



**HAL**  
open science

## Conductance in discrete dynamical systems

S. Fernandes, C. Grácio, C. Ramos

► **To cite this version:**

S. Fernandes, C. Grácio, C. Ramos. Conductance in discrete dynamical systems. *Nonlinear Dynamics*, 2010, 61 (3), pp.435-442. 10.1007/s11071-010-9660-3 . hal-00567029

**HAL Id: hal-00567029**

**<https://hal.science/hal-00567029>**

Submitted on 18 Feb 2011

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Conductance in discrete dynamical systems

S. Fernandes · C. Grácio · C. Ramos

Received: 6 March 2009 / Accepted: 12 January 2010 / Published online: 18 February 2010  
© Springer Science+Business Media B.V. 2010

**Abstract** We will cover results related to the study of the conductance on digraphs arising from discrete dynamical systems. Several definitions of conductance are known and we present a study which gives a comparison between them in order to choose the appropriate definition to applications. The study makes a strong use of symbolic dynamics. As an application, we analyze the mechanical system of the nonlinear pendulum.

**Keywords** Discrete dynamical systems · Symbolic dynamics · Invariants · Conductance

## 1 Introduction

Our main concern is the search for invariants which can distinguish nonequivalent dynamics, in the topological sense, considering diverse discrete dynamical systems, arising from the iterates of a map in the interval.

Many nonlinear dynamical systems, either discrete or continuous, can be transformed into difference equations or iterated maps of the interval. In turn,

we can obtain a symbolic or combinatorial description of the original system which may be very useful to compute measures of the complexity of the system; see [18].

We use the context of graph theory (see [3], [14], and [15]) and discrete dynamical systems to establish the analogy between some concepts ([6] and [5]): edge and vertex expansion versus mixing time, graph isoperimetric numbers versus conductance, and the Laplacian of a graph versus the discrete Laplacian.

Our interest in conductance has begun with the notion of rapid mixing. It is known that an ergodic Markov chain converges to a stationary distribution, the equilibrium. The time of convergence, the mixing time, differs from one system to another and techniques to know how long the chain must evolve to attain some given proximity of the stationary distribution are of great importance in problems like coupling and synchronization. Intuitively, a random walk is rapidly mixing if there are no bottlenecks or funnels. In graph theory, there are several works relating the rate of convergence with the isoperimetric number or conductance and the second eigenvalue of the Laplacian of a graph (see [2]).

Consider a discrete dynamical system described by the iterates of a Markov map  $f$  in the interval  $I$  and let  $(I_i)_{i=1,\dots,n}$  be the corresponding Markov partition. For each such map, there is a finite topological Markov chain or Markov subshift according to the following definitions.

---

All authors are supported by the CIMA-UE.

---

S. Fernandes (✉) · C. Grácio · C. Ramos  
Departamento de Matemática da Universidade de Évora,  
R. Romão Ramalho, 59, 7000-671, Évora, Portugal  
e-mail: saf@uevora.pt

A subshift is a pair  $(\Lambda, \sigma)$ , where  $\Lambda$  is the space of admissible sequences in some alphabet, with the product topology, and  $\sigma$  is the shift map  $\sigma(\xi_1\xi_2\xi_3\dots) = (\xi_2\xi_3\dots)$ , for  $(\xi_1\xi_2\xi_3\dots) \in \Lambda$ , with  $\sigma(\Lambda) = \Lambda$ . A Markov subshift is a particular case of a subshift and is characterized as follows: Let  $\Sigma = \{1, 2, \dots, n\}$  be the alphabet set and consider  $\Sigma^{\mathbb{N}}$  endowed with the product topology. The  $n$ -full shift is  $(\Sigma^{\mathbb{N}}, \sigma)$ . The Markov subshift determined by  $A = (a_{ij})_{i,j=1}^n$  (a 0–1 transition matrix) is the pair  $(\Sigma_A, \sigma)$ , where  $\Sigma_A$  is the  $\sigma$ -invariant closed subset of  $\Sigma^{\mathbb{N}}$  given by  $\Sigma_A = \{(\xi_i)_{i \geq 1} \in \Sigma^{\mathbb{N}} : a_{\xi_i \xi_{i+1}} = 1\}$ .

In our case, the Markov subshift  $(\Sigma_{A_f}, \sigma)$  is obtained from the alphabet  $\Sigma = \{1, 2, \dots, n\}$  and the transition matrix  $A_f$  defined by

$$(A_f)_{ij} := \begin{cases} 1 & \text{if } f(I_i) \supseteq I_j \\ 0 & \text{otherwise.} \end{cases}$$

There is a unique Markov chain associated with the subshift  $\Sigma_A$ , via the Parry measure; see [19], which is induced by the Perron eigenvectors of the matrix  $A$ . This Markov chain is characterized by a stochastic matrix  $P$  which depends only on the matrix  $A$  and the referred eigenvectors.

Probability distributions in the state space are given by row vectors, so that if the initial distribution is  $p(0)$  then the  $t$ -step distribution is given by  $p(t) = p(0)P^t$ .

