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Bochner-almost periodicity for stochastic processes

Fazia BEDOUHENE*, Omar MELLAH† and Paul RAYNAUD DE FITTE‡

February 16, 2011

Abstract

We compare several notions of almost periodicity for continuous processes defined on the time interval $I = \mathbb{R}$ or $I = [0, +\infty)$ with values in a separable Banach space \mathbb{E} (or more generally a separable completely regular topological space): almost periodicity in distribution, in probability, in quadratic mean, almost sure almost periodicity, almost sure equi-almost periodicity. In the deterministic case, all these notions reduce to Bochner-almost periodicity, which is equivalent to Bohr-almost periodicity when $I = \mathbb{R}$, and to asymptotic Bohr-almost periodicity when $I = [0, +\infty)$.

1 Introduction

Since the creation of the theory of almost periodicity by Harald Bohr in the 1920s, many concepts of almost periodicity have proved useful, in particular those of Stepanov, Besicovitch, and Weyl. In the context of stochastic processes, each of these notions gives rise to several possible definitions, such as: almost periodicity in distribution, in probability, in quadratic mean, almost sure etc (not to mention e.g. processes with almost periodic covariance functions and many others, see [23]). In this paper we study the relationships between some of these definitions for Bochner almost periodicity. Bochner-almost periodicity coincides with Bohr-almost periodicity when the time

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interval I is the whole line \mathbb{R} [7], but it only amounts to asymptotic Bohr-almost periodicity when $I = [0, +\infty)$. It has the advantage over Bohr's definition that it does not depend on the uniform structure of the state space but only on its topology (see Remark 2.1 in this paper).

Our work completes that of C. Tudor [23] who compared different types of Bohr (or Bochner) almost periodicity for continuous processes in the case when $I = \mathbb{R}$.

As an application of our study, it is possible to compare previous results proving almost periodicity in the Bohr sense of the solution of some stochastic differential equations. For example, in the works of C. Tudor and his collaborators (see in particular [15, 22, 1, 10]), almost periodicity in distribution of the solutions is proved. On the other hand, in a series of papers (in particular [2, 3, 4]) Bezandry and Diagana prove almost periodicity in quadratic mean, which amounts to almost periodicity in probability plus uniform integrability of the square of the norms. We show that Bochner-almost periodicity in distribution implies Bochner-almost periodicity in probability, and that both notions are equivalent for processes with uniformly continuous paths.

The paper is organized as follows: The next section is devoted to the case when the state space is metrizable. An extension of the comparison between almost periodicity in distribution and in probability in the completely regular case is given in Section 3 for possible applications in nonmetrizable locally convex topological vector spaces. This extension is done by considering completely regular spaces as inverse limits of metrizable spaces and using the results of section 2. Some complements on criteria of almost periodicity and asymptotic almost periodicity on metric spaces and in completely regular spaces are given in the Appendix (Section 4).

Notations and terminology In all what follows, we are given

- a separable completely regular topological space \mathbb{E} (the state space), e.g. \mathbb{E} is a separable metrizable space,
- an interval I , which is \mathbb{R} or \mathbb{R}^+ (the time interval),
- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- a continuous stochastic process X on \mathbb{E} defined on $\Omega \times I$.

We use the following notations:

- We denote by $\mathcal{C}(I; \mathbb{E})$ the space of continuous functions from I to \mathbb{E} .
- The space of trajectories of the process X is the space of continuous functions from I to \mathbb{E} , which we endow with the topology of uniform convergence on compact subsets of I and denote by $\mathcal{C}_k = \mathcal{C}_k(I; \mathbb{E})$. The definition

of \mathcal{C}_k seems to rely on the choice of a uniformity on \mathbb{E} (e.g. of a metric in the metrizable case), but actually the topology of \mathcal{C}_k is independent of this choice, see [12, Theorem 8.2.6].

- For any uniform space F , we denote $\mathcal{C}_u(I; F)$ the space of continuous functions from I to F endowed with the topology of uniform convergence on I .
- For any function $x : I \rightarrow \mathbb{E}$, and for every $t \in I$, $\tilde{x}(t)$ is the *translation mapping* $x(t + \cdot) : I \rightarrow \mathbb{E}$, $s \mapsto x(t + s)$.
- For any topological space F , we denote by $\mathcal{P}(\mathcal{C}_k)$ the space of Radon probabilities on F endowed with the *topology of narrow convergence*, which is the coarsest topology such that the mappings $\mu \mapsto \mu(f)$ are continuous for all bounded continuous functions $f : F \rightarrow \mathbb{R}$.
- We denote by law (U) the law of a random variable U . The law of the process X is the law of the \mathcal{C}_k -valued random variable $X = \tilde{X}(0)$.

As we concentrate mainly on the study of Bochner-almost periodicity for stochastic processes, *by default, in this paper, “almost periodic” means “Bochner-almost periodic”*.

2 Metrizable case

In this section, \mathbb{E} is a Polish space i.e. \mathbb{E} is separable and its topology can be defined by a distance $d_{\mathbb{E}}$ such that $(\mathbb{E}, d_{\mathbb{E}})$ is complete.

Before we give the different definitions of almost periodicity for stochastic processes, let us recall the corresponding definitions for deterministic functions.

2.1 Almost periodicity in the deterministic case

Let $d_{\mathbb{E}}$ be a distance on \mathbb{E} which is compatible with the topology of \mathbb{E} , and let $x : I \rightarrow (\mathbb{E}; d_{\mathbb{E}})$ be a continuous function. We say that x is *Bochner-almost periodic* if the set $\{\tilde{x}(t), t \in I\} = \{x(t + \cdot), t \in I\}$ is totally bounded in the space $\mathcal{C}_u(I, \mathbb{E})$.

We say that x is *Bohr-almost periodic* (resp. *asymptotically Bohr-almost periodic*) if for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ (resp. there exists $l(\varepsilon) > 0$ and $T = T(\varepsilon) \geq 0$) such that any interval of length $l(\varepsilon)$ contains at least an ε -almost period, that is, a number τ for which

$$d_{\mathbb{E}}(x(t + \tau), x(t)) \leq \varepsilon, \text{ for all } t \in I = \mathbb{R} \text{ (resp. for all } t \geq T \text{ such that } t + \tau \geq T).$$

In the first case (Bohr almost periodicity), we say that the set of ε -almost periods of x is *relatively dense* in I . If \mathbb{E} is a vector space, a function $x : \mathbb{R} \rightarrow$

\mathbb{E} is asymptotically Bohr-almost periodic iff it has the form $x = y + z$, where y is Bohr-almost periodic and z is a continuous function with $\lim_{\pm\infty} z = 0$, see e.g. [19].

The following statements are equivalent (see [7, 10] and Corollary 4.3):

1. x is Bochner-almost periodic.
2. x satisfies *Bochner's double sequence criterion*, that is, for every pair of sequences (α'_n) and (β'_n) in I , there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ respectively with same indexes such that, for every $t \in I$, the limits

$$(1) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} x(t + \alpha_n + \beta_m) \text{ and } \lim_{n \rightarrow \infty} x(t + \alpha_n + \beta_n),$$

exist and are equal.

3. x is almost periodic in Bohr's sense if $I = \mathbb{R}$, or asymptotically almost periodic in Bohr's sense if $I = \mathbb{R}^+$.

Remark 2.1 (i) Bochner's double sequence criterion shows that Bochner almost periodicity is topological, i.e. it doesn't depend on the choice of $d_{\mathbb{E}}$, provided that $d_{\mathbb{E}}$ is compatible with the topology of \mathbb{E}

(ii) A striking property of Bochner's double sequence criterion is that the limits in (1) exist in any of the three modes of convergences: pointwise, uniform on compact intervals and uniform on I (with respect to $d_{\mathbb{E}}$). This criterion has thus the advantage that it allows to establish uniform convergence by checking pointwise convergence.

