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BIJECTIVE COUNTING OF MAPS BY GIRTH AND DEGREES I: RESTRICTED BOUNDARY CONDITIONS

OLIVIER BERNARDI* AND ÉRIC FUSY†

ABSTRACT. For each positive integer d , we present a bijection between the set of planar maps of girth d inside a d -gon and a set of decorated plane trees. The bijection has the property that each face of degree k in the map corresponds to a vertex of degree k in the tree, so that maps of girth d can be counted according to the degree distribution of their faces. More precisely, we obtain for each integer d an explicit expression for the multivariate series $F_d(x_d, x_{d+1}, x_{d+2}, \dots)$ counting rooted maps of girth d inside a d -gon, where each variable x_k marks the number of inner faces of degree k . The series F_1 (corresponding to maps inside a loop) was already computed bijectively by Bouttier, Di Francesco and Guitter, but for $d \geq 2$ the expression of F_d is new. As special cases, we recover several known bijections (bipartite maps, loopless triangulations, simple triangulations, simple quadrangulations, etc.).

Our strategy is based on the use of a “master bijection”, introduced by the authors in a previous paper, between a class of oriented planar maps and a class of decorated trees. We obtain our bijections for maps of girth d by specializing the master bijection. Indeed, by defining some “canonical orientations” for maps of girth d , it is possible to identify the class of maps of girth d inside a d -gon with a class of oriented maps on which the master bijection specializes nicely. The same strategy was already used in a previous article in order to count d -angulations of girth d , and what we present here is a very significant extension of those results.

1. INTRODUCTION

A map is a connected graph embedded without edge-crossing in the oriented sphere. There is a very rich literature on the enumeration of maps, going back to the seminal work of Tutte [23, 24] via generating functions. The approach of Tutte applies to many map families (triangulations, bipartite maps, 2-connected maps, etc.) but involves some technical calculations (the *quadratic method* or its generalizations [8]; see also [14] for a more analytic approach). For many families of maps, the generating function turns out to be algebraic, and to have a simple expression in terms of the generating function of a family of trees. Enumerative results for maps can alternatively be obtained by a matrix integral approach [11] or an algebraic approach [17].

The bijective approach to the enumeration of maps is more recent. It was greatly developed by Schaeffer [21] after initial constructions by Cori and Vauquelin [13], and Arquès [1]. This approach usually gives more transparent proofs of the enumerative formulas (and algorithmic byproducts) and has proved surprisingly powerful

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for studying the metric properties of maps and solving statistical mechanics models on maps. Most of the bijections for maps are between a class of maps and a class of “decorated” plane trees. These bijections can be divided into two categories: (A) bijections in which the decorated tree is a spanning tree of the map (and the “decorations” are part of the edges not in the spanning trees), and (B) bijections in which the decorated plane tree associated to a map M has vertices of two colors black and white corresponding respectively to the faces and vertices of the map (these bicolored trees are called *mobiles* in several articles)¹. The first bijection of type A is Schaeffer’s construction for Eulerian maps [20]. The first bijection of type B is Schaeffer’s construction for quadrangulations [21] (which can be seen as a reformulation of [13]) which was extended by Bouttier, Di Francesco and Guitter [10]. Bijections of both types require to first endow the maps with a “canonical structure” (typically an orientation) characterizing the class of maps: *Schnyder woods* for simple triangulations, *2-orientations* for simple quadrangulations, *Eulerian orientations* for Eulerian maps, etc. For several classes of maps, there exists both a bijection of type A and of type B. For instance, the bijections [20] and [10] both allow one to count bipartite maps.

In a recent paper [7] the authors proceeded to group the bijections of type B under one roof by introducing a “master bijection” Φ between a general class of oriented maps and a class of *mobiles* (decorated bicolored plane trees). It seems that all the known bijections of type B (for maps without statistical mechanics model) can be seen as specializations of the master bijection Φ , which is actually a slight extension of a bijection introduced by the first author in [2] and subsequently reformulated in [4] (where it was already specialized to recover the bijection for bipartite maps presented in [10]). In [7] the master bijection was applied to count bijectively *d-angulations* of *girth* d (maps with faces of degree d with no cycle of length less than d). This was a success in the sense that, for $d \geq 5$ the obtained enumerative results were new and seem difficult to obtain by a non-bijective approach (a former limitation of the bijective approach to maps was that the first step in order to design a bijection was to guess the correct family of decorated plane trees using exact enumerative formulas obtained by a non-bijective approach).

In the present paper, still relying on the master bijection Φ , we obtain a considerable extension of the results in [7]. Namely, for each positive integer d we describe a bijection between plane maps (i.e., maps with a root-face) of *girth* d and *outer degree* d (the degree of the root-face is d) and a specific family of mobiles. The bijection is such that each face of degree i in the map ($i \geq d$ since the girth is d) corresponds to a black vertex of degree i in the mobile. For $d \geq 3$, the bijection specializes to the one in [7] when all faces have degree d . Enumerating the mobiles then yields for each d an explicit system of algebraic equation defining the series $F_d(x_d, x_{d+1}, x_{d+2}, \dots)$ counting rooted maps with outer face of degree d and girth d , where x_i marks the number of inner faces of degree i in each $i \geq d$. For $d = 1$ (corresponding to unconstrained rooted maps) the system defining F_1 had already been proved (bijectively) by Bouttier, Di Francesco and Guitter in [9].

¹This classification comes with two subtleties. First, there are two *dual versions* for bijections of type B: in one version the decorations of the mobiles are some dangling half-edges, while in the dual version the decorations are some labellings of the vertices; see [4, Sec.7]. Second, it sometimes happen that a bijection of type A can be identified with a “degenerate form” of a bijections of type B in which all the white vertices of the mobiles are leaves; see Section 6.

Our bijection generalizes several known bijections. For the case $d = 1$ it coincides with the one described by Bouttier, Di Francesco and Guitter in [9]. In the case $d = 2$ the maps counted by F_2 exactly correspond to rooted loopless maps (since an outer face of degree 2 can be collapsed into an edge, which then becomes the root-edge), and in that case, our bijection has two interesting specializations: rooted bipartite maps (with control on the degrees of the faces) and loopless triangulations. It turns out that for these two specializations the mobiles take a particularly simple form, and can easily be identified with some non-bicolored plane trees. Under this identification, it is possible to see that our bijection coincides with some earlier bijections introduced by Schaeffer in the dual setting: for bipartite maps our bijection coincides with the bijection [20] for Eulerian maps and for loopless triangulations our bijection coincides with the bijection [21, Sec.2.3.4] for bridgeless cubic maps (this bijection is also described and extended in [18]). Hence, our construction in the case $d = 2$ forms a unifying bridge between the two above mentioned constructions by Schaeffer. In the case $d = 4$, we obtain a (seemingly new) closed formula for the number of simple bipartite maps with given number of faces of each degree; however our proof is only valid if at least one of the faces has degree 4. Lastly, a slight reformulation allows us to include the case $d = 0$ in our constructions, recovering a bijection described in [10] for vertex-pointed maps.

Relation with article II in preparation [6]. In the present article, a limitation in the enumeration of maps by girth and degrees is that there must be at least one face (the root-face) of degree equal to the girth (this is not a real limitation for $d = 1$, or $d = 2$ since in these case the outer face of degree 1 or 2 can be collapsed leaving a rooted maps of girth d without constraint on the root-face). This limitation will be dropped in the future article [6].

In another article in preparation [5] we will show that Φ can be specialized to count d -angulations with no non-facial cycle of length $\leq d$, generalizing the bijections described in [15] and [16] for $d = 3$ (irreducible triangulations) and $d = 4$ (irreducible quadrangulations).

Outline of the paper. We first give some terminology on maps and orientations in Section 2, and then recall in Section 3 the master bijection introduced in [7] between a set $\tilde{\mathcal{O}}$ of (weighted) oriented maps and a set of (weighted) mobiles. Our strategy is then to obtain for each positive integer d a bijection for the class \mathcal{C}_d of maps of girth d and outer degree d by specializing the master bijection. In order to do so, one has to show that maps in \mathcal{C}_d are characterized by the existence of a “canonical” orientation which allows one to identify \mathcal{C}_d with a subfamily of $\tilde{\mathcal{O}}$. In [7] it was shown that, for $d \geq 3$, d -angulations of girth d are characterized by the existence of a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}$. Moreover, in the bipartite case $d = 2b$, one can consider slightly simpler orientations called $b/(b-1)$ -orientations.

We present our main results in Section 4, starting with the bipartite case which is simpler. More specifically, in Subsection 4.1 we extend the definition of $b/(b-1)$ -orientations to bipartite maps of outer degree $2b$, for $b \geq 1$. We show that a bipartite map M of outer degree $2b$ admits a $b/(b-1)$ -orientation if and only if it has girth $2b$, and each such map M has a unique $b/(b-1)$ -orientation in $\tilde{\mathcal{O}}$, called *canonical $b/(b-1)$ -orientation of M* . The set of canonical $b/(b-1)$ -orientations is then shown to be amenable to a specialization of the master bijection: for each $b \geq 1$ we obtain a bijection between bipartite plane maps of girth $2b$ and outer degree $2b$, and a

specific family of mobiles. Through this bijection, each inner face of degree $2i$ in the map corresponds to a black vertex of degree $2i$ in the associated mobile. The general case of plane maps of girth $d \geq 1$ and outer degree d is treated similarly in Subsection 4.2. The mobiles are slightly more complicated in this case, but it is still true that each inner face of degree i in the map corresponds to a black vertex of degree i in the associated mobile.

In Section 5, we enumerate the map families by decomposing the associated mobiles. The main result is, for each $d \geq 1$, an explicit system of algebraic equations characterizing the series $F_d(x_d, x_{d+1}, x_{d+2}, \dots)$ of rooted maps of outer degree d and girth d , where each variable x_i counts the inner faces of degree i .

In Section 6, we take a closer look at the cases $b = 1, 2$ and $d = 1, 2$, and we also describe a slight reformulation which allows us to include the further case $d = 0$. The relation between our bijections and the bijections described in [9, 10, 21, 20] are explained. Further enumeration results are also given (rooted loopless maps counted by edges, simple bipartite maps counted by the number of faces in each degree).

In Section 7, we give the missing proofs about the existence of $b/(b-1)$ -orientations and $d/(d-2)$ -orientations.

2. MAPS AND ORIENTATIONS

This section gathers definitions about maps and orientations.

Maps. A *map* is a connected planar graph embedded (without edge-crossing) in the oriented sphere and considered up to continuous deformation. The *faces* are the connected components of the complementary of the graph. A *plane tree* is a map without cycles (it has a unique face). The numbers v , e , f of vertices, edges and faces of a map are related by the *Euler relation*: $v - e + f = 2$. Cutting an edge e at its middle point gives two *half-edges*, each incident to an endpoint of e (they are both incident to the same vertex if e is a loop). A *corner* is the angular section between two consecutive half-edges around a vertex. The *degree* of a vertex or face x , denoted $\deg(x)$, is the number of incident corners. A *d -angulation* is a map such that every face has degree d . *Triangulations* and *quadrangulations* correspond to the cases $d = 3$ and $d = 4$ respectively. The *girth* of a graph is the minimum length of its cycles. Obviously, a map of girth d does not have faces of degree less than d . Note that a map is *simple* (has no loop nor multiple edges) if and only if it has girth at least 3. A graph is *bipartite* if its vertices can be bicolored in such a way that every edge connects two vertices of different colors. Clearly, the girth of a bipartite graph is even. Lastly, it is easy to see that a planar map is bipartite if and only if every face has even degree.

A *plane map* (also called *face-rooted map*) is a map with a marked face, called the *root-face* (we think of the root-face as the infinite face in the planar embedding). A *rooted map* (also called *corner-rooted map*) is a map with a marked corner, called the *root*; in that case the *root-face* and *root-vertex* are the face and vertex incident to the root. The *outer degree* of a plane (or rooted) map is the degree of the root-face. The faces distinct from the root-face are called *inner faces*. The vertices, edges, and corners are called *outer* if they are incident to the root-face and *inner* otherwise.

Bi-orientations and weighted bi-orientations. A *bi-orientation* of a map G is a choice of an orientation for each half-edge of G : each half-edge can be either *ingoing* (oriented toward the vertex), or *outgoing* (oriented toward the middle of the edge). For $i \in \{0, 1, 2\}$, we call *i -way* an edge with exactly i ingoing half-edges. Our convention for representing 0-way, 1-way, and 2-way edges is given in Figure 1 (top line). The ordinary notion of *orientation* corresponds to bi-orientations having only 1-way edges. The *indegree* of a vertex v of G is the number of ingoing half-edges incident to v .

A *plane bi-orientation* O is a plane map together with a bi-orientation. A *directed path* of O is a path $P = (v_0, \dots, v_k)$ of G such that $\{v_i, v_{i+1}\}$ is either 2-way or 1-way from v_i to v_{i+1} for $i \in \{0, \dots, k-1\}$. The orientation O is said to be *accessible* from a vertex v if any vertex is reachable from v by a directed path. A *clockwise circuit* of O is a simple cycle C of G such that each edge of C is either 2-way or 1-way with the interior of C on its right. A *counterclockwise circuit* is defined similarly. The orientation O is said to be *minimal* if it has no counterclockwise circuit.