Suppose that the obtained Markov chain is aperiodic (this is true whenever  $A$  is aperiodic itself), so that there is a unique stationary distribution  $\pi$ , with

- (a)  $\pi_i > 0$ , for all  $i$ ;
- (b)  $\sum_i \pi_i = 1$ ;
- (c)  $\pi P = \pi$ .

Moreover, suppose that the chain is ergodic and so

$$\lim_{t \rightarrow \infty} (P^t)_{ij} \rightarrow \pi_j \quad \text{for all } i, j.$$

For each such dynamical system, there is an underlying graph  $G$ , whose vertex set  $V$  is the set of intervals of the Markov partition and whose edges  $E$  correspond to nonzero entries of the transition matrix  $A_f = (a_{ij})$ .

Denote by  $V = \{1, 2, \dots, n\}$  the set of vertices. Then

- if  $a_{ij} = 1$ , there is an edge that joins the vertex  $i$  with the vertex  $j$ .
- if  $a_{ij} = 0$ , there is no edge between these vertices.

The quantities we are looking for, in discrete dynamical systems, can be viewed as quantities in graph theory. We will introduce them and present a study of some ordered tree of piecewise linear Markov maps, described below.

Bezrukov in a survey on isoperimetric problems on graphs; see [2], presented four different versions of isoperimetric numbers on graphs. They were established to determine the edge expansion/congestion of the graph and so to prove rapid mixing. This edge expansion, in a weighted version, is also called conductance of the graph.

All these definitions were set for a simple, non-weighted, connected graph  $G = (V, E)$ , where  $V$  is the set of vertices and  $E$  the set of edges. Let  $|V| = n$  denote the number of vertices in  $G$ ,  $\text{deg}(i)$  denote the degree of the vertex  $i$  and for  $A \subset V$ ,  $\text{Vol}(A) = \sum_{i \in A} \text{deg}(i)$ . Let  $\theta_G(A) = \{(i, j) \in E : i \in A, j \notin A\}$  and let  $|A|$  denote the number of vertices in  $A$ . The four definitions are as follows:

$$i_1(G) = \min_{\emptyset \neq A \subset V} \frac{|\theta_G(A)|}{\min\{|A|, n - |A|\}};$$

$$i_2(G) = \min_{\emptyset \neq A \subset V} \frac{|\theta_G(A)|}{\min\{\text{Vol}(A), \text{Vol}(\bar{A})\}};$$

$$i_3(G) = \min_{\emptyset \neq A \subset V} \frac{|\theta_G(A)| n}{|A| \cdot |\bar{A}|};$$

$$i_4(G) = \min_{\emptyset \neq A \subset V} \frac{|\theta_G(A)|}{|A| \cdot \log(\frac{n}{|A|})}.$$

In [6] was introduced the conductance of a discrete dynamical system using the adapted weighted version of  $i_1$  as was done in [3] for the correspondent weighted graph

$$\Phi_1 = \min_{\emptyset \neq U \subset V} \frac{\sum_{i \in U, j \in \bar{U}} \pi_i P_{ij}}{\min\{\sum_{i \in U} \pi_i, \sum_{i \in \bar{U}} \pi_i\}}. \tag{1}$$

In probabilistic terms, this ratio measures the ability of the system in equilibrium to leave small parts of the state space, conditioned by belonging to them. It allowed differentiating systems from the point of view of the fluidity of the system. In fact, this definition captures the existence of small parts from where is difficult to escape.

The adaptation of the other three definitions is now the following:

$$\Phi_2 = \min_{\emptyset \neq U \subset V} \frac{\sum_{i \in U, j \in \bar{U}} \pi_i P_{ij}}{\min\{\sum_{i \in U, j \in V} (P_{ij} + P_{ji}), \sum_{i \in \bar{U}, j \in V} (P_{ij} + P_{ji})\}};$$

$$\begin{aligned} \Phi_3 &= \min_{\emptyset \neq U \subset V} \frac{\sum_{i \in U, j \in \bar{U}} \pi_i P_{ij}}{\sum_{i \in U} \pi_i \cdot \sum_{i \in \bar{U}} \pi_i}; \\ \Phi_4 &= \min_{\emptyset \neq U \subset V} \frac{\sum_{i \in U, j \in \bar{U}} \pi_i P_{ij}}{\sum_{i \in U} \pi_i \cdot \log\left(\frac{1}{\sum_{i \in U} \pi_i}\right)}. \end{aligned} \tag{2}$$

In fact, the difference between them is the measure of the subsets  $U \subset V$ . In the definition of  $\Phi_2$  we measure a subset of states  $U$  using the weight in the edges in and out of  $U$ . The definition of  $\Phi_3$  provides a similar measure with that of  $\Phi_1$  but allows avoiding unbalanced partitions.

The need of these new definitions came from the existence of certain systems where  $\Phi_1$  is constant, in spite of the dynamics being different (for an example, see [7]). These examples are strongly supported by the theory of symbolic dynamics which we briefly describe below.

