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ON CUBIC HYPERSURFACES OF DIMENSION SEVEN AND EIGHT

ATANAS ILIEV AND LAURENT MANIVEL

ABSTRACