, that belong to the two families *fbd1* and *fbd2* respectively and they have the same value of Jacobi constant C , are different. So, the family *fbd2* can not join the family *fbd1* in order to form a bubble (Contopoulos 1983a). Thus, it is topologically necessary that the evolution of the family *fbd2* should present a similar bifurcation pattern and dynamical structure of spirals as the previous mentioned cases of the families of simple and double periodic orbits *fb1* and *fbd1*. So, each branch of the characteristic of the family *fbd2* spirals into its focal point with $C_\infty=3$. The terminating point corresponds to the initial conditions of a limiting asymptotic non-symmetric orbit spiraling into the Langrangian point L_4 . These results help us to comprehend the motions around the point L_4 of Lagrange.

3.2 Evolution and stability of the main family *fl2*

1. Simple periodic orbits

The second branch of the characteristic curve does not appear in Figure 4. As C decreases, at the $1 \rightarrow 2$ critical point -first bifurcation point- (C_1, x_1, \dot{x}) (Tables IV, V) bifurcates a new stable family of non-symmetric double periodic orbits-family *fld1*. After that, for smaller values of C , the family *fld1* loses its stability at the $2 \rightarrow 4$ bifurcation point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table V) and so on (Figure 4). The Feigenbaum scenario appears.

We followed the main family after its first bifurcation point. As C decreases, the characteristic of the family continues to the left and upwards (the absolute value of x increases) in the x - C diagram while the period continuously increases (Figure 4). The family becomes critically stable at the second bifurcation point (C_2, x_2, \dot{x}) (Tables IV). At this $1 \rightarrow 2$ transition point a new stable family of double non-symmetric periodic orbits-family *fld2*-with two branches towards larger values of C is generated. For somewhat larger C the new family *fld2* also becomes unstable at the $2 \rightarrow 4$ bifurcation point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table V, Figure 6d) and so on. Thus, we have here the Feigenbaum scenario (Figure 4).

As C decreases the main family is stable until the first turning point $(C_{t1}, x_{t1}, \dot{x})$ at which it has minimum C . After this point, the characteristic of the family turns to the right and upwards. As C increases further the absolute value of x decreases and the family becomes critically stable again at the second turning point $(C_{t2}, x_{t2}, \dot{x})$ and it has maximum C . With C decreasing the family has a stable segment until the third $1 \rightarrow 2$ bifurcation point (C_3, x_3, \dot{x}) (Tables IV,V). At this point a new stable family of double non-symmetric periodic orbits emanates, while its two branches are towards smaller value of C . The new family becomes critically unstable at the $2 \rightarrow 4$ transition point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table V) and so a Feigenbaum scenario appears.

With C decreasing beyond the third $1 \rightarrow 2$ bifurcation point the family becomes stable again at the fourth $1 \rightarrow 2$ bifurcation point (C_4, x_4, \dot{x}) , giving birth to a Feigenbaum sequence towards larger values of C . As C decreases further, the family *fl2* has a stable segment until the third turning point $(C_{t3}, x_{t3}, \dot{x})$ at which the family has minimum C and so on. We stopped the numerical calculations at the point $(C_{st}, x_{st}, \dot{x})$ (Table IV) which is after the fifth turning point $(C_{t5}, x_{t5}, \dot{x})$.

The same phenomenon occurs in the other branch of the main family. Thus the evolution of the family *fl2* has a similar bifurcation pattern and dynamical structure as the families *fb1* and *fld1*. Namely, as the Jacobi constant changes each branch of the main family spirals around its focal-terminating-point at $C_\infty=3$, $x_{up}=-2.279677$, $x_l=2.28174$. As we approach the focal point of the family infinite changes of stability to instability and vice versa appear while each bifurcation point produces infinite inverse Feigenbaum sequences and so on. Therefore there exist infinite spirals and each one of them produces infinite new inner spirals and so on. That is, a basic pattern is continuously reproduced. The stable parts along the characteristic of the family are smaller and smaller as we approach to the terminating point.

In Figure 5 we give a set of simple periodic orbits. We found that with C decreasing, a cusp near the Lagrangian point L_4 appears which turns to a loop surrounding this point (Figures 5b,c). As C varies further, a new inner cusp appears which evolves to a new inner loop surrounding the point L_4 (Figure 5d) and so on. Thus, as we approach the terminating point of the family we pass through an infinite set of self-similar orbits, every member of this set reproducing a basic pattern. The terminating point of the family corresponds to an infinite non-symmetric self-similar orbit, with infinite period and $C_\infty=3$, approaching asymptotically the triangular point L_4 .

II. Families of double periodic orbits *fld1* and *fld2*

We follow now the evolution of the family *fld1* beyond the $2 \rightarrow 4$ its bifurcation point (Table V, Figure 6a). As C decreases, the two branches of its characteristic continue to the left in the x - C diagram (Figure 4). The upper branch turns upwards while the lower one extends initially downwards and then turns upwards, that is, they have a similar behavior as the previous case of the family *fld1*. The two branches seem to pass respectively above and below the spirals of the main family. We stopped the evolution of the family at the point $(C_{st}, x_{up}, \dot{x}, x_l)$ because of the very large degree of stochasticity and so of the very large computing time needed. The instability parameter reaches an absolutely large value (Table V, Figure 6c).

Figure 6 shows a set of non-symmetric double periodic orbits. We see that the orbits (a), (b), (c) of the family *fld1* present a similar evolution as the orbits of the main family *fl2* (Figure 5) as well as the orbits of the families *fb1* and *fld1* (Figures 2, 3). That is, with decreasing C three cusps on the orbits appear near the triangular libration points L_4, L_5 , which evolve to three small loops. As C decreases further the loops grow and enclose these points. It seems that as C varies

new inner loops continuously appear on the orbits surrounding the Lagrangian points. We conclude that the family has a similar bifurcation pattern and dynamical structure as the previous cases of the families *fl2*, *fb1* and *fd1*. Each branch of the characteristic of the family will approach asymptotically its focal point, with $C_\infty=3$, spiralling around it.

Also, the evolution of the family *fld2* shows that its two branches extend to the right and in general downwards in the x - C diagram (Figure 4). The two branches are close to the characteristic of the main family and inside the two branches of the previous family *fld1*. We stopped the evolution of the family *fld2* at the point $(C_{st}, x_{up}, \dot{x}, x_l)$ because of the very large computing time needed (Figure 6e). The instability parameter reaches an absolutely large value (Table V).

In Figure 6 we give two non-symmetric double periodic orbits of the family *fld2*. We see that as C increases the two loops grow and enclose the Lagrangian point L_4 . The periodic orbits (c) and (e) (Figure 6) of the families *fld1* and *fld2* respectively have different shapes. Thus, the two families do not seem to join each other in order to form a bubble. Thus, it is topological necessity the two families to have a similar bifurcation pattern as in the previous case of the families *fd1*, *fl2* and *fb1*.

3.3 Evolution and stability of the main family *fc1*

I. Simple periodic orbits

As C increases the stability index of the main family decreases and at the point (C_1, x_1, \dot{x}) (Tables VI, VII) we have the $1 \rightarrow 2$ critical point (first bifurcation point) bifurcates a new family of double non-symmetric periodic orbits- family *fd1*- which is also initially stable. After this point, as C increases, the family of double periodic orbits becomes unstable at the $2 \rightarrow 4$ bifurcation point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table VII). We have here the Feigenbaum scenario.