A \mathbb{Z} -*weighted bi-orientation* is a bi-orientation where each half-edge h has a *weight* which is a positive integer if h is ingoing and a non-positive integer if h is outgoing. The *weight* of an edge is then defined as the sum of the weights of its half-edges. The *weight* of a vertex v is the sum of the weights of the ingoing half-edges incident to v . The *weight* of a face f , denoted $w(f)$, is the sum of the weights of the outgoing half-edges incident to f and having f on their right. We call *admissible* a \mathbb{Z} -weighted plane bi-orientation such that outer half-edges have non-negative weights, outer edges and outer vertices have weight 1. This condition implies that the contour of the outer face is a simple directed cycle made of 1-way edges with weights 0 and 1 on the half-edges. Lastly, we define $\tilde{\mathcal{O}}$ as the set of \mathbb{Z} -weighted plane bi-orientations which are admissible, minimal, and accessible from every outer vertex.

3. MASTER BIJECTION BETWEEN BI-ORIENTED MAPS AND MOBILES

In this section we recall the “master bijection” Φ described in [7] (where it is denoted Φ_-) between \mathbb{Z} -weighted bi-orientations in $\tilde{\mathcal{O}}$ and a set of *mobiles*. This bijection will be specialized in Section 4 to plane maps of girth d and outer degree d for any positive integer d .

A *mobile* is a plane tree with vertices colored either black or white, and where the black vertices can be incident to some dangling half-edges called *buds*. Buds are represented by outgoing arrows as in Figure 2 (right). The *degree* of a black vertex is its number of incident half-edges (including the buds). The *excess* of a mobile is the number of half-edges incident to a white vertex minus the number of buds. A \mathbb{Z} -*weighted mobile* is a mobile where each non-bud half-edge h carries a *weight* which is a positive integer if h is incident to a white vertex and a non-positive integer if h is incident to a black-vertex. The *weight* of a vertex is the sum of weights of all its incident (non-bud) half-edges.

We now define the mapping Φ on \mathbb{Z} -weighted bi-orientations (see Figure 2 for an example).

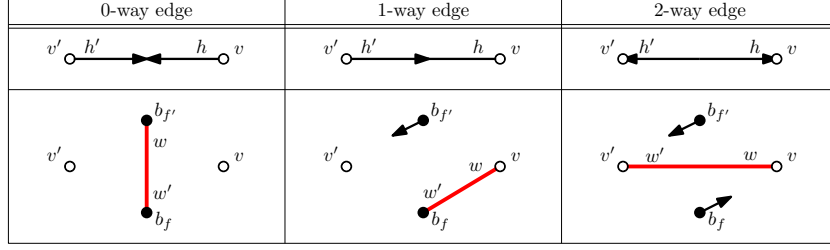


FIGURE 1. Local transformation of 0-way, 1-way and 2-way edges during the bijection Φ .

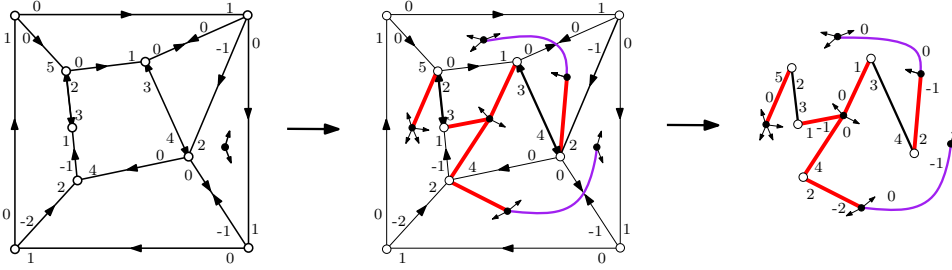


FIGURE 2. Master bijection Φ applied to a bi-orientation in $\tilde{\mathcal{O}}$.

Definition 1. Let O be a \mathbb{Z} -weighted bi-orientation in $\tilde{\mathcal{O}}$ with root-face f_0 . We view the vertices of O as white and place a black vertex b_f in each face f of O . The embedded graph $\Phi(O)$ with black and white vertices is obtained as follows:

- Return all the edges of the outer-face (which is a clockwise directed cycle of 1-way edges).
- For each edge e perform the following operation represented in Figure 1. Let h and h' be the half-edges of e with respective weight w and w' . Let v and v' be respectively the vertices incident to h and h' , let c , c' be the corners preceding h , h' in clockwise order around v , v' , and let f , f' be the faces containing these corners.
 - If e is 0-way, then create an edge between the black vertices b_f and $b_{f'}$ across e , and give weight w and w' to the half-edges incident to $b_{f'}$ and b_f respectively. Then, delete the edge e .
 - If e is 1-way with h being the ingoing half-edge, then create an edge joining the black vertex b_f to the white vertex v in the corner c , and give weight w and w' to the half-edges incident to v and b_f respectively. Then, glue a bud on $b_{f'}$ in the direction of c' , and delete the edge e .
 - If e is 2-way, then glue buds on b_f and $b_{f'}$ in the direction of the corners c and c' respectively (and leave intact the weighted edge e).
- Delete the black vertex b_{f_0} , the white outer vertices of O , and the edges between them (no other edge or bud is incident to these vertices).

The following theorem is proved in [7]:

Theorem 2. *The mapping Φ is a bijection between the set $\tilde{\mathcal{O}}$ of \mathbb{Z} -weighted bi-orientations and the set \mathcal{M} of \mathbb{Z} -weighted mobiles of negative excess, with the following parameter-correspondences:*

bi-orientation in $\tilde{\mathcal{O}}$		mobile in \mathcal{M}
inner vertex	\longleftrightarrow <i>same weight</i>	white vertex
inner face	\longleftrightarrow <i>same degree</i> \longleftrightarrow <i>same weight</i>	black vertex
inner edge	\longleftrightarrow <i>same weight</i>	edge
	$\left\{ \begin{array}{l} 0\text{-way} \longleftrightarrow \text{black-black} \\ 1\text{-way} \longleftrightarrow \text{black-white} \\ 2\text{-way} \longleftrightarrow \text{white-white} \end{array} \right\}$	
outer degree d	\longleftrightarrow	excess $-d$

For O a \mathbb{Z} -weighted bi-orientation in $\tilde{\mathcal{O}}$ of outer degree d , and $T = \Phi(O)$ the corresponding mobile, we call *exposed* the d buds of the mobile $T = \Phi(O)$ created by applying the local transformation to the outer edges of O (which have preliminarily been returned). The following additional claim, already proved in [7] will be useful for counting purposes.

Claim 3. *Let O be a \mathbb{Z} -weighted bi-orientation in $\tilde{\mathcal{O}}$ of outer degree d , and let $T = \Phi(O)$ be the corresponding mobile. There is a bijection between the set $\tilde{\mathcal{O}}$ of all corner-rooted maps obtained from O by marking an outer corner, and the set $\tilde{\mathcal{T}}$ of all mobiles obtained from T by marking an exposed bud. Moreover, there is a bijection between the set T_{\rightarrow} of mobiles obtained from T by marking a non-exposed bud, and the set $T_{\rightarrow\circ}$ of mobiles obtained from T by marking a half-edge incident to a white vertex.*

Before we close this section we recall from [7] how to recover the map starting from a mobile (this description will be useful in Section 6 to compare our bijection with other known bijections). Let T be a mobile (weighted or not) with non-positive excess δ . The corresponding *fully blossoming* mobile T' is obtained from T by inserting a *fake* black vertex at the middle of each white-white edge, and then for each black vertex v (fake or not) inserting a dangling half-edge called *stem* in each corner preceding a black-white edge in clockwise order around v . A fully blossoming mobile is represented in solid lines in Figure 3. Turning in counterclockwise direction around the mobile T' , one sees a sequence of buds and stems. The *partial closure* of T' is obtained by drawing an edge from each bud to the next available stem in counterclockwise order around T' (these edges can be drawn without crossings). This leaves $|\delta|$ buds unmatched (since the excess δ is equal to the number of stems minus the number of buds). The *complete closure* of T' is obtained by first creating a *root-vertex* in the face containing the unmatched buds and joining it to all the unmatched buds, and then deleting all the white-white and black-white edges of the mobile T and erasing the fake black vertices (these were at the middle of some edges); see Figure 3. Lastly, we denote by $\Psi(T)$ the plane map which is the *dual* of the complete closure C , that is, $\Psi(T)$ is obtained by placing a vertex inside each face of C and an edge across each edge of C (the root-face of $\Psi(T)$ is the one containing the root-vertex of C).

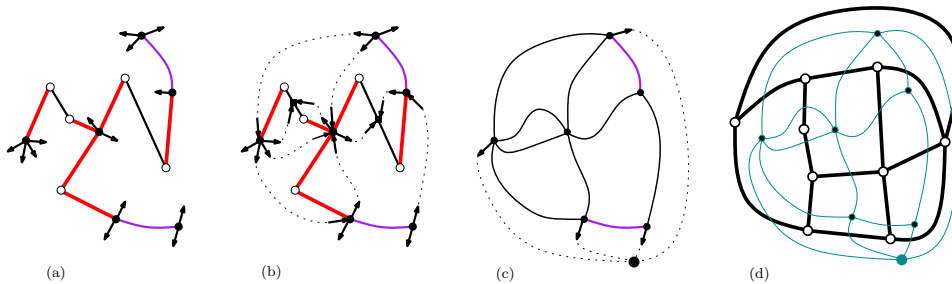


FIGURE 3. The inverse mapping Ψ . (a) A mobile T . (b) The fully blossoming mobile T' (drawn in solid lines with buds represented as outgoing arrows, and stems represented by ingoing arrows) and its partial closure (drawn in dashed lines). (c) The complete closure of T' . (d) The dual map $\Psi(T)$.

Proposition 4 ([7]). *Let O be a plane (weighted) bi-orientation in $\tilde{\mathcal{O}}$, and let $T = \Phi(O)$ be the mobile associated to O by the master bijection. Then the complete closure $M = \Psi(T)$ is the map underlying O .*

4. BIJECTIONS FOR MAPS OF FIXED GIRTH WITH CONTROL ON FACE-DEGREES

In this section, we present our main bijective results. For each positive integer d , we consider the class \mathcal{C}_d of plane maps of outer degree d and girth d . We define some \mathbb{Z} -weighted bi-orientations that characterize the maps in \mathcal{C}_d . This allows us to identify the class \mathcal{C}_d with a subfamily of $\tilde{\mathcal{O}}$. We then specialize the master bijection and obtain a bijection for maps in \mathcal{C}_d with control on the number of inner faces in each degree $i \geq d$. For the sake of clarity we start with the bipartite case, where the orientations and bijections are simpler.

4.1. Bipartite case. We start with the definition of the \mathbb{Z} -weighted bi-orientations that characterize the bipartite maps in \mathcal{C}_{2b} .

Definition 5. *Let b be a positive integer and let M be a bipartite plane map of outer degree $2b$ having no face of degree less than $2b$. A $b/(b-1)$ -orientation of M is an admissible \mathbb{Z} -weighted bi-orientation such that every outgoing half-edge has weight 0 or -1 and*

- (i) *each inner edge has weight $b-1$,*
- (ii) *each inner vertex has weight b ,*
- (iii) *each inner face f has degree and weight satisfying $\deg(f)/2 + w(f) = b$.*

Figure 4 shows $b/(b-1)$ -orientations for $b = 2$ and $b = 3$. Figure 8 (middle) shows the case $b = 1$. Observe that for $b \geq 2$ the $b/(b-1)$ -orientations have no 0-way edges, while for $b \leq 2$ they have no 2-way edges. For $2b$ -angulations, Condition (iii) of $b/(b-1)$ -orientation implies that the weight of every face is 0, hence the weight of every outgoing half-edge is 0. Thus, Definition 5 of $b/(b-1)$ -orientation generalizes the definition given in [7] for $2b$ -angulations.