### 2 Symbolic dynamics

Symbolic dynamics is an important tool to study piecewise monotone interval maps. The main idea is to encode the itineraries of the images of the critical (or discontinuity) points of a piecewise monotone map  $f$  on an interval in symbolic sequences. These symbolic sequences are called *kneading sequences*, after [13]. The set of kneading sequences for a map  $f$  is called the *kneading invariant* of  $f$  and is denoted by  $\mathcal{K}(f)$ . This kneading invariant is a topological invariant and can be used to obtain topological information on the discrete dynamical system induced by  $f$ , such as the topological entropy,  $h_{\text{top}}(f)$ .

We are particularly interested in the unimodal and bimodal classes of interval maps. The unimodal class is formed by continuous maps on the interval with one critical point. The bimodal class is formed by continuous interval maps with two critical points. In the bimodal case, it is necessary to distinguish between two types of maps: the type  $(+, -, +)$  and type  $(-, +, -)$ , according to the subintervals where the bimodal map is increasing  $(+)$  or decreasing  $(-)$ .

By a result of Parry, see [16], for each unimodal map  $f$  there is only one piecewise linear map  $F$  (by semiconjugation) defined on the interval  $[0, 1]$  with slope  $\pm s$ , so that  $h_{\text{top}}(f) = h_{\text{top}}(F) = \log s$ . The topological entropy is a complete invariant for the piecewise linear unimodal family. Equivalently, the knead-

ing invariant is a complete invariant for the same family.

Similarly, given a bimodal map  $f$  there is a piecewise linear map  $F$  (also by semiconjugation) defined on  $[0, 1]$  with slope  $\pm s$ , so that  $h_{\text{top}}(f) = h_{\text{top}}(F) = \log s$ . Nevertheless, in the bimodal case, the map  $F$  is not unique. In fact, there is a one-parameter family of piecewise linear maps in those conditions. The parameter for this family, denoted by  $r(f)$ , is a topological invariant and it was introduced in [1] to distinguish between isentropic (equal entropy) dynamical behaviors in the bimodal class. Thus, for a piecewise linear bimodal map  $f$ , the values  $h_{\text{top}}(f)$  and  $r(f)$  form a complete set of topological invariants. Several studies on this new invariant followed, such as [4], [8], [17], and [12]. This new topological invariant was related with conductance in [9] and [6], which arise naturally in the context of reversible Markov chains.

In the following subsections, we give a brief review of symbolic dynamics for both unimodal and bimodal maps.

#### 2.1 Unimodal maps

Let  $f_a$  be a one-parameter family of surjective unimodal maps on an interval. The address of  $x \in [0, 1]$  is defined by

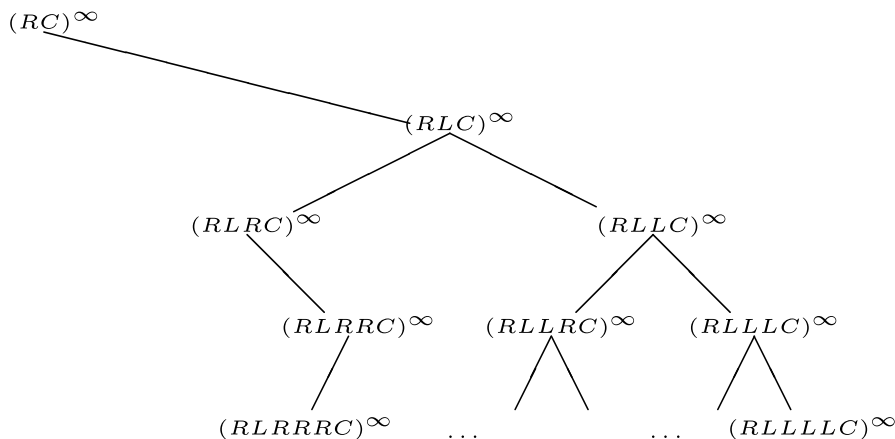
$$\text{ad}(x) = \begin{cases} L & \text{if } x \in [0, c] \\ C & \text{if } x = c \\ R & \text{if } x \in ]c, 1], \end{cases}$$

where  $c$  is the critical point of  $f_a$ . The itinerary of a point  $x$  is defined by