We continued the characteristic of the family *fc1* beyond its first bifurcation point (Figure 7). The other branch of the characteristic is outside of the x - C diagram of Fig.7. As C increases the characteristic curve extends to the right and downwards in the x - C diagram (the absolute value of x decreases) and at the second $1 \rightarrow 2$ bifurcation point (C_2, x_2, \dot{x}) (Tables VI, VII) the family becomes critically stable again. In this point a new stable family of non-symmetric double periodic orbits bifurcates-family *fd2*-from the main family with two branches towards smaller values of C . As C decreases this new family becomes critically unstable at the $2 \rightarrow 4$ transition point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table VII, Figure 9c) and so the Feigenbaum scenario appears (Figure 7).

The main family is stable until the first turning point $(C_{t1}, x_{t1}, \dot{x})$ and it has maximum C . As C decreases the characteristic curve turns to the left and downwards in the x - C diagram. With C decreasing further the absolute value of x increases and the family becomes critically stable again at the second turning point $(C_{t2}, x_{t2}, \dot{x})$ and it has minimum C . After the second turning point, C increases with the absolute value of x increasing while the family has a small stable segment until the third $1 \rightarrow 2$ bifurcation point (C_3, x_3, \dot{x}) (Tables VI, VII), at which a new stable family of non-symmetric double periodic orbits with two branches towards larger values of C bifurcates. This stable family loses its stability at the $2 \rightarrow 4$ transition point $(C_{2 \rightarrow 4}, x_{up}, \dot{x}, x_l)$ (Table VII) and so on. Here a Feigenbaum scenario appears.

After the third bifurcation point, as C increases, the main family becomes stable again at the point (C_4, x_4, \dot{x}) . At this fourth $1 \rightarrow 2$ bifurcation point a bifurcated new stable family of double

non-symmetric periodic orbits extends towards smaller values of C . This stable family loses its stability and the Feigenbaum phenomenon appears.

After the $1 \rightarrow 2$ transition point the family $fc1$ becomes critically unstable again at the third turning point $(C_{t3}, x_{t3}, \dot{x})$ at which we have a maximum C and so on. We stopped the numerical calculations at the point $(C_{st}, x_{st}, \dot{x})$ (Table VI). This point is after the fifth $1 \rightarrow 2$ bifurcation point at which we found the usual Feigenbaum scenario.

Thus the evolution of the main family presents a similar bifurcation pattern and dynamical structure as in the cases of the families $fb1$ and $fb2$. That is, as C varies, each branch of the family spirals into its focal point at $C_\infty = 3$, $x_{up} = -0.214405$ and $x_l = 0.215312$. Therefore, infinite spirals appear and each one of them generates infinite new inner spirals and so on. Each member of these infinite sets of the spirals reproduces a basic bifurcation pattern. So, we found a new route to Chaos

Figure 8 shows four simple periodic orbits. The last orbit (d) has two loops surrounding the Lagrangian point L_5 . As C varies new inner cusps appear which evolve into new inner loops surrounding the point L_5 and so on. Thus, we have here self-similar orbits. The terminating point of the family corresponds to an infinite non-symmetric self-similar orbit, with infinite period, approaching asymptotically the triangular point L_5 ($C_\infty = 3$).

II. Families of double periodic orbits $fcd1$ and $fcd2$

We continued the family $fcd1$ beyond the $2 \rightarrow 4$ its bifurcation point—first bifurcation point of this family (Table VII, Figure 9a). The two branches of its characteristic curve extend to the right in the x - C diagram. As C increases, the upper branch turns initially upwards in x and then downwards, as in the previous mentioned cases of the families of double periodic orbits. The lower branch moves continuously downwards while the period of the orbits initially decreases and then increases. The two branches are outside and far from the characteristic of the main family (Figure 7). So, it seems that the two branches will pass outside of the spirals of the main family as in the previous case of the family $fb1$. Because of the very large computing time needed, we stopped at the point $(C_{st}, x_l, \dot{x}, x_{up})$ (Table VII, Figure 9b).

On the contrary, the evolution of the family $fcd2$ (Tables VI, VII) shows that its two branches extend to the left and upwards for the upper branch and downwards initially and then upwards for the lower one, while the period continuously increases. The two branches of the family are close to the characteristic of the main family and inside of the two branches of the previous family $fcd1$ (Figure 7). This family becomes more and more unstable, as C decreases. We did not follow further the family because of the very large degree of stochasticity and consequently of the very large computing time needed. We stopped at the point $(C_{st}, x_{up}, \dot{x}, x_l)$ (Table VII, Figure 9d)

Figure 9 shows some double non-symmetric periodic orbits along the characteristic of the two families $fcd1$ and $fcd2$. It seems that the two families $fcd1$ and $fcd2$ present similar bifurcation structure and dynamical behavior as the previously mentioned families of double periodic orbits of the main family $fb1$. Comparing the variation of the periods of the orbits of the two families (Table VII) as well as the different shape of their orbits b and d of Figure 9, it seems that they do not join each other in order to form a bubble. Thus, it is topologically necessary that the families should have a similar bifurcation pattern as in the previous case of the main family $fb1$ as well as in the case of the double periodic orbits of the family $fb1$. That is, each bifurcated family seems to turn around its focal point with $C_\infty = 3$, while infinite bifurcation points and so infinite inverse Feigenbaum sequences and infinite spirals appear. Consequently, in

the case $fc1$ we found a new mechanism which leads to a large degree of stochasticity. The conclusions help us to comprehend the motions around the point L_5 of Lagrange.

4. Conclusions

We studied systematically cases of non-symmetric periodic orbits of the families b , l , c and n of the planar restricted 3-body problem and in the case of equal masses of the primaries. The study gives a new interesting dynamical structure of the phase space and the topological characteristics change not only quantitatively but also qualitatively.

For the main families of simple non-symmetric periodic orbits $fb1$, $fl2$, $fc1$, and fz_1 that bifurcate at the points $b1$, $l2$, $c1$, (Hénon 1965) and z_1 (Pinotsis 1986) respectively, we found the evolution and their stability. In their first bifurcation point these families present a similar bifurcation structure. We did not find the termination point of the family fz_1 because of the very large computing time required.

As regards the main families $fb1$, $fl2$, $fc1$, as the Jacobi constant C varies each branch of the characteristic of a main family of simple non-symmetric periodic orbits spirals around its focal point-termination point in x - at which the Jacobi constant is $C_\infty=3$. As the characteristic of each main family approaches asymptotically its termination point infinite changes from stability to instability and vice versa and so infinite bifurcation points appear. So, infinite successive very small stable and very large unstable segments along the characteristic of each main family are formed. The stable parts along the characteristic of each family are smaller and smaller as we approach to the terminating point. Each bifurcation point generates infinite inverse Feigenbaum sequences. That is, each bifurcating family presents the same phenomenon and so on. Thus, infinite sets of inverse Feigenbaum sequences and infinite sets of spirals appear. Each member of these infinite sets of the spirals reproduces a basic bifurcating pattern. Thus, the families are strongly unstable and so the regions near the Lagrangian points L_4 , L_5 are mostly chaotic.

Every bifurcating family of double non-symmetric periodic orbits produces also the same phenomenon, i.e. infinite inverse Feigenbaum sequences and infinite spirals that have infinite different focal points. Also, each bifurcated family of higher multiplicity presents the same phenomenon. That is, each bifurcated family leads to spiral around its focal point in x with $C_\infty=3$. Each spiral has infinite changes of stability to instability and vice versa. Infinite bifurcation points appear and each one of them generates infinite inverse Feigenbaum sequences and so on. Therefore, we have infinite spirals with infinite different focuses and each one of them generates infinite new inner spirals and so on. Thus, infinite sets of spirals appear and each member of these infinite sets of the spirals reproduces a basic bifurcating pattern.