(iii) Asymptotic Bohr-almost periodicity is strictly weaker than Bohr-almost periodicity, as shows the following example provided by Ruess and Summers [19]:

$$x(t) = \frac{1}{1+t} \cos[\ln(1+t)], \quad t \in \mathbb{R}^+.$$

2.2 Almost periodicity and asymptotic almost periodicity in distribution

Following Tudor's [23] terminology, we say that X is *Bochner-almost periodic in distribution*, APD for short, (resp. *Bohr-almost periodic in distribution*, resp. *asymptotically Bohr-almost periodic in distribution*) if the mapping $t \mapsto \text{law}(\tilde{X}(t))$, $I \rightarrow \mathcal{P}(\mathcal{C}_k)$ is Bochner-almost periodic (resp. Bohr-almost

periodic in distribution, resp. asymptotically Bohr-almost periodic in distribution). Other definitions of almost periodicity in distribution are possible, see [1, 15, 23]: for example we say that X has *almost periodic finite dimensional distributions* (APFD for short) if, for all finite sequences (t_1, \dots, t_n) in I , the mapping $t \mapsto \text{law}(X(t_1 + t), \dots, X(t_n + t))$ is almost periodic. Likewise, if the mapping $t \mapsto \text{law}(X(t))$ is almost periodic, we say that X has *almost periodic one-dimensional distributions* (APOD for short).

Bochner-almost periodicity in distribution as defined here is proved for some solutions of stochastic differential equations in [15, 22, 1, 10].

Proposition 2.2 *If X is APD then almost all its trajectories are uniformly continuous on I .*

Proof Suppose that X is almost periodic in distribution, then the family $(\tilde{X}(r))_{r \in I}$ is tight in \mathcal{C}_k . By [5, Theorem 7.3] or [24, Theorem 4], this implies that, for each interval $[a, b] \subset I$, for every $\varepsilon > 0$ and for every $\eta > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}\left\{ \sup_{\substack{|t-s| < \delta \\ t, s \in [a, b]}} d_{\mathbb{E}}(\tilde{X}(r)(t), \tilde{X}(r)(s)) > \eta \right\} < \varepsilon$$

for all $r \in I$. Equivalently,

$$\mathbb{P}\left\{ \sup_{\substack{|t-s| < \delta \\ t, s \in [a, b]}} d_{\mathbb{E}}(X(r+t), X(r+s)) > \eta \right\} < \varepsilon$$

for all $r \in I$. Therefore

$$\mathbb{P}\left\{ \sup_{|t-s| < \delta} d_{\mathbb{E}}(X(t), X(s)) > \eta \right\} < \varepsilon.$$

In particular, for each $n \in \mathbb{N}^*$, there exists $\delta_n > 0$ such that

$$\mathbb{P}\left\{ \sup_{|t-s| < \delta_n} d_{\mathbb{E}}(X(t), X(s)) > \frac{1}{n} \right\} < \frac{1}{n}.$$

Let

$$\Omega_n = \left\{ \sup_{|t-s| < \delta_n} d_{\mathbb{E}}(X(t), X(s)) \leq \frac{1}{n} \right\} \text{ and } \Omega_0 = \limsup_n \Omega_n.$$

For each $\omega \in \Omega_0$, the mapping $t \mapsto X(t)(\omega)$ is uniformly continuous. Furthermore, we have $\mathbb{P}(\Omega_n) \geq 1 - \frac{1}{n}$, thus $\mathbb{P}(\Omega_0) = 1$ since $\mathbb{P}(\limsup_n \Omega_n) \geq \limsup_n \mathbb{P}(\Omega_n)$. \square

Corollary 2.3 *If the trajectories of X are uniformly continuous, the following properties are equivalent:*

- (1) X is APFD.
- (2) X is APD.

Before we give the proof of Corollary 2.3 we recall the following definition: For $f \in C_b(\mathbb{E})$, (where $C_b(\mathbb{E})$ is the Banach space of continuous and bounded functions defined on \mathbb{E}),

$$|f|_L = \sup \left\{ \frac{f(x) - f(y)}{d_{\mathbb{E}}(x, y)}, ; x \neq y \right\}, \quad |f|_{\infty} = \sup_{x \in \mathbb{E}} |f(x)|,$$

and $|f|_{BL} = \max\{|f|_{\infty}, |f|_L\}$.

For $\mu, \nu \in \mathcal{P}(E)$, we define

$$d_{BL}(\mu, \nu) = \sup_{|f|_{BL} \leq 1} \int_{\mathbb{E}} f d(\mu - \nu).$$

The metric d_{BL} on $\mathcal{P}(E)$ is complete and compatible with the topology of narrow convergence.

Proof of Corollary 2.3 Clearly (2) \Rightarrow (1). Assume that X is APFD. Let (α'_n) and (β'_n) be two sequences in I and for $t_1, t_2, \dots, t_k, t \in I$ define (using notations of [23])

$$\mu_t^{t_1, \dots, t_k} := \text{law}((X(t_1 + t), \dots, X(t_k + t))).$$

By a diagonal procedure we can find subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that, for every $k \geq 1$, for all $q_1, q_2, \dots, q_k \in \mathbb{Q} \cap I$ (where \mathbb{Q} is the set of rational numbers), and for every $t \in I$, the limits

$$\lim_n \mu_{t+\alpha_n+\beta_n}^{q_1, \dots, q_k} \quad \text{and} \quad \lim_n \lim_m \mu_{t+\alpha_n+\beta_m}^{q_1, \dots, q_k}$$

exist and are equal. We have, for all $t_1, t_2, \dots, t_k, t \in I$, and for all $q_1, q_2, \dots, q_k \in \mathbb{Q} \cap I$,

$$\begin{aligned} d_{BL}(\mu_{t+\alpha_n+\beta_n}^{q_1, \dots, q_k}, \mu_{t+\alpha_n+\beta_n}^{t_1, \dots, t_k}) &= \sup_{\|f\|_{BL} \leq 1} \int_{\mathbb{E}^k} f d(\mu_{t+\alpha_n+\beta_n}^{q_1, \dots, q_k} - \mu_{t+\alpha_n+\beta_n}^{t_1, \dots, t_k}) \\ &\leq \int_{\Omega} d_{\mathbb{E}^k} \left[(X(q_1 + t + \alpha_n + \beta_n), \dots, X(q_k + t + \alpha_n + \beta_n)), \right. \\ &\quad \left. (X(t_1 + t + \alpha_n + \beta_n), \dots, X(t_k + t + \alpha_n + \beta_n)) \right] d\mathbb{P}. \end{aligned}$$

Thus, by uniform continuity of the trajectories of X , if $(q_1, \dots, q_k) \rightarrow (t_1, \dots, t_k)$, then

$$d_{BL}\left(\mu_{t+\alpha_n+\beta_n}^{q_1, \dots, q_k}, \mu_{t+\alpha_n+\beta_n}^{t_1, \dots, t_k}\right) \rightarrow 0$$

uniformly with respect to $t \in I$ and $n \geq 0$. By a classical result on inversion of limits, we deduce that, for all $k \geq 1$ and $t_1, \dots, t_k, t \in I$, the limits

$$\lim_n \mu_{t+\alpha_n+\beta_n}^{t_1, \dots, t_k} \text{ and } \lim_n \lim_m \mu_{t+\alpha_n+\beta_m}^{t_1, \dots, t_k}$$

exist and are equal. Therefore, to show that the limits

$$\lim_n \text{law}\left(\tilde{X}(t + \alpha_n + \beta_n)\right) \text{ and } \lim_n \lim_m \text{law}\left(\tilde{X}(t + \alpha_m + \beta_n)\right)$$

exist and are equal, it is enough to prove that $(\tilde{X}(t))_{t \in I}$ is tight in \mathcal{C}_k . Since the trajectories of X are uniformly continuous, we can find for each $\omega \in \Omega$ and for each $\eta > 0$, a number $\delta(\omega) > 0$, such that, for all $t, s \in I$,

$$(2) \quad \sup_{|t-s| < \delta(\omega)} d_{\mathbb{E}}(X(t)(\omega), X(s)(\omega)) < \eta.$$

For each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$(3) \quad \mathbb{P}\{\delta > \delta_\varepsilon\} > 1 - \varepsilon.$$

Indeed, suppose that (3) is false, then there exists $\varepsilon_0 > 0$ such that for every $\delta^* > 0$, we have $\mathbb{P}\{\delta \leq \delta^*\} > \varepsilon_0$. In particular, for each $m \in \mathbb{N}$ $\mathbb{P}\{\delta \leq \frac{1}{m}\} > \varepsilon_0$. Let $A_m = \{\delta \leq \frac{1}{m}\}$, we have

$$\mathbb{P}\{\delta = 0\} = \inf_m \mathbb{P}\{\delta \leq \frac{1}{m}\} > \varepsilon_0 > 0,$$

which contradicts $\delta(\omega) > 0$ for every $\omega \in \Omega$. Thus, from (2) and (3), we deduce that there exists δ_ε such that

$$\mathbb{P}\left\{\sup_{|t-s| < \delta_\varepsilon} d_{\mathbb{E}}(X(t), X(s)) > \eta\right\} \leq \varepsilon.$$

Therefore, for every interval $[a, b] \subset I$ and every $r \in I$, we have

$$(4) \quad \mathbb{P}\left\{\sup_{\substack{|t-s| < \delta_\varepsilon \\ t, s \in [a, b]}} d_{\mathbb{E}}(\tilde{X}(r)(t), \tilde{X}(r)(s)) > \eta\right\} \leq \varepsilon.$$

Since X is APFD, the family $(X(t))_{t \in I} = (\tilde{X}(t)(0))_{t \in I}$ is tight, thus by (4) and [24, 5] we conclude that $(\tilde{X}(t))_{t \in I}$ is tight in \mathcal{C}_k .

