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Michèle Grillot, Philippe Grillot

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Abstract

In this paper we study solutions of the quasi-linear parabolic equations $\frac{\partial u}{\partial t} - \Delta_p u = a(x)|u|^{q-1}u$ in $(0,T)\times\Omega$ with Robin boundary condition $\frac{\partial u}{\partial \nu}|\nabla u|^{p-2} = b(x)|u|^{r-1}u$ in $(0,T)\times\partial\Omega$ where Ω is a regular bounded domain in $\mathbb{R}^N$, $N \geq 3$, $q > 1$, $r > 1$ and $p \geq 2$. Some sufficient conditions on a and b are obtained for those solutions to be bounded or blowing up at a finite time. Next we give the asymptotic behavior of the solution in special cases.

Keywords : quasilinear parabolic equation, blow-up, asymptotic behavior, Robin boundary condition.

1 Introduction

Let Ω be a regular bounded domain in $\mathbb{R}^N$, $N \geq 3$, $q > 1$, $r > 1$ and $p \geq 2$. We consider a continuous function a on $\overline{\Omega}$ and a continuous function b on $\partial\Omega$, the boundary of Ω. We study the solutions of the following equation :

$$\frac{\partial u}{\partial t} - \Delta_p u = a(x)|u|^{q-1}u \tag{1}$$

in $(0,T)\times\Omega$ where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ denotes the p-laplacian of u, subject to the Robin boundary condition :

$$\frac{\partial u}{\partial \nu}|\nabla u|^{p-2} = b(x)|u|^{r-1}u \tag{2}$$

in $(0,T)\times\partial\Omega$ where $\dfrac{\partial u}{\partial \nu}$ denotes the normal derivative of u on $\partial\Omega$, ν is the unit outward normal to $\partial\Omega$.

In this paper, some sufficient conditions are obtained for solutions of (1)-(2) to be bounded or tending to infinity at a finite time. Those conditions depend on a, b, p, q and r. In special cases we can study the asymptotic behavior of classical solutions. A function u of (t,x) is said to be a classical function in $(0,T)\times\Omega$ if u is uniformly continuous in the closure of $(0,T)\times\Omega$ and the functions $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i^2}$ are continuous in $(0,T)\times\Omega$.

The problem of global existence of the solutions of (1)-(2) arises from many branches of mathematics and applied mathematics and has been discussed by many authors in particular contexts: see for example [3], [5], [7] and [13] for a = constant, $b = 0$ and $p = 2$; [4] for $a \leq -a_0 < 0$, $b \geq 0$, $p = 2$ and $u(0,.)$ is small enough; [9] for $b = 0$ and $p = 2$ and other particular cases.

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The problem of asymptotic behavior of the solutions of (1)-(2) was also studied in specific cases: see [1] and [12] for $a =$ constant, $b = 0$ and $p = 2$ and [9] for $a \leq 0$, $b = 0$ and $p = 2$.

Our results here are not exhaustive but are improvements on the previous studies.

The aim of the second section is to study the conditions which imply that any solution of (1)-(2) blows up at a finite time. Our first result is the following :

Theorem 1 *Assume one of the following conditions*
(H1) $q = r$ *and* $\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(x)d\sigma > 0$
or (H2) $q \neq r$, $\int_\Omega a(x)\,dx > 0$ *and* $b \geq 0$
or (H3) $q \neq r$, $\int_{\partial\Omega} b(x)\,d\sigma > 0$ *and* $a \geq 0$.
Then there exists no positive solution of (1)-(2) *on* $(0,\infty)\times\Omega$.

If we add some assumptions on u, we can extend this result :

Theorem 2 *Assume one of the following conditions*
(H1') $q = r$ *and* $\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(x)d\sigma = 0$
or (H2') $q \neq r$, $\int_\Omega a(x)\,dx = 0$ *and* $b \geq 0$
or (H3') $q \neq r$, $\int_{\partial\Omega} b(x)\,d\sigma = 0$ *and* $a \geq 0$.
Then there exists no positive bounded solution of (1)-(2) *on* $(0,\infty)\times\Omega$.

The next natural condition to be envisaged is $\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(x)d\sigma < 0$. But this condition does not insure the global existence of a solution of (1)-(2). The third part of this paper proposes conditions on a and b such that $\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(x)d\sigma < 0$ and for which the solution of (1)-(2) blows up at a finite time. This result is based on a Keller-Osserman type estimation which is an extension of a result of [8]:

Proposition 1 *Assume* $p \geq 2$, $q > p-1$ *(no condition on* r*) and that there exists a constant* $a_0 > 0$ *such that for all* $x \in \overline{\Omega}$ *:* $a(x) \leq -a_0$, *then there exists a constant* $C = C(p,q,N) > 0$ *such that for all solution* u *of* (1)-(2) *on a set* $(0,T)\times\Omega$ *and all* $(t,x) \in (0,T)\times\Omega$ *:*

$$|u(t,x)| \leq C\left[a_0^{-\frac{1}{q-p+1}} d(x)^{-\frac{p}{q-p+1}} + a_0^{-\frac{1}{q-1}} t^{-\frac{1}{q-1}}\right] . \tag{3}$$

where $d(x)$ *denotes the distance from* x *to the boundary of* Ω.

We deduce from this proposition the next result :

Corollary 1 *Assume* $p \geq 2$, $q > \max(p-1, r)$ *and* $q(p-1) < p(r-1)+1$, *then there exists functions* a *and* b *such that* $\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(x)d\sigma < 0$ *and for which there exists no positive solution of* (1)-(2) *on* $(0,\infty)\times\Omega$.

Then we consider in the fourth part of the paper a stronger condition than $\int_\Omega a(x)\,dx + \int_{\partial\Omega} b(x)d\sigma < 0$, that is $a \leq 0$ and $b \leq 0$. In this case, comparing any solution of (1)-(2) with the corresponding solution of the quasi-linear heat equation, we notice that any solution of (1)-(2) is global. Then we study the asymptotic behavior of those solutions. We start proving that u tends to 0 at infinity. This predictable result was proved in [11] for $p = 2$ and we give the proof when $p > 2$.

Proposition 2 *Assume that* $p \geq 2$, $r > 1$, *(no condition on* q*),* $a \leq 0$ *and* $b \leq 0$ *hold and* b *is non identically equal to* 0 *(*a *can be identically equal to* 0*). Let* u *be a solution of* (1)-(2) *in* $(0,\infty)\times\Omega$. *Then*

$$\lim_{t\to+\infty} u(t,x) = 0 \tag{4}$$

uniformly on $\overline{\Omega}$.

Next we give an *a priori* estimate :

Proposition 3 *Assume that $p \geq 2$, $r > p-1$ (no condition on q), $a \leq 0$ and $b \leq 0$ hold and b is non identically equal to 0 (a can be identically equal to 0). Let u be a solution of (1)-(2) in $(0,+\infty)\times\Omega$. Then there exists $t_0 > 0$ and $C > 0$ such that for all $(t,x) \in (t_0,\infty)\times\Omega$:*

$$|u(t,x)| \leq Ct^{-\frac{1}{r-1}}.$$

Moreover if $q \geq r$, we obtain :

$$|u(t,x)| \leq Ct^{-\frac{1}{r-1}} \leq Ct^{-\frac{1}{q-1}}.$$

The main results of this part are the followings :

Theorem 3 *Assume $p \geq 2$, $q = r > p-1$, $a \leq 0$ and $b \leq 0$ hold and b is non identically equal to 0 (a can be identically equal to 0). Let u be a positive solution of (1)-(2) in $(0,+\infty)\times\Omega$. Then*

$$\lim_{t\to+\infty} t^{\frac{1}{q-1}}u(t,x) = L(r,a,b)$$

uniformly in $\overline{\Omega}$ where

$$L(r,a,b) = \left(\frac{r-1}{|\Omega|}\left(-\int_\Omega a(x)dx - \int_{\partial\Omega} b(x)dx\right)\right)^{-\frac{1}{r-1}} \tag{5}$$

and $|\Omega| = \int_\Omega dx$.