The orbits of each main family near the corotation –Lagrangian points L_4 , L_5 present a new dynamical structure (Figures 2, 5, 8). As C changes along the characteristic of each main family a cusp close to the Lagrangian point L_4 or L_5 on the orbits emanates that evolves to loop surrounding one of these points. As C changes continuously, approaching asymptotically its terminating point, a new cusp inside the loop appears that turns to new inner loop enclosing one of the triangular points L_4 or L_5 and so on. This process reproduces continuously a basic pattern. As we approach the terminating point, an infinite set of self-similar orbits is constructed. The same phenomenon presents the families of double non-symmetric periodic orbits (Figures 3b,c and 6b,c) as well as the orbits of each bifurcated family of higher multiplicity. So, the terminating point of each bifurcating family corresponds to a non-symmetric self-similar orbit with infinite period and $C_\infty=3$, that approaches asymptotically the point. Thus, the termination

principle for a family of periodic orbits, which states that either the period of an orbit or its size or the energy goes to infinity (Szebehely 1967), is satisfied.

Bubbles do not exist in this case of families of periodic orbits that terminate at L_4 or L_5 . In general, there do not exist bubbles in the case $\mu=0.5$. We have found indications that bubbles are formed after a critical value of μ .

The maximum absolute value of the instability parameter of each family is very large of order 10^6 . This value grows further as we approach the focus of its characteristic as well as the multiplicity of the bifurcating families increases. Therefore, there must be large chaotic regions of stochastic orbits around these periodic orbits.

This mechanism of the infinite sets of inverse Feigenbaum sequences and infinite sets of spirals of each (main) family are generated near corotation points L_4 or L_5 . Each multiplicity family terminates after infinite spirals at its focal point- constant point in x - with the value of Jacobi constant $C_\infty=3$. The Lagrangian points L_4 , L_5 as well as the regions near them are strongly unstable.

This mechanism produces, in general, a very large degree of stochasticity that generates large chaotic regions near the corotation points (L_4 or L_5). This is a new route to chaos. We have thus a new mechanism that produces very large degree of stochasticity. These conclusions help us to comprehend better the motions around the corotation points L_4 and L_5 . We anticipate that this mechanism will appear also in other rotating dynamical systems of 2-degrees of freedom.

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References

- Bardin, B.: On motions near the Lagrange equilibrium point L_4 in the case of Routh's critical mass ratio. *Celest. Mech. Dyn. Astron.* **82**, 163 (2002)
- Bier, M. and Bountis, T.: Remerging Feigenbaum Trees in Dynamical Systems. *Phys. Lett.*, **A104**, 239 (1984)
- Bruno, A.D. and Varin, V.P.: The limit problems for the equation of oscillations of a satellite, *Celest. Mech. Dyn. Astron.* **67**:1, 1-40 (1997)
- Bruno, A.D.: Families of periodic solutions to the Beletsky equation, *Cosmic Research* **40**, 3, 274-295 (2002)
- Contopoulos, G.: Inverse Feigenbaum Sequences in Hamiltonian Systems: *Lett. Nuovo Cim.* **37**, 149 (1983a)
- Contopoulos, G.: Infinite Bifurcations, Gaps and Bubbles in Hamiltonian Systems: *Physica D* **8**, 142 (1983b)
- Contopoulos, G.: The generation of spiral characteristics, *Celest. Mech. Dyn. Astron.*, **50**, 251 (1991)
- Contopoulos, G.: Order and Chaos, p.33: in *Galactic Dynamics and N-Body Simulations*, G. Contopoulos, N. SpyrouL, Blahos (Eds), Springer Verlag (1993)
- Contopoulos, G.: *Order and Chaos in Dynamical Astronomy*, Springer Verlag, N.York (2002)
- Contopoulos, G. and Zikides, M.: Periodic Orbits and Ergodic Components of a Resonant Dynamical System: *Astron. Astrophys.* **90**, 198 (1980)

- Contopoulos, G. and Pinotsis, A. D.: Infinite Bifurcations in the Restricted Three-Body Problem. *Astron. Astrophys.* **133**, 49 (1984)
- Couillet, P. and Tresser, C.: Iterations d' Endomorphismes et Groupe de Renormalisation: *J. Phys. Coll. (Paris)*, Volume 39, Number C5, (1978)
- Eckmann, J.P.: Roads to Turbulence in Dissipative Dynamical Systems. *Rev. Mod. Phys.*, **53**, 643 (1981)
- Efthymiopoulos, C.: Formal Integrals and Nekhoroshev stability in a mapping model for the Trojan asteroids. *Celest. Mech. Dyn. Astron.* **92**, 29 (2005)
- Erdi, B., Forgacs-Dajika, E., Nagy, I., Rajnai, R.: A parametric study of stability and resonances around L_4 in the elliptic restricted three-body problem. *Celest. Mech. Dyn. Astron.* **104**, 145 (2009)
- Hadjidemetriou, J.: *Periodic Orbits and Stability*, Programme Univer. Europ. Erasmus, Thessaloniki (1989)
- Heggie, D.C.: On the Bifurcations of a Certain Family of Periodic Orbits. *Celest. Mech.*, **29**, 207 (1983)
- Hénon, M.: *Exploration Numerique du probleme restreint, (II) Masses egales, stabilite des orbites periodiques*; *Ann. Astrophys.* **28**, 992-1007 (1965b)
- Henrard, J., Navarro, J.F.: Families of periodic orbits emanating from homoclinic in the restricted problem of three bodies. *Celest. Mech. Dyn. Astron.* **89**, 285 (2004)
- Feigenbaum, M. J.: Quantitative Universality for a Class of Nonlinear Transformations, *J. Stat. Phys.*, **19**, 25 (1978)
- Magenat, P.: Periodic orbits in triaxial galactic models. *Astron. Astrophys.* **108**, 89 (1982)
- Pinotsis, A. D. : *Bifurcations, stability and universality of families of periodic orbits in the restricted three-body problem*; *Astron. Astrophys.*, **159**, 231-238 (1986)
- Pinotsis, A. D. : *Successive bifurcations and evolution of double and quadruple periodic orbits in the restricted three-body problem*; *Astron. Astrophys.*, **174**, 317-322 (1987)
- Pinotsis, A. D. : *Bifurcations and instabilities in the restricted three-body problem*; in *Long Term Dynamical Behaviour of Natural andn Artificial N-Body Systems*, ed. A. E. Roy, 465-469, Kluver Academic Publishers (1988)
- Sandor, Z., Erdi, B., Efthymiopoulos, C.: The phase space structure around L_4 in the restricted three-body problem. *Celest. Mech. Dyn. Astron.* **78**, 113 (2000)
- Sicardy, B.: Stability of the triangular Lagrange points beyond Gascheaus's value. *Celest. Mech. Dyn. Astron.* (2010).
- Szebehely, V.: *Theory of orbits*, Academic Press, New York (1967)

Figure captions

Figure 1: Evolution of the families of non-symmetric simple (main family $fb1$), double, quadruple,... periodic orbits schematically (the actual values are given in the Tables). () stable, (....) unstable orbits. The families of double periodic orbits $fbd1$ and $fbd2$, emanated at the $1 \rightarrow 2$ first and second bifurcation points of the family $fb1$. The two branches of the

characteristic curve of the family $fbd2$ remain always inside the two branches of the characteristic curve of the family $fd1$.