□

Corollary 2.4 *Assume that $I = \mathbb{R}^+$. If X is an homogeneous Feller process with uniformly continuous trajectories, the following properties of X are equivalent:*

- (1) X is APOD.
- (2) X is APFD.
- (3) X is APD.

Proof The equivalence (2) \Leftrightarrow (3) follows from Corollary 2.3. Clearly (2) \Rightarrow (1). There remains to show that (1) \Rightarrow (2).

Using again Tudor's notation [23], let us set

$$\mu_t := \text{law}(X(t)) \text{ and } \mu_t^{t_1, \dots, t_k} := \text{law}((X(t_1 + t), \dots, X(t_k + t))).$$

Suppose that X is APOD, then for every sequence $(\alpha'_n) \subset \mathbb{R}^+$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that for each function $g \in C_b(\mathbb{E})$ ($C_b(\mathbb{E})$ is the Banach space of continuous and bounded functions, $g : \mathbb{E} \rightarrow \mathbb{R}$) the limit

$$\lim_n \mu_{t+\alpha_n}(g)$$

exists uniformly with respect to $t \in \mathbb{R}^+$. To show that the limit

$$\lim_n \mu_{t+\alpha_n}^{t_1, \dots, t_k}$$

exists uniformly with respect to $t \in \mathbb{R}^+$, it is enough to show it for $k = 2$ and $t_1 < t_2$. Let $f \in C_b(\mathbb{E} \times \mathbb{E})$, we have

$$\mu_{t+\alpha_n}^{t_1, t_2}(f) = \int_{\mathbb{E}} \mu_0(dx_0) \int_{\mathbb{E}} p_{t_1+t+\alpha_n}(x_0, dx_1) \int_{\mathbb{E}} p_{t_2-t_1}(x_1, dx_2) f(x_1, x_2),$$

where $p_t(x, \cdot)$ is the transition function for a time-homogeneous Markov process X . Since X is a Feller process, the function

$$x_1 \mapsto g(x_1) := \int_{\mathbb{E}} p_{t_2-t_1}(x_1, dx_2) f(x_1, x_2)$$

is in $C_b(\mathbb{E})$. Therefore

$$\mu_{t+\alpha_n}^{t_1, t_2}(f) = \int_{\mathbb{E}} \mu_0(dx_0) \int_{\mathbb{E}} p_{t_1+t+\alpha_n}(x_0, dx_1) g(x_1) = \mu_{t+t_1+\alpha_n}(g)$$

converges uniformly with respect to $t \in \mathbb{R}^+$, hence X is APFD. \square

2.3 Almost periodicity and asymptotic almost periodicity in probability or in p -mean

We say that X is *Bochner-almost periodic in probability* if $\{\tilde{X}(t), t \in I\}$ is totally bounded with respect to the topology of uniform convergence in probability, which amounts to say that X satisfies Bochner's double sequence criterion for the convergence in probability. Recall from Remark 2.1(ii) that this property depends only on the topology of convergence in probability, not on any metric which is compatible with this topology, thus it is independent of the metric $d_{\mathbb{E}}$ on \mathbb{E} .

We say that X is *Bohr-almost periodic (resp. asymptotically Bohr-almost periodic) in probability* if for any $\varepsilon > 0$ and $\eta > 0$, there exists $l = l(\varepsilon, \eta) > 0$ (resp. there exists $l = l(\varepsilon, \eta) > 0$ and $T = T(\varepsilon, \eta)$) such that any interval of length l contains at least a number τ for which

$$P\{d_{\mathbb{E}}(X(t + \tau), X(t)) > \eta\} \leq \varepsilon, \text{ for all } t \in \mathbb{R} \text{ (resp. for all } t \geq T, t + \tau \geq T).$$

Precupanu [16, 17] showed that most of the basic properties of Bohr-almost periodic functions can be formulated accordingly for \mathbb{R} -valued almost periodic in probability processes, particularly, the usual approximation theorem with random trigonometric polynomials is valid for stochastic processes.

Now, let us consider the case when \mathbb{E} is a normed space and $I = \mathbb{R}$. We denote by $\|\cdot\|_{\mathbb{E}}$ the norm of \mathbb{E} . Let $p \geq 1$. We say that X is *almost periodic in p -mean* if the mapping $X : \mathbb{R} \rightarrow L^p(\Omega; \mathbb{E})$ is almost periodic with respect to the metric induced by $\|\cdot\|_p = \left(\int \|\cdot\|_{\mathbb{E}}^p dP\right)^{\frac{1}{p}}$.

Proposition 2.5 *Let $I = \mathbb{R}$. If X is almost periodic in p -mean, then it is almost periodic in probability. Conversely, if X is almost periodic in probability and the family $\{\|X(t)\|_{\mathbb{E}}^p, t \in \mathbb{R}\}$ is uniformly integrable, then X is p -mean almost periodic.*

Proof This is a direct consequence of Bochner's double sequence criterion and Vitali Theorem. \square

Almost periodicity in quadratic mean (using Bohr's definition, but with $I = \mathbb{R}$) is proved for some solutions of stochastic differential equations in [2, 3, 4].

Counterexample 2.6 (almost periodicity in probability and uniform continuity of the trajectories) Almost periodicity in probability of a process X does not imply that its trajectories are uniformly continuous. For example, let (δ_n) be a sequence of random variables on the probability

space $([0, 1], dt)$ (dt is the Lebesgue measure) such that $\delta_n := 1_{[\frac{n}{2^k}, \frac{n+1}{2^k}]}$ for $2^{k-1} \leq n < 2^k$, $k \in \mathbb{N}$. For each $n \geq 1$, let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f_n(t) = \begin{cases} nt & \text{if } t \in [0, \frac{1}{2}] \\ n(1-t) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

and let $(X_t)_{t \geq 0}$ be defined by

$$X_t(\omega) = \begin{cases} f_{[t]}(t - [t]) & \text{if } \delta_{[t]}(\omega) = 1 \\ 0 & \text{if } \delta_{[t]}(\omega) = 0 \end{cases}$$

where $[t]$ is the integer part of t . The trajectories of X are continuous on \mathbb{R}^+ , but almost none is uniformly continuous, nor bounded, since $\mathbb{P}(\limsup \delta_n = 1) = 1$. By Proposition 2.2, the process X is not almost periodic in distribution. On the other hand, we have, for every $\varepsilon > 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}(X_t > \varepsilon) \leq \lim_{t \rightarrow \infty} \mathbb{P}(\delta_{[t]} = 1) = 0$$

which means that X is asymptotically Bohr-almost periodic in probability thus it is Bochner almost periodic in probability.

A variant of this example, with uniformly bounded trajectories, is obtained by replacing X by $\min(X, 1)$.

2.4 Almost sure almost periodicity properties

We can give at least three different definitions:

- (a) The stochastic process X is *almost surely almost periodic* if there exists a measurable subset $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and for every $\omega \in \Omega_1$, the trajectory $t \mapsto X(t)(\omega)$ is almost periodic.
- (b) The stochastic process X satisfies *Bochner's almost sure uniform double sequence criterion* if, for every pair of sequences (α'_n) and (β'_n) in I , there exists a measurable subset $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ respectively, with same indexes (independent of ω) such that, for every $t \in I$, the limits

$$(5) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} X(t + \alpha_n + \beta_m)(\omega) \text{ and } \lim_{n \rightarrow \infty} X(t + \alpha_n + \beta_n)(\omega)$$

exist and are equal for all $\omega \in \Omega_1$. (In this case, Ω_1 depends on the pair of sequences (α'_n) and (β'_n) .)