Proposition 6. *Let b be a positive integer. Let M be a bipartite plane map of outer degree $2b$ having no face of degree less than $2b$. Then M admits a $b/(b-1)$ -orientation*

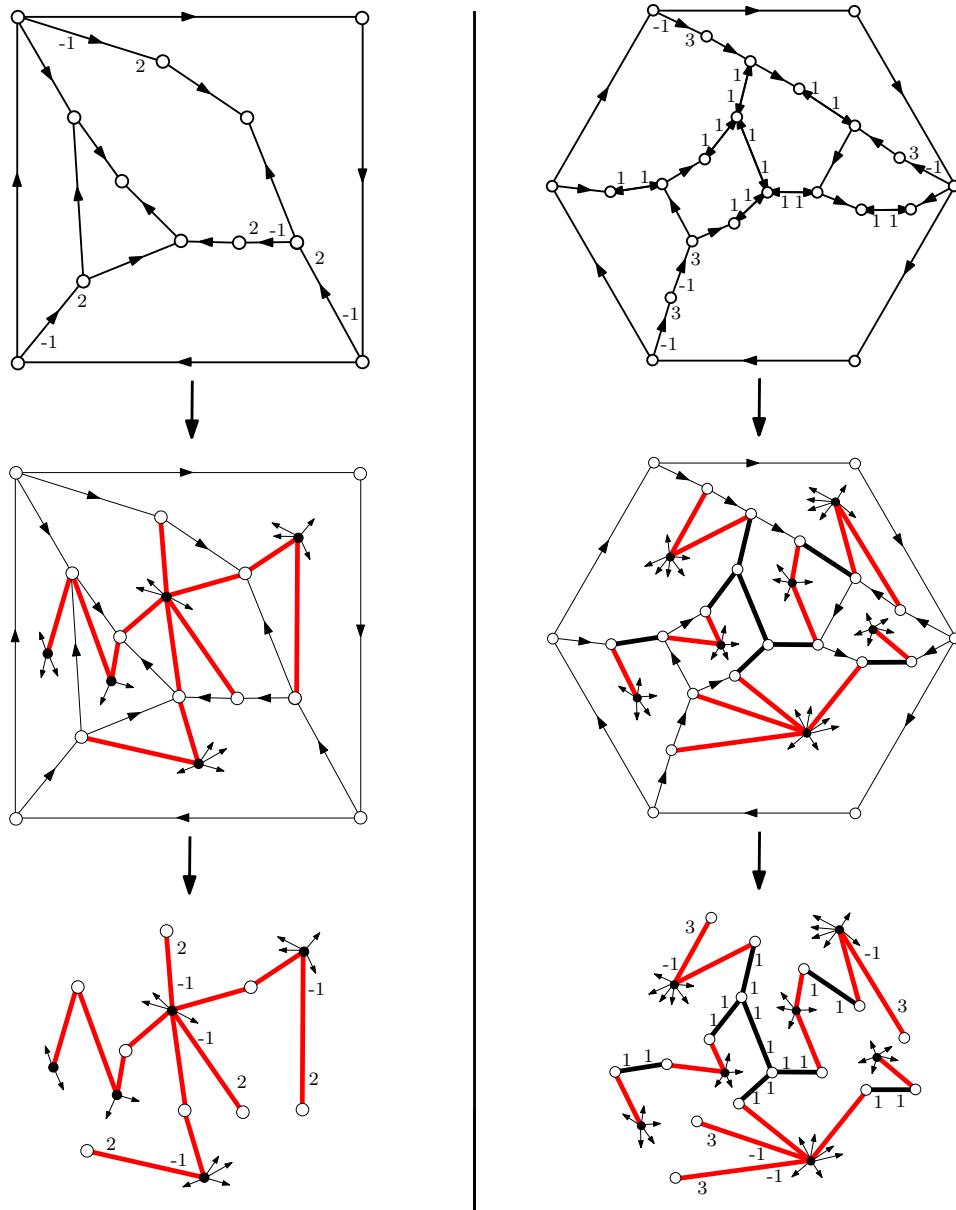


FIGURE 4. Bijections for bipartite maps in the cases $b = 2$ (left) and $b = 3$ (right). Top: a plane bipartite map of girth $2b$ and outer degree $2b$ endowed with its canonical $b/(b-1)$ -orientation. The weights are not indicated for 1-way edges with weight 0 on the outgoing half-edge. Bottom: the associated bibranching mobiles. The weights are not indicated for black-white edges with weight 0 at the black extremity.

if and only if M has girth $2b$. In this case, M has a unique $b/(b-1)$ -orientation in $\tilde{\mathcal{O}}$, called its canonical $b/(b-1)$ -orientation.

The proof of Proposition 6 (which extends a result given in [7] for $2b$ -angulations) is delayed to Section 7. We now define the class of \mathbb{Z} -weighted mobiles that we will show to be in bijection with bipartite maps in \mathcal{C}_{2b} .

Definition 7. For $b \geq 1$, a b -bibranching mobile is a \mathbb{Z} -weighted mobile such that half-edges incident to black vertices have weight 0 or -1 and

- (i) each edge has weight $b - 1$,
- (ii) each white vertex has weight b ,
- (iii) each black vertex v has degree and weight satisfying $\deg(v)/2 + w(v) = b$; equivalently a black vertex adjacent to ℓ white leaves has degree $2b + 2\ell$.

Examples of b -bibranching mobiles are given in Figure 4. Observe that for $b \geq 2$, the b -bibranching mobiles have no black-black edges, while for $b \leq 2$ they have no white-white edges.

Claim 8. For $b \geq 1$, any b -bibranching mobile has excess $-2b$.

Proof. Let T be a b -bibranching mobile. Let e be the number of edges and β be the number of buds. Let v_b and v_w be the number of black and white vertices respectively. Let h_b and h_w be the number of non-bud half-edges incident to black and white vertices respectively. By definition, the excess δ of the mobile is $\delta = h_w - \beta$. Now, by Condition (iii) on black vertices, one gets $(h_b + \beta)/2 + S = b v_b$, where S is the sum of weights of the half-edges incident to black vertices. By Conditions (i) and (ii), one gets $e(b - 1) = b v_w + S$. Eliminating S between these relations gives $2e(b - 1) + h_b + \beta = b(v_b + v_w)$. Lastly, plugging $v_b + v_w = e - 1$ and $2e = h_b + h_w$ in this relation, yields $h_w - \beta = -2b$. \square

By Proposition 6, the set of bipartite maps in \mathcal{C}_{2b} can be identified with the subset of $b/(b-1)$ -orientations in $\tilde{\mathcal{O}}$. Thus, it remains to show that the master bijection Φ induces a bijection between the $b/(b-1)$ -orientations in $\tilde{\mathcal{O}}$ and the b -bibranching mobiles. In view of the parameter-correspondence induced by the master bijection Φ (Theorem 2), it is clear that Conditions (i), (ii), (iii) of the $b/(b-1)$ -orientations correspond respectively to Conditions (i), (ii), (iii) of the b -bibranching mobiles. Thus, by Theorem 2, the master bijection Φ induces a bijection between the set of $b/(b-1)$ -orientations in $\tilde{\mathcal{O}}$ and the set of b -bibranching mobiles of excess $-2b$. Moreover, by Claim 8 the constraint on the excess is redundant. We conclude:

Theorem 9. For any positive integer b , bipartite plane maps of girth $2b$ and outer degree $2b$ are in bijection with b -bibranching mobiles. Moreover, each inner face of degree $2i$ in the map corresponds to a black vertex of degree $2i$ in the mobile.

Figure 4 illustrates the bijection on two examples ($b = 2$, $b = 3$). The cases $b = 1$ and $b = 2$ are examined in more details in Section 6. In particular we explain the relation between our bijection in the case $b = 1$ and the bijection described by Schaeffer in [20].

4.2. General case. We now treat (in a similar fashion) the case of general (not necessarily bipartite) maps.

Definition 10. Let d be a positive integer and let M be a plane map of outer degree d having no face of degree less than d . A $d/(d-2)$ -orientation of M is an admissible \mathbb{Z} -weighted bi-orientation such that every outgoing half-edge has weight 0 , -1 or -2 and

- (i) each inner edge has weight $d - 2$,
- (ii) each inner vertex has weight d ,
- (iii) each inner face f has degree and weight satisfying $\deg(f) + w(f) = d$.

Figure 4 shows some $d/(d-2)$ -orientations for $d = 3$ and $d = 5$. The cases $d = 1$ and $d = 2$ are represented in Figures 7 and 8 respectively. For d -angulations, Condition (iii) of $d/(d-2)$ -orientations implies that the weight of every outgoing half-edge is 0 . In this case Definition 5 of $d/(d-2)$ -orientations restricts to the one given in [7] for d -angulations.

Proposition 11. Let d be a positive integer. Let M be a plane map of outer degree d having no face of degree less than d . Then, M admits a $d/(d-2)$ -orientation if and only if M has girth d . In this case, M has a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}$, which is called the canonical $d/(d-2)$ -orientation of M .

The proof of Proposition 11 (which extends a result given in [7] for d -angulations) is delayed to Section 7.

Remark 12. Let $b \geq 1$ and $d = 2b$. We call *even* the $d/(d-2)$ -orientations such that every weight is even. Clearly, doubling the weight of half-edges on inner edges in a $b/(b-1)$ -orientation gives an even $d/(d-2)$ -orientation, and any even $d/(d-2)$ -orientation is obtained in this way. Moreover, an even $d/(d-2)$ -orientation is in $\tilde{\mathcal{O}}$ if and only if the corresponding $b/(b-1)$ -orientation is in $\tilde{\mathcal{O}}$. Hence, Proposition 6 shows that the canonical $d/(d-2)$ -orientation of a map in \mathcal{C}_d is even if and only if the map is bipartite. In this case, the canonical orientation can be “simplified” to a $b/(b-1)$ -orientation (by dividing the weights of half-edges on inner edges by 2).

Similarly as in Section 4.1 we now define the mobiles corresponding to canonical $d/(d-2)$ -orientations via the master bijection.

Definition 13. For a positive integer d , a d -branching mobile is a \mathbb{Z} -weighted mobile such that half-edges incident to black vertices have weight 0 , -1 or -2 and

- (i) each edge has weight $d - 2$,
- (ii) each white vertex has weight d ,
- (iii) each black vertex v has degree and weight satisfying $\deg(v) + w(v) = d$.

Observe that for $d \geq 3$, the d -branching mobiles have no black-black edges, while for $d \leq 3$ they have no white-white edges. By very similar arguments as in Claim 8 one proves:

Claim 14. Any d -branching mobile has excess $-d$.

By Proposition 11, the class \mathcal{C}_d of maps can be identified with the subset of $d/(d-2)$ -orientations in $\tilde{\mathcal{O}}$. Moreover, as in the bipartite case, it is easy to see from Theorem 2 that the master bijection Φ induces a bijection between the set of $d/(d-2)$ -orientations in $\tilde{\mathcal{O}}$ and the set of d -branching mobiles. We conclude:

Theorem 15. For any positive integer d , plane maps of girth d and outer degree d are in bijection with d -branching mobiles. Moreover, each inner face of degree i in the map corresponds to a black vertex of degree i in the mobile.

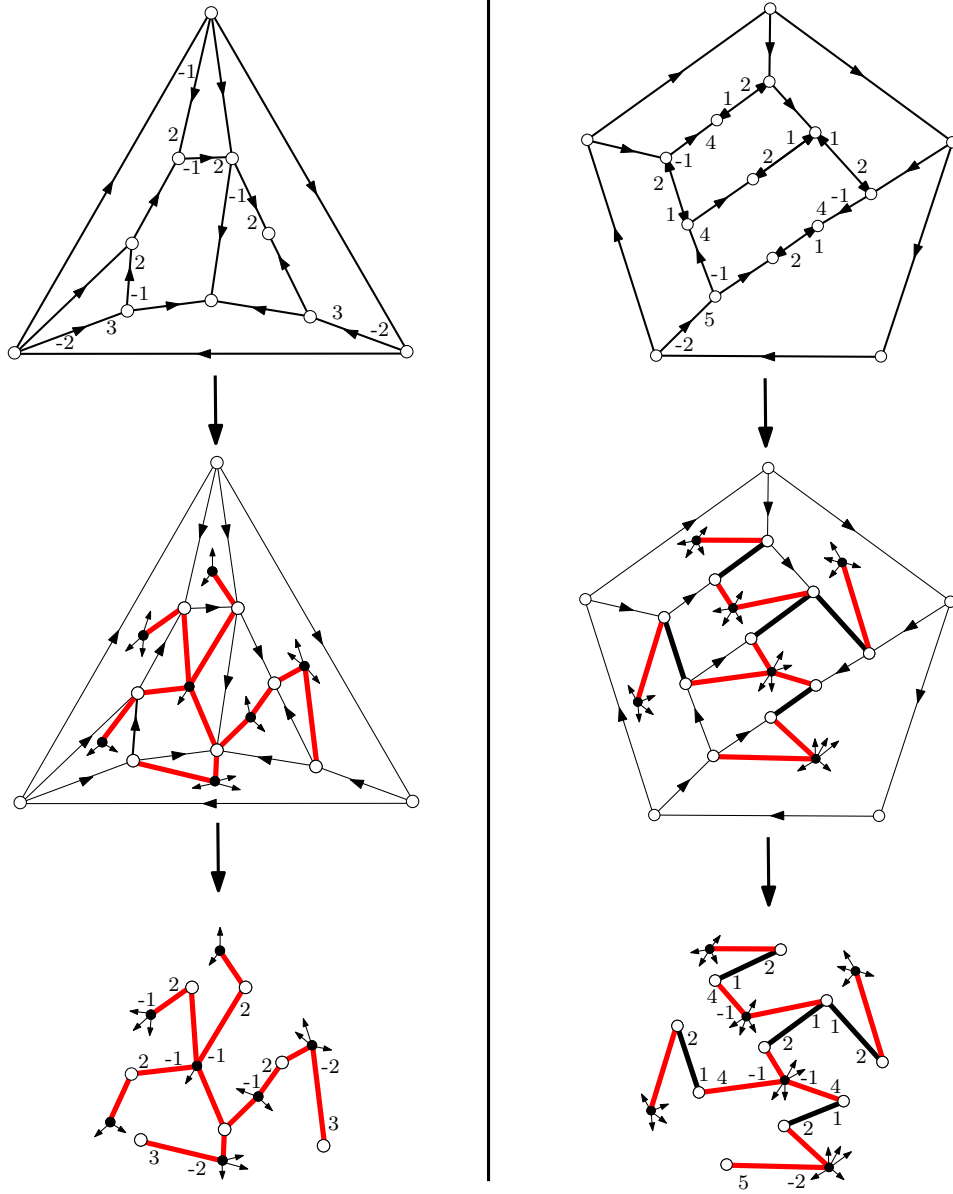


FIGURE 5. Bijections obtained for some maps in \mathcal{C}_d for $d = 3$ (left) and $d = 5$ (right). Top: a plane map of girth d and outer degree d endowed with its canonical $d/(d-2)$ -orientation. The weights are not indicated for 1-way edges with weight 0 on the outgoing half-edge. Bottom: the associated mobile. The weights are not indicated for black-white edges with weight 0 at the black extremity.