Figure 2: A set of simple periodic orbits in the x-y plane: (a) A symmetric periodic orbit of the original family b in the neighborhood of the bifurcation point $b1$. Also, three non-symmetric periodic orbits along the characteristic of the main family $fb1$. We see a change of the form of the cusps or of the loops as C varies: (b) near the first turning point, (c) near the second turning point, and (d) at the point at which we stopped the numerical calculations.

Figure 3: Some double non-symmetric periodic orbits in the x-y plane: Along the characteristic curve of the family $fd1$: (a) near the $2 \rightarrow 4$ bifurcation point, (b) at the point $C=3.047539$, $x=-1.498219454462$, $\dot{x} = -0.03068701204$, $T=13.3517$ and (c) at the point at which we stopped the numerical calculations: Also, along the characteristic of the family $fd2$: (d) near the $2 \rightarrow 4$ bifurcation point, and (e) at the point at which we stopped the numerical calculations. The orbits (b) and (e) of the two families $fd1$ and $fd2$ correspond to the same C .

Figure 4: Evolution of the families of non-symmetric simple (main family $fl2$), double, quadruple,... periodic orbits schematically (the actual values are given in the Tables). (—) stable, (....) unstable orbits. The families of double periodic orbits $fd1$ and $fd2$ emanated at the $1 \rightarrow 2$ first and second bifurcation points of the family $fl2$. The two branches of the characteristic curve of the family $fd2$ remain inside the two branches of the characteristic curve of the family $fd1$.

Figure 5: Simple periodic orbits: (a) A symmetric periodic orbit of the original family l near the bifurcation point $l2$. Also three non-symmetric periodic orbits of the main family $fl2$: (b) a cusp appears near the point $C=3.19$, $x=-2.079926229183$, $\dot{x} = 0.191954854072$, $T=10.8154$, (c) near the first turning point and (d) near the fifth turning point.

Figure 6: Double non-symmetric periodic orbits: Orbits belonging to the family $fd1$: (a) near the bifurcation point $2 \rightarrow 4$, (b) three cusps appear near the point $C=3.1230853$, $x=-2.13360173108$, $\dot{x} = 0.19734606655$ and (c) at the point at which we stopped the numerical calculations: Orbits along the characteristic of the family $fd2$: (d) near the $2 \rightarrow 4$ bifurcation point of the family $fd2$ and (e) at the point at which we stopped the numerical calculations.

Figure 7: Evolution of the families of non-symmetric simple (main family $fc1$), double, quadruple,... periodic orbits schematically (the actual values are given in the Tables). (—) stable, (....) unstable orbits. The families of double periodic orbits $fd1$ and $fd2$ emanated at the $1 \rightarrow 2$ first and second bifurcation points of the family $fc1$. The two branches of the characteristic curve of the family $fd2$ remain inside the two branches of the characteristic curve of the family $fd1$.

Figure 8: A set of simple periodic orbits in the x-y plane: (a) A symmetric periodic orbit of the original family c in the neighborhood of the bifurcation point $c1$. Also, three non-symmetric periodic orbits along the characteristic of the main family $fc1$. We see a change of the form of the cusps or of the loops as C varies: (b) near the first turning point, (c) near the second turning point and (d) at the point at which we stopped the numerical calculations.

Figure 9: Some double non-symmetric periodic orbits in the x-y plane: Orbits belonging to the family $fd1$: (a) near the bifurcation point $2 \rightarrow 4$ and (b) at the point at which we stopped the numerical calculations: Orbits along the characteristic of the family $fd2$: (c) near the $2 \rightarrow 4$ bifurcation point and (d) at the point at which we stopped the numerical calculations.

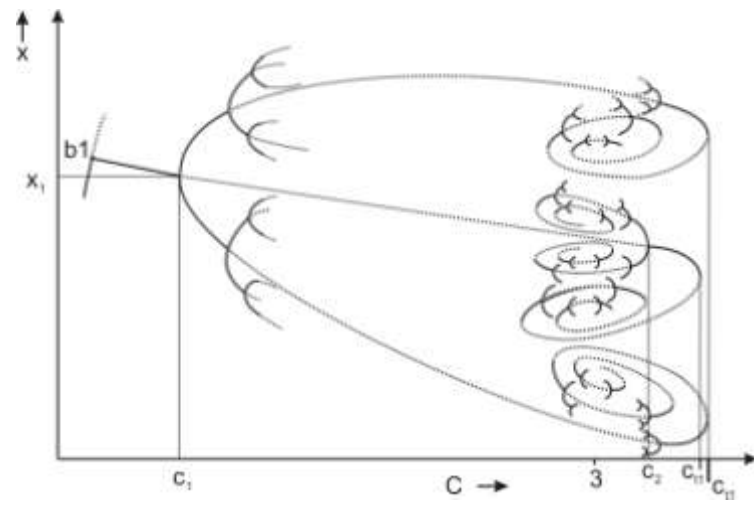
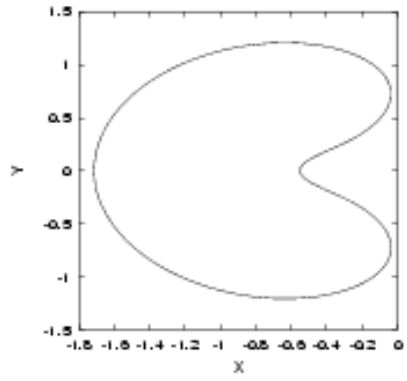
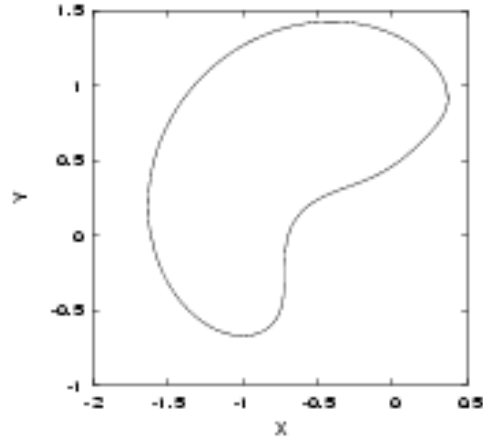


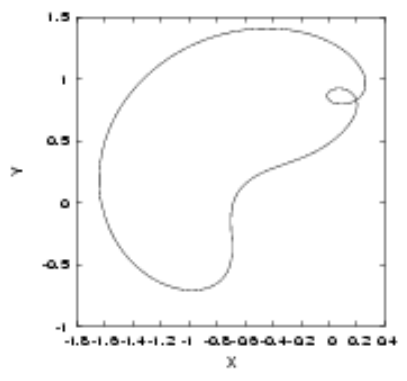
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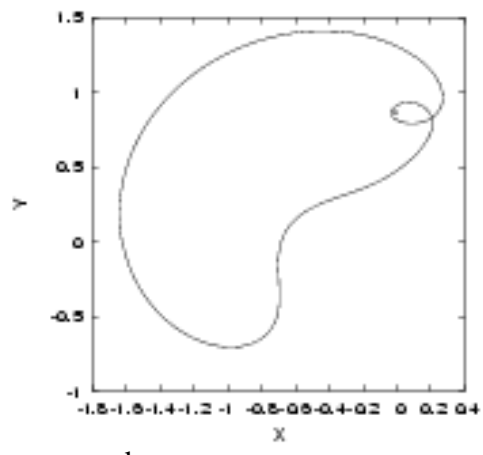
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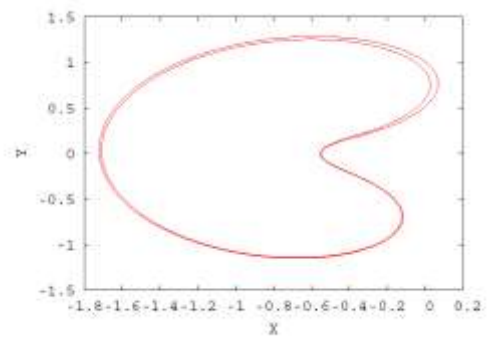