(c) We say that X is *almost surely equi-almost periodic in Bochner's sense* if there exists a measurable subset $\Omega_1 \subset \Omega$ such that $P(\Omega_1) = 1$ and the family of trajectories $\{X(\omega)\}_{\omega \in \Omega_1}$ is equi-almost periodic in Bochner's sense, that is, if the family of translation mappings $\{\tilde{X}(t)(\omega)\}_{\omega \in \Omega_1}$ is totally bounded on $\mathcal{C}_u(I, \mathbb{E})$.

Obviously (c) \Rightarrow (b) and (c) \Rightarrow (a). The converse implications are false, see Counterexamples 2.7 and 2.14.

If $\{X(\omega)\}_{\omega \in \Omega_1}$ is a totally bounded subset of $\mathcal{C}_k(I; \mathbb{E})$, then X is almost surely equi-almost periodic in the Bochner sense iff it satisfies Bochner's uniform double sequence criterion, with the additional assumption that the set Ω_1 in (c) is independent of (α'_n) and (β'_n) . For the proof of this property and connections with equi-almost periodicity in the Bohr's sense, see Theorem 4.2 in the Appendix.

Counterexample 2.7 (almost sure almost periodicity and almost sure equi-almost periodicity) The converse to (c) \Rightarrow (a) is false. For example, the process X defined by $X(t)(w) = e^{i\frac{t}{w}}$, for every $w \in]0, 1]$ and $t \in I$ is almost surely almost periodic, but it is not almost periodic in probability (thus it is not almost surely equi-almost periodic). Indeed, since $|X| \leq 1$ then to prove that X is not almost periodic in probability, it suffices to show that X is not almost periodic in quadratic mean. Let τ be an ε -almost period i.e. $E|X_{t+\tau} - X_t|^2 \leq \varepsilon$, for all $t \in I$. Let $A(\tau) := E|X_{t+\tau} - X_t|^2$,

$$\begin{aligned} A(\tau) &= \int_0^1 |e^{i\frac{t+\tau}{w}} - e^{i\frac{t}{w}}|^2 dw = \int_0^1 |e^{i\frac{\tau}{w}} - 1|^2 dw = 4 \int_0^1 \left| \frac{e^{i\frac{\tau}{2w}} - e^{-i\frac{\tau}{2w}}}{2i} \right|^2 dw \\ &= 2 \int_0^1 \sin^2\left(\frac{\tau}{w}\right) dw = 2|\tau| \int_{|\tau|}^{\infty} \frac{1 - \cos(2u)}{u^2} du \\ &= 2|\tau| \sum_{k=0}^{\infty} \int_{|\tau|+k\pi}^{|\tau|+(k+1)\pi} \frac{1 - \cos(2u)}{u^2} du \geq 2|\tau| \int_{|\tau|+\pi}^{\infty} \frac{1}{u^2} du = 2\frac{|\tau|}{|\tau| + \pi}. \end{aligned}$$

Thus if $|\tau| \geq \frac{\varepsilon\pi}{2 - \varepsilon}$, we get $A(\tau) > \varepsilon$, which shows that the set of ε -almost periods is not relatively dense in I , thus X is not almost periodic in quadratic mean.

2.5 Comparison between almost periodicities

Lemma 2.8 *Let $(X_n(t))$ be a sequence of processes which converges in probability uniformly with respect to $t \in I$. There exists a subsequence $(X_{n_k}(t)) \subset (X_n(t))$ which converges almost surely for each $t \in I$.*

Proof Assume that $(X_n(t))$ converges in probability to a process $(X(t))$ uniformly with respect to $t \in I$, let us denote $Y_n(t) = d(X_n(t), X(t))$. Then for each $\varepsilon > 0$ and $\eta > 0$, there exists $N = N(\varepsilon, \eta)$ such that for all $t \in I$ and $n \geq N$, we have

$$P(Y_n(t) > \varepsilon) < \eta.$$

Let (η_k) be a real positive sequence such that

$$\sum_{k \geq n} \eta_k \text{ converges to zero as } n \rightarrow \infty.$$

To each $\varepsilon > 0$ and $\eta_k > 0$, there corresponds $N_k = N(\varepsilon, \eta_k)$ such that, for all $t \in I$, $P(Y_{N_k}(t) > \varepsilon) < \eta_k$. Let us show that $(Y_{N_k}(t))$ converges almost surely for every $t \in I$. We have

$$P(\sup_{k \geq n} Y_{N_k}(t) > \varepsilon) = P(\bigcup_{k \geq n} \{Y_{N_k}(t) > \varepsilon\}) \leq \sum_{k \geq n} \eta_k$$

and the right hand term converges to zero when $n \rightarrow \infty$. Thus $(Y_{N_k}(t))$ converges almost surely to zero for every $t \in I$. Consequently $(X_{N_k}(t))$ converges almost surely to $(X(t))$ for every $t \in I$. \square

Theorem 2.9 *The following properties of X are equivalent:*

- (1) X satisfies Bochner's almost sure uniform double sequence criterion.
- (2) X is almost periodic in probability.

Proof Obviously $1 \Rightarrow 2$.

Assume that X is almost periodic in probability. Let (α'_n) and (β'_n) be two sequences in I . There exist subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that for each $t \in I$, the limits in probability

$$P - \lim_n X(\alpha_n + \beta_n + t) \text{ and } P - \lim_n \lim_m X(\alpha_n + \beta_m + t)$$

exist and are equal. By Remark 2.1(ii) these limits exist uniformly with respect to $t \in I$. The previous Lemma implies that there exist subsequences (still denoted with the same way) $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that, for each $t \in I$, the limits

$$\lim_n X(\alpha_n + \beta_n + t) \text{ and } \lim_n \lim_m X(\alpha_n + \beta_m + t)$$

exist and are equal almost surely. \square

Let us now compare almost periodicity in distribution with the other almost periodicity properties.

Remark 2.10 In all preceding definitions of almost periodicity, it is only in the definition of almost periodicity in distribution that the choice of a topology on the space $\mathcal{C}(I, \mathbb{E})$ of trajectories plays a role.

Theorem 2.11 (A general comparison result) *Consider the following properties of X :*

- (a) X is APD.
- (b) X satisfies Bochner's almost sure uniform double sequence criterion.
- (c) X is almost periodic in probability.

Then

- (1) (a) \Rightarrow (b) \Leftrightarrow (c).
- (2) If we assume that almost all trajectories of X are uniformly continuous, the equivalence (a) \Leftrightarrow (b) \Leftrightarrow (c) holds true.

Proof

Proof of (1): The equivalence (b) \Leftrightarrow (c) has been shown in Theorem 2.9. There remains to show that (a) \Rightarrow (b).

Assume (a) and let us prove (b). We only need to check Bochner's double sequence criterion. Let (α'_n) and (β'_n) be two sequences in I . There exist subsequences (α_n) and (β_n) with same indexes such that for every $t \in I$ the limits

$$(6) \quad \lim_n \lim_m \text{law} \left(\tilde{X}(t + \alpha_n + \beta_m) \right) \text{ and } \lim_n \text{law} \left(\tilde{X}(t + \alpha_n + \beta_n) \right)$$

exist and are equal. This means in particular that, for each n , the limit $\lim_m \text{law} \left(\tilde{X}(t + \alpha_n + \beta_m) \right)$ exists. Note that for every $t, s \in I$, $\tilde{X}(t)(s) = \tilde{X}(0)(t + s)$ thus (6) implies

$$\lim_n \lim_m \text{law} \left(\tilde{X}(\alpha_n + \beta_m) \right) \text{ and } \lim_n \text{law} \left(\tilde{X}(\alpha_n + \beta_n) \right)$$

exist and are equal. We now use a Skorokhod representation theorem due to Blackwell and Dubins [6]:

Theorem (Blakwell-Dubins)¹ *Let \mathbb{U} be a Polish space, and let $\mathcal{P}(\mathbb{U})$ be the space of probability measures on the Borel σ -algebra of \mathbb{U} , endowed with the narrow convergence. There exists a probability space $(\underline{\Omega}, \underline{\mathcal{F}}, \underline{\mathbb{P}})$ and, for each $\mu \in \mathcal{P}(\mathbb{U})$, a measurable mapping $\mathfrak{X}(\mu) : \underline{\Omega} \rightarrow \mathbb{U}$ such that, if (μ_n)*

is a sequence in $\mathcal{P}(\mathbb{U})$ which narrowly converges to some $\mu \in \mathcal{P}(\mathbb{U})$, then $(\mathfrak{X}(\mu_n))$ converges a.s. to $\mathfrak{X}(\mu)$.