Figure 5 illustrates the bijection on two examples ($d = 3$, $d = 5$). As explained in Remark 12, when $d = 2b$ in even, the canonical $b/(b-1)$ -orientation of a bipartite map can be identified with its canonical $d/(d-2)$ -orientation (by doubling every weight). Hence, the bijection of Theorem 15 is an extension of the bijection of Theorem 9. For all $d \geq 3$, the bijection of Theorem 15 extends the bijection given in [7] for d -angulations of girth d . The cases $d = 1$ and $d = 2$ of Theorem 15 are examined in more details in Section 6, in particular the relation between our bijection in the case $d = 1$ and the bijection described by Bouttier, Di Francesco and Guitter in [9] (we also show a link with another bijection described by Bouttier, Di Francesco and Guitter in [10]).

5. GENERATING FUNCTIONS

In this section we give, for each positive integer d , a system of equations specifying the generating function F_d of rooted maps of girth d and outer degree d counted according to the number of inner face of each degree.

We first set some notations. For any integers p, q we denote by $[p..q]$ the set of integers $\{k \in \mathbb{Z}, p \leq k \leq q\}$. If $G(x)$ is a (Laurent) formal power series in x , we denote by $[x^k]G(x)$, the coefficient of x^k in $G(x)$. For all non-negative integer j we define the polynomial h_j in the variables w_1, w_2, \dots by:

$$(1) \quad h_j(w_1, w_2, \dots) := [t^j] \frac{1}{1 - \sum_{i>0} t^i w_i} = \sum_{r \geq 0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j}} w_{i_1} \cdots w_{i_r}.$$

Note that $h_0 = 1$. In other words, h_j is the (polynomial) generating function of integer compositions of j where the variable w_i marks the number of parts of size i .

Let d be a positive integer. By Theorem 9 and Claim 3, counting rooted plane maps of girth d and outer degree d reduces to counting d -branching mobiles rooted at an exposed bud. To carry out the latter task we simply write the generating function equation corresponding to the recursive decomposition of trees at the root. We call *planted d -branching mobile* a mobile with a dangling half-edge that can be obtained as one of the two connected components obtained after cutting a d -branching mobile M at the middle of an edge. Recall that the half-edges of a d -branching mobiles have weight in $[-2..d]$. For j in $[-2..d]$, we denote by \mathcal{W}_j the family of planted d -branching mobiles of root-weight $d - 2 - j$. We denote by $W_i \equiv W_i(x_d, x_{d+1}, x_{d+2} \dots)$ the generating function of \mathcal{W}_i , where for $i \geq d$ the variable x_i marks the black vertices of degree i . We now consider the recursive decomposition of planted mobiles and obtain equations characterizing the W_i 's.

Let j be in $[-2..d-3]$, and let T be a mobile in \mathcal{W}_j . Since $d - 2 - j > 0$, the vertex w incident to the dangling half-edge of T is white. Let e_1, \dots, e_r be the edges incident to w . For all $i = 1 \dots r$, let T_i be the planted mobile T_i , the part not containing w , obtained by cutting the edge e_i in the middle, and denote by $\alpha(i) > 0$ the weight of the half-edge of e_i incident to w (so that T_i is in $\mathcal{W}_{\alpha(i)}$). Since the white vertex w has weight d , one gets the constraint $\sum_i \alpha(i) = j + 2$. Conversely any sequence of planted mobiles T_1, \dots, T_r in $\mathcal{W}_{\alpha(1)}, \dots, \mathcal{W}_{\alpha(r)}$ such that $\alpha(i) > 0$ and $\sum_i \alpha(i) = j + 2$ gives a planted mobile in \mathcal{W}_j . Thus for all j in $[-2..d-3]$,

$$W_j = \sum_{r>0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j+2}} W_{i_1} \cdots W_{i_r} = h_{j+2}(W_1, \dots, W_{d-1}).$$

Note that $W_{-1} = h_1(W_1) = W_1$ whenever $d > 1$, and that the special case $W_{-2} = 1$ is coherent with our convention $h_0 = 1$.

Let j be in $[d - 2 .. d]$, and let T be a mobile in \mathcal{W}_j . The vertex v incident to the dangling edge of T is black, and is incident to some buds and to some edges. If the half-edge incident to v has weight $i \in \{0, -1, -2\}$ the planted mobile not containing v obtained by cutting e in its middle is in \mathcal{W}_i . Therefore the condition on black vertices of d -branching mobiles gives for all j in $[d - 2 .. d]$,

$$W_j = [u^{j+1}] \sum_{i \geq d} x_i (u + uW_0 + W_{-1} + u^{-1})^{i-1}.$$

where the summand u in the parenthesis correspond to the buds incident to the vertex v , while the summands uW_0 , W_{-1} and $u^{-1} = u^{-1}W_{-2}$ correspond respectively to the half-edges of weight 0, -1 , -2 incident to v . We summarize:

Theorem 16. *Let d be a positive integer, and let $F_d \equiv F_d(x_d, x_{d+1}, x_{d+2}, \dots)$ be the generating function of rooted maps of girth d with outer degree d , where each variable x_i counts the number of inner faces of degree i . Then,*

$$(2) \quad F_d = W_{d-2} - \sum_{j=-2}^{d-3} W_j W_{d-2-j},$$

where $W_{-2} = 1, W_{-1}, W_0, \dots, W_d$ are the unique formal power series satisfying:

$$(3) \quad \begin{cases} W_j = h_{j+2}(W_1, \dots, W_{d-1}) & \text{for all } j \text{ in } [-2 .. d-3], \\ W_j = [u^{j+1}] \sum_{i \geq d} x_i (u + uW_0 + W_{-1} + u^{-1})^{i-1} & \text{for all } j \text{ in } [d-2 .. d], \end{cases}$$

where the polynomials h_j are defined by (1). In particular, for any finite set $\Delta \subset \{d, d+1, d+2, \dots\}$, the specialization of F_d obtained by setting $x_i = 0$ for all i not in Δ is algebraic (over the field of rational function in $x_i, i \in \Delta$).

For $d = 1$, Theorem 16 gives exactly the system of equations obtained by Bouttier, Di Francesco and Guitter in [9]. Observe that for any integer $d \geq 2$ the series W_{-1} and W_1 are equal (since $h_1 = w_1$), so the number of unknown series is $d + 1$ in these cases. Observe additionally that for $d \geq 1$ the series W_d is not needed to define the other series W_0, W_1, \dots, W_{d-1} . Lastly, under the specialization $\{x_d = x, x_i = 0 \forall i > d\}$ one gets $W_{d-1} = W_d = 0$ and $W_{d-2} = x(1 + W_0)^{d-1}$; in this case we recover the system of equations given in [7] for the generating function of rooted d -angulations of girth d .

Proof. The fact that the solution of the system (3) is unique is clear. Indeed, it is easy to see that the series the second equation shows that the W_{-1}, W_0, \dots, W_d have no constant terms, and from this it follows that the coefficients of these series are uniquely determined by induction on the total degree.

We now prove (2). By Theorem 15 and Claim 3 (first assertion) the series F_d is equal to the generating function of d -branching mobiles rooted at an exposed bud (where x_i marks the black vertices of degree i). Moreover by the second assertion of Claim 3, this is equal to the difference between the generating function B_d of d -branching mobiles rooted at a bud, and the generating function H_d of d -branching mobiles with a marked half-edge incident to a white vertex. Lastly, $B_d = W_{d-2}$ because d -branching mobiles rooted at a bud identifies with mobiles in \mathcal{W}_{d-2} , and

$H_d = \sum_{i=-2}^{d-3} W_i W_{d-2-i}$ since d -branching mobiles with a marked half-edge incident to a white vertex are in bijection (by cutting the edge) with ordered pairs (T, T') of planted d -branching mobiles in $\mathcal{W}_i \times \mathcal{W}_{d-2-i}$ for some i in $[-2..d-3]$. \square

Remark 17. The expression (2) for the series F_d is not suitable for an asymptotic analysis of the coefficients of F_d because of the minus sign. However, it is possible to obtain a *positive* expression (i.e., without minus sign) for the series $G_{q,d}(x_d, x_{d+1}, \dots)$ of rooted maps of girth d with a root-face of degree q and another marked face of degree d , where the variable x_i counts non-root faces of degree i . Indeed, from the bijection of Theorem 15, it is easy to see that these maps are in bijection with d -branching mobiles with a marked corner incident to a black vertex of degree q . This yields

$$(4) \quad G_{q,d}(x_d, x_{d+1}, \dots) = [u^d](u + uW_0 + W_{-1} + u^{-1})^q.$$

Moreover, by considering maps with two marked corners (one in a face of degree q and one in a face of degree d), it is easy to see that $\frac{\partial}{\partial x_q} F_d = \frac{d}{q x_d} G_{q,d}$. However, all the results and proofs presented here (both for obtaining (2) or (4)) require one of the faces of the map to be of degree d . This limitation shall be overcome in the future article [6].

We now explore the simplifications occurring in the bipartite case.

Theorem 18. *Let $b \geq 1$, and let $E_b \equiv F_{2b}(x_{2b}, 0, x_{2b+2}, 0, x_{2b+4}, \dots)$ be the generating function of rooted bipartite maps of girth $2b$ with outer degree $2b$, where each variable x_{2i} marks the number of inner faces of degree $2i$. Then,*

$$(5) \quad E_b = V_{b-1} - \sum_{j=-1}^{b-2} V_j V_{b-j-1},$$

where $V_{-1} = 1, V_0, \dots, V_b$ are the unique formal power series satisfying:

$$(6) \quad \begin{cases} V_j = h_{j+1}(V_1, \dots, V_{b-1}) & \text{for all } j \text{ in } [-1..b-2], \\ V_j = \sum_{i \geq b} x_{2i} \binom{2i-1}{i-j-1} (1+V_0)^{i+j} & \text{for all } j \text{ in } \{b-1, b\}. \end{cases}$$

Theorem 18 can be obtained by a direct counting of b -bibranching mobiles (which are a bit simpler than d -branching mobiles). However in the proof below we derive Theorem 18 as a consequence of Theorem 16

Proof. Equation (5) and (6) are obtained respectively from (2) and (3) simply by setting for all integer i , $x_{2i+1} = 0$, $W_{2i} = V_i$, $W_{2i+1} = 0$. Hence we only need to prove that the series W_i defined by (2) satisfy for all i , $W_{2i+1}(x_{2b}, 0, x_{2b+2}, 0, \dots) = 0$. This property holds because one can show that *every monomial in the series $W_{2i+1}(x_{2b}, x_{2b+1}, x_{2b+2}, \dots)$, $i \in \mathbb{Z}$ contains at least one variable x_r with r odd*, by a simple induction on the total degree of these monomials. \square

6. SPECIAL CASES $d \in \{1, 2\}$, AND $b \in \{1, 2\}$, AND THE CASE $d = 0$.

In this section, we analyze in more details the bijections obtained in the particular cases $d = 0, 1$, or 2 and $b = 1$ or 2 .

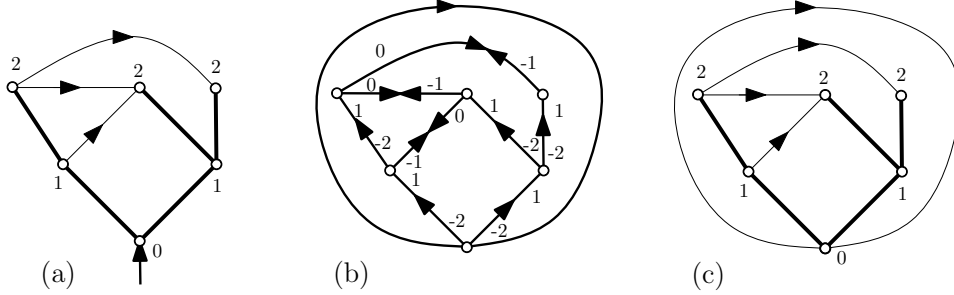


FIGURE 6. (a) A rooted map. (b) The corresponding map M in \mathcal{C}_1 endowed with its canonical $1/(-1)$ -orientation. (c) The root-distances and the rightmost BFS-tree (indicated by thick lines).

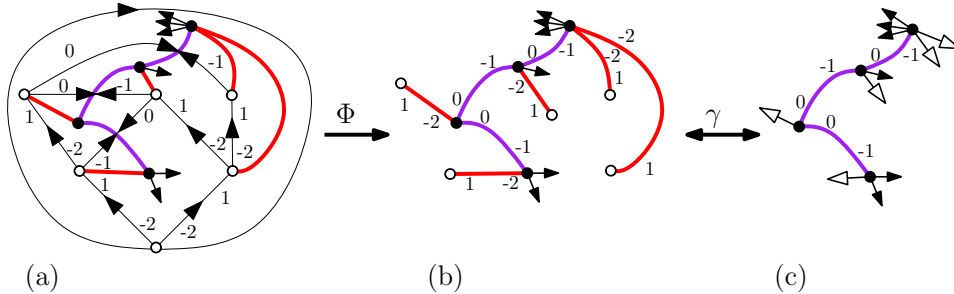


FIGURE 7. Bijection in the case $d = 1$ and its relation with [9]. (a) The bijection Φ . (b) The resulting 1-branching mobile B . (c) The well-charged tree $\gamma(B)$.