Theorem 4 *Assume that $p \geq 2$, $q > r > p-1$, $a \leq 0$ and $b \leq 0$ hold and b is non identically equal to 0 (a can be identically equal to 0). Let u be a positive solution of (1)-(2) in $(0,\infty)\times\Omega$. Then*

$$\lim_{t\to+\infty} t^{1/(r-1)}u(t,x) = L(r,b) \tag{6}$$

uniformly in $\overline{\Omega}$ where

$$L(r,b) = \left(\frac{1-r}{|\Omega|}\left(\int_{\partial\Omega} b(x)d\sigma\right)\right)^{-\frac{1}{r-1}} \quad . \tag{7}$$

The fifth part is devoted to the case $p = 2$ in which we deal with solutions which can change sign and we make precise the behavior of the solutions.

Theorem 5 *Assume that $p = 2$, $q = r$, $a \leq 0$ and $b \leq 0$ hold and b is non identically equal to 0 (a can be identically equal to 0). Let u be a solution of (1)-(2) in $(0,\infty)\times\Omega$. Then $t^{1/(r-1)}u(t,x)$ converges uniformly in $\overline{\Omega}$ to some limit l as t goes to infinity where $l \in \{0, L(r,a,b), -L(r,a,b)\}$.*

Theorem 6 *Assume that $p = 2$, $q > r > 1$, $a \leq 0$ and $b \leq 0$ and b is non identically equal to 0 (a can be identically equal to 0). Let u be a solution of (1)-(2) in $(0,\infty)\times\Omega$. Then $t^{\frac{1}{r-1}}u(t,x)$ converges uniformly in $\overline{\Omega}$ to some limit l as t goes to infinity where $l \in \{0, L(r,b), -L(r,b)\}$.*

Corollary 2 *Assume the assumptions of theorem 5 or 6 hold and*

$$\lim_{t\to\infty} t^{\frac{1}{r-1}} u(t,x) = L \tag{8}$$

where L is given by (5) if $q = r$ and by (7) if $q > r$. Then

$$\lim_{t\to\infty} t^{\frac{r}{r-1}} \frac{\partial u}{\partial t} = -\frac{L}{r-1} \tag{9}$$

uniformly in $\overline{\Omega}$.

Finally, we study the case where $t^{\frac{1}{r-1}} u(t,x)$ tends to 0 as t goes to infinity.

Theorem 7 *Assume the assumptions of theorem 5 and that u is a solution of (1)-(2) in $(0,\infty)\times\Omega$ such that*

$$\lim_{t\to\infty} t^{\frac{1}{q-1}} \|u(t,.)\|_{L^\infty(\Omega)} = 0. \tag{10}$$

Then there exists $\psi \in Ker(-\Delta + \lambda_1 I)$ such that

$$\lim_{t\to\infty} e^{\lambda_1 t} u(t,x) = \psi(x) \tag{11}$$

uniformly in Ω, where λ_1 is the first nonzero eigenvalue of $-\Delta$ in $W^{1,2}(\Omega)$ with the Neumann boundary condition : $\partial\psi/\partial\nu = 0$.

The proof of this theorem is not written here because it is sufficient to follow the similar proof of theorem 1.11 in [9] which uses a technical lemma introduced by Chen-Matano-Véron [2].

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2 The cases where the solutions blow-up

We begin to proof theorems 1 and 2 adapting the idea of [9].

Proof of theorem 1 : Let u be a positive solution of (1)-(2) in $(0,T)\times\Omega$. Multiplying equation (1) by u^{-r} and integrating on $(0,s)\times\Omega$ with $s < T$, we obtain

$$\int_0^s\int_\Omega \frac{d}{dt}\left(\frac{1}{1-r}u^{1-r}(t,x)\right) dx\,dt \;-\; \int_0^s\int_\Omega (\Delta_p u)(t,x)u^{-r}(t,x)\,dx\,dt \;=\; \int_0^s\int_\Omega a(x)u^{q-r}\,dx\,dt$$

Using Green inequality, it implies

$$\frac{1}{1-r}\int_\Omega u^{1-r}(s,x)\,dx \;-\; \frac{1}{1-r}\int_\Omega u^{1-r}(0,x)\,dx \;+\; \int_0^s\int_\Omega |\nabla u|^{p-2}\nabla u(t,x).\nabla(u^{-r})(t,x)\,dx\,dt$$

$$-\int_0^s\int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}(t,x)(u^{-r})(t,x)\,d\sigma\,dt \;=\; \int_0^s\int_\Omega a(x)u^{q-r}\,dx\,dt$$

we deduce that

$$\frac{1}{1-r}\int_\Omega u^{1-r}(s,x)\,dx \;-\; \frac{1}{1-r}\int_\Omega u^{1-r}(0,x)\,dx \;-\; r\int_0^s\int_\Omega |\nabla u|^p(t,x)u^{-1-r}(t,x)\,dx\,dt$$

$$- \int_0^s \int_{\partial\Omega} b(x)\, d\sigma\, dt \;= \int_0^s \int_\Omega a(x) u^{q-r}\, dx\, dt$$

and then

$$\int_\Omega u^{1-r}(s,x)\, dx \;=\; \int_\Omega u^{1-r}(0,x)\, dx \;+ r(1-r) \int_0^s \int_\Omega |\nabla u|^p(t,x) u^{-1-r}(t,x)\, dx\, dt$$

$$+\; (1-r)s \left(\int_{\partial\Omega} b(x)\, d\sigma \right) + (1-r) \int_0^s \int_\Omega a(x) u^{q-r}\, dx\, dt.$$

If we assume (H1), then

$$\int_\Omega u^{1-q}(s,x)\, dx \;=\; \int_\Omega u^{1-q}(0,x)\, dx \;+ q(1-q) \int_0^s \int_\Omega |\nabla u|^p(t,x) u^{-1-q}(t,x)\, dx\, dt$$

$$+\; (1-q)s \left(\int_{\partial\Omega} b(x)\, d\sigma \;+ \int_\Omega a(x)\, dx \right)$$

$$\leq \int_\Omega u^{1-q}(0,x)\, dx \;+\; (1-q)s \left(\int_{\partial\Omega} b(x)\, d\sigma \;+ \int_\Omega a(x)\, dx \right). \tag{12}$$

If we assume (H3), then

$$\int_\Omega u^{1-r}(s,x)\, dx \;\leq\; \int_\Omega u^{1-r}(0,x)\, dx \;+ r(1-r) \int_0^s \int_\Omega |\nabla u|^p(t,x) u^{-1-r}(t,x)\, dx\, dt$$

$$+\; (1-r)s \left(\int_{\partial\Omega} b(x)\, d\sigma \right). \tag{13}$$

Letting s go to infinity in (12) or (13) implies a contradiction. Therefore the blow-up time for u is finite. If we assume (H2), we multiply equation (1) by u^{-q}, integrate on $(0,s)\times\Omega$ with $s<T$ and use Green inequality to obtain

$$\int_\Omega u^{1-q}(s,x)\, dx \;\leq\; \int_\Omega u^{1-q}(0,x)\, dx \;+ q(1-q) \int_0^s \int_\Omega |\nabla u|^p(t,x) u^{-1-q}(t,x)\, dx\, dt$$

$$+\; (1-q)s \left(\int_\Omega a(x)\, dx \right)$$

which leads us to the same conclusion.