Proof of Theorem 2.11 (continuation) Let us apply Blackwell-Dubins Theorem to the Polish space \mathcal{C}_k . For every $t \in I$, we denote

$$\tilde{Y}(t) = \mathfrak{X}\left(\text{law}\left(\tilde{X}(t)\right)\right) \text{ and } Y = \tilde{Y}(0) = \mathfrak{X}\left(\text{law}\left(\tilde{X}(0)\right)\right) = \mathfrak{X}(\text{law}(X)).$$

Then all limits in

$$(7) \quad \lim_n \lim_m \tilde{Y}(\alpha_n + \beta_m) \text{ and } \lim_n \tilde{Y}(\alpha_n + \beta_n)$$

exist and are equal a.e., that is, almost all trajectories of Y belong to the set \mathfrak{E} of $x \in \mathcal{C}_k$ such that all limits in

$$(8) \quad \lim_n \lim_m \tilde{x}(\alpha_n + \beta_m) \text{ and } \lim_n \tilde{x}(\alpha_n + \beta_n)$$

exist and are equal. Let us check that this is a property of $\text{law}(Y)$. Let $d_{\mathcal{C}}$ be a distance on \mathcal{C}_k which is compatible with the topology of \mathcal{C}_k , and denote, for $s, s' \in I$ and $\varepsilon > 0$

$$(9) \quad A(s, \tilde{y}(s'), \varepsilon) = \{x \in \mathcal{C}_k; d_{\mathcal{C}}(\tilde{x}(s), \tilde{y}(s')) < \varepsilon\}.$$

It is easy to see that $A(s, \tilde{y}(s'), \varepsilon)$ is a Borel subset of \mathcal{C} . Indeed, we have

$$A(s, \tilde{y}(s'), \varepsilon) = \{x \in \mathcal{C}_k; d_{\mathcal{C}}(\tau_s(\tilde{x}(0)), \tilde{y}(s')) < \varepsilon\},$$

which is open by continuity for every $t \in I$ of the mapping

$$\tau_t : \begin{cases} \mathcal{C}_k & \rightarrow \mathcal{C}_k \\ x = \tilde{x}(0) & \mapsto \tilde{x}(t) = x(t + \cdot). \end{cases}$$

Using similar arguments, we show that the set

$$\mathfrak{D} = \{x \in \mathcal{C}_k; \text{all limits in (8) exist}\}$$

is a Borel subset of \mathcal{C}_k . For $\varepsilon > 0$, $s \in I$, and $n \geq 1$, let

$$\begin{aligned} \mathfrak{B}(n, \varepsilon) &= \left\{x \in \mathcal{C}_k; \limsup_m d_{\mathcal{C}}(\tilde{x}(\alpha_n + \beta_m), \tilde{x}(\alpha_n + \beta_n)) < \varepsilon\right\} \\ &= \cup_{M \geq 1} \cap_{m \geq M} A(\alpha_n + \beta_m, \alpha_n + \beta_n, \varepsilon) \end{aligned}$$

¹As was noticed by P.A. Meyer, the proof of Blackwell and Dubins does not apply to filters or generalized sequences, contrarily to that given later by Fernique [13], see details in [18, page 762]. But the sequential result proved by Blackwell and Dubins is sufficient for our purpose in this paper.

where $A(s, s', \varepsilon)$ is an open subset of \mathcal{C}_k defined by:

$$(10) \quad A(s, s', \varepsilon) = \{x \in \mathcal{C}_k; d_{\mathcal{C}}(\tilde{x}(s), \tilde{x}(s')) < \varepsilon\}.$$

Let

$$\mathfrak{C} = \bigcap_{K \geq 1} \bigcup_{N \geq 1} \bigcap_{n \geq N} \mathfrak{B}(n, 1/K).$$

The set \mathfrak{C} is a Borel subset of \mathcal{C}_k , and a function $x \in \mathcal{C}_k$ belongs to \mathfrak{C} iff

$$(\forall K \geq 1), (\exists N \geq 1), (\forall n \geq N), (\exists M \geq 1), (\forall m \geq M), \\ d_{\mathcal{C}}(\tilde{x}(\alpha_n + \beta_m), \tilde{x}(\alpha_n + \beta_n)) < \frac{1}{K}.$$

Thus, $x \in \mathfrak{D} \cap \mathfrak{C}$ iff the limits in (8) exist and are equal. Thus the limits in (7) exist and are equal a.e. iff $\mathbf{P}\{Y \in \mathfrak{D} \cap \mathfrak{C}\} = 1$. As $\text{law}(X) = \text{law}(Y)$, and by continuity, for each $t \in I$, of the mapping

$$\pi_t : \begin{cases} \mathcal{C}_k & \rightarrow \mathbb{E} \\ x & \mapsto x(t), \end{cases}$$

we deduce that there exists a measurable subset $\Omega_1 \subset \Omega$ such that $\mathbf{P}(\Omega_1) = 1$ and, for every $t \in I$, the limits

$$(11) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} X(t + \alpha_n + \beta_m)(\omega) \text{ and } \lim_{n \rightarrow \infty} X(t + \alpha_n + \beta_n)(\omega),$$

exist and are equal for every $\omega \in \Omega_1$. Thus X satisfies Bochner's almost sure uniform double sequence criterion, which proves (b).

Proof of (2): There only remains to show that (c) \Rightarrow (a).

Assume (c) and let (α'_n) and (β'_n) be two sequences in I . There exist subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that for each $t, s \in I$, the limits in probability

$$\mathbf{P} - \lim_n X(\alpha_n + \beta_n + t + s) \text{ and } \mathbf{P} - \lim_n \lim_m X(\alpha_n + \beta_m + t + s)$$

exist and are equal. Hence, for every $(t_1, t_2, \dots, t_k) \in I^k$,

$$\mathbf{P} - \lim_n (X(\alpha_n + \beta_n + t_1 + s), X(\alpha_n + \beta_n + t_2 + s), \dots, X(\alpha_n + \beta_n + t_k + s))$$

and

$$\mathbf{P} - \lim_n \lim_m (X(\alpha_n + \beta_m + t_1 + s), X(\alpha_n + \beta_m + t_2 + s), \dots, X(\alpha_n + \beta_m + t_k + s))$$

exist and are equal, thus

$$\lim_n \text{law} (X(\alpha_n + \beta_n + t_1 + s), \dots, X(\alpha_n + \beta_n + t_k + s))$$

and

$$\lim_n \lim_m \text{law} ((X(\alpha_n + \beta_m + t_1 + s), \dots, X(\alpha_n + \beta_m + t_k + s)))$$

exist and are equal, that is, X is APFD, and by Corollary 2.3 we deduce (a). \square

Corollary 2.12 *Every continuous strictly stationary process is almost periodic in probability.*

Corollary 2.13 *If X is almost surely equi-almost periodic, then X is almost periodic in distribution and in probability.*

Proof Indeed, since X is almost surely equi-almost periodic, almost all its trajectories are uniformly continuous, and we can apply Theorem 2.11-(2). \square

Counterexample 2.14 (almost sure equi-almost periodicity and almost periodicity in distribution) The converse to Corollary 2.13 is false. For example let U be an integer-valued random variable with $P(U = n) > 0$ for each integer n , and let $X(t) = U$ for every $t \in I$. Then X is almost periodic in distribution (it is even stationary), and it has almost periodic (constant) trajectories. But X is not almost surely equi-almost periodic, since the family of its trajectories is not almost surely equibounded.

3 Extension to completely regular state spaces

Topological preliminaries We now assume that \mathbb{E} is a separable completely regular space. Recall that a Hausdorff topological space is completely regular iff it is uniformizable, that is, iff its topology can be defined by a family of semi-metrics. Such a family defines a uniformity. The notions of uniform continuity, Cauchy sequences, Cauchy nets and totally bounded sets are defined relatively to a given uniformity. For more details see e.g. [12].