6.1. Case $d=1$ (general rooted maps) and relation with [9]. The class of maps \mathcal{C}_1 (plane maps of girth 1 and outer degree 1) easily identifies with the class of rooted maps (without girth constraint): indeed the root-face which is a loop can be thought as marking a corner and then be deleted; see Figure 6. Thus, Theorem 15 gives a bijection between the set of rooted maps and a set of decorated plane trees (the 1-branching mobiles). As shown below, this bijection coincides with the bijection described by Bouttier, Di Francesco and Guitter in [9].

We first characterize the canonical $1/(-1)$ -orientations. Let M be a map in \mathcal{C}_1 . We call *root-vertex* the vertex incident to the root-face. We call *root-distance* of a vertex v the minimum length of the paths joining the root-vertex and v . A spanning tree of M is a *BFS-tree* (or *bread-first-search tree*) if the root-distance of any vertex is the same in the map and in the tree. The root-distances and a BFS-tree are shown in Figure 6. Let T be a BFS-tree and let e be an edge not in T . The edge e creates a cycle with T which separates two regions of the plane. We call *left-to-right orientation* of e the orientation such that the region on the left of e contains the root-face. The outgoing and ingoing half-edges of e are then called *left* and *right* half-edges. It is easy to see (see e.g. [4]) that there exists a unique BFS-tree, called *rightmost BFS-tree*, such that the root-distance does not

decrease along edges not in T traversed left-to-right. The following characterization of canonical $1/(-1)$ -orientations is illustrated in Figure 6.

Proposition 19. *Let M be a map in \mathcal{C}_1 and let T be its rightmost BFS-tree. Then, the unique $1/(-1)$ -orientation of M in $\tilde{\mathcal{O}}$ is obtained as follows:*

- Every edge in T is 1-way, oriented from parent to child with weight -2 on the outgoing half-edge and weight 1 on the ingoing half-edge.
- Every inner edge e not in T is 0-way. If the root-distance is the same (resp. different) for the two endpoints of e , then a weight 0 is given to the left (resp. right) half-edge and a weight -1 is given to the right (resp. left) half-edge.

Proof. Let O be the \mathbb{Z} -weighted bi-orientation described in Proposition 19. We want to prove that O is a $1/(-1)$ -orientation in $\tilde{\mathcal{O}}$ (the uniqueness is granted by Proposition 11). The orientation O is clearly accessible and minimal, hence O is in $\tilde{\mathcal{O}}$. Moreover, O clearly satisfies Conditions (i) and (ii) of $1/(-1)$ -orientations. Thus, it only remains to check Condition (iii) for an inner face f , that is, we want to show $\deg(f) + w(f) = 1$. By definition, $\deg(f) + w(f) = \sum_c \epsilon(c)$ where the sum runs over the corners of f and the contribution $\epsilon(c)$ of a corner c is 1 if c follows an ingoing half-edge in clockwise around its vertex and $1 + w$ if c follows an outgoing half-edge of weight w . We now compare $\epsilon(c)$ with the difference $\delta(c)$ between the root-distance of c and the root-distance of the next corner c' in clockwise order around f . Let e be the unique edge incident to f that is not in T and has f on its right when traversed left-to-right. It is easy to check that if the edge from c to c' around f is e (resp. distinct from e), then $\epsilon(c) = \delta(c) + 1$ (resp. $\epsilon(c) = \delta(c)$). Since $\sum_c \delta(c) = 0$, this gives $\sum_c \epsilon(c) = 1$, hence $\deg(f) + w(f) = 1$. \square

We now examine the 1-branching mobiles. The following characterization follows immediately from the definitions.

Proposition 20. *The 1-branching mobiles are the \mathbb{Z} -weighted mobiles such that:*

- every white vertex is a leaf, every black vertex v satisfies $\deg(v) + w(v) = 1$,
- there is no white-white edge, the black-black edges have a weight of -1 on one half-edge and 0 on the other, the black-white edges have weight -2 at the black extremity and 1 at the white extremity.

A 1-branching mobile is represented in Figure 7 (b). We now explain the relation with the *well-charged trees* considered in [9]. We call *charged tree* a plane tree with two types of dangling half-edges called *white arrows* and *black arrows*. The *charge* of a subtree is the number of white arrows minus the number of black arrows. A *well-charged tree* is a charged tree such that cutting any edge gives two subtrees of charge 0 and -1 respectively. A well-charged tree mobile is represented in Figure 7 (c). In [9] it was proved that rooted maps (with a loop as root-face) are in bijection with well-charged trees. We now explain the relation between 1-branching mobiles and well-charged trees. For a 1-branching mobile B , we denote by $\gamma(B)$ the charged tree obtained by replacing white leaves and buds respectively by white arrows and black arrows. The mapping γ is represented in Figure 7. It is easy to check that for any black-black edge e of B , the charges of the subtrees obtained by deleting the edge e from $\gamma(B)$ are equal to the weights of the half-edges of e incident to these subtrees. Hence, $\gamma(B)$ is a well-charged tree. It is then easy to check that γ is a bijection between 1-branching mobiles and well-charged trees. In addition, the

bijection described in [9] coincides with ours via duality. Indeed to obtain the map from the tree, the closure operations, described respectively in Proposition 4 and in [9] are the same (and to obtain the tree from the map both constructions rely on the rightmost BFS-tree in a very similar way).

6.2. Case $d=2$ (loopless maps) and relation with [21, Theo.2.3.4]. The class of maps \mathcal{C}_2 (plane maps of girth 2 and outer degree 2) easily identifies with the class of loopless planar maps with a marked edge (indeed the outer face of degree 2 can be collapsed into a marked edge). Hence, for $d = 2$, Theorem 15 gives a bijection between edge-marked loopless maps and 2-branching mobiles. These bijections are illustrated in Figure 8. We now examine the 2-branching mobiles. It is easy to see that the weights on the half-edges of 2-branching mobiles are *redundant* (that is, they are determined uniquely by the configuration of the edge). Up to forgetting these weights, the 2-branching mobiles are the bicolored plane trees such that

- there is no white-white edges, and white vertices have degree 1 or 2,
- black vertices adjacent to ℓ white leaves are incident to a total of $\ell + 2$ buds or black-black edges.

We first consider the further specialization to loopless triangulations; see the right column of Figure 8. Theorem 15 yields a bijection between edge-marked loopless triangulations and 2-branching mobiles such that each black vertex has degree 3. These are the mobiles with no white-white edge such that every white vertex has degree 2, and every black vertex has degree 3 and is adjacent to a unique white vertex. Another way of describing this subclass of 2-branching mobiles, is to say that they are (unicolored) binary trees endowed with a perfect matching of the inner nodes. These are exactly the *blossoming trees* shown to be in bijection with *bridgeless cubic maps* (these are the dual of loopless triangulations) in Schaeffer's Ph.D. thesis [21, Theo.2.3.4]². Moreover, the bijection of Schaeffer coincides with ours via duality: to obtain the map from the tree, the closure operations described respectively in Proposition 4 and in [21, Theo.2.3.4] are the same.

We now return to the general case of the bijection for $d = 2$; see the left column of Figure 8. We show how to recover the formula given in [25] (where it is obtained by a substitution approach) for the number of rooted loopless maps with n edges. First observe that, up to collapsing the root-face of degree 2 into an edge, rooted maps in \mathcal{C}_2 identify with rooted loopless maps with at least one edge (without constraint on the degree of the root-face). Hence, the multivariate series F_2 counts rooted loopless maps (with at least one edge), where x_i marks the number of faces of degree i . In that case, with the notation $R := 1 + W_0$ and $S := W_{-1} = W_1$ the system (3) can be written as

$$(7) \quad \begin{cases} R &= 1 + [u^1]X, \\ S &= [u^2]X, \\ X &= \sum_j x_j (uR + S + u^{-1})^{j-1}, \end{cases}$$

and

$$F_2 = R - 1 - S^2 - [u^3]X.$$

²See also [18] where Schaeffer's bijection is extended to nonseparable bridgeless *near-cubic* maps (maps where the root-vertex has arbitrary degree and the other vertices have degree 3).

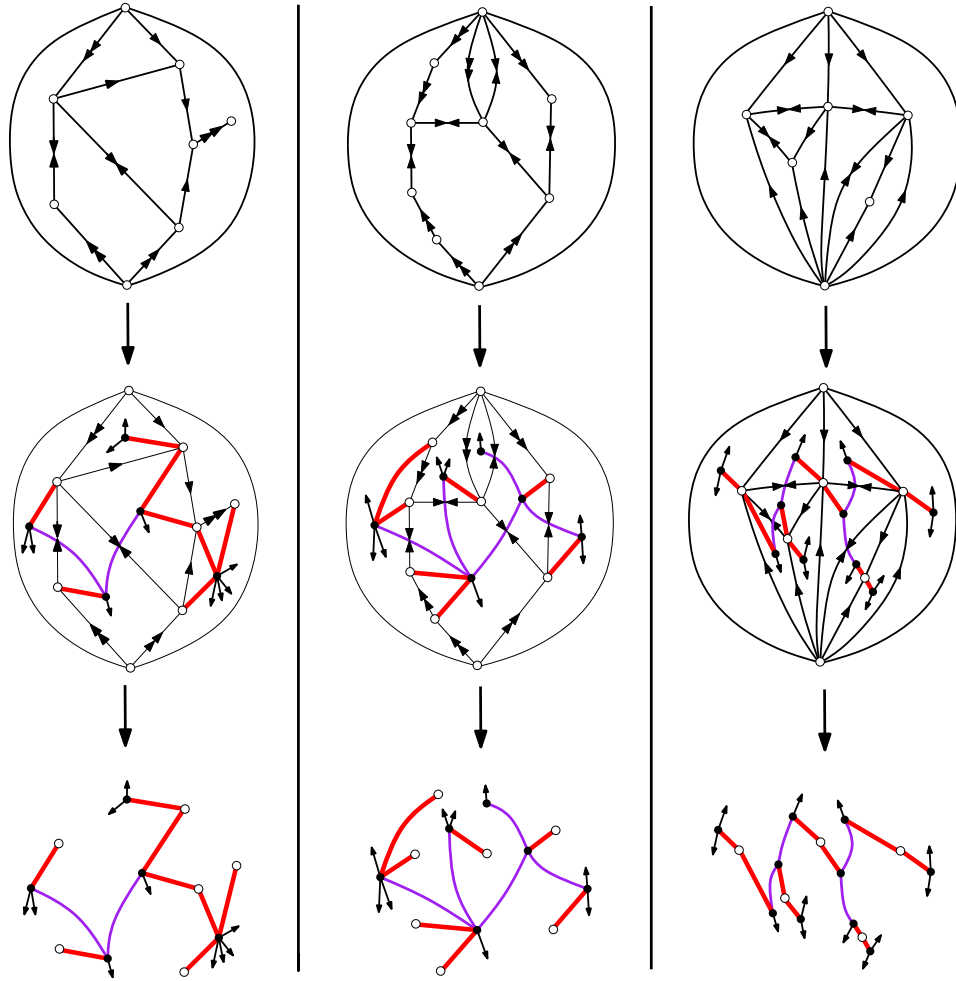


FIGURE 8. Bijection for $d = 2$ (loopless maps) on 3 examples. The canonical $2/0$ -orientations are represented with the following conventions: 1-way edges with weights -2 and 2 on the half-edges are indicated by a double arrow, while 1-way edges with weights -1 and 1 on the half-edges are indicated by a simple arrow. The example in the middle column is bipartite hence gives a 1-bibranching mobile (in that case all the white vertices are leaves). The example in the right column has all its inner faces of degree 3 (in that case all the white vertices have degree 2).

Consider the specialization $x_i = t^i$ in F_2 , which gives the series of rooted loopless maps according to the number of half-edges. First, note that

$$B_k := [u^k] \sum_j t^j (uR + S + u^{-1})^j$$

is the series of Motzkin paths ending at height k where down steps, horizontal steps, and up steps have respective weights t , tS , and tR . Such paths ending at height 0 are called *Motzkin bridges*, those ending at 0 and having non-negative height all the way are called *Motzkin excursions*. Let $B := B_0$ be the series of Motzkin bridges and M the series of Motzkin excursions. It is a classical exercise to show:

$$(i) M = 1 + tSM + t^2RM^2, \quad (ii) B = 1 + tSB + 2t^2RMB \quad \text{and} \quad (iii) B_k = B(tRM)^k.$$

In particular, (iii) shows that under the specialization $x_i = t^i$ one gets

$$(iv) R = 1 + t^2BMR, \quad \text{and} \quad (v) S = t^3BM^2R^2.$$

So we have a system of four equations $\{(i), (ii), (iv), (v)\}$ for the unknown series $\{M, B, R, S\}$, and this system has clearly a unique power series solution. With the help of a computer algebra system, one can extract the first coefficients and then guess and check that the solution is $\{M = \alpha, B = \alpha^2, R = \alpha, S = t^3\alpha^6\}$, where the series $\alpha \equiv \alpha(t)$ is specified by $\alpha = 1 + t^2\alpha^4$. Hence, under the specialization $x_i = t^i$, the expression of F_2 becomes $F_2 = \alpha^2(2 - \alpha) - 1$. We summarize:

Proposition 21. *Let c_n be the number of rooted loopless maps with n edges and let $C(t) = \sum_{n \geq 0} c_n t^n$. Then, $C(t) = \alpha^2(2 - \alpha)$, where $\alpha \equiv \alpha(t)$ is the unique formal power series satisfying $\alpha = 1 + t\alpha^4$. Hence, by the Lagrange inversion formula,*

$$(8) \quad c_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Formula (8) was already obtained in [25] using a substitution approach, but no completely bijective proof is known (here we rely on a bijection, but also do a few algebraic manipulations). The sequence $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ appears recurrently in combinatorics, for instance it also counts rooted simple triangulations with $n+3$ vertices [22, 19], and intervals in the n th Tamari lattice [12, 3].