$$it_a(x) = \text{ad}(x) \text{ad}(f_a(x)) \text{ad}(f_a^2(x)) \dots$$

The kneading sequence corresponds to the itinerary of the point 1, i.e.,  $\mathcal{K}(f_a) = it_a(f_a(c))$ . The sign function  $\varepsilon : \bigcup_{k \geq 1} \{L, C, R\}^k \rightarrow \{-1, 0, 1\}$  is defined by  $\varepsilon(P) = \prod_{i=1}^k \varepsilon(P_i)$  with  $\varepsilon(L) = 1$ ,  $\varepsilon(C) = 0$ ,  $\varepsilon(R) = -1$ , where  $P = P_1 \dots P_k \in \{L, C, R\}^k$ . Consider the set  $\{L, C, R\}$  ordered by  $L < C < R$ . Let us define an induced order relation in  $\{L, C, R\}^{\mathbb{N}}$ . Consider two sequences  $P_1 P_2 \dots$  and  $Q_1 Q_2 \dots$  in  $\{L, C, R\}^{\mathbb{N}}$ . There is a number  $k = 0, 1, \dots$  so that  $P_i = Q_i$  for  $i < k$ . Then we set  $P_1 P_2 \dots < Q_1 Q_2 \dots$  if and only if  $P_k < Q_k$  and  $\varepsilon(P_1 \dots P_{k-1}) = 1$  or  $Q_k < P_k$  and  $\varepsilon(P_1 \dots P_{k-1}) = -1$ . The sequences  $S \in \{L, C, R\}^{\mathbb{N}}$

**Fig. 1** Kneading sequences tree  $\mathcal{T}_{KS}$  of the unimodal maps up to period 6



which occur as itineraries are precisely those  $S \prec \mathcal{K}(f_a)$ .

The set of values  $a$ , for which  $\mathcal{K}(f_a) = it_a(f_a(c))$  is periodic, is countable and, what is more important, in this case, we can associate a transition matrix to  $f_a$ ; see [10]. Moreover, the periodic kneading sequences have a complete combinatorial description using a tree structure  $\mathcal{T}_{KS}$ ; see [10] or Fig. 1.

Consider the periodic kneading sequence  $\mathcal{K}(f_a) = (K_1 K_2 \dots K_n)^\infty$ . Let  $x_i$  be the points such that

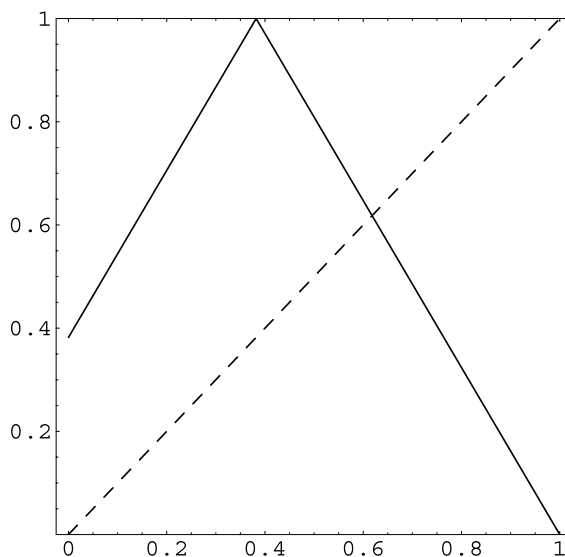
$$it_a(x_i) = \sigma^i(\mathcal{K}(f_a)) = (K_i K_{i+1} \dots K_{(i-1) \bmod n})^\infty.$$

Let  $\rho$  be the permutation which orders the points  $x_i$  in the real line, i.e.,  $x_{\rho(1)} < x_{\rho(2)} < \dots < x_{\rho(n)}$ . These points induce a partition in  $[0, 1]$ , defined by  $I_i = [x_{\rho(i)}, x_{\rho(i+1)}]$ ,  $i = 1, \dots, n$ . The Markov subshift  $(\Sigma_{A_{f_a}}, \sigma)$  is obtained as mentioned above.

The growth number of  $f_a, s$ , is given by the spectral radius of  $A_{f_a}, \lambda(A_{f_a})$ . The topological entropy of  $f_a$  is given by  $h_{top}(f_a) = \log s = \log \lambda(A_{f_a,b})$ ; see [1]. Now, let  $F_s$  be the one-parameter family of piecewise linear maps

$$F_s(x) = \begin{cases} sx + 2 - s & \text{if } x \in [0, (s - 1)s^{-1}] \\ s(1 - x) & \text{if } x \in [(s - 1)s^{-1}, 1], \end{cases}$$

which are a subclass of the unimodal class; see Fig. 2. Every surjective unimodal map  $f_a$  is topological semi-conjugated to a particular map  $F_s$  for which  $s = e^{h_{top}(f_a)} = e^{h_{top}(F_s)}$ . The maps  $f_a$  and  $F_s$  have the same growth number  $s$  and the same topological entropy  $\log s$ . Therefore, from the topological point of view, it is sufficient to analyze the family  $F_s$  and the symbolic



**Fig. 2** Unimodal map of the family  $F_s$  for  $s = \frac{1+\sqrt{5}}{2}$

dynamics described above applies in the same way to the family  $F_s$ .