Let \mathfrak{U} be a uniformity (a set of entourages) on \mathbb{E} which is compatible with the topology of \mathbb{E} , and let \mathfrak{D} be an upwards filtering set of semimetrics

on \mathbb{E} which induces this uniformity. For every $d \in \mathfrak{D}$, we denote by $\check{\mathbb{E}}_d$ the quotient metric space $(\mathbb{E}/d, \check{d})$, and by $\check{\mathcal{C}}_k^d$ the space $\mathcal{C}_k(I; \check{\mathbb{E}}_d)$. We denote by p_d the projection mapping:

$$p_d : \begin{cases} \mathbb{E} & \rightarrow \check{\mathbb{E}}_d \\ x & \mapsto \check{x}_d \end{cases}$$

which extends to \mathcal{C}_k :

$$p_d : \begin{cases} \mathcal{C}_k & \rightarrow \check{\mathcal{C}}_k^d \\ x & \mapsto p_d \circ x =: \check{x}_d(\cdot) \end{cases}$$

These mappings are continuous and the uniform space \mathbb{E} (resp. \mathcal{C}_k) is the limit of the inverse system $\{\check{\mathbb{E}}_d\}_{d \in \mathfrak{D}}$ (resp. $\{\check{\mathcal{C}}_k^d\}_{d \in \mathfrak{D}}$).

Almost periodicity for deterministic functions A function x in $\mathcal{C}(I, \mathbb{E})$ is said to be *Bochner-almost periodic*, if the set $\{x(t + \cdot), t \in I\}$ of its translation mappings is totally bounded on $\mathcal{C}_u(I, \mathbb{E})$.

A function x in $\mathcal{C}(\mathbb{R}; \mathbb{E})$ is said to be *Bohr-almost periodic* if for each entourage V in \mathfrak{U} , there exists a real number $l = l(V) > 0$ such that every interval $[a, a + l]$ contains at least one point τ satisfying

$$(x(t + \tau), x(t)) \in V, \text{ for all } t \in \mathbb{R}.$$

We say that a function $x \in \mathcal{C}(\mathbb{R}^+; \mathbb{E})$ is *asymptotically Bohr-almost periodic* if, for each entourage $V \in \mathfrak{U}$, there exist a real number $l = l(V) > 0$ and $T = T(V) > 0$ with the property that any interval $[a, a + l] \subset \mathbb{R}^+$ contains a number τ such that

$$(x(t + \tau), x(t)) \in V, \text{ for all } t \geq T \text{ such that } t + \tau \geq T.$$

The relationships between these notions of almost periodicity are explained in Lemma 4.4 in the Appendix.

Almost periodicity in distribution vs in probability for stochastic processes Let $\mathcal{P}(\mathbb{E})$ (resp. $\mathcal{P}(\mathcal{C}_k)$) be the space of Radon probabilities on \mathbb{E} (resp. \mathcal{C}_k) endowed with the initial topology of narrow convergence, that is the coarsest topology such that the mappings $\mu \rightarrow \mu(f)$ are continuous for all measurable bounded function $f : \mathbb{E} \rightarrow \mathbb{R}$ (resp. $f : \mathcal{C}_k \rightarrow \mathbb{R}$). For more details on this topology, see [21].

For each projection mapping p_d , we can define a mapping from $\mathcal{P}(\mathcal{C}_k)$ to $\mathcal{P}(\check{\mathcal{C}}_k^d)$ (or from $\mathcal{P}(\mathbb{E})$ to $\mathcal{P}(\check{\mathbb{E}}_d)$) by

$$\mu \rightarrow (p_d)_\#(\mu) = \check{\mu}^d$$

where $(p_d)_\#(\mu)$ denotes the image of μ by p_d that is, $\check{\mu}^d(f) = \mu(f \circ p_d)$ for every measurable bounded function $f : \check{\mathcal{C}}_k^d \rightarrow \mathbb{R}$.

We say that the process X is *almost periodic in distribution* (resp. *asymptotically almost periodic in distribution*) in Bohr's or Bochner sense if the mapping

$$\begin{cases} I & \rightarrow \mathcal{P}(\mathcal{C}_k) \\ t & \mapsto \text{law}(\tilde{X}(t)) \end{cases}$$

is almost periodic (resp. asymptotically almost periodic) in Bohr's or Bochner sense in $\mathcal{P}(\mathcal{C}_k)$.

We say that X is *Bochner-almost periodic in probability* if the set $\{\tilde{X}(t), t \in I\}$ of translation mappings is totally bounded in probability.

We say that X is *Bohr-almost periodic* (resp. *asymptotically Bohr-almost periodic*) in probability if for any entourage V and $\eta > 0$, there exists $l = l(V, \eta) > 0$ (resp. there exists $l = l(V, \eta) > 0$ and $T = T(V, \eta)$) such that any interval of length l contains at least a number τ for which

$$P\{(X(t + \tau), X(t)) \notin V\} \leq \eta, \text{ for all } t \in \mathbb{R} \text{ (resp. for all } t \geq T, t + \tau \geq T).$$

In order to establish a result similar to Theorem 2.11, we need to characterize total boundedness in inverse limits of uniform spaces.

Lemma 3.1 *Let $\{\mathbb{F}_\rho\}_{\rho \in \Sigma}$ be an inverse system of uniform spaces where Σ is directed set, and let*

$$\mathbb{F} = \varprojlim \{\mathbb{F}_\rho\}$$

the projective limit satisfying $\mathbb{F}_\rho = p_\rho(\mathbb{F})$, for every projection mapping p_ρ from \mathbb{F} to \mathbb{F}_ρ . Then, \mathbb{F} is totally bounded if and only if for each $\rho \in \Sigma$, \mathbb{F}_ρ is totally bounded.

Proof It is straightforward that, if \mathbb{F} is totally bounded, \mathbb{F}_ρ is also totally bounded for each $\rho \in \Sigma$.

Conversely, assume that for each $\rho \in \Sigma$, \mathbb{F}_ρ is totally bounded. Let $\widehat{\mathbb{F}}_\rho$ be the completion of \mathbb{F}_ρ , then $\widehat{\mathbb{F}}_\rho$ is a compact space and $\{\widehat{\mathbb{F}}_\rho, \rho \in \Sigma\}$ is

an inverse system (see [21, second proof of Prokhorov theorem, page 78]). Hence the projective limit

$$\widehat{\mathbb{F}} := \varprojlim \widehat{\mathbb{F}}_\rho$$

is a compact space (see [8, page I.64, Proposition 8]). On the other hand, using [12, Proposition 2.5.6. page 100] one can see that $\overline{\mathbb{F}}_\rho, \rho \in \Sigma$, is an inverse system satisfying

$$\varprojlim \overline{\mathbb{F}}_\rho = \overline{\mathbb{F}},$$

where $\overline{\mathbb{F}}_\rho$ is the closure of \mathbb{F}_ρ in $\widehat{\mathbb{F}}_\rho$ and $\overline{\mathbb{F}}$ is the closure of \mathbb{F} in $\widehat{\mathbb{F}}$, which yields $\overline{\mathbb{F}} = \widehat{\mathbb{F}}$, thus \mathbb{F} is totally bounded. \square

Now, we can state and prove our main result in the case of a process with values in a completely regular space \mathbb{E} .

Theorem 3.2 *Consider the following properties of X :*

- (a) X is almost periodic in distribution .
- (b) X is almost periodic in probability.

Then

- (1) (a) \Rightarrow (b)
- (2) If almost all trajectories of X are uniformly continuous, then (b) \Rightarrow (a).

Proof Let $X^d = p_d \circ X$. Put $\tilde{\mu}_t = \text{law}(\tilde{X}(t))$ and $\tilde{\mu}_t^d = \text{law}(\tilde{X}^d(t))$.

Assume that X is almost periodic in distribution, i.e. the set $\{\tilde{\mu}_{t+}; t \in I\}$ of translation mappings is a totally bounded subset of the space of functions on I with values in the space $\mathcal{P}(\mathcal{C}_k)$. Then by Lemma 3.1 the set $\{\tilde{\mu}_{t+}^d; t \in I\}$ of translation mappings is a totally bounded subset of the space of functions on I with values in the space $\mathcal{P}(\check{\mathcal{C}}_k^d)$ for every $d \in \mathfrak{D}$, which means that, for each $d \in \mathfrak{D}$, the process X^d with values in the space $\check{\mathcal{C}}_k^d$ is almost periodic in distribution.

For $d \in \mathfrak{D}$, let $(\widehat{\mathbb{E}}_d, \check{d})$ be the completion of the metric space $\check{\mathbb{E}}_d$, then X^d is almost periodic in distribution in $\widehat{\mathbb{E}}_d$. In view of Theorem 2.11, we deduce that it is almost periodic in probability in $\widehat{\mathbb{E}}_d$, hence it is almost periodic in probability in $\check{\mathbb{E}}_d$. It remains to show that X is almost periodic in probability in \mathbb{E} .