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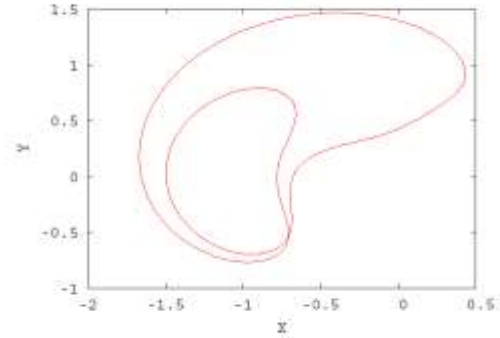


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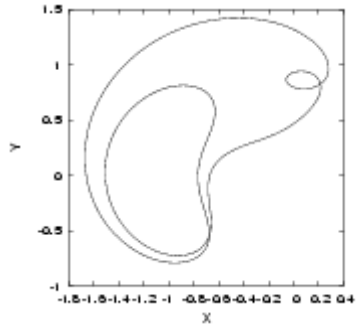
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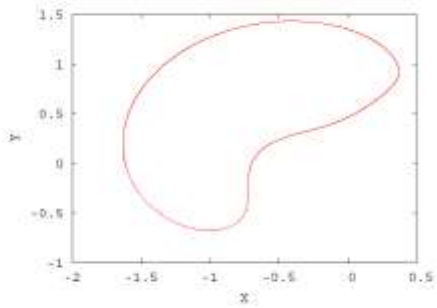
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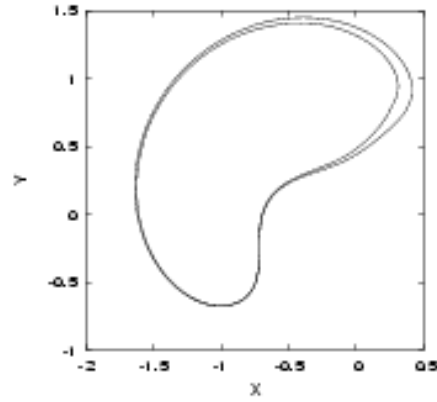
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Figure 3

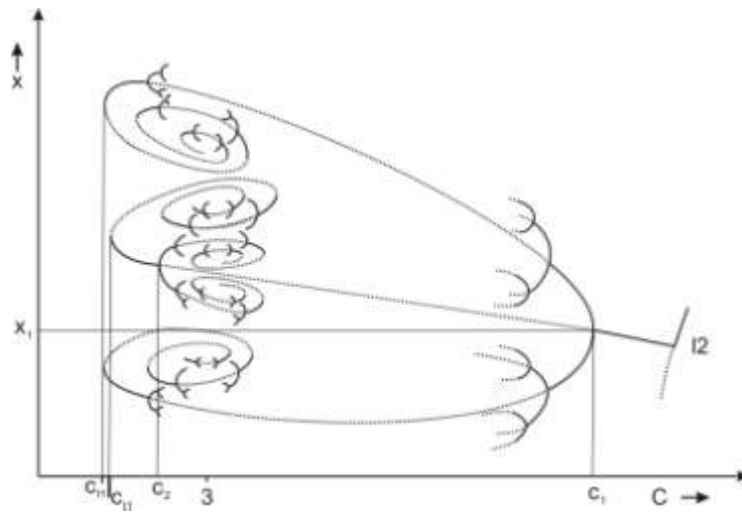
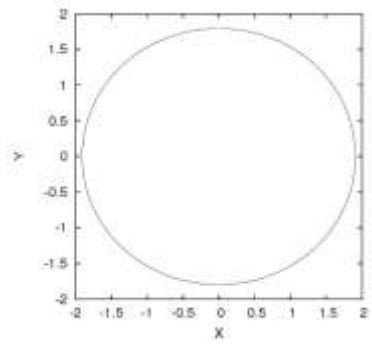
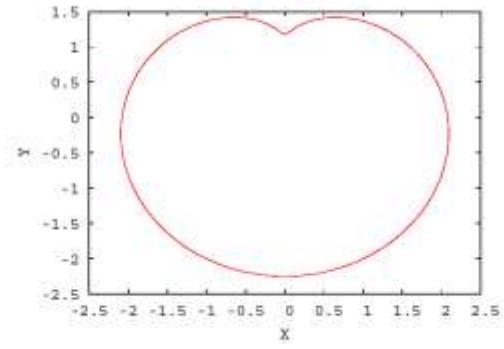


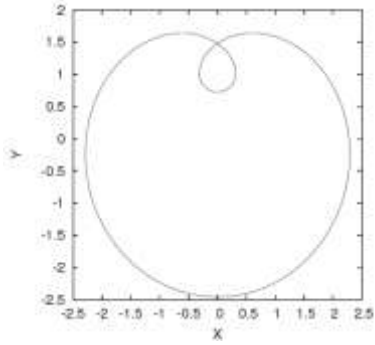
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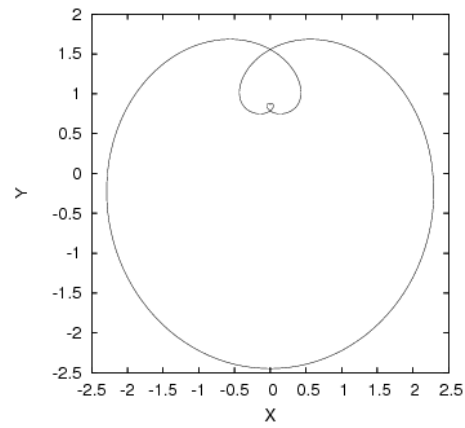
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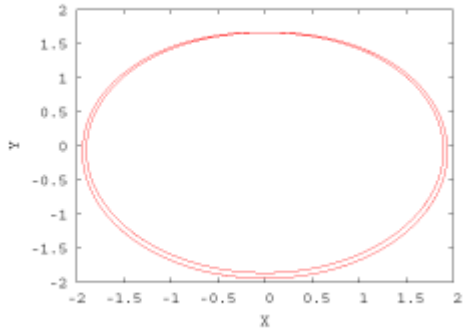


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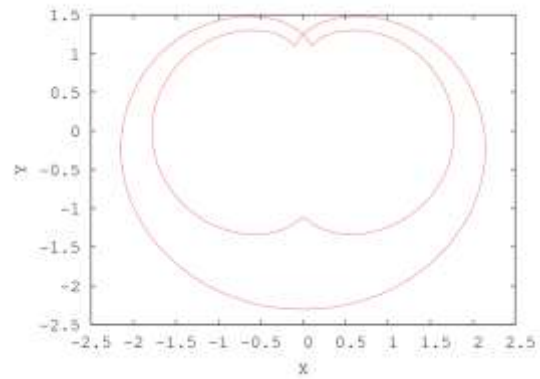