### 2.2 Bimodal maps

Let us consider the family  $f_{a,b}$  of surjective bimodal maps on an interval. As for the unimodal family, let  $F_{s,r}$  be the two-parameter family of piecewise linear maps

$$F_{s,r}(x) = \begin{cases} sx + 1 - sr & \text{if } x \in [0, r] \\ -s(x - r - s^{-1}) & \text{if } x \in [r, r + s^{-1}] \\ s(x - r - s^{-1}) & \text{if } x \in [r + s^{-1}, 1] \end{cases}$$

which are a subclass of the bimodal class. As we said previously, for every family of bimodal maps  $f_{a,b}$

there is a one-parameter family of piecewise linear maps with constant slope  $s = e^{h_{\text{top}}(f_{a,b})}$ . If  $f_{a,b}$  is surjective and of the type  $(+, -, +)$ , there is a unique representative in the family  $f_{a,b}$  which is topological semiconjugated to a particular map in the family  $F_{s,r}$  with  $s = e^{h_{\text{top}}(f_{a,b})}$  and  $r(f_{a,b})$  is defined in any of the following [1], [17], or [4]. Nevertheless, in the Markov case  $r(f_{a,b})$  can be obtained from the Perron eigenvector of the associated transition matrix as will be explained below.

The symbolic dynamics for the bimodal family (including the piecewise linear family  $F_{s,r}$ ) is developed as follows (see [11]). Let  $f_{a,b}$  with critical points  $c_1$  and  $c_2$ . The address of  $x \in [0, 1]$  is given by

$$\text{ad}(x) = \begin{cases} L & \text{if } x \in [0, c_1[ \\ A & \text{if } x = c_1 \\ M & \text{if } x \in ]c_1, c_2[ \\ B & \text{if } x = c_2 \\ R & \text{if } x \in ]c_2, 1]. \end{cases}$$

The itinerary of a point  $x$ , with respect to the family  $f_{a,b}$ , is defined as in the unimodal case by

$$it_{a,b}(x) = \text{ad}(x) \text{ad}(f_{a,b}(x)) \text{ad}(f_{a,b}^2(x)) \dots$$

The bimodal sign function is defined by  $\varepsilon : \bigcup_{k \geq 1} \{L, A, M, B, R\}^k \rightarrow \{-1, 0, 1\}$ , with  $\varepsilon(P) = \prod_{i=1}^k \varepsilon(P_i)$ , for  $P = P_1 \dots P_k \in \{L, A, M, B, R\}^k$  and  $\varepsilon(L) = 1$ ,  $\varepsilon(A) = 0$ ,  $\varepsilon(M) = -1$ ,  $\varepsilon(B) = 0$ ,  $\varepsilon(R) = 1$ . Consider the set  $\{L, A, M, B, R\}$  ordered by  $L < A < M < B < R$ . Let us define an induced order relation in  $\{L, A, M, B, R\}^{\mathbb{N}}$ . Consider two sequences  $P_1 P_2 \dots$  and  $Q_1 Q_2 \dots$  of  $\{L, A, M, B, R\}^{\mathbb{N}}$ . There is a number  $k = 0, 1, \dots$  such that  $P_i = Q_i$  for  $i < k$ . Then we set  $P_1 P_2 \dots < Q_1 Q_2 \dots$  if and only if  $P_k < Q_k$  and  $\varepsilon(P_1 \dots P_{k-1}) = 1$  or  $Q_k < P_k$  and  $\varepsilon(P_1 \dots P_{k-1}) = -1$ .

We denote the bimodal kneading invariant by

$$\mathcal{K}(f_{a,b}) = (\mathcal{K}_1, \mathcal{K}_2) = (it_{a,b}(f_{a,b}(c_1)), it_{a,b}(f_{a,b}(c_2))).$$

The sequences  $S \in \{L, A, M, B, R\}^{\mathbb{N}}$  which occur as itineraries are those  $\mathcal{K}_2 < S < \mathcal{K}_1$ . The reason is that  $c_1$  is a maximal point and  $c_2$  is a minimal point for the map  $f_{a,b}$ .

The set of pairs  $(a, b)$  so that  $it_{a,b}(f_{a,b}(c_1))$  and  $it_{a,b}(f_{a,b}(c_2))$  are periodic is a countable set. In this case, we can associate a transition matrix to  $f_{a,b}$ ;

see [11]. Let  $\{X_i\}_{i=1}^{n_0+n_1}$  be the union of the sets  $\{\sigma^i(\mathcal{K}_1)\}_{i=1}^{n_0}$  and  $\{\sigma^j(\mathcal{K}_2)\}_{j=1}^{n_1}$ , where  $n_0$  is the period of  $\mathcal{K}_1$  and  $n_1$  is the period of  $\mathcal{K}_2$ . Denote by  $x_i$  the points in the interval so that  $X_i = it_{a,b}(x_i)$ . There is a permutation  $\rho$  which orders the set  $\{x_i\}_{i=1}^{n_0+n_1}$ , i.e.,

$$x_{\rho(1)} < x_{\rho(2)} < \dots < x_{\rho(n_0+n_1)}.$$

Consider the subintervals

$$I_i = [x_{\rho(i)}, x_{\rho(i+1)}], \quad i = 1, \dots, n_0 + n_1.$$

The Markov subshift  $(\Sigma_{A_{f_{a,b}}}, \sigma)$  is obtained from the alphabet  $\Sigma = \{1, 2, \dots, n\}$ , with  $n = n_0 + n_1$ , and the transition matrix  $A_{f_{a,b}}$  defined by

$$(A_{f_{a,b}})_{ij} := \begin{cases} 1 & \text{if } f_{a,b}(I_i) \supseteq I_j \\ 0 & \text{otherwise.} \end{cases}$$

As in the unimodal case, the growth number of  $f_{a,b}$  is equal to the spectral radius of the transition matrix  $A_{f_{a,b}}$ ,  $\lambda(A_{f_{a,b}})$ , and the topological entropy  $h_{\text{top}}(f_{a,b})$  is given by  $h_{\text{top}}(f_{a,b}) = \log \lambda(A_{f_{a,b}})$ .