Let $d \in \mathfrak{D}$, since X^d is almost periodic in probability in $\check{\mathbb{E}}_d$, the set $\{\tilde{X}^d(t), t \in I\}$ is totally bounded on the space $\mathcal{C}_u(I, L_0^d)$ (where $L_0^d = L_0(\Omega, \check{\mathbb{E}}_d)$),

the space of $\check{\mathbb{E}}_d$ -valued random variables). By Lemma 3.1, $\{\tilde{X}(t), t \in I\}$ is totally bounded on the space $\mathcal{C}_u(I, L_0)$ (where $L_0 = L_0(\Omega, \mathbb{E})$, the space of \mathbb{E} -valued random variables). This means that X is almost periodic in probability in \mathbb{E} .

Conversely, assume that X is almost periodic in probability and that almost all its trajectories are uniformly continuous. Then almost all trajectories of X^d are uniformly continuous and by Lemma 4.4 X^d is almost periodic in probability in $\check{\mathbb{E}}_d$ for each $d \in \mathfrak{D}$, hence X^d is almost periodic in probability in the completion space $\widehat{\check{\mathbb{E}}}_d$ (of $\check{\mathbb{E}}_d$) for each $d \in \mathfrak{D}$. In view of Theorem 2.11 we get that X^d is almost periodic in distribution in $\widehat{\check{\mathbb{E}}}_d$, i.e. the set $\{\tilde{\mu}_{t+}^d; t \in I\}$ of translation mappings is totally bounded on the space of functions on I with values in the space $\mathcal{P}(\check{\mathcal{C}}_k^d)$. By Prokhorov Theorem on inverse systems of Radon measures (see [21, Theorem 21, page 74]), it is easy to see that the set $\{\tilde{\mu}_{t+}^d; t \in I\}$ is an exact inverse system and that its projective limit is the set $\{\tilde{\mu}_{t+}; t \in I\}$ of translation mappings. Now, applying Lemma 3.1 to this set, we obtain that it is totally bounded, that is, X is almost periodic in distribution. \square

4 Appendix : Almost periodicity and asymptotic almost periodicity in completely regular spaces

First we consider the case when \mathbb{E} is a metric space.

Bochner's criteria of almost periodicity for functions with values in a metric space Let (\mathbb{E}, d) be a metric space (not necessarily complete). A subset $\mathcal{A} \subset I$ is said to be *relatively dense* if there exists a real number $l > 0$ such that for each number $a \in I$, $[a, a + l] \cap \mathcal{A} \neq \emptyset$. Any such number l is called an *inclusion length* of \mathcal{A} .

A subset \mathcal{H} of $\mathcal{C}(I, \mathbb{E})$ is said to be *equi-almost periodic* (respectively *equi-asymptotically almost periodic*) *in Bohr's sense* if, for every $\varepsilon > 0$, there exists a relatively dense set $\mathcal{A} = \mathcal{A}(\mathcal{H}, \varepsilon)$ in I (resp. there exist a relatively dense set $\mathcal{A} = \mathcal{A}(\mathcal{H}, \varepsilon)$ in I and a number $r \geq 0$) such that, for every $\tau \in \mathcal{A}$ and $h \in \mathcal{H}$,

$$d(h(t + \tau), h(t)) < \varepsilon \text{ for all } t \in \mathbb{R} \text{ (resp. for all } t \geq r, t + \tau \geq r).$$

A subset \mathcal{H} of $\mathcal{C}(I, \mathbb{E})$ is said to be *equinormal* if, from every sequence (α'_n) in I , one may extract a subsequence (α_n) such that the sequence $h(t + \alpha_n)$ converges uniformly with respect to $t \in I$ and $h \in \mathcal{H}$.

The following results are adaptations of those obtained by Ruess and Summers [20] in the case of locally convex spaces. We only consider the case $I = \mathbb{R}^+$, the almost periodic case follows exactly in the same way with obvious minor changes.

Lemma 4.1 *Let \mathcal{H} be a subset of $\mathcal{C}(\mathbb{R}^+, \mathbb{E})$. Suppose that \mathcal{H} is a totally bounded in $\mathcal{C}_k(\mathbb{R}^+, \mathbb{E})$ and equi-asymptotically almost periodic. Then, \mathcal{H} is uniformly equicontinuous on \mathbb{R}^+ and $\mathcal{H}(\mathbb{R}^+)$ is totally bounded in \mathbb{E} .*

Proof Let $\varepsilon > 0$, since \mathcal{H} is equi-asymptotically almost periodic in Bohr's sense, there exist a number $r = r(\varepsilon) > 0$ and a relatively dense set $\mathcal{A} = \mathcal{A}(\mathcal{H}, \varepsilon)$ in $[r, +\infty[$ such that, for each $\tau \in \mathcal{A}$ and for every $h \in \mathcal{H}$,

$$d(h(t + \tau), h(t)) < \varepsilon \text{ for all } t \geq r \text{ such that } t + \tau \geq r.$$

Let $N = \max(r, l)$, where l is an inclusion length of \mathcal{A} . Choose $\tau_k \in [kN, (k+1)N]$, $k = 1, 2, \dots$. Let us denote by $\mathcal{H}/_{[0, 5N]}$ the restriction of all functions of \mathcal{H} to the interval $[0, 5N]$. Then $\mathcal{H}/_{[0, 5N]}$ is totally bounded in $\mathcal{C}([0, 5N]; \mathbb{E})$. Hence, by Ascoli Theorem, we get that $\mathcal{H}/_{[0, 5N]}$ is uniformly equicontinuous. Let $\delta \in [0, \frac{N}{2}]$ such that $d(h(t_1), h(t_2)) < \frac{\varepsilon}{3}$ whenever $h \in \mathcal{H}$ and $t_1, t_2 \in [0, 5N]$ with $|t_1 - t_2| < \delta$. Now, suppose that $t_1, t_2 > 4N$, with $|t_1 - t_2| < \delta$. Taking $k \in \mathbb{N}$ such that $t_1, t_2 \in [kN, (k+2)N]$, let $s_i = t_i - \tau_{k-2}$, $i = 1, 2$. Since $s_1, s_2 \in [N, 4N]$ and $|s_1 - s_2| < \delta$, we have

$$\begin{aligned} d(h(t_1), h(t_2)) &< d(h(s_1 + \tau_{k-2}), h(s_1)) + d(h(s_1), h(s_2)) \\ &\quad + d(h(s_2), h(s_2 + \tau_{k-2})) < \varepsilon \end{aligned}$$

for any h in \mathcal{H} , which implies that \mathcal{H} is indeed uniformly equicontinuous on \mathbb{R}^+ . To verify that $\mathcal{H}(\mathbb{R}^+)$ is totally bounded in \mathbb{E} , we start again from the equicontinuity of \mathcal{H} to obtain a finite cover $\{T_i\}_{i=1}^n$ of $[0, 3N]$ and $t_i \in T_i$, $i = 1, \dots, n$ such that for every $h \in \mathcal{H}$

$$d(h(t), h(t_i)) \leq \frac{\varepsilon}{2} \text{ whenever } t \in T_i, i = 1, \dots, n.$$

If $t > 3N$, let us choose $k \in \mathbb{N}$ so that $t \in [kN, (k+1)N]$. Setting $s = t - \tau_{k-2}$, we then have that $s \in [N, 3N]$ whence $s \in T_i$ for some $i \in 1, \dots, n$. Therefore, given any $h \in \mathcal{H}$, we get

$$(12) \quad d(h(t), h(t_i)) < d(h(s + \tau_{k-2}), h(s)) + d(h(s), h(t_i)) < \varepsilon.$$

Since for each $i = 1, \dots, n$ the set $\mathcal{H}(t_i)$ is totally bounded, we deduce in view of inequality (12) that $\mathcal{H}(\mathbb{R}^+)$ is totally bounded. □

Theorem 4.2 *Let \mathcal{H} be a subset of $\mathcal{C}(\mathbb{R}^+; \mathbb{E})$. The following conditions are equivalent:*

1. *The set $\tilde{\mathcal{H}}^+ = \{\tilde{h}(t), h \in \mathcal{H}, t \in \mathbb{R}^+\}$ is a totally bounded subset of $\mathcal{C}_u(\mathbb{R}^+; \mathbb{E})$ i.e. \mathcal{H} is Bochner-equi-almost periodic.*
2. *(a) \mathcal{H} is a totally bounded subset in $\mathcal{C}_k(\mathbb{R}^+, \mathbb{E})$, and
(b) \mathcal{H} satisfies Bochner's uniform double sequence criterion.*
3. *(a) \mathcal{H} is a totally bounded subset in $\mathcal{C}_k(\mathbb{R}^+, \mathbb{E})$, and
(b) \mathcal{H} is equinormal.*
4. *(a) \mathcal{H} is a totally bounded subset in $\mathcal{C}_k(\mathbb{R}^+, \mathbb{E})$, and
(b) \mathcal{H} is equi-asymptotically almost periodic in Bohr's sense.*

Proof From [7], we have the equivalence between 2 and 3. From [25, Theorem5, and Theorem6 page 56] and Lemma 4.1, we deduce the equivalence between 3 and 4.