Remark : Moreover, we deduce from (12) that the blow-up time T' of u satisfies

$$T' < \left(\int_\Omega u^{1-q}(0,x)\, dx \right) \left((q-1) \left(\int_{\partial\Omega} b(x)\, d\sigma \;+ \int_\Omega a(x)\, dx \right) \right)^{-1}.$$

The other cases are similar.

Proof of theorem 2 : **Step 1** Assume that there exists a nonzero nonnegative continuous solution v of

$$\begin{cases} -\Delta_p v = a(x) v^q \quad \text{in} \;\; \Omega \\ \dfrac{\partial v}{\partial \nu} |\nabla v|^{p-2} = b(x) v^r \quad \text{sur} \;\; \partial\Omega. \end{cases} \tag{14}$$

We deduce from the strong maximum principle that v is positive. Therefore, multiplying (14) by v^{-q}, we obtain

$$\int_\Omega -\Delta_p v(x)\, v^{-q}(x) \;=\; \int_\Omega a(x)\, dx,$$

the Green inequality and the Robin boundary condition of (14) imply

$$\int_\Omega |\nabla v|^{p-2}\nabla v(x)\, \nabla(v^{-q})(x) \;=\; \int_{\partial\Omega} b(x) v^{r-q} d\sigma \;+\; \int_\Omega a(x)\, dx$$

If we assume (H1'), then

$$-q\int_\Omega |\nabla v(x)|^p\, v^{-1-q}(x)\, dx \;=\; 0$$

From the equation of problem (14) the only constant which is a solution is zero. We conclude that there exists no nonzero nonnegative continuous solution of (14). This is similar if we assume (H2') or (H3').

Step 2 Assume that u is a bounded positive solution of (1)-(2) on $(0,\infty)\times\Omega$. Then we claim that there exists a sequence (t_n) tending to infinity such that $u(t_n,.)$ converges to 0 in $C(\overline{\Omega})$.
Since there exists a constant $M_1 > 0$ such that for all $(x,t)\in(0,\infty)\times\Omega$,

$$0 < u(t,x) \le M_1.$$

The standard quasi-linear regularity theory [10] implies that there exists a constant $M_2 > 0$ such that for any $T \ge 2$, we have

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}([T-1,T+1]\times\overline{\Omega})} \le M_2 \tag{15}$$

for some $\alpha\in(0,1)$. Therefore, the ω-limit set of the trajectory of u in $C(\overline{\Omega})$, defined by

$$\Gamma^+ = \bigcap_{t>0}\left(\overline{\bigcup_{\tau>t}\{u(\tau,.)\}}^{C(\overline{\Omega})}\right),$$

is nonempty. Multiplying (1) by $v = \partial u/\partial t$ and integrating on $(\varepsilon,t)\times\Omega$ for $0<\varepsilon<t$, we obtain

$$\int_\varepsilon^t\int_\Omega \frac{\partial u}{\partial t}(t,x)v(t,x)\, dxdt - \int_\varepsilon^t\int_\Omega \Delta_p u(t,x)v(t,x)\, dxdt = \int_\varepsilon^t\int_\Omega a(x)u^q(t,x)v(t,x)\, dxdt.$$

Using Green inequality, we deduce

$$\int_\varepsilon^t\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2(t,x)\, dxdt + \int_\varepsilon^t\int_\Omega |\nabla u|^{p-2}\nabla u(t,x).\nabla v(t,x)\, dxdt$$

$$-\int_\varepsilon^t\int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}(t,x)v(t,x)\, d\sigma dt = \int_\varepsilon^t\int_\Omega a(x)u^q(t,x)v(t,x)\, dxdt.$$

that is

$$\int_\varepsilon^t\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2(t,x)\, dxdt + \frac{1}{p}\int_\varepsilon^t\int_\Omega \frac{\partial}{\partial t}|\nabla u|^p(s,x)\, dxdt - \int_\varepsilon^t\int_{\partial\Omega} b(x)u^r(s,x)v(s,x)\, d\sigma dt$$

$$= \int_\varepsilon^t \int_\Omega \frac{\partial}{\partial t}\left(\frac{a(x)}{q+1}u^{q+1}(s,x)\right)\,dxdt$$

therefore

$$\int_\varepsilon^t \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2(s,x)\,dxdt = \left[\int_\Omega \left(-\frac{1}{p}|\nabla u|^p(s,t) + \frac{a(x)}{q+1}u^{q+1}(s,x)\right)\,dx\right]_\varepsilon^t + \left[\int_{\partial\Omega} b(x)\frac{u^{r+1}(s,x)}{r+1}\,d\sigma\right]_\varepsilon^t.$$

As $u(t,.)$ is bounded in $W^{1,p}(\Omega)\cap C(\overline{\Omega})$ independently of $t \geq \varepsilon$, we deduce that $\int_\varepsilon^\infty \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2(s,x)\,dxdt$ is finite. Thus there exists a sequence (t_n) tending to infinity and a continuous nonnegative weak solution w of (14) such that $\lim\limits_{t_n\to\infty} \frac{\partial u}{\partial t}(t_n,.) = 0$ in $L^2(\Omega)$ and $\lim\limits_{t_n\to\infty} u(t_n,.) = w(.)$ uniformly in $\overline{\Omega}$. Step 1 allows us to conclude that $w = 0$.

Step 3 As in the proof of theorem 1, we multiply equation (1) by u^{-q} and integrate on $(0,t_n)\times\Omega$, we obtain

$$\int_0^{t_n}\int_\Omega \frac{d}{dt}\left(\frac{1}{1-q}u^{1-q}(t_n,x)\right)\,dx\,dt \;-\; \int_0^{t_n}\int_\Omega (\Delta_p u)(t,x)u^{-q}(t,x)\,dx\,dt \;= t_n\int_\Omega a(x)\,dx.$$

Because of Green equality and the condition on the functions a and b, we get :

$$-\frac{1}{q-1}\int_\Omega u^{1-q}(t_n,x)\,dx + \frac{1}{q-1}\int_\Omega u^{1-q}(0,x)\,dx - q\int_0^{t_n}\int_\Omega |\nabla u|^p(t,x)(u^{-1-q})(t,x)\,dx\,dt =$$

$$\int_0^{t_n}\int_{\partial\Omega} b(x)u^{r-q}(t,x)\,d\sigma\,dt + t_n\int_\Omega a(x)\,dx$$

and thus

$$\frac{1}{q-1}\int_\Omega u^{1-q}(0,x)\,dx \;\geq\; \frac{1}{q-1}\int_\Omega u^{1-q}(t_n,x)\,dx + \int_0^{t_n}\int_{\partial\Omega} b(x)u^{r-q}(t,x)\,d\sigma\,dt + t_n\int_\Omega a(x)\,dx \tag{16}$$

If we assume (H1') or (H2'), then the second member of (16) tends to infinity because of step 2. This contradiction implies that there exists no global bounded positive solution. The case (H3') is similar.