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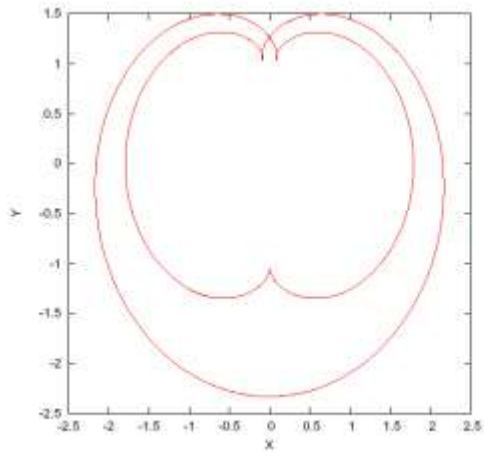
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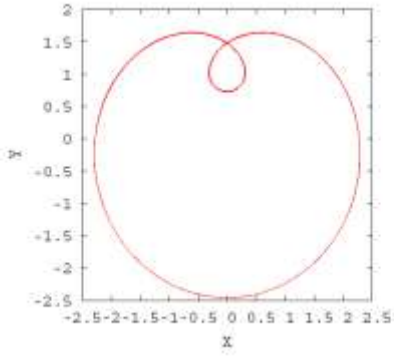
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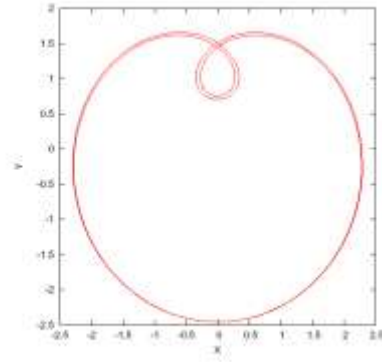
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Figure 6

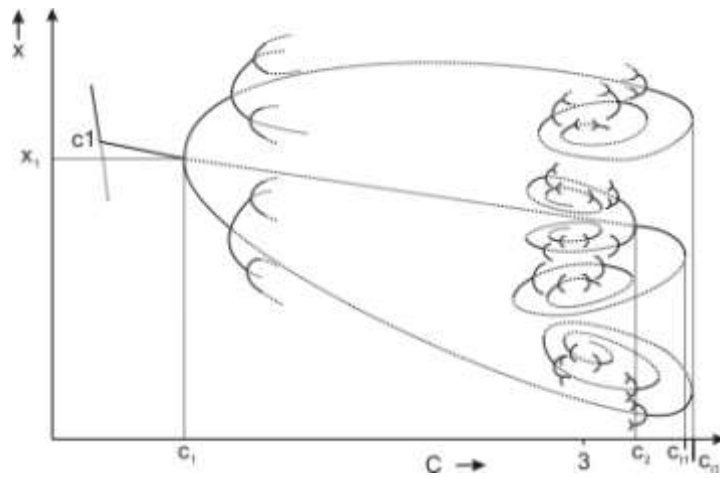


Figure 7

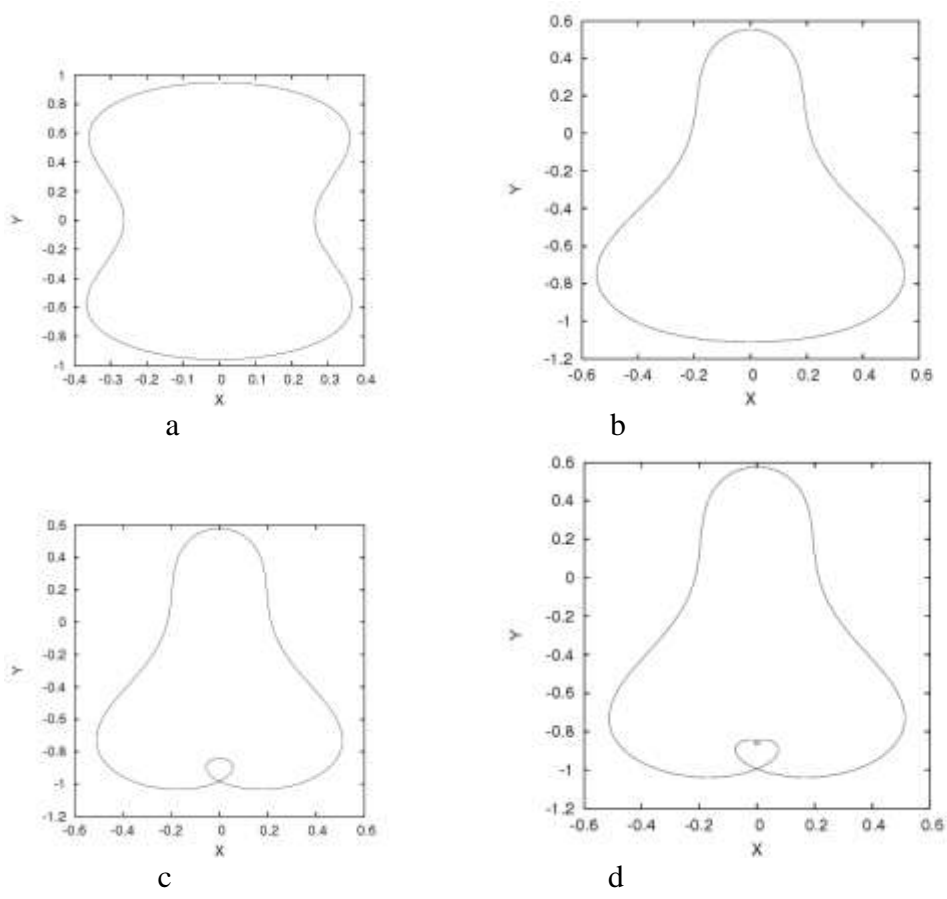
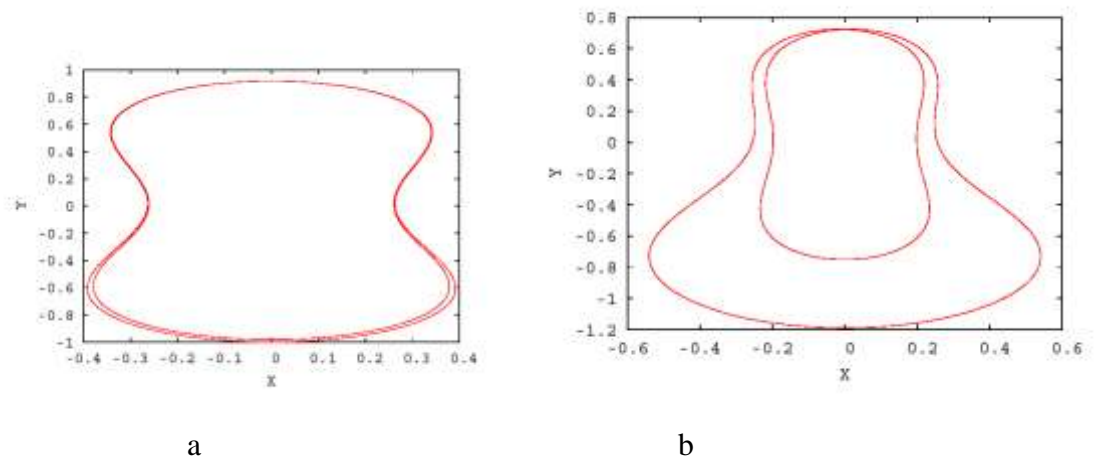


Figure 8



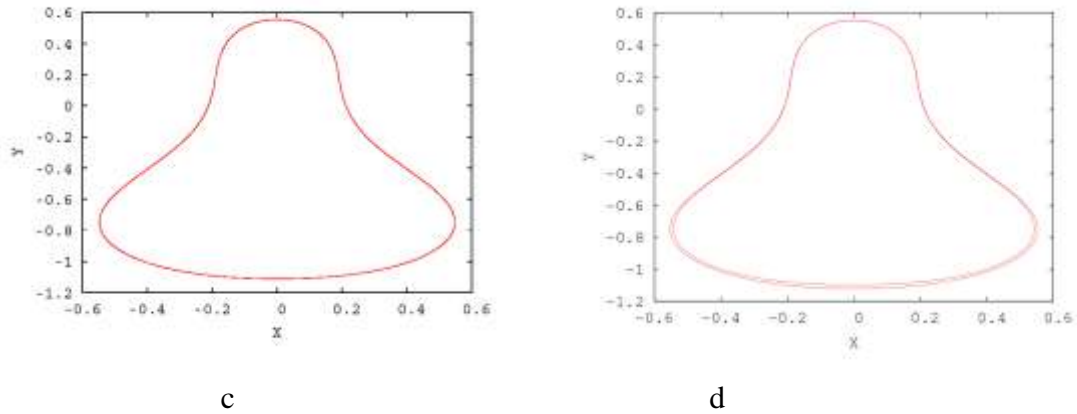


Figure 9

“Electronic Supplementary Material”

Please remove all the Tables for the paper of **A. D. Pinotsis**.