6.3. Case $b=1$ (general bipartite maps) and relation with [20]. Bipartite maps have girth at least 2. Hence, in the case $b=1$, Theorem 9 gives a bijection between bipartite plane maps of outer degree 2 and 1-bibranching mobiles. This bijection is represented in the middle column of Figure 8.

First, it is easy to see that the weights on the half-edges of 1-bibranching mobiles are redundant. Up to forgetting these weights, the 1-bibranching mobiles are the bicolored plane trees such that white vertices are leaves, and any black vertex adjacent to ℓ white leaves has degree $2+2\ell$. The 1-bibranching mobiles actually coincide with the *blossoming trees* defined by Schaeffer [20] in his bijection with rooted *Eulerian maps* (maps with vertices of even degree; these are the dual of bipartite maps). Indeed, in blossoming trees non-leave vertices have even degree and each vertex of degree $2i$ carries $i-1$ dangling half-edges called *stems*, while in 1-bibranching mobiles each black vertex of degree $2i$ is adjacent to $i-1$ white leaves. Moreover the bijection of Schaeffer coincides with ours via duality: to obtain the map from the tree, the closure operations described respectively Proposition 4 and in [20] are the same.

The following formula (originally due to Tutte [23]) for the number of rooted bipartite maps with n_i faces of degree $2i$ can be obtained via the counting of

blossoming trees as was done by Schaeffer in [20]:

$$(9) \quad 2 \frac{e!}{v!} \prod_{i=1}^k \frac{1}{n_i!} \binom{2i-1}{i-1}^{n_i}.$$

6.4. Case $b=2$ (simple bipartite maps). In the case $b = 2$, Theorem 9 gives a bijection for simple plane bipartite maps of outer degree 4. An example is shown in Figure 4. We now describe the 2-bibranching mobiles and proceed to count them according to the degree of their black vertices. It is easy to see that the weights on the half-edges of 2-bibranching are redundant, and that up to forgetting these weights the 2-bibranching mobiles are the bicolored plane trees such that

- there is no black-black nor white-white edges,
- the white vertices have degree 1 or 2, and any black vertex adjacent to ℓ white leaves has degree $4 + 2\ell$.

We will prove the following result via the counting of 2-bibranching mobiles:

Proposition 22 (simple bipartite maps with at least one quadrangular face). *Let $k \geq 2$, and let n_2, n_3, \dots, n_k be non-negative integers with $n_2 > 0$. The number of rooted simple bipartite maps with n_i faces of degree $2i$ for $i = 2 \dots k$ is*

$$(10) \quad 2 \frac{(e+n-3)!}{(e-1)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i},$$

where $n = \sum_i n_i$ is the number of faces, and $e = \sum_i i n_i$ is the number of edges.

In the future article [6], we shall show that (10) holds even when $n_2 = 0$.

Proof. Let $C(n_2, n_3, \dots, n_k)$ be the number of rooted simple bipartite maps of outer degree 4 with n_i inner faces of degree $2i$. It is easy to see (by considering maps with two marked corners, the first being in a face of degree 4) that if n_2 is positive, then the number of simple rooted maps with n_i faces of degree $2i$ is $4C(n_2 - 1, n_3, \dots, n_k)/(2e)$. It remains to prove that for all integers n_2, \dots, n_k

$$(11) \quad C(n_2, n_3, \dots, n_k) = \frac{4(e+n)!}{(e+2)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i},$$

where $n = \sum_i n_i$ and $e = \sum_i i n_i$.

By Theorem 9 and Claim 3, one gets

$$C(n_2, n_3, \dots, n_k) = B(n_2, \dots, n_k) - H(n_2, \dots, n_k),$$

where $B(n_2, \dots, n_k)$ (resp. $H(n_2, \dots, n_k)$) is the number of 2-bibranching mobiles with n_i black vertices of degree $2i$ with a marked bud (resp. with a marked half-edge incident to a white vertex). The numbers b and h of buds and half-edge incident to a white vertex in a 2-bibranching mobile with n_i black vertices of degree $2i$ satisfy $b + h = 2 \sum_i i n_i = 2e$, and $b - h = 4$ (since 2-bibranching mobiles have excess -4). Hence, $b = e + 2$ and

$$C(n_2, n_3, \dots, n_k) = B(n_2, \dots, n_k) - \frac{h}{b} B(n_2, \dots, n_k) = \frac{4}{e+2} B(n_2, \dots, n_k).$$

Moreover,

$$B(n_2, \dots, n_k) = \frac{(e+n)!}{(e+1)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i},$$

since $\frac{(e+n)!}{(e+1)!} \prod_i \frac{1}{n_i!}$ is the number of rooted (at a leaf) plane trees with n_i vertices of degree $i+2$ (the black vertices) and a total of $e+2$ leaves (the buds), and $\binom{2i-1}{i-2}$ is the number of ways of inserting the $i-2$ white leaves around a vertex of degree $i+2$. This proves (11) hence Proposition 22. \square

6.5. The case $d = 0$. We show here that a slight reformulation of the master bijection, orientations, and mobiles allows us to include the case $d = 0$, thereby recovering a bijection for vertex-pointed maps obtained by Bouttier, Di Francesco and Guitter in [10].

In this section, for $k \geq 0$, a k -pointed map is either (for $k > 0$) a plane map whose root-face is a k -gon, or (for $k = 0$) a map with a pointed vertex; the root face (for $k > 0$) or the pointed vertex (for $k = 0$) is called the *marked k -gon*. Edges, vertices, and faces outside of the marked k -gon are said to be *inner*. A *weighted bi-orientation* on a k -pointed map is a bi-orientation of its inner edges, the ingoing half-edges have nonnegative weight and outer half-edges have nonpositive weight. The bi-orientation is called *admissible* if every half-edge outside the k -gon and incident to a vertex of the k -gon is outgoing; it is called *accessible* if every inner vertex can be reached from the k -gon; and it is called *minimal* if every circuit has the k -gon on its left. For $k \geq 0$ let $\tilde{\mathcal{O}}_k$ be the family of weighted bi-orientations on k -pointed maps that are admissible, accessible, and minimal. A *weighted mobile* is a mobile with weights on half-edges, those incident to black vertices have nonpositive weight and those incident to white vertices have nonnegative weight. The master bijection [7] yields for all $k \geq 0$ a bijection between $\tilde{\mathcal{O}}_k$ and the family of weighted mobiles of excess k . The parameter correspondence is the same as in Theorem 2.

We now reformulate $d/(d-2)$ -orientations so as to include the case $d = 0$. For $d \geq 0$, a $d/(d-2)$ -orientation is an admissible weighted bi-orientation of a d -pointed map such that ingoing half-edges have weight d or positive, outgoing half-edges have weight in $\{-2, -1, 0\} \setminus \{d\}$, inner vertices have weight d , inner edges have weight $d-2$, and each inner face f has degree and weight satisfying $\deg(f) + w(f) = d$. Note that for $d \geq 1$ this is exactly Definition 10 (up to forgetting the weights and orientations on outer edges).

Proposition 11 can be reformulated as follows to include the case $d = 0$: “For $d \geq 0$ a d -pointed map admits a $d/(d-2)$ -orientation if and only if it has no cycle of length smaller than d . In that case it admits a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}_d$, called its canonical $d/(d-2)$ -orientation”. We now proceed to prove the case $d = 0$ of this statement. Define a *vertex-labelling* of a vertex-pointed map M as a labelling of its inner vertices by values in \mathbb{Z} such that the label-difference of two adjacent vertices is at most 1 (in absolute value) and the label of the pointed vertex is 0. It is easy to see that the *distance labelling* (each vertex is labelled by its distance to the pointed vertex) is the unique vertex-labelling such that each inner vertex has a neighbor of smaller label. Clearly a vertex-labelling induces a bi-orientation on M (an edge $\{v, v'\}$ with $\text{label}(v') = \text{label}(v) + 1$ is oriented 1-way from v to v' , an edge $\{v, v'\}$ with $\text{label}(v') = \text{label}(v)$ is 0-way). Putting weights $(-1, -1)$ on 0-way edges and weights $(-2, 0)$ on 1-way edges (weight -2 at the tail) one clearly gets a $0/(-2)$ -orientation, and the mapping is invertible; hence the vertex-labellings of M correspond bijectively to the $0/(-2)$ -orientations of M . Note that a $0/(-2)$ is accessible if and only if each inner vertex has a neighbor of smaller label in the associated vertex-labelling, that is, it is the distance labelling. Hence

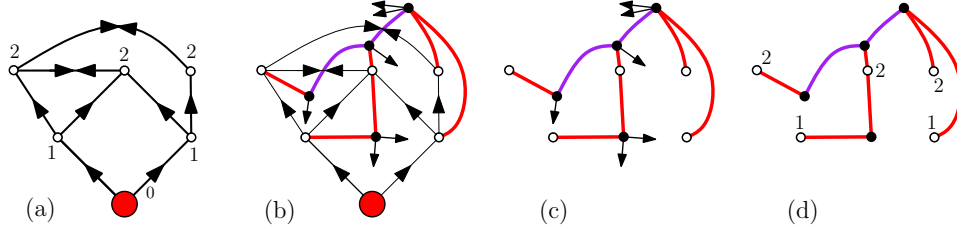


FIGURE 9. Bijection in the case $d = 0$. (a) The canonical $0/(-2)$ -orientation, which is just the geodesic orientation. (b) The master bijection. (c) The 0 -branching mobile (without the weights). (d) The corresponding labelled mobile.

the unique accessible $0/(-2)$ -orientation is the geodesic bi-orientation (edges on a geodesic path are 1-way and edges not on any geodesic path are 0-way), where 1-way edges receive weights $(-2, 0)$ (weight -2 at the tail) and 0-way edges receive weights $(-1, -1)$. The geodesic orientation is minimal (it is even acyclic), hence we have proved the restatement of Proposition 11 for $d = 0$.

We can also reformulate d -branching mobiles so as to include $d = 0$; for $d \geq 0$ a d -branching mobile is a weighted mobile where half-edges at black vertices have weight in $\{-2, -1, 0\} \setminus \{d\}$, half-edges at white vertices have weight d or positive, edges have weight $d - 2$, white vertices have weight d , and each black vertex has degree and weight satisfying $\deg(v) + w(v) = d$. Again for $d \geq 1$ this coincides with Definition 13. Claim 14 is easily checked to hold also for $d = 0$, which yields the following slight restatement of Theorem 15 now holding for all $d \geq 0$ (Figure 9(b)-(c) shows the bijection for $d = 0$):

Proposition 23. *For any nonnegative integer d , d -pointed maps with no cycle of length smaller than d are in bijection with d -branching mobiles; each inner face of degree i in the map corresponds to a black vertex of degree i in the mobile.*

The bijection for $d = 0$ is easily checked to identify with the one by Bouttier Di Francesco Guitter [10] (that this bijection can be recovered by a master construction using orientations was already observed in [4, Section 7]). Given a vertex-pointed map M endowed with its geodesic orientation, the construction in [10] consists in drawing the mobile (without the buds) associated with the geodesic orientation and recording on each white vertex the label of distance from the root-vertex. One obtains this way a bijection between vertex-pointed maps and labelled mobiles satisfying the following property, where the *jump* of a corner c incident to a white vertex v is the difference $\delta(c)$ between the label of v and the label of the next white vertex in clockwise direction around the mobile: “each jump $\delta(c)$ satisfies $\delta(c) \geq -1$ ”. In our case ($d = 0$) we draw the mobile (0 -branching mobile) associated with the canonical $0/(-2)$ -orientation, which we have proved to be the geodesic orientation. The difference with [10] is that we include buds in the mobile and do not record the labels of distance from the root-vertex. The weights on 0 -branching mobiles imply that each black vertex has as many buds as white neighbors. Hence, if we call *balanced* an unweighted mobile satisfying this property, then a 0 -branching mobile yields a balanced mobile, and the mapping is easily checked to be invertible. There is also an easy identification between the balanced mobiles and the labelled

mobiles in [10]: given a labelled mobile L , one obtains a balanced mobile by adding, for each corner c incident to a white vertex, a total of $\delta(c) + 1$ buds in the next corner incident to a black vertex in clockwise direction around the mobile (and then forgetting all the labels), see Figure 9(c)-(d). Again this mapping is easily shown to be invertible. To summarize the labelled mobiles in [10] identify to our 0-branching mobiles and the bijection in [10] coincides with ours (case $d = 0$) up to this identification.