3 The case where : $\int_{\partial\Omega} b(x)d\sigma + \int_\Omega a(x)dx < 0$

Proof of proposition 1 : Let $x_0 \in \Omega$ and $t_1 > 0$ be fixed. Set $k = d^2(x_0)/t_1$ and $r = |x - x_0|$. We introduce the function w defined in $D := \{(x,t) \text{ such that} |x-x_0|^2 < kt,\ 0 < t \leq t_1\}$ by :

$$w(t,x) = \frac{C}{(kt - r^2)^{\frac{p}{q-p+1}}}$$

with $C > 0$ a constant to be determined such that w becomes a super-solution of (1) in D. First $w = \infty$ on the parabolic boundary on D. On the other hand, a straightforward computation gives:

$$\frac{\partial w}{\partial t} - \Delta_p w - a(x)|w|^{q-1}w = -\frac{pCk}{q-p+1}(kt-r^2)^{-\frac{q+1}{q-p+1}} - \left(\frac{2Cp}{q-p+1}\right)^{p-1} r^{p-2}\times$$

$$(kt-r^2)^{-\frac{(q+1)(p-1)}{q-p+1}}\left[(p-1)+\frac{2qpr^2}{q-p+1}(kt-r^2)^{-1}+N\right]$$

$$-a(x)C^q(kt-r^2)^{-\frac{qp}{q-p+1}}.$$

Recall that $a \le -a_0 < 0$ on $\overline{\Omega}$. Thus

$$\frac{\partial w}{\partial t} - \Delta_p w - a(x)|w|^{q-1}w \ge C(kt-r^2)^{-\frac{qp}{q-p+1}} \times \left[a_0 C^{q-1} - \frac{pk}{q-p+1}(kt-r^2)^{\frac{q(p-1)-1}{q-p+1}}\right.$$

$$\left. - \left(\frac{2p}{q-p+1}\right)^{p-1} C^{p-2} r^{p-2}\left[(p-1+N)(kt-r^2)+\frac{2qp}{q-p+1}r^2\right]\right].$$

Since $kt - r^2 \le kt \le kt_1 = d^2(x_0)$ and $r = |x - x_0| \le d(x_0)$, we obtain :

$$\frac{\partial w}{\partial t} - \Delta_p w - a(x)|w|^{q-1}w \ge C(kt-r^2)^{-\frac{qp}{q-p+1}}\times$$

$$\left[\frac{a_0 C^{q-1}}{3} - \frac{p}{q-p+1}\frac{d(x_0)^{2+2\frac{q(p-1)-1}{q-p+1}}}{t_1}\right.$$

$$+\frac{a_0 C^{q-1}}{3} - \left(\frac{2p}{q-p+1}\right)^{p-1}(p-1+N)C^{p-2}d(x_0)^p$$

$$\left.+\frac{a_0 C^{q-1}}{3} - \left(\frac{2p}{q-p+1}\right)^{p-1} C^{p-2}\left(\frac{2pq}{q-p+1}\right)d(x_0)^p\right].$$

Therefore we are looking for a constant C such that

$$\left\{\begin{array}{lll} C^{q-1} & \ge \frac{3}{a_0}\frac{p}{q-p+1}\frac{d(x_0)^{\frac{2p(q-1)}{q-p+1}}}{t_1} \\ C^{q-p+1} & \ge \frac{3}{a_0}\left(\frac{2p}{q-p+1}\right)^{p-1}(N+p-1)d(x_0)^p \\ C^{q-p+1} & \ge \frac{3}{a_0}\left(\frac{2p}{q-p+1}\right)^{p} q d(x_0)^p \end{array}\right. \tag{17}$$

Finally there exists a constant $C > 0$ under the form :

$$C = K(q,p,N)\left[a_0^{-\frac{1}{(q-1)}} d(x_0)^{\frac{2p}{q-p+1}} t_1^{-\frac{1}{(q-1)}} + a_0^{-\frac{1}{q-p+1}} d(x_0)^{\frac{p}{q-p+1}}\right]$$

such that w is a super-solution of (1). The maximum principale implies for all $(t,x) \in D$:

$$u(t,x) \le w(t,x)$$

and in particular :

$$u(t_1,x_0) \le K(q,p,N)\left[a_0^{-\frac{1}{q-1}} t_1^{-\frac{1}{q-1}} + a_0^{-\frac{1}{q-p+1}} d(x_0)^{-\frac{p}{q-p+1}}\right].$$

The same holds for $-u$ and we obtain (3).

Proof of corollary 1 : Let u be a positive solution of (1)-(2) in $(0,T)\times\Omega$. Multiplying equation (1) by u^{-r} and integrating on $(\eta,s)\times\Omega$ with $0<\eta<s<T$, we obtain as in section 1 :

$$\frac{1}{r-1}\int_\Omega u^{1-r}(s,x)dx=\frac{1}{r-1}\int_\Omega u^{1-r}(\eta,x)dx$$

$$-r\int_\eta^s\int_\Omega|\nabla u|^p(t,x)u^{-1-r}(t,x)dxdt-(s-\eta)\int_{\partial\Omega}b(x)d\sigma-\int_\eta^s\int_{\partial\Omega}a(x)u^{q-r}(t,x)dxdt. \quad (18)$$

Since $q>r$, if the function a satisfies for all $x\in\overline{\Omega}$:

$$-a_1=\min_{\overline{\Omega}}a\le a(x)\le -a_0 \quad (19)$$

with $a_0>1$ then proposition 1 implies :

$$-a(x)u(t,x)^{q-r}\le\tilde{C}a_1a_0^{-\frac{q-r}{q-1}}\left(d(x)^{-\frac{p}{q-p+1}}+\eta^{-\frac{1}{q-1}}\right)^{q-r} \quad (20)$$

with $\int_\Omega d(x)^{-\frac{p(q-r)}{q-p+1}}dx<\infty$ for p,q and r such that $\frac{p(q-r)}{q-p+1}<1$ i.e $p(q-r)<q-p+1$ or $q(p-1)<p(r-1)+1$. We deduce from (18) that

$$\frac{1}{r-1}\int_\Omega u^{1-r}(s,x)dx\le\frac{1}{r-1}\int_\Omega u^{1-r}(\eta,x)dx-(s-\eta)\left[\int_{\partial\Omega}b(x)d\sigma-\overline{C}a_1a_0^{-\frac{q-r}{q_1}}\right]. \quad (21)$$

It remains to prove that we can find functions a and b such that :

$$\int_{\partial\Omega}b(x)d\sigma+\int_\Omega a(x)dx<0 \quad (22)$$

$$\forall x\in\overline{\Omega}:-a_1\le a(x)\le -a_0\ with\ a_0>1 \quad (23)$$

$$\int_{\partial\Omega}b(x)d\sigma-\overline{\Omega}a_1a_0^{-\frac{q-r}{q-1}}>0. \quad (24)$$

If (24) holds, then we obtain a contradiction as s tends to infinity in (21) and the corollary is proved. The conditions (22)-(24) are satisfied if (23) and the following condition hold :

$$\overline{C}a_1a_0^{-\frac{q-r}{q-1}}<\int_{\partial\Omega}b(x)d\sigma<a_0|\Omega|\le-\int_\Omega a(x)dx \quad (25)$$

then we can take $a_1=2a_0$ and a_0 sufficiently large such that $2\overline{C}a_0^{-\frac{q-r}{q-1}}<|\Omega|$. After that we choose a and b satisfying (25) which end the proof of the corollary.