There remains to show that 4 is equivalent to 1.

Assume 4, then in view of Lemma 4.1 the subset $\mathcal{H}(\mathbb{R}^+)$ is totally bounded in \mathbb{E} , thus $\tilde{h}^+(t)$ is totally bounded in \mathbb{E} for each $t \in \mathbb{R}^+$. We have also in view of Lemma 4.1 that $\tilde{\mathcal{H}}^+$ is uniformly equicontinuous. Thus by Ascoli Theorem [9, Theorem2, page X.17], it is totally bounded in $\mathcal{C}_u(\mathbb{R}^+; \mathbb{E})$.

Conversely, it is obvious that if $\tilde{\mathcal{H}}^+$ is a totally bounded subset of $\mathcal{C}_u(\mathbb{R}^+; \mathbb{E})$ then \mathcal{H} is a totally bounded subset of $\mathcal{C}_k(\mathbb{R}^+; \mathbb{E})$. There remains to show that \mathcal{H} is equi-asymptotically almost periodic in Bohr's sense. For every $\varepsilon > 0$, we can construct a finite cover $\{T_i\}_{i=1}^n$ of \mathbb{R}^+ and $t_i \in T_i$, $i = 1, \dots, n$ such that, for every $\omega \in \mathbb{R}^+$, we have

$$d(h(\omega + t), h(\omega + t_i)) \leq \varepsilon, \text{ whenever } t \in T_i, i = 1, \dots, n.$$

Let $r = l > \max\{t_1, t_2, \dots, t_n\}$, and let us set

$$\mathcal{A} = \left[\bigcup_1^n (T_i - t_i) \right] \cap [r, +\infty[.$$

Let us check that $\mathcal{A} \cap [t, t + l] \neq \emptyset$ for all $t \geq r$, i.e. that \mathcal{A} is a relatively dense set in $[r, +\infty[$. Let $t \geq r$, since $\{T_i\}_{i=1}^n$ is a cover of \mathbb{R}^+ and $t + l \in \mathbb{R}^+$, there exists $i \in 1, \dots, n$ such that $t + l \in T_i$. Observe that $t \leq t + l - t_i \leq t + l$ and then $t + l - t_i \in \mathcal{A} \cap [t, t + l]$. Now, for a given $t \geq r$ and $\tau \in \mathcal{A}$, let us

choose $i \in \{1, \dots, n\}$ and $s \in T_i$ such that $\tau = s - t_i$. Since $t - t_i \geq 0$ we have, for all $t \in [r, +\infty[$ and for all $h \in \mathcal{H}$.

$$d(h(t + \tau), h(t)) = d(h(t - t_i + s), h((t - t_i) + t_i)) < \varepsilon.$$

Hence \mathcal{H} is equi-asymptotically almost periodic in Bohr's sense. \square

Corollary 4.3 *Let $x \in \mathcal{C}(\mathbb{R}^+; \mathbb{E})$. The following two statements are equivalent:*

1. *The set $\tilde{\mathcal{H}}^+(x) = \{\tilde{x}(t), t \in \mathbb{R}^+\}$ is totally bounded in $\mathcal{C}_u(\mathbb{R}^+; \mathbb{E})$.*
2. *x is asymptotically almost periodic in Bohr's sense.*

Almost periodicity and asymptotic almost periodicity in completely regular spaces Let \mathbb{E} be a separable completely regular space, let \mathfrak{U} be a uniformity on \mathbb{E} which is compatible with the topology of \mathbb{E} , and \mathfrak{D} a family of semimetrics on \mathbb{E} which induce the uniformity. We denote by \mathbb{E}_d the semimetric space $(\mathbb{E}; d)$.

Let \mathbb{E} be a separable completely regular space. Let \mathfrak{U} be a uniformity on \mathbb{E} and let \mathfrak{D} be a family of semimetrics on \mathbb{E} which induces the uniformity \mathfrak{U} . For each $d \in \mathfrak{D}$, we denote by \mathbb{E}_d the semimetric space $(\mathbb{E}; d)$.

Lemma 4.4 *Let x in $\mathcal{C}(I; \mathbb{E})$. For each $d \in \mathfrak{D}$, let $x_d = i_d \circ x$, where $i_d: \mathbb{E} \rightarrow \mathbb{E}_d$ is the canonical injection. Then,*

1. *x is Bohr-almost periodic (resp. Bohr-asymptotically almost periodic) if and only for each semimetric d in \mathfrak{D} , the function x_d is Bohr-almost periodic (resp. Bohr-asymptotically almost periodic).*
2. *x is Bochner-almost periodic if and only for each semimetric d in \mathfrak{D} , the function x_d is Bochner-almost periodic.*

Proof

1. Assume that x is Bohr-almost periodic. Let d be a semimetric in \mathfrak{D} and let $\varepsilon > 0$. The set $V = \{(x, y) \in \mathbb{E} \times \mathbb{E}; d(x, y) < \varepsilon\}$ is in \mathfrak{U} . Since x is almost periodic, there exists a real number $l = l(V) > 0$ such that every interval $[a, a + l]$ contains at least one point τ satisfying

$$(x(t + \tau), x(t)) \in V, \text{ for every } t \in I$$

hence

$$d(x(t + \tau), x(t)) < \varepsilon, \text{ for every } t \in I$$

which means that x_d is almost periodic.

Conversely, assume that for each d in \mathfrak{D} , the function f_d is almost periodic i.e. for each $\varepsilon > 0$ there exists a real number $l = l(\varepsilon, d) > 0$ such that every interval $[a, a + l]$ contains at least one point τ such that

$$d(x(t + \tau), x(t)) < \varepsilon, \text{ for every } t \in I.$$

Let V in \mathfrak{U} , then there exists a semimetric d in \mathfrak{D} such that

$$\{(x, y); d(x, y) < \varepsilon\} \subset V.$$

In particular, we have

$$(x(t + \tau), x(t)) \in V, \text{ for every } t \in I,$$

and thus x is almost periodic.

2. In view of Lemma 3.1, the equivalence we wish to prove is a direct consequence of the fact that the set $\{\tilde{x}(t), t \in I\}$ of translation mappings associated to x is totally bounded in the space $\mathcal{C}_u(I; \mathbb{E})$ if and only if, for each $d \in \mathfrak{D}$, the set $\{x_d(t + \cdot), t \in I\}$ of translation mappings associated to x_d is totally bounded in the space $\mathcal{C}_u(I; \mathbb{E}_d)$.

□

Remark 4.5 By passing to the quotient spaces $\check{\mathbb{E}}_d$, the previous Lemma reduces the study of almost periodicity and asymptotic almost periodicity in uniform spaces to that of almost periodicity and asymptotic almost periodicity in metric spaces. Thanks to Lemma 4.4, we can extend to uniform spaces the equivalence between Bohr's almost periodicity (resp. asymptotic almost periodicity) and Bochner's almost periodicity. This result is well known in the case when $I = \mathbb{R}$, see [14]. In the case when $I = \mathbb{R}^+$ and esp is a locally convex space, it is also well-known, see [20].

Theorem 4.6 *Let \mathbb{E} be a uniform space (nonnecessarily complete) and $f : I \rightarrow \mathbb{E}$ a continuous function. Then*

1. *If $I = \mathbb{R}$, the following statements are equivalent:*
 - (a) *x is Bohr-almost periodic.*

(b) x is Bochner-almost periodic.

2. If $I = \mathbb{R}^+$, the following conditions are equivalent:

(a) x is Bohr-asymptotically almost periodic.

(b) x is Bochner-almost periodic.

Proof This is a direct consequence of Lemma 4.4 and of the corresponding result in (semi)metric spaces. \square

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