7. PROOFS

In this section we prove Propositions 6 and 11. We start with the proof that having girth d (resp. $2b$) is necessary to admit a $d/(d-2)$ -orientations (resp. $b/(b-1)$ -orientation). Then we prove the existence and uniqueness of a $b/(b-1)$ -orientations for bipartite maps in \mathcal{C}_{2b} for $b \geq 2$. Then we prove the existence and uniqueness of a $d/(d-2)$ -orientations for maps in \mathcal{C}_d for $d \geq 2$. Lastly we prove the cases $b = 1$ and $d = 1$.

7.1. Necessity of girth d or $2b$.

Lemma 24. *Let d be a positive integer. Let M be a plane map of outer degree d . If M admits a $d/(d-2)$ -orientation, then it has girth d .*

Proof. Since the outer degree is d the girth is at most d (not that M cannot be a tree because $d/(d-2)$ -orientation are admissible, so that the outer face is a simple cycle). We now consider a cycle C of length ℓ and want to prove that $\ell \geq d$. Let v , e , and n be the numbers of vertices, edges, and faces strictly inside of C . Let S (resp. S') be the sum of weight of the outgoing half-edges inside C (resp. strictly inside C). Clearly, $S \leq S' \leq 0$. Conditions (i) and (ii) of $b/(b-1)$ -orientations give $dv = (d-2)e - S'$ (since both side are the sum of weights of the vertices strictly inside C), hence $dv \leq (d-2)e - S$. Moreover, Condition (iii) of $b/(b-1)$ -orientations gives $dn = S + \sum_f \deg(f)$, where the sum is over the faces strictly inside C . Since $\sum_f \deg(f) = 2e + \ell$, we get $S = dn - 2e - \ell$. Thus, $dv \leq (d-2)e - dn + 2e + \ell$, that is, $\ell \geq d(v - e + n) = d$, where the last equation is the Euler relation. \square

Corollary 25. *Let b be a positive integer and let M be a bipartite map of outer degree $2b$. If M admits a $b/(b-1)$ -orientation, then it has girth $2b$.*

Proof. If M admits a $b/(b-1)$ -orientation, then doubling the weights of half-edges on inner edges gives a $d/(d-2)$ -orientation for $d = 2b$. Hence M has girth $d = 2b$ by Lemma 24. \square

7.2. Existence and uniqueness of a $b/(b-1)$ -orientations for $b \geq 2$. In this subsection, we fix an integer $b \geq 2$ and a bipartite map M of girth $2b$ and outer degree $2b$, and we prove the existence of a unique $b/(b-1)$ -orientation of M in $\tilde{\mathcal{O}}$ (thereby completing the proof of Proposition 6 for $b \geq 2$). In order to prove this result, we consider some orientations on a related map denoted $M+Q$.

We call *inner-quadrangulation* of M the map Q obtained from M by inserting a vertex v_f , called *face-vertex*, in each inner face f of M , adding an edge from v_f to each corner around f , and finally deleting the edges (but not the vertices) of M . Inner faces of Q have degree 4 and correspond to the inner edges of M . We denote by $M+Q$ the map obtained by superimposing M and Q .

Definition 26. We call regular orientation of $M+Q$, an admissible \mathbb{Z} -weighted orientation such that the weight of every outgoing half-edge is 0 and

- (i) inner edges of M have weight $b-1$, edges of Q have weight 1,
- (ii) inner vertices of M have weight b ,
- (iii) any face-vertex v has a weight and degree satisfying $w(v) - \deg(v)/2 = b$.

For an edge e of Q with endpoint in M denoted v , we call M -edge of e the edge of M following e in clockwise order around v . A regular orientation of $M+Q$ is called *coherent* if for all edge e of Q directed toward its endpoint v in M , the corresponding M -edge e' is a 1-way edge oriented toward v (observe that in this case v is an inner vertex and e' is an inner edge of M). Given a coherent regular orientation X of $M+Q$, we denote by $\sigma(X)$ the \mathbb{Z} -weighted orientation of M obtained by

- keeping the bi-orientation of every edge e of M in X
- keeping the weights on the half-edges, except if e is the M -edge of an edge of Q directed toward its endpoint in M , in which case the weights 0 and $b-1$ are replaced by -1 and b respectively.

We claim that the orientation $\sigma(X)$ is a $b/(b-1)$ -orientation of M . Indeed, Conditions (i), (ii), (iii) of regular orientations correspond respectively to Conditions (i), (ii), (iii) of $b/(b-1)$ -orientations. Moreover, the mapping σ is easily seen to be surjective and invertible. We summarize:

Lemma 27. *The mapping σ is a bijection between the coherent regular orientations of $M+Q$ and the $b/(b-1)$ -orientations of M .*

Proving the existence and uniqueness of a $b/(b-1)$ -orientation of M in $\tilde{\mathcal{O}}$ now reduces to proving the existence and uniqueness of a coherent regular orientation whose image by σ is in $\tilde{\mathcal{O}}$. We start by proving the existence of a regular orientation of $M+Q$. Roughly speaking, we will use the fact that M can be completed into a $2b$ -angulation of girth $2b$, and appeal to the following result proved in [7].

Lemma 28. *Any $2b$ -angulation of girth $2b$ admits a $b/(b-1)$ -orientation which is accessible from every outer vertex.*

We will also need the following result.

Claim 29. *Let A be a $2b$ -angulation endowed with a $b/(b-1)$ -orientation. Let C be a simple cycle of length $2c$ having only inner vertices. Then, the sum S of weights of the ingoing half-edge incident to C but strictly outside of the region enclosed by C is $b+c$.*

Proof. By the Euler relation (and the incidence relation between faces and edges), the numbers v and e of vertices and edges inside of C (including vertices and edges on C) satisfy $(b-1)e + b + c = bv$. Moreover $bv = S + (b-1)e$ since both side equals the sum of weights of ingoing half-edge incident to vertices inside C . \square

Lemma 30. *There exists a regular orientation of $M+Q$ which is accessible from every outer vertex of M .*

Proof. The proof is illustrated in Figure 10. The first step is to complete M into a $2b$ -angulation of girth $2b$, by adding vertices and edges inside each inner face of M . We make use of the following basic fact: “for each even integer j , there exists a plane map L_j of girth $2b$, whose outer face is a simple cycle of length j , and whose

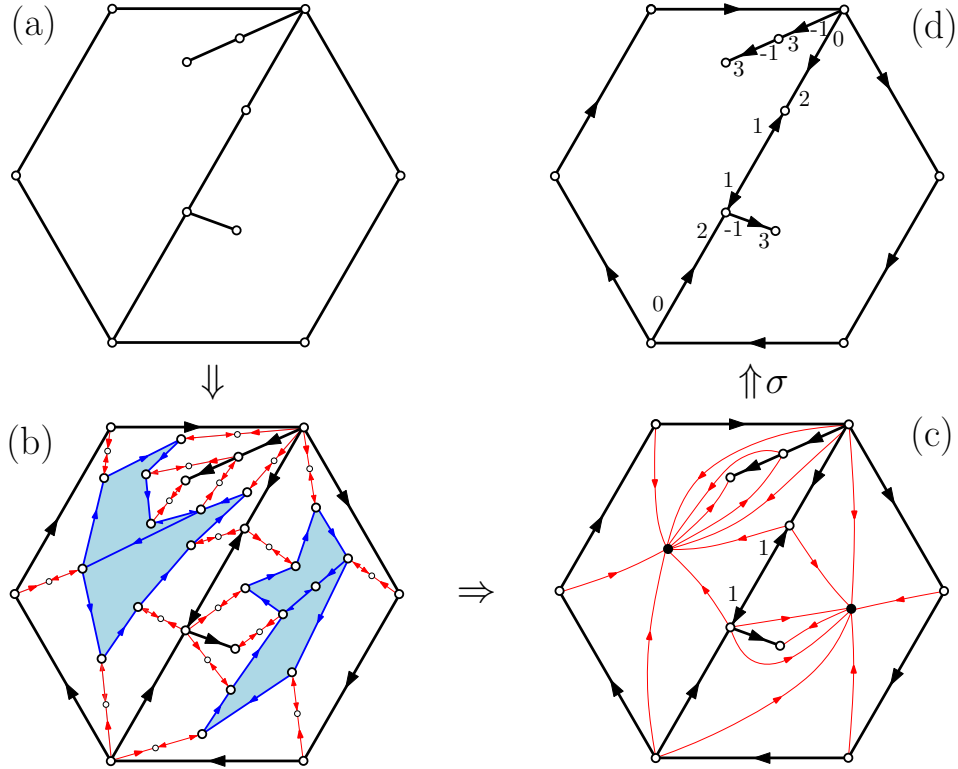


FIGURE 10. Process for constructing a $b/(b-1)$ -orientation of M ($b = 3$ here). In (b) M is completed into a $2b$ -angulation A of girth $2b$, and A is endowed with a $b/(b-1)$ -orientation. In (c) the $b/(b-1)$ -orientation of A is shrunk into a regular orientation X of $M+Q$. In (d) the map M gets the $b/(b-1)$ -orientation $\sigma(X)$.

inner faces have degree $2b$ ". Now, for each inner face f of M , we consider the sequence of the corners $c_1, \dots, c_{\deg(f)}$ in clockwise order around f (note that the incident vertices of these corners might not be all distinct). We "throw" a simple path P_i of length $(b-1)$, called a *transition-path*, from each corner c_i toward the interior of f . Denote by v_i the vertex at the free extremity of each path P_i . We then connect the vertices v_i along a simple cycle C_f (of length $\deg(f)$), the order of the vertices along the cycle corresponding to the order of the corners around f . Then we patch a copy of the plane map $L_{\deg(f)}$ inside C_f . Doing this for each inner face of M we obtain a $2b$ -angulation A , which is easily checked to be of girth $2b$. By Lemma 28, A admits a $b/(b-1)$ -orientation O . Recall that the half-edges of O have non-negative weights (this is an easy consequence of Condition (iii) of $b/(b-1)$ -orientations). Let P be a transition path, and let w, w' be the weights of the half-edges h, h' at the extremities of P . We claim that $\{w, w'\} = \{0, 1\}$. Indeed, the fact that the weights of each edge of P is $b-1$, and the weights of the two half-edges incident to each inner vertex of P add up to b ensures that $w + w' = 1$ (and the two weights are non-negative). We then perform the following operations to obtain an orientation of $M+Q$:

- (1) we shrink each transition-path P into a single 1-way edge of weight 1 by only keeping the extremal half-edges h, h' of weight w, w' ,
- (2) we contract the cycle inserted inside each inner face (and the map L_j contained therein) into a single vertex (which becomes a face-vertex of Q).

We claim that the orientation X of $M+Q$ obtained is a regular orientation. Indeed, the weights of the vertices and edges of M are the same as in the $b/(b-1)$ -orientation O , that is, inner (resp. outer) vertices of M has weight b (resp. 1) and inner (resp. outer) edges of M have weight $b-1$ (resp. 1). Moreover Claim 29 implies that any face-vertex of v has weight $w(v) = \deg(v)/2 + b$. Thus X is a regular orientation of $M+Q$, and it only remains to show that it is accessible from every outer vertex. Now, the orientation O is accessible from the outer vertices, and moreover the operations for going from O to X can only increase the accessibility between vertices. Thus, X is accessible from the outer vertices. \square

We now examine the properties of regular orientations.

Proposition 31. *The regular orientations of $M+Q$ are all accessible from the outer vertices. Moreover there exists a unique minimal regular orientation of $M+Q$, and this orientation is coherent.*

We use the following general result proved in [7].

Lemma 32. *Let G be a plane map with vertex set V and edge set E . Let α be a function from $V \cup E$ to the \mathbb{N} . Suppose that there exists an α -orientation, that is, a \mathbb{Z} -weighted bi-orientation such that the weight of any outgoing half-edge is 0, the weight of any vertex v is $\alpha(v)$, and the weight of any edge e is $\alpha(e)$. In this case, there exists a unique minimal α -orientation. Moreover, if an α -orientation is accessible from a vertex v , then the other α -orientations are also accessible from v .*

Proof of Proposition 31. Lemma 32 directly ensures that regular orientations are accessible from the outer vertices, and grants the existence and uniqueness of a minimal regular orientation X . It only remains to prove that the minimal orientation X is coherent. Let $e = \{a, v\}$ be an edge of Q oriented toward its endpoint v in M . Let $\epsilon = \{u, v\}$ be the M -edge of e , and let f be the face of M containing e . We want to show that ϵ is oriented 1-way toward v . Assume this is not the case, that is, the weight i of the half-edge incident to u is positive. Let $e' = \{a, u\}$ be the edge preceding e in clockwise order around a . Since ϵ can be traversed from v to u , the edge e' must be oriented away from a (otherwise the triangle $\{a, v, u\}$ would form a counterclockwise circuit, in contradiction with the minimality of X). Let ϵ' be the M -edge of e' . Since the vertex u has total weight b , with contribution $i > 0$ by the edge ϵ and contribution 1 by the edge e' , we conclude that the weight of ϵ' at u is at most $b-2$, hence ϵ' is not 1-way toward v . By the same arguments, the edge e'' preceding e' in clockwise order around a is also oriented away from a . Continuing in this way around the face f we reach the contradiction that all edges incident to a are oriented away from a . \square

We now complete the proof of the existence and uniqueness of the $b/(b-1)$ -orientation of M in $\tilde{\mathcal{O}}$ by proving that the image of a coherent orientation X by the mapping σ is in $\tilde{\mathcal{O}}$ if and only if X is the minimal regular orientation of $M+Q$.