4 Asymptotic behavior of global solutions

The remaining of this paper is devoted to the study of the asymptotic behaviour of the global solutions of (1)-(2). We begin with a lemma :

Lemma 1 *Let $\psi \in C^0(\overline{\Omega})$ and $A < 0$. Then there exists $B_0 > 0$ and $S_0 > 0$ such that for all $x \in \Omega$, $B > B_0$ and $S > S_0$:*

$$B + S^A \psi(x) > 0\,.$$

Proof : Since $\psi \in C^0(\overline{\Omega})$, there exists $\zeta > 0$ such that $B + S^A\psi(x) > B - \zeta S^A$ and

$$B - \zeta S^A > 0 \;\Leftrightarrow\; BS^{-A} > \zeta$$

Since $-A > 0$, we can choose $S_0 = 1$ and $B_0 = 2\zeta$.

Proof of proposition 2 : We treat only the case $p > 2$. (see [11] for $p = 2$ where we don't need the following parameters γ and $\tilde{\gamma}$) and b non identically zero (we can adapt the proof of [9] for $b = 0$). We are looking for a supersolution w of (1)-(2) of the form

$$w(t,x) = \gamma t^{-\lambda} + \tilde{\gamma}\psi(x)t^{-\mu} \tag{26}$$

where γ, $\tilde{\gamma}$, λ, μ and ψ are to be determined. By a straightforward computation, we obtain

$$\frac{\partial w}{\partial t} - \Delta_p w - a(x)w^q = -\gamma\lambda t^{-\lambda-1} - \tilde{\gamma}\mu t^{-\mu-1}\psi(x) - \tilde{\gamma}^{p-1}t^{-\mu(p-1)}\Delta_p\psi(x) - a(x)\left(\gamma t^{-\lambda} + \tilde{\gamma}\psi(x)t^{-\mu}\right)^q. \tag{27}$$

We choose $-\lambda - 1 = -\mu(p-1)$ that is $\mu = (\lambda+1)/(p-1)$ and $\tilde{\gamma}^{p-1} = \gamma$ to have

$$\frac{\partial w}{\partial t} - \Delta_p w - a(x)w^q = t^{-\lambda-1}\left[\gamma(-\lambda - \Delta_p\psi(x)) - \frac{\lambda+1}{p-1}\gamma^{\frac{1}{p-1}}t^{\lambda-\frac{\lambda+1}{p-1}}\psi(x)\right.$$

$$\left. -a(x)t^{\lambda+1-\lambda q}\left(\gamma + \gamma^{\frac{1}{p-1}}\psi(x)t^{\lambda-\frac{\lambda+1}{p-1}}\right)^q\right].$$

Then we look for a solution ψ of

$$\begin{cases} -\lambda - \Delta_p\psi = \alpha & in \quad \Omega \\ \dfrac{\partial\psi}{\partial\nu}|\nabla\psi|^{p-2} = g & on \quad \partial\Omega \end{cases} \tag{28}$$

where $\alpha \in (0,\infty)$ and g are to be determined so that w would be a supersolution of (1)-(2). Remark if ψ is a solution of (28) then $\psi + C$ is also a solution of (28). Thus we assume that ψ is positive. If we choose $\lambda - \frac{\lambda+1}{p-1} < 0$ that is

$$\lambda < \frac{1}{p-2} \tag{29}$$

so that $t^{\lambda-\frac{\lambda+1}{p-1}} \to 0$ when t tends to $+\infty$, and

$$\alpha \geq \frac{\lambda+1}{p-1}\,, \tag{30}$$

we obtain since $a \leq 0$:

$$\begin{array}{rcl} \dfrac{\partial w}{\partial t} - \Delta_p w - a(x)w^q & \geq & t^{-\lambda-1}\left[\gamma\alpha - \frac{\lambda+1}{p-1}\gamma^{\frac{1}{p-1}}t^{\lambda-\frac{\lambda+1}{p-1}}\psi(x)\right] \\ & \geq & t^{-\lambda-1}\left[\frac{\lambda+1}{p-1}\gamma^{\frac{1}{p-1}}\left(\gamma^{\frac{p-2}{p-1}} - \psi(x)t^{\lambda-\frac{\lambda+1}{p-1}}\right)\right]. \end{array} \tag{31}$$

By lemma 1, there exists $t_0 > 0$ and $\gamma_0 > 0$ such that for all $t \geq t_0$, $x \in \overline{\Omega}$ and $\gamma \geq \gamma_0$:

$$\gamma^{\frac{p-2}{p-1}} - \psi(x)t^{\lambda-\frac{\lambda+1}{p-1}} > 0.$$

Thus $\dfrac{\partial w}{\partial t} - \Delta_p w - a(x)w^q \geq 0$ for all $t \geq t_0$, $x \in \Omega$ and $\gamma \geq \gamma_0$. On the other hand, the boundary condition leads us to have

$$\frac{\partial w}{\partial \nu}|\nabla w|^{p-2} - bw^r = \gamma t^{-\lambda-1}g - bt^{-\lambda r}\left(\gamma + \gamma^{\frac{1}{p-1}}\psi(x)t^{\lambda-\frac{\lambda+1}{p-1}}\right)^r \geq 0\ . \tag{32}$$

Then we choose $g = \beta b$ and condition (32) is equivalent to

$$-bt^{-\lambda r}\left[\gamma^{\frac{r}{p-1}}\left(\frac{\gamma^{\frac{p-2}{p-1}}}{2} + \frac{\gamma^{\frac{p-2}{p-1}}}{2} + \psi(x)t^{\lambda-\frac{\lambda+1}{p-1}}\right)^r - \beta\gamma t^{-\lambda-1+\lambda r}\right] \geq 0 \tag{33}$$

But $\frac{\gamma^{\frac{p-2}{p-1}}}{2} + \psi(x)t^{\lambda-\frac{\lambda+1}{p-1}} > 0$, and since $b \leq 0$, we have

$$\begin{aligned}
&\frac{\partial w}{\partial \nu}|\nabla w|^{p-2} - bw^r \\
&\geq -bt^{-\lambda r}\left[\gamma^{\frac{r}{p-1}}\left(\frac{\gamma^{\frac{p-2}{p-1}}}{2}\right)^r - \beta\gamma t^{-\lambda-1+\lambda r}\right] \\
&\geq -bt^{-\lambda r}\gamma\left[\frac{\gamma^{r-1}}{2^r} - \beta t^{-\lambda-1+\lambda r}\right]
\end{aligned} \tag{34}$$

which is positive by lemma 1 with t_0 and γ_0 depending on β, under the condition $\lambda r - \lambda - 1 < 0$ that is

$$\lambda < \frac{1}{r-1}\ . \tag{35}$$

The compatibility condition for this nonlinear Neumann problem leads us to have

$$\alpha = -\lambda + \beta\left(\frac{-1}{mes(\Omega)}\int_{\partial\Omega} b(x)d\sigma\right) \tag{36}$$

Thus we first choose from (29) and (35) : $0 < \lambda < \min\left(\frac{1}{p-2}, \frac{1}{r-1}\right)$. Next from (30) and (36), we choose β large enough such that $-\lambda + \beta\left(\frac{-1}{mes(\Omega)}\int_{\partial\Omega} b(x)d\sigma\right) \geq \frac{\lambda+1}{p-1}$. Then we define α by (36) and finally we obtain ψ solution of (28) and t_0 and γ_0 from lemma 1 such that $\dfrac{\partial w}{\partial t} - \Delta_p w - a(x)w^q \geq 0$ and $\dfrac{\partial w}{\partial \nu}|\nabla w|^{p-2} - bw^r \geq 0$ for all $t \geq t_0$, $x \in \Omega$ and $\gamma \geq \gamma_0$. It remains to treat the initial data. We take γ large enough such that

$$u(t_0, x) \leq \gamma t_0^{-\lambda} \leq w(t_0, x)$$

on $\overline{\Omega}$ and we conclude, from the comparison principle that

$$u(x, t) \leq w(t, x) \tag{37}$$

for all $(t, x) \in [t_0, \infty) \times \overline{\Omega}$. In the same way, we prove that $-u(t, x) \leq w(t, x)$ for all $(t, x) \in [t_0, \infty) \times \overline{\Omega}$ wich implies (4).