The numerical invariants for  $f_{a,b}$ ,  $s = s(f_{a,b})$  and  $r = r(f_{a,b})$  can be recovered using the Perron eigenvector of  $A_{f_{a,b}}$ . Let  $u = (u_i)_{i=1}^{n_0+n_1}$  be the right Perron eigenvector of  $A_{f_{a,b}}$ . Then

$$r = \sum_{i=1}^{n_L} u_i \quad \text{and} \quad s^{-1} = \sum_{i=n_L}^{n_L+n_M} u_i,$$

where  $n_L$  (respectively,  $n_M, n_R$ ) is the number of symbols  $L$  (respectively,  $M, R$ ) in the pair  $\mathcal{K}(f_{a,b})$ ; see [4].

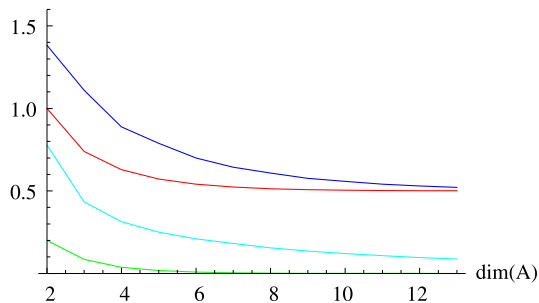
### 3 Conductance and topological entropy

We shall present now some families of unimodal and bimodal maps given by their kneading invariants  $\mathcal{K}(f_{a_k})$  or  $\mathcal{K}(f_{a_k, b_k})$ . The idea is to follow some paths in the tree of trajectories, increasing the periods of the critical points and so the length of the transition matrix. We expected to obtain, in the limit, information about the maps with no periodic itinerary of the critical points. Results concerning the rate of mixing were established in [8].

Consider the unimodal family

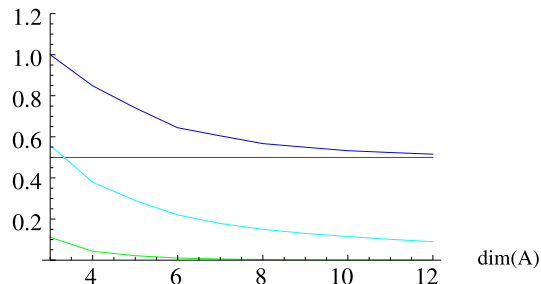
$$(RL^k C)^\infty$$

$\phi_1, \phi_2, \phi_3, \phi_4$



**Fig. 3** (Color online) The four conductances  $\Phi_2 < \Phi_4 < \Phi_1 < \Phi_3$  for the family  $(RL^kC)^\infty$

$\phi_1, \phi_2, \phi_3, \phi_4$



**Fig. 4** (Color online) The four conductances  $\Phi_2 < \Phi_4 < \Phi_1 < \Phi_3$  for the family  $(R(L^kBL)^\infty, (L^{k+1}B)^\infty)$

indexed by  $k$ . This family of maps corresponds to the branch most on the right of the tree in Fig. 1. When  $k$  increases, we go down (increasing the period) and right (increasing the topological entropy) on the tree. In Fig. 3 we can observe the behaviour of the four quantities  $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ , when  $k$  increases.

To observe the families of bimodal maps, we adopted a similar strategy even though we do not have a binary ordered tree. We consider the families

$$(R(L^kBL)^\infty, (L^{k+1}B)^\infty), \tag{3}$$

$$((RM^kA)^\infty, (LM^kB)^\infty). \tag{4}$$

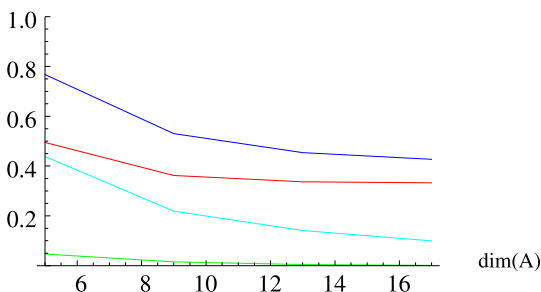
The maps in (3) have no periodic itineraries. However, after a transient block, they become periodic. This kind of trajectory is called eventually periodic. This family corresponds to isentropic maps, see [4], with constant  $\Phi_1$  (see [6]) and indicates the need of other invariants. In this case, the use of  $\Phi_2, \Phi_3$  or  $\Phi_4$  is completely justified as can be seen in Fig. 4.