Title of the paper: **Infinite Feigenbaum Sequences and Spirals in the Vicinity of the Lagrangian Periodic Solutions.**

Table I

Bifurcations of the main family *fb1*. C_1, C_2, \dots , Jacobi constant of the bifurcation points. C_{t1}, C_{t2}, \dots , Jacobi constant of the turning points.

Initial conditions of the critical points			Period T	Stability index A
$C_{in}=2.27078$	$x_{up}=-1.7155626399$ $x_l=-0.5496542824$	$\dot{x}_{bl}=0$	5.7902	+1
$C_1=2.284816$	$x_1=-1.7154767053$	$\dot{x}=-0.0384865989$	5.8113	-1
$C_2=3.053810501$	$x_2=-1.61536591$	$\dot{x}=-0.18809097$	8.9442	-1
$C_{t1}=3.05381119$	$x_{t1}=-1.61534262$	$\dot{x}=-0.18803613$	8.9488	+1
$C_{t2}=2.993115816307$	$x_{t2}=-1.626211247202$	$\dot{x}=-0.1806600851015$	12.2168	+1
$C_3=2.99311581766$	$x_3=-1.6262116309338$	$\dot{x}=-0.1806609429056$	12.2173	-1
$C_4=3.00084455473514$	$x_4=-1.6249085675$	$\dot{x}=-0.181644983$	15.5392	-1
$C_{t3}=3.00084455473767$ 2	$x_{t3}=-1.6249085509296$	$\dot{x}=-0.1816449703712$	15.5393	+1
$C_{t4}=2.998958724736589$ 8	$x_{t4}=-1.62506982337$	$\dot{x}=-0.181525103213$	18.8499	+1
$C_5=2.998958724736637$	$x_5=-1.62506982346$	$\dot{x}=-0.1815251034061$	18.85	-1
$C_{st}=3.0000123276$	$x_{st}=-1.6250503904397$	$\dot{x}=-0.1815406045172$	21.9283	-3.89×10^5

Table II
Bifurcations of the family of double periodic orbits $fbd1$. C_1, C_2, \dots , Jacobi constant of the bifurcation points. C_{t1}, C_{t2}, \dots , Jacobi constant of the turning points.

Initial conditions of the critical points			Period T	Stability index A
$C_{in}= 2.284816$	$x_1=-1.7154767053$	$\dot{x}=-0.0384865989$	5.8113	-1
$C_1=2.286493$	$x_{up}=-1.72113211$ $x_l=-1.708763228629$	$\dot{x}=-0.04267198044$	11.6179	-1
$C_2=3.056031450829$	$x_l=-1.49728175567$ $x_{up}=-1.6509765232$	$\dot{x}=-0.02958157413$	13.8063	-1
$C_{t1}=3.05603145088715$	$x_l=-1.4972818162$ $x_{up}=-1.650976169619$	$\dot{x}=-0.02958148958$	13.8064	+1
$C_{t2}=2.9928415643453027$	$x_l=-1.510426741038$ $x_{up}=-1.660705844398$	$\dot{x}=-0.0288617082239$	17.1041	+1
$C_3=2.99284156434545$	$x_l=-1.5104267399485$ $x_{up}=-1.660482771437$	$\dot{x}=-0.0288617097337$	17.1042	-1
$C_{st}=2.99794649$	$x_l=-1.5093075839134$ $x_{up}=-1.660211636989$	$\dot{x}=-0.0290258657503$	18.6567	-1.05×10^6

Table III
Bifurcations of the first three families of double periodic orbits $fbd1$, $fbd2$ and $fbd3$

Initial conditions of the bifurcation points of the family $fbd1$			Period T	Stability index A
$C_{1 \rightarrow 2}=2.284816$	$x_1=-1.7154767053$	$\dot{x}=-0.0384865989$	5.8113	-1

$C_{2 \rightarrow 4}=2.286493$	$x_{up}=-1.72113211$ $x_f=-1.70876322863$	$\dot{x}=-0.04267198044$	11.6179	-1
$C_{4 \rightarrow 8}=2.28668105$	$x_f=-1.70761$ $x_{up}=-1.709725$	$\dot{x}=-0.03423224$	23.2361	-1
Initial conditions of the bifurcation points of the family $fb2$			Period T	Stability index A
$C_{1 \rightarrow 2}=3.053810501$	$X_2=-1.61536591$	$\dot{x}=-0.18809097$	8.9442	-1
$C_{2 \rightarrow 4}=3.053810329$	$x_f=-1.6153884368924$ $x_{up}=-1.6168196461756$	$\dot{x}=0.1881101288858$	17.8885	-1
$C_{4 \rightarrow 8}=3.053810307$	$x_f=-1.6153876697247$ $x_{up}=-1.616819721$	$\dot{x}=-0.1881118816314$	35.7768	-1
$C_{st}=3.047539$	$x_{up}=-1.62115333$ $x_f=-1.6127423$	$\dot{x}=-0.19184434$	17.9617	-7.4×10^4
Initial conditions of the bifurcation points of the family $fb3$			Period T	Stability index A
$C_{1 \rightarrow 2}=2.99311581766$	$x_3=-1.6262116309338$	$\dot{x}=-0.1806609429056$	12.2173	-1
$C_{2 \rightarrow 4}=2.993115818$	$x_{up}=-1.626211997351$ $x_f=-1.626211277421$	$\dot{x}=-0.180661248628$	24.4347	-1

Table IV
Bifurcations of the main family $fl2$. C_1, C_2, \dots , Jacobi constant of the bifurcation points.
 C_{t1}, C_{t2}, \dots , Jacobi constant of the turning points.