Proof of the proposition 3 : the only difference with the proof of proposition 2 is that we take $\lambda = 1/(r-1)$ in (35) and also choose γ such that $\gamma \geq (e^r\beta)^{\frac{1}{(r-1)}}$ in (34).

Proof of theorems 3 and 4 : In this proof we denote by L the constant $L(r,a,b)$ defined in (5) or $L(r,b)$ defined in (7), we shall precise later. Let u be a positive solution of (1)-(2) in $(0,\infty)\times\Omega$.

Step 1: Supersolution.
Let $\varepsilon>0$. We look for a supersolution of (1)-(2) of the form :

$$w(t,x)=\left(L+\frac{\varepsilon}{2}\right)t^{-\frac{1}{r-1}}+\left(L+\frac{\varepsilon}{2}\right)^{\frac{r}{p-1}}\psi(x)t^{-\frac{r}{(p-1)(r-1)}}.$$

where ψ is to be determined. A straightforward computation gives :

$$\frac{\partial w}{\partial t}-\Delta_p w-a(x)|w|^{q-1}w=t^{-\frac{1}{r-1}-1}\left[-\frac{1}{r-1}\left(L+\frac{\varepsilon}{2}\right)-\right.$$

$$\left.-\frac{r}{(p-1)(r-1)}\left(L+\frac{\varepsilon}{2}\right)^{\frac{r}{p-1}}\psi(x)t^{-\frac{r-p+1}{(p-1)(r-1)}}-\left(L+\frac{\varepsilon}{2}\right)^{r}\Delta_p\psi\right]-a(x)|w|^{q-1}w. \tag{38}$$

Then we consider the following Neumann boundary value problem

$$\begin{cases} -\frac{1}{r-1}\left(L+\frac{\varepsilon}{2}\right)-\left(L+\frac{\varepsilon}{2}\right)^{r}\Delta_p\psi-a(x)\zeta\left(L+\frac{\varepsilon}{2}\right)^{r}=\eta \text{ in } \Omega \\ \frac{\partial\psi}{\partial\nu}|\nabla\psi|^{p-2}=b \text{ on } \partial\Omega \end{cases} \tag{39}$$

where $\zeta=1$ if $q=r$ and $\zeta=0$ if $q>r$. Remark if ψ is a solution of (39), then $\psi+C$ is also a solution of (39). Thus we assume that ψ is choosen positive. Therefore (38) becomes :

$$\frac{\partial w}{\partial t}-\Delta_p w-a(x)|w|^{q-1}w=t^{-\frac{1}{r-1}-1}\left[\eta-\frac{r}{(p-1)(r-1)}\left(L+\frac{\varepsilon}{2}\right)^{\frac{r}{p-1}}\psi(x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right.$$

$$\left.+a(x)\zeta\left(L+\frac{\varepsilon}{2}\right)^{r}-a(x)\left(L+\frac{\varepsilon}{2}\right)^{q}t^{-\frac{q-r}{r-1}}\left(1+\left(L+\frac{\varepsilon}{2}\right)^{\frac{r-p+1}{p-1}}\psi(x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right)^{q}\right] \tag{40}$$

Now there exists $t_0=t_0(\varepsilon,\psi)$ and there exists a uniformly bounded positive function M on $[t_0,\infty)\times\overline{\Omega}$ such that

$$\left(1+\left(L+\frac{\varepsilon}{2}\right)^{\frac{r-p+1}{p-1}}\psi(x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right)^{q}=1+M(t,x)t^{-\frac{r-p+1}{(p-1)(r-1)}} \tag{41}$$

for all $(t,x)\in[t_0,\infty)\times\overline{\Omega}$. We distiguish two cases :

First case : $q=r$. Thus $\zeta=1$, (40) and (41) imply

$$\frac{\partial w}{\partial t}-\Delta_p w-a(x)|w|^{r-1}w=t^{-\frac{1}{r-1}-1}\left[\eta-\frac{r}{(p-1)(r-1)}\left(L+\frac{\varepsilon}{2}\right)^{\frac{r}{p-1}}\psi(x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right.$$

$$\left.-a(x)\left(L+\frac{\varepsilon}{2}\right)^{r}M(t,x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right]. \tag{42}$$

Next, the Neumann compatibility condition is:

$$-\frac{|\Omega|}{r-1}\left(L+\frac{\varepsilon}{2}\right)-\left(L+\frac{\varepsilon}{2}\right)^{r}\int_{\partial\Omega}b(x)d\sigma-\left(L+\frac{\varepsilon}{2}\right)^{r}\int_{\Omega}a(x)dx=\eta|\Omega|$$

which is equivalent with $L = L(r, a, b)$ defined in (5) to :

$$\frac{1}{r-1}\left(L+\frac{\varepsilon}{2}\right)^r\left[L^{1-r}-\left(L+\frac{\varepsilon}{2}\right)^{1-r}\right]=\eta. \tag{43}$$

Second case : $q > r$. Thus $\zeta = 0$, (40) and (41) imply

$$\frac{\partial w}{\partial t}-\Delta_p w-a(x)|w|^{q-1}w=t^{-\frac{1}{r-1}-1}\left[\eta-\frac{r}{(p-1)(r-1)}\left(L+\frac{\varepsilon}{2}\right)^{\frac{r}{p-1}}\psi(x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right.$$

$$\left.-a(x)\left(L+\frac{\varepsilon}{2}\right)^q t^{-\frac{q-r}{r-1}}\left(1+M(t,x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right)\right]. \tag{44}$$

Next, the Neumann compatibility condition is:

$$-\frac{|\Omega|}{r-1}\left(L+\frac{\varepsilon}{2}\right)-\left(L+\frac{\varepsilon}{2}\right)^r\int_{\partial\Omega}b(x)d\sigma=\eta|\Omega|$$

which is equivalent with $L = L(r, b)$ defined in (7) to (43).
On the other hand we consider the boundary condition for both cases. Using (39) and (41), we obtain

$$\frac{\partial w}{\partial \nu}|\nabla w|^{p-2}-b|w|^{r-1}w=-bM(t,x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\geq 0.$$