Another example of bimodal maps, this time with periodic itineraries of critical points, is plotted in Fig. 5.

As we can see in these examples the behavior of all definitions of conductance is similar but converges to different values. While the definitions 1 and 3 supply nonnull values, the other two seem to converge always to zero.

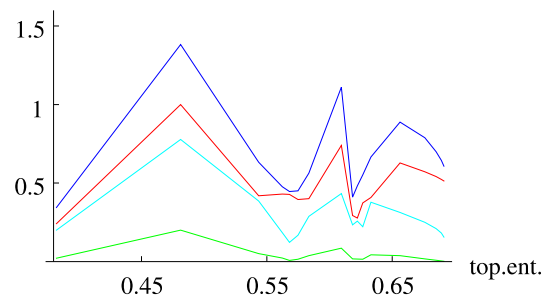
The variation of conductance with the topological entropy  $h_{\text{top}}(f_a)$  can be followed in Fig. 6 where are plotted the results for all admissible unimodal maps of the tree  $\mathcal{T}_{KS}$  with period smaller than 8. The plot consists of the pairs  $(h_{\text{top}}(f_a), \Phi)$  for each  $f_a$ . We can

$\phi_1, \phi_2, \phi_3, \phi_4$



**Fig. 5** (Color online) The four conductances  $\Phi_2 < \Phi_4 < \Phi_1 < \Phi_3$  for the family  $((RM^kA)^\infty, (LM^kB)^\infty)$

$\phi_1, \phi_2, \phi_3, \phi_4$



**Fig. 6** (Color online) The four conductances  $\Phi_2 < \Phi_4 < \Phi_1 < \Phi_3$  versus the topological entropy for all admissible maps of the tree  $\mathcal{T}_{KS}$  up to period 8

observe their variation with the topological entropy  $h_{\text{top}}(f_a) = \log(\lambda(A_{f_a}))$ .

We can establish the following conclusions.



**Conclusion 1** *The quantities  $\Phi_1, \Phi_2, \Phi_3,$  and  $\Phi_4$  defined by (1) and (2) vary with the growth number of the interval map, and consequently with the topological entropy.*

**Conclusion 2** *The quantities  $\Phi_2$  and  $\Phi_4,$  defined by (2), tend to zero when the period of the itinerary of the critical points tends to infinity. Otherwise, the quantities  $\Phi_1$  and  $\Phi_3$  tend to values depending on the dynamics.*

This last assertion implies that we cannot use  $\Phi_2$  and  $\Phi_4$  to characterize, in the limit, the maps with nonperiodic itineraries of the critical points, unlike the case of the first and third definition, as was proved in [6].

We can also observe that the conductance decreases when the period of the critical point increases and this imply the minima and the maxima in Fig. 6.

#### 4 Conductance in a mechanical system

In the third section, we started our study with the kneading invariants characterizing classes of dynamical systems which can be reduced to iterated maps of the interval, either unimodal or bimodal. From each kneading invariant, we can obtain an equivalent interval map of constant slope, determining numerical values, one value (topological entropy) in the unimodal case, or two independent values (topological entropy and the invariant  $r$ ) in the bimodal case. Moreover, we can obtain also a subshift of finite type characterized by a transition matrix  $A,$  and a digraph from which we compute different kinds of invariants such as the conductance. When we have a dynamical system given by a differential equation, we can obtain the above mentioned invariants (topological entropy,  $r,$  conductance, ...) using an appropriate Poincaré section. Let us analyze an example of a mechanical system. The mechanical system we consider is the nonlinear pendulum as in [17]. The differential equation is given by

$$\ddot{\theta} + a\dot{\theta} + \sin \theta = b \cos \omega t,$$

with parameters  $a$  (damping factor),  $b$  (amplitude),  $\omega$  (frequency), and the angle variable  $\theta.$  As it was shown in [17], there is a region of the parameters for

which the dynamics is transformed, under an appropriate Poincaré section, to an unimodal interval map (very close to a quadratic map). Therefore, varying the parameters  $a, b,$  and  $\theta,$  we have a continuous variation of the critical value. Therefore, every unimodal kneading sequence is realized by a certain choice of the parameters  $a, b, \theta$  (many different values can lead to the same kneading sequence). In particular, we can experimentally determine the values for which we obtain the studied sequence  $(RL^kC)^\infty.$  For those values, the conductances for the pendulum are shown in Fig. 3. It was also shown that there is a region in the parameter space where the dynamics is approximately transformed, under the appropriate Poincaré section, to a bimodal interval map. We have described in Sect. 2 the procedure to obtain the transition matrix for a bimodal map of type  $(+, -, +).$  For the pendulum, the obtained bimodal map is of type  $(-, +, -);$  nevertheless the procedure is similar, in terms of obtaining the transition matrix. Here, we present the following example. For  $a = 0.5, b = 1.1,$  and  $\theta = 2/3,$  the attractor is a periodic orbit and the corresponding kneading invariant is  $((LMMBRMA)^\infty, (RMALMMB)^\infty)$  (the orbits of the critical points in this case coincide). Then the transition matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the corresponding conductances are

$$\begin{aligned} \Phi_1 &\approx 0.117033; \\ \Phi_2 &\approx 0.00823569; \\ \Phi_3 &\approx 0.202558; \\ \Phi_4 &\approx 0.00199775. \end{aligned}$$