Initial conditions of the critical points			Period T	Stability index A
$C_{in}=3.552593$	$x_{up}=-1.907961968103$ $x_f=1.9079619681$	$\dot{x}=0$	10.5675	+1
$C_1=3.53401$	$X_1=-1.91796399$	$\dot{x}=0.05286984$	10.5765	-1
$C_2=2.855512$	$x_2=-2.2764855257$	$\dot{x}=0.2132893264$	11.8966	-1
$C_{t1}=2.85545487$	$x_{t1}=-2.27808186$	$\dot{x}=0.21300771$	11.9193	+1
$C_{t2}=3.01707925685$	$x_{t2}=-2.27955289$	$\dot{x}=0.19140455$	15.3749	+1
$C_3=3.01707913$	$x_3=-2.2795231$	$\dot{x}=0.19140926$	15.3783	-1
$C_4=2.997887875$	$x_4=-2.2796852$	$\dot{x}=0.19413788$	18.6593	-1
$C_{t3}=2.997887874567$	$x_{t3}=-2.2796858$	$\dot{x}=0.19413779$	18.6599	+1
$C_{t4}=3.000260188348925$	$x_{t4}=-2.27967606904$	$\dot{x}=0.19380232$	21.9771	+1
$C_5=3.000260188348$	$x_5=-2.27967606$	$\dot{x}=0.19380233$	21.9772	-1
$C_6=2.9999679358126972$	$x_6=-2.2796774342327$	$\dot{x}=0.1938436770669$	25.2887	-1
$C_{t5}=2.9999679358126962$	$x_{t5}=-2.2796774343397$	$\dot{x}=0.1938436770498$	25.2888	+1

8				
$C_{st}=2.99997945696$	$x_{up}=-2.27968469583$	$\dot{x}=0.19384087578$	26.2157	4.7×10^5

Table V
Bifurcations of the families of double periodic orbits *fld1*, *fld2* and *fld3*

Initial conditions of the bifurcation points of the family <i>fld1</i>			Period T	Stability index A
$C_{1 \rightarrow 2}=3.53401$	$X_1=-1.91796399$	$\dot{x}=0.05286984$	10.5765	-1
$C_{2 \rightarrow 4}=3.5317879$	$x_{up}=-1.9359955675679$ $x_l=-1.897372374$	$\dot{x}=0.0611642364043$	21.1842	-1
$C_{4 \rightarrow 8}=3.53153746$	$x_{up}=-1.9372144299$ $x_l=-1.93525442$	$\dot{x}=0.061036773178$	42.3717	-1
$C_{st}=3.071987$	$x_{up}=-2.15553929883$ $x_l=-1.79270089598$	$\dot{x}=0.20318136029$	25.3679	$-6.25 \cdot 10^4$
Initial conditions of the bifurcation points of the family <i>fld2</i>			Period T	Stability index A
$C_{1 \rightarrow 2}=2.855512$	$x_2=-2.2764855257$	$\dot{x}=0.2132893264$	11.8966	-1
$C_{2 \rightarrow 4}=2.8555266$	$X_{up}=-2.2769878265466-$ $x_l=-2.2700686916169$	$\dot{x}=0.212988032694$	23.7933	-1
$C_{4 \rightarrow 8}=2.85552832$	$x_{up}=-2.2771457736215$ $x_l=-2.2746043598865$	$\dot{x}=0.2129308405946$	47.5859.	-1
$C_{st}=2.947418671875$	$X_{up}=-2.21282188725$ $x_l=-2.28353496416$	$\dot{x}=0.22731611007$	24.5566	$-1.42 \cdot 10^4$
Initial conditions of the bifurcation points of the family <i>fld3</i>			Period T	Stability index A
$C_{1 \rightarrow 2}=3.01707913$	$x_3=-2.2795231$	$\dot{x}=0.19140926$	15.3783	-1
$C_{2 \rightarrow 4}=3.017079098089889$	$x_{up}=-2.27953209496$ $x_l=-2.2768103614415$	$\dot{x}=0.1914041540738$	30.7566	-1

Table VI
**Bifurcations of the main family $fc1$. C_1, C_2, \dots , Jacobi constant of the bifurcation points.
 C_{t1}, C_{t2}, \dots , Jacobi constant of the turning points.**

Initial conditions of the critical points			Period T	Stability index A
$C_{in}=2.6217$	$x_{up}=-0.2624399239$ $x_l=0.262412784$	$\dot{x}=0$	5.2617	+1
$C_1=2.6249439$	$X_2=-0.262203$	$\dot{x}=0.03937928$	5.271	-1
$C_2=3.0433717756$	$x_2=-0.21174171$	$\dot{x}=0.27078923$	7.9827	-1
$C_{t1}=3.043371908421$	$x_{t1}=-0.21173066$	$\dot{x}=0.27073376$	7.9849	+1
$C_{t2}=2.994695024137$	$x_{t2}=-0.21476724$	$\dot{x}=0.25165819$	11.2868	+1
$C_3=2.99469502439$	$x_3=-0.21476741$	$\dot{x}=0.25165904$	11.2871	-1
$C_4=3.000654379316$	$x_4=-0.21438139001$	$\dot{x}=0.25403459$	14.6037	-1
$C_{t3}=3.00065437931646$	$x_{t3}=-0.2143813874$	$\dot{x}=0.25403457867$	14.6038	+1
$C_{t4}=2.9999193730694$	$x_{t4}=-0.214428857034$	$\dot{x}=0.253742667486$ 7	17.91518	+1
$C_5=2.999919373069410877$	$x_5=-0.214428857076$	$\dot{x}=0.253742667696$	17.9152	-1
$C_{st}=2.99993004195$	$x_{st}=-0.214431457$	$\dot{x}=0.253763457842$ 2	18.4213	-3.4×10^5

Table VII
Bifurcations of the families of double periodic orbits $fcd1$, $fcd2$ and $fcd3$

Initial conditions of the bifurcation points of the family $fcd1$			Period T	Stability index A
$C_{1 \rightarrow 2}=2.6249439$	$X_1=-0.262203$	$\dot{x}=0.03937928$	5.271	-1
$C_{2 \rightarrow 4}=2.6253365$	$x_{up}=-0.2633431725$	$\dot{x}=0.04477541497$	10.5362	-1

	$x_f = -0.26070513131$			
$C_{4 \rightarrow 8} = 2.62538195$	$x_f = -0.2604342$ $x_{up} = -0.26092077$	$\dot{x} = 0.03373525$	21.0721	-1
$C_{st} = 2.789904985$	$x_f = -0.195501410821$ $x_{up} = -0.2535792185$	$\dot{x} = 0.018973376764$	9.9598	$-2.19 \cdot 10^4$
Initial conditions of the bifurcation points of the family $fcd2$			Period T	Stability index A
$C_{1 \rightarrow 2} = 3.0433717756$	$x_2 = -0.21174171$	$\dot{x} = 0.27078923$	7.9827	-1
$C_{2 \rightarrow 4} = 3.043371741$	$x_{up} = -0.2117482633$ $x_f = -0.21173527$	$\dot{x} = 0.27081565$	15.9655	-1
$C_{4 \rightarrow 8} = 3.0433717373$	$x_{up} = -0.211748507159$ $x_f = -0.21167271047$	$\dot{x} = 0.270816005452$	31.9309	-1
$C_{st} = 3.0430907835$	$x_{up} = -0.21237236357$ $x_f = -0.20813463864$	$\dot{x} = 0.27317910053$	15.9697	$-1.69 \cdot 10^4$
Initial conditions of the bifurcation points of the family $fcd3$			Period T	Stability index A
$C_{1 \rightarrow 2} = 2.99469502439$	$x_3 = -0.21476741$	$\dot{x} = 0.25165904$	11.2871	-1
$C_{2 \rightarrow 4} = 2.99469502443929914$	$x_{up} = 0.2147675008222$ $x_f = -0.2130428044011$	$\dot{x} = 0.251659407344$ 2	22.5741	-1

Table VIII
Bifurcations of the family of double periodic orbits fz_l

Initial conditions of the bifurcation points			Period T	Stability index A
$C_{in} = 2.2330236$	$X = -0.5233143$	$\dot{x} = 0$	12.2682	+1
$C_{2 \rightarrow 4} = 2.234761$	$X = -0.52225233$	$\dot{x} = 0.02279438$	12.3055	-1
$C_{4 \rightarrow 8} = 2.234933$	$x = -0.52176309$	$\dot{x} = 0.02353585$	24.6135	-1
$C_{st} = 2.285669$	$x = -0.51127768$	$\dot{x} = 0.07348204$	12.8737	$-6.25 \cdot 10^4$