Finally for ε given, we first choose $\eta > 0$ defined by (43). Therefore ψ is determined from (39). Then there exists $T > t_0$ such that

$$\eta-\frac{r}{(p-1)(r-1)}\left(L+\frac{\varepsilon}{2}\right)^{\frac{r}{p-1}}\psi(x)t^{-\frac{r-p+1}{(r-1)(p-1)}}-a(x)\left(L+\frac{\varepsilon}{2}\right)^q t^{-\frac{r-p+1}{(r-1)(p-1)}}M(t,x)\geq 0$$

if $q = r$ and such that

$$\eta-\frac{r}{(p-1)(r-1)}\left(L+\frac{\varepsilon}{2}\right)^{\frac{r}{p-1}}\psi(x)t^{-\frac{r-p+1}{(p-1)(r-1)}}-a(x)\left(L+\frac{\varepsilon}{2}\right)^q t^{-\frac{q-r}{r-1}}\left(1+M(t,x)t^{-\frac{r-p+1}{(p-1)(r-1)}}\right)\geq 0$$

if $q > r$ so that, because of (42) if $q = r$ and (44) if $q > r$:

$$\frac{\partial w}{\partial t}-\Delta_p w-a(x)|w|^{q-1}w\geq 0$$

on $(T, \infty)\times\overline{\Omega}$ and

$$\frac{\partial w}{\partial \nu}|\nabla w|^{p-2}-b|w|^{r-1}w\geq 0$$

on $(T, \infty)\times\partial\Omega$. From proposition 2, there exists $\tau \geq T$ such that

$$u(t,x)\leq w(T,x)$$

for all $(t, x) \in [\tau, \infty)\times\overline{\Omega}$. We apply the comparison principle to $(t, x) \mapsto w(T + t, x)$ and $(t, x) \mapsto u(t + \tau, x)$ on $(0, \infty)\times\Omega$ and we conclude that

$$u(t+\tau,x)\leq w(t+T,x)$$

for all $(t,x) \in (0,\infty) \times \overline{\Omega}$, which implies that there exists $t_1 > \tau$ such that

$$(t+T)^{\frac{1}{r-1}} u(t+\tau,x) \leq L + \varepsilon$$

for all $(t,x) \in (t_1,\infty) \times \overline{\Omega}$ and

$$\limsup_{t \to +\infty} t^{\frac{1}{r-1}} u(t,x) \leq L \tag{45}$$

uniformly on $\overline{\Omega}$.

Step 2: Subsolution. The proof is similar to step 1.
Let $\varepsilon > 0$. We look for a subsolution of (1)-(2) of the form

$$w(t,x) = \left(L - \frac{\varepsilon}{2}\right) t^{-\frac{1}{r-1}} + \left(L - \frac{\varepsilon}{2}\right)^{\frac{r}{p-1}} \psi(x) t^{-\frac{r}{(p-1)(r-1)}}.$$

We keep (39)-(40) replacing $L + \frac{\varepsilon}{2}$ by $L - \frac{\varepsilon}{2}$ but now we choose ψ negative in Ω, then M is also negative. The Neumann compatibility condition leads us to choose η defined by

$$\eta = \frac{1}{r-1} \left(L - \frac{\varepsilon}{2}\right)^r \left[L^{1-r} - \left(L - \frac{\varepsilon}{2}\right)^{1-r} \right] < 0. \tag{46}$$

As in step 1, we deduce that $\dfrac{\partial w}{\partial t} - \Delta_p w - a(x)|w|^{q-1}w \leq 0$ on a set of the form $(T,\infty) \times \Omega$ and $\dfrac{\partial w}{\partial \nu} |\nabla w|^{p-2} - b|w|^{r-1}w \leq 0$ on $(T,\infty) \times \partial\Omega$.
Since w tends to 0 when t tends to $+\infty$ uniformly in $\overline{\Omega}$, there exists $\tilde{T} \geq T$ such that

$$u(T+1,x) \geq w(\tilde{T},x) \quad \forall x \in \overline{\Omega}.$$

We apply the comparison principle to $(t,x) \mapsto u(t+T+1,x)$ and $(t,x) \mapsto w(t+\tilde{T},x)$ on $(0,\infty) \times \Omega$ and we conclude that

$$u(t+T+1,x) \geq w(t+\tilde{T},x)$$

for all $(t,x) \in (0,\infty) \times \overline{\Omega}$, which implies that there exists $T_1 > \tilde{T}$ such that

$$(t+\tilde{T})^{\frac{1}{r-1}} u(t+T+1,x) \geq L - \frac{\varepsilon}{2} + \left(L - \frac{\varepsilon}{2}\right)^{\frac{r}{p-1}} \psi(x) (t+\tilde{T})^{-\frac{r-p+1}{(p-1)(r-1)}} \geq L - \varepsilon$$

for all $(t,x) \in (T_1,\infty) \times \overline{\Omega}$ and we conclude that

$$\liminf_{t \to +\infty} t^{\frac{1}{q-1}} u(t,x) \geq L.$$

uniformly in $\overline{\Omega}$ which ends the proof of theorem 3 and 4.

5 The case of the Laplacian

The next result needs no sign assumption on $a(x)$ and $b(x)$. We introduce the mean function $\overline{u}$ of u definied by

$$\overline{u}(t) = \frac{1}{|\Omega|} \int_\Omega u(t,x)dx \tag{47}$$

Lemma 2 *Let u be a solution of (1)-(2) in $(0,\infty)\times\Omega$. Assume that there exists a constant $K>0$ such that*

$$|u(t,x)|\leq Kt^{-\frac{1}{r-1}} \quad \textit{and} \quad |u(t,x)|\leq Kt^{-\frac{1}{q-1}} \tag{48}$$

in $[0,\infty)\times\overline{\Omega}$. Then there exists a constant $C>0$ such that

$$\|u(t,.)-\overline{u}(t)\|_{L^\infty(\Omega)}\leq C(t+1)^{-\frac{r}{r-1}} \tag{49}$$

for all $t\geq 0$

Proof : We don't direcly compare the function u and its mean function but we introduce the function v definie by $v(t,x)=t^{\frac{1}{r-1}}u(t,x)$ for $t>0$. This function satisfies

$$\left\{\begin{array}{l}\dfrac{\partial v}{\partial t}-\Delta v-t^{-\frac{q-1}{r-1}}a(x)|v|^{q-1}v-\dfrac{1}{(r-1)t}v=0 \quad in \quad (0,\infty)\times\Omega\\ \dfrac{\partial v}{\partial \nu}-\dfrac{b}{t}|v|^{r-1}v=0 \quad on \quad (0,\infty)\times\partial\Omega\end{array}\right. \tag{50}$$

and if we set $w=v-\overline{v}$, we obtain

$$\left\{\begin{array}{l}\dfrac{\partial w}{\partial t}-\Delta w+f=0 \quad in \quad (0,\infty)\times\Omega\\ \dfrac{\partial w}{\partial \nu}-\dfrac{b}{t}|v|^{r-1}v=0 \quad on \quad (0,\infty)\times\partial\Omega\end{array}\right. \tag{51}$$

where f is definied by

$$f(t,x)=-\frac{1}{t}\left[\frac{1}{r-1}(v(t,x)-\overline{v}(t))-\frac{1}{|\Omega|}\int_{\partial\Omega}b(x)|v|^{r-1}(t,x)v(t,x)d\sigma\right]$$

$$-t^{-\frac{q-1}{r-1}}\left[a(x)|v|^{q-1}v(t,x)-\frac{1}{|\Omega|}\int_\Omega a(x)|v|^{q-1}v(t,x)dx\right]\ . \tag{52}$$