Lemma 33. *The minimal regular orientation of $M+Q$ is mapped by σ to a $b/(b-1)$ -orientation of M that lies in $\tilde{\mathcal{O}}$.*

Proof. Let X be the minimal regular orientation of $M+Q$, and let $Y = \sigma(X)$. We want to prove that Y is minimal and accessible from every outer vertex. The minimality of Y is obvious since the orientation Y is the restriction of the minimal orientation X . We now consider an outer vertex v_0 and prove that Y is accessible from v_0 . Let k be the number of inner faces of M , and let b_1, \dots, b_k be the face-vertices of $M+Q$. Let H_0 to be the underlying bi-orientation of X (forgetting the weights), and for $i \in \{1, \dots, k\}$ let H_i be the bi-orientation obtained from H_0 by deleting the face-vertices b_1, \dots, b_i and their incident edges. Note that H_k is the underlying bi-orientation of Y . Recall that H_0 is accessible from v_0 . We will now show that H_i is accessible from v_0 by induction on i . We assume that H_{i-1} is accessible from v_0 and suppose for contradiction that a vertex w is not accessible from v_0 in H_i . In this case, each directed path P from v_0 to w in H_{i-1} goes through b_i . Let P be such a path, and let e_0 and e_1 be the edges arriving at and leaving b_i along P . We call *left-degree* of P the number of edges of Q between e_0 and e_1 in clockwise order around b_i . We choose P so as to minimize the left-degree. Call P_0 the portion of P before b_i , and P_1 the portion of P after b_i . Let u be the origin of e_0 , let v be the end of e_1 and let ϵ be the M -edge of e_1 ; see Figure 11. Since the regular orientation X is coherent, the edge ϵ is oriented 1-way toward v . Call v' the origin of ϵ . Note that $v' \neq u$ (if $v' = u$ one could pass by ϵ , thus avoiding b_i , to go from v_0 to w) and that the edge $e'_1 = \{b_i, v'\}$ preceding e_1 in clockwise order around b_i must be directed from v' to b_i (otherwise one could replace in P the portion $u \rightarrow b_i \rightarrow v$ by $u \rightarrow b_i \rightarrow v' \rightarrow v$, yielding a path with smaller left-degree, a contradiction). Since H_{i-1} is accessible from v_0 , there exists a directed path P' in H_{i-1} from v_0 to v' . We can choose P' in such a way that it shares an initial portion with P but does not meet again P once it leaves it. Note that v' is not accessible from v_0 in H_i (if it was, so would be v , hence so would be w), so P' has to pass by b_i , so the portion of P' before b_i equals P_0 . Let e' be the edge taken by P' when it leaves b_i and let P'_1 be the portion of P' after e' . Note that e' can not be strictly between e_0 and e_1 in clockwise order around b_i (otherwise by a similar argument as above, it would be possible to produce a path from v_0 to w with smaller left-degree than in P). Since P'_1 can not meet P_0 again, it has to form a counterclockwise circuit together with the two edges e'_1 and e' , see Figure 11. We reach a contradiction. This concludes the proof that H_i is accessible from v_0 . By induction on i , the bi-orientation H_k underlying Y is accessible from v_0 . Thus Y is accessible from every outer vertex and is in $\tilde{\mathcal{O}}$. \square

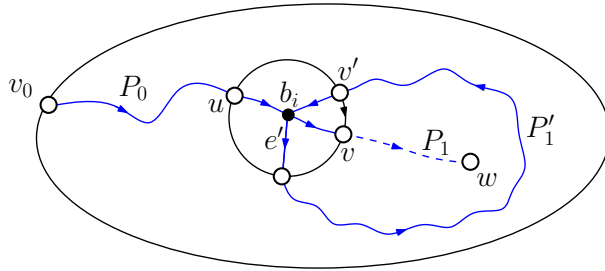


FIGURE 11. The situation in the proof of Lemma 33.

Lemma 34. *If X is a coherent regular orientation of $M+Q$ which is not minimal, then its image by σ is not in $\tilde{\mathcal{O}}$.*

Proof. Suppose for contradiction that the $b/(b-1)$ -orientation $Y = \sigma(X)$ is in $\tilde{\mathcal{O}}$. Since X is not minimal, it has a simple counterclockwise circuit C . By choosing C to enclose no other counterclockwise circuit, we can assume that C has no *chordal path* (a directed path strictly inside of C connecting two vertices of C). Since Y has no counterclockwise circuit, C must contain at least one edge e of Q oriented toward its endpoint v in M . Since X is coherent, the M -edge ϵ of e is oriented 1-way toward v . Hence the edge ϵ lies strictly inside C . Let v_0 be an outer vertex. Since X is accessible from v_0 (by Proposition 31), there exists an oriented path in X from v_0 to v ending at the edge ϵ . The portion P of the path inside C is a chordal path for C , yielding a contradiction. \square

With Lemmas 33 and 34 we have proved that a $b/(b-1)$ -orientation of M is in $\tilde{\mathcal{O}}$ if and only if it is the image of a minimal regular orientation of $M+Q$. With Proposition 31 we have established the existence and uniqueness of the minimal regular orientation of $M+Q$. This concludes the proof that if a bipartite map M of outer degree $2b$ has girth $2b$, then it admits a unique $b/(b-1)$ -orientation in $\tilde{\mathcal{O}}$. The necessity of girth $2b$ was established in Corollary 25. This concludes the proof of Proposition 6 for $b \geq 2$. The case $b = 1$ will be treated in Subsection 7.4.

7.3. Existence and uniqueness of a $d/(d-2)$ -orientations for $d \geq 2$. In this subsection, we fix an integer $d \geq 1$ and a map M of girth d and outer degree d . We will prove that if $d \geq 2$, then there exists of a unique $d/(d-2)$ -orientation of M in $\tilde{\mathcal{O}}$ (thereby completing the proof of Proposition 11 for $d \geq 2$). This will be done by a reduction to the bipartite case as illustrated in Figure 12.

We denote by M' the map obtained from M by inserting a vertex v_e , called an *edge-vertex*, in the middle of each edge e of M . Clearly M' is bipartite, has outer degree $2d$ and girth $2d$. Given a \mathbb{Z} -weighted orientation of M' , the *induced orientation* on M is the \mathbb{Z} -weighted bi-orientation of M obtained by contracting the edge-vertices and their two incident half-edges (the two other half-edges get glued together).

Claim 35. *Let X be a $d/(d-1)$ -orientation of M' , and let Y be the induced orientation on M . Then for any inner edge of M the weights i, j on the half-edges satisfy either $i < d, j < d$ and $i + j = d - 2$, or $\{i, j\} = \{-1, d\}$ in which case the edge is called special.*

Proof. This is an easy consequence of the fact that the edge-vertex v_e has weight d , and the two edges of M' incident to v_e have weight $d - 1$. \square

Let X be a $d/(d-1)$ -orientation of M' , and let Y be the induced orientation on M . We denote by $\tau(X)$ the admissible \mathbb{Z} -weighted bi-orientation of M obtained from Y by replacing the weights -1 of special inner edges by -2 .

Lemma 36. *The mapping τ is a bijection between the $d/(d-1)$ -orientations of M' and the $d/(d-2)$ -orientations of M . Moreover, a $d/(d-1)$ -orientation X is in $\tilde{\mathcal{O}}$ if and only if $\tau(X)$ is in $\tilde{\mathcal{O}}$*

Proof. Let X be a $d/(d-1)$ -orientation. Claim 35 implies that $\tau(X)$ satisfy Conditions (i) and (ii) of $d/(d-2)$ -orientations. Moreover Condition (iii) follows from the fact that the weight of faces is preserved by τ . Thus $\tau(X)$ is a $d/(d-2)$ -orientation.

	outer edges	inner edges	
M'			
M			

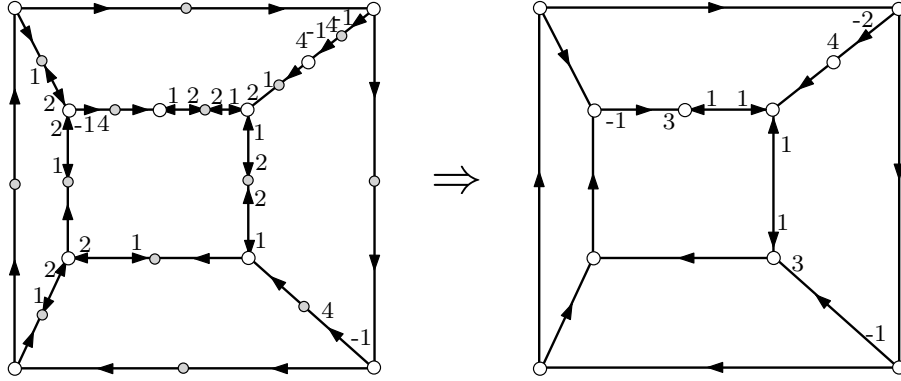


FIGURE 12. Top: The mapping τ between $d/(d-1)$ -orientation of the bipartite map M' (obtained from M by inserting a vertex in the middle of each edge) and $d/(d-2)$ -orientations of M . Bottom: example in the case $d = 4$.

We now prove that τ is a bijection. Observe that a $d/(d-1)$ -orientation of M' is completely determined by its contraction (since the weight of every edge is fixed). Thus, the mapping τ is invertible. Again it is easily checked that the inverse of τ maps $d/(d-2)$ -orientations of M to $d/(d-1)$ -orientations of M' .

It remains to prove the second assertion. Let X be any $d/(d-1)$ -orientation of M' . For any inner edge $e = \{u, v\}$ of M , it is easy to see that the edge e can be traversed from u to v in $\tau(X)$ (that is, e is 2-way or 1-way toward e) if and only if the path of M' made of the edges $e_1 = \{u, v_e\}$ and $e_2 = \{v_e, v\}$ can be traversed from u to v in X . Therefore, the orientation $\tau(X)$ is minimal if and only if X is minimal. Moreover, for any edge $e = \{u, v\}$ of M , it is either possible to traverse $e_1 = \{u, v_e\}$ from u to v_e , or to traverse $e_2 = \{v_e, v\}$ from v to v_e (since the weight of v is positive). Hence, the orientation $\tau(X)$ is accessible from a vertex v_0 if and only if X is accessible from v_0 . Thus a $d/(d-1)$ -orientation X is in $\tilde{\mathcal{O}}$ if and only if $\tau(X)$ is in $\tilde{\mathcal{O}}$ \square

We now suppose that $d \geq 2$ and prove that M admits a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}$. Since $d \geq 2$, it has been proved in subsection 7.2 that M' admits a unique $d/(d-1)$ -orientation in $\tilde{\mathcal{O}}$. Therefore, Lemma 36 implies that the map M admits a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}$. This concludes the proof that for all $d \geq 2$, maps of girth d and outer degree d admit a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}$. The necessity of having girth has been proved in Lemma 24. This concludes the proof of Proposition 11 in the case $d \geq 2$.

7.4. Existence and uniqueness for $b = 1$ and $d = 1$. In this subsection we prove the cases $b = 1$ of Proposition 6 and the case $d = 1$ of Proposition 11.

We first show that the cases $d \geq 2$ of Proposition 11 (which have already been proved) actually implies all the cases $b \geq 1$ of Proposition 6. Let $b \geq 1$, let $d = 2b$ and let M be a bipartite map of outer degree $2b$ and girth $2b$. Recall that an *even* $d/(d-2)$ -orientation of M is a $d/(d-2)$ -orientation where all weights on half-edges of inner edges are even. One immediately checks that $b/(b-1)$ -orientations of M identify to even $d/(d-2)$ -orientations of M (by doubling the weight of half-edges on inner edges). Thus, in order to recover Proposition 6 from Proposition 11 we only have to show that the canonical $d/(d-2)$ -orientation of M is even. Let T be the d -branching mobile associated with M . By definition the weights of the half-edges of any edge of T add up to $2b - 2$ which is even. Let A be the set of edges of T which have odd weights on both half-edges. One easily checks that each vertex of T is incident to an even number of edges from A (because white vertices have even weight and black vertex have even degree), hence A forms an Eulerian subgraph of T . Since T is a tree, A must be empty. So we have shown that the canonical $d/(d-2)$ -orientation of M is even, that is, identifies to a $b/(b-1)$ -orientation. This establishes Proposition 6, including the case $b = 1$.

Lastly, we establish the case $d = 1$ of Proposition 11 by using the same strategy as in Subsection 7.3. We want to prove that a map M of girth 1 and outer degree 1 admits a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}$. We consider the bipartite map M' obtained by inserting a vertex at the middle of each edge. As we have just proved (case $b = 1$), the bipartite map M' admits a unique $1/0$ -orientation in $\tilde{\mathcal{O}}$. Therefore, Lemma 36 implies that the map M admits a unique $1/-1$ -orientation in $\tilde{\mathcal{O}}$. This concludes the proof that for $d = 1$, maps of girth d and outer degree d admit a unique $d/(d-2)$ -orientation in $\tilde{\mathcal{O}}$. This concludes the proof of Proposition 11, including the case $d = 1$.

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