*Remark 3* In this case  $\Phi_4 < \Phi_2,$  in the contrary of results in Sect. 3, where  $\Phi_2 < \Phi_4.$  This can be related with the fact that the obtained bimodal map is of type  $(-, +, -)$  instead of type  $(+, -, +)$  considered in Sect. 3, but this statement lacks confirmation.



The time of convergence to the equilibrium differs from one system to another and the conductance detects the existence of bottlenecks or funnels, which delay the evolution to the stationarity. Therefore, a small conductance implies a greater mixing time. This phenomenon has been addressed in several works (see [6]), for  $\Phi_1$ , but we can see that it is true for any of the settings considered as they have similar behaviour.

As we concluded in Sect. 3, conductance independently of the topological classification, can give us an indication on how fast the system converges to the attractor. It is an independent measure regarding topological entropy or growth number, since it varies with it. Furthermore, it varies monotonically with the period of the attractive orbit.

## References

- Almeida, P., Lampreia, J.P., Sousa Ramos, J.: Topological invariants for bimodal maps. In: Iteration Theory (Batschuns, 1992), pp. 1–8. World Sci. Publishing, River Edge (1996)
- Bezrukov, S.L.: Edge isoperimetric problems on graphs. *Theor. Comput. Sci.* **307**(3), 473–492 (2003)
- Bollobás, B.: Volume estimates and rapid mixing, flavours of geometry. *MSRI Publ.* **31** (1997)
- Correia Ramos, C., Martins, N., Severino, R., Sousa Ramos, J.: Noncommutative topological dynamics. *Chaos Solitons Fractals* **27**(1), 15–23 (2006)
- Fernandes, S., Grácio, C., Sousa Ramos, J.: Laplacians and the Cheeger constants for discrete dynamical system. In: Difference Equations, Special Functions and Orthogonal Polynomials, pp. 213–223. World Sci. Publ., Hackensack (2007)
- Fernandes, S., Sousa Ramos, J.: Conductance, Laplacian and mixing rate in discrete dynamical systems. *Nonlinear Dyn.* **44**(1–4), 117–126 (2006)
- Fernandes, S., Sousa Ramos, J.: Spectral invariants and conductance in iterated maps. In: Iteration Theory (ECIT '04), *Grazer Math. Ber.*, vol. 350, pp. 69–81. Karl-Franzens-Univ. Graz, Graz (2006)
- Fernandes, S., Sousa Ramos, J.: Second eigenvalue of transition matrix associated to iterated maps. *Chaos Solitons Fractals* **31**(2), 316–326 (2007)
- Grácio, C., Sousa Ramos, J.: The first eigenvalue of the Laplacian and the conductance of a compact surface. *Nonlinear Dyn.* **44**(1–4), 243–250 (2006)
- Lampreia, J.P., Rica da Silva, A., Sousa Ramos, J.: Tree of topological Markov chains. In: CFMC-E4/84, Lisbon (1984). <http://www.math.ist.utl.pt/~sramos/tree84>
- Lampreia, J.P., Sousa Ramos, J.: Symbolic dynamics of bimodal maps. *Port. Math.* **54**(1), 1–18 (1997)
- Martins, N., Severino, R., Sousa Ramos, J.: Isentropic real cubic maps. *Dynamical systems and functional equations (Murcia, 2000)*. *Int. J. Bifur. Chaos Appl. Sci. Eng.* **13**(7), 1701–1709 (2003)
- Milnor, J., Thurston, W.: On iterated maps of the interval. *Dynamical systems (College Park, MD, 1986–87)*. In: *Lecture Notes in Math.*, vol. 1342, pp. 465–563. Springer, Berlin (1988)
- Mohar, B.: The Laplacian spectrum of graphs. *Graph Theory, Comb. Appl.* **2**, 871–898 (1991)
- Mohar, B.: Some Applications of Laplace Eigenvalues of Graphs. University of Ljubljana, Ljubljana (1997). Preprint series, vol. 35
- Parry, W.: Symbolic dynamics and transformations of the unit interval. *Trans. Am. Math. Soc.* **122**, 368–378 (1966)
- Ramos, M.M., Correia Ramos, C., Severino, R., Sousa Ramos, J.: Topological invariants of a chaotic pendulum. *Int. J. Pure Appl. Math.* **10**(2), 209–226 (2004)
- Sousa Ramos, J.: Introduction to nonlinear dynamics of electronic systems: tutorial. *Nonlinear Dyn.* **44**(1–4), 3–14 (2006)
- Walters, P.: *An Introduction to Ergodic Theory*. Springer, Berlin (2007)