Note that f is bounded because of (48). We introduce the solution z of the following heat equation:

$$\left\{\begin{array}{l}\dfrac{\partial z}{\partial t}-\Delta z=0 \quad in \quad (0,\infty)\times\Omega\\ \dfrac{\partial z}{\partial \nu}=\dfrac{b}{t}|v|^{q-1}v \quad on \quad (0,\infty)\times\partial\Omega\end{array}\right. \tag{53}$$

and set $W=w-z$. Then

$$\left\{\begin{array}{l}\dfrac{\partial W}{\partial t}-\Delta W+f=0 \quad in \quad (0,\infty)\times\Omega\\ \dfrac{\partial W}{\partial \nu}=0 \quad on \quad (0,\infty)\times\partial\Omega.\end{array}\right. \tag{54}$$

Introducing the continuous semigroup of contractions of $L^2(\Omega)$ generated by the Laplacian with Neumann boundary data and its restriction to $L^\infty(\Omega)\cap(Ker(-\Delta))^\perp$, the results of [6] and [9] p.128 lead us to the existence of a constant $G>0$ such that

$$\|W(t+1,.)\|_{L^\infty(\Omega)}\leq\frac{G}{t}$$

for all t near infinity. We conclude that this is the same to w and we obtain (49).

The **proof of theorems 5 and 6** is exacly the same as in [9], using both theorems 3 and 4 and lemma 2.

Proof of corollary 2 : Let $v(t,x) = t^{\frac{r}{r-1}} u(t,x)$. Then

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = f & \text{in} \quad (0,\infty)\times\Omega \\ \dfrac{\partial v}{\partial \nu} = b t^{\frac{r}{r-1}} u^r & \text{on} \quad (0,\infty)\times\partial\Omega \end{cases} \tag{55}$$

where $f(t,x) = \frac{r}{r-1} t^{\frac{1}{r-1}} u(t,x) + a(x) t^{\frac{r}{r-1}} (u(t,x))^q$. Now we introduce

$$w = v - \overline{v} = v - \frac{1}{|\Omega|}\int_\Omega v(t,x)\,dx$$

and

$$g = f - \overline{f}.$$

Thus w is solution of

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w = g - \dfrac{t^{\frac{r}{r-1}}}{|\Omega|}\displaystyle\int_{\partial\Omega} b(x)(u(t,x))^q\,d\sigma & \text{in} \quad (0,\infty)\times\Omega \\ \dfrac{\partial w}{\partial \nu} = b t^{\frac{r}{r-1}} u^q & \text{on} \quad (0,\infty)\times\partial\Omega. \end{cases}$$

As in the proof of theorem 2, we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial w}{\partial t}\right) - \Delta\left(\dfrac{\partial w}{\partial t}\right) = G & \text{in} \quad (0,\infty)\times\Omega \\ \dfrac{\partial}{\partial \nu}\left(\dfrac{\partial w}{\partial t}\right) = B & \text{on} \quad (0,\infty)\times\partial\Omega \end{cases}$$

with

$$G(t,x) = \frac{\partial}{\partial t}\left(f - \overline{f}\right) - \frac{r}{r-1}\frac{t^{\frac{r}{r-1}-1}}{|\Omega|}\int_{\partial\Omega} b(x)(u(t,x))^q\,d\sigma$$

$$-q\frac{t^{\frac{r}{r-1}}}{|\Omega|}\int_{\partial\Omega} b(x)(u(t,x))^{q-1}\frac{\partial u}{\partial t}(t,x)\,d\sigma,$$

$$B(t,x) = \frac{r}{r-1} t^{\frac{1}{r-1}} b(x)(u(t,x))^q + q t^{\frac{r}{r-1}} b(x)(u(t,x))^{q-1}\frac{\partial u}{\partial t}(t,x),$$

and

$$\frac{\partial f}{\partial t}(t,x) = \frac{1}{t}\left(\frac{r}{(r-1)^2} t^{\frac{1}{r-1}} u(t,x) + \frac{r}{r-1} t^{\frac{r}{r-1}}\frac{\partial u}{\partial t}(t,x)\right.$$

$$\left. + \frac{r}{r-1} a(x) t^{\frac{r}{r-1}} (u(t,x))^q + q a(x) t^{\frac{r}{r-1}} u^{q-1}\frac{\partial u}{\partial t}\right).$$

As a consequence of classical estimates for parabolic equations, there exists $t_0 > 0$ such that $t^{\frac{r}{r-1}}\frac{\partial u}{\partial t}(t,x)$ remains bounded in $(t_0,\infty)\times\overline{\Omega}$ and since $q \geq r$, we have $|t^{\frac{r}{r-1}} u(t,x)^q| \leq |t^{\frac{1}{r-1}} u(t,x)|^q$ for all $(t,x) \in (t_0,\infty)\times\overline{\Omega}$ for t_0 sufficiently large. Therefore, because of (8), there exists a positive constant M such that

$$|G(t,x)| \leq \frac{M}{t}$$

for all $(t,x) \in (t_0,\infty) \times \overline{\Omega}$ and

$$|B(t,x)| \leq \frac{M}{t}$$

for all $(t,x) \in (t_0,\infty) \times \partial\Omega$. Thus, as in the proof of lemma 2, we deduce

$$\left\| \frac{\partial v}{\partial t}(.,t) - \frac{\partial \overline{v}}{\partial t}(t) \right\|_{L^\infty(\overline{\Omega})} = \left\| \frac{\partial w}{\partial t}(.,t) \right\|_{L^\infty(\overline{\Omega})} \leq \frac{C}{t} \tag{56}$$

for t large enough and some positive constant C. By the definition of v, we have :

$$\frac{\partial v}{\partial t} = \frac{r}{r-1} t^{\frac{1}{r-1}} u + t^{\frac{r}{r-1}} \frac{\partial u}{\partial t} \tag{57}$$

and because of (55), we have

$$\frac{\partial \overline{v}}{\partial t}(t) = \overline{f} + \frac{1}{|\Omega|} \int_{\partial\Omega} b(x) (t^{\frac{1}{r-1}} u(t,x))^r \, d\sigma$$

$$= \frac{r}{r-1} t^{\frac{1}{r-1}} \overline{u} + \frac{t^{-\frac{q-r}{r-1}}}{|\Omega|} \int_\Omega a(x) (t^{\frac{1}{r-1}} u(t,x))^q \, dx + \frac{1}{|\Omega|} \int_{\partial\Omega} b(x) (t^{\frac{1}{r-1}} u(t,x))^r \, d\sigma \ .$$

We distinguish two cases.

First case : $q = r$. From (8), we obtain

$$\lim_{t\to\infty} \frac{\partial \overline{v}}{\partial t} = \frac{q}{q-1} L + \frac{1}{|\Omega|} \left(\int_\Omega a(x) L^q dx + \int_{\partial\Omega} b(x) L^q \, d\sigma \right) = L. \tag{58}$$

Second case : $q > r$. From (8), we obtain

$$\lim_{t\to\infty} \frac{\partial \overline{v}}{\partial t} = \frac{r}{r-1} L + \frac{1}{|\Omega|} \left(\int_\Omega b(x) L^r d\sigma \right) = L. \tag{59}$$

Because of (56), (57) and (58) or (59) we get

$$\lim_{t\to\infty} t^{\frac{r}{r-1}} \frac{\partial u}{\partial t} = L - \frac{r}{r-1} L = -\frac{L}{r-1}$$

uniformly in $\overline{\Omega}$ which is (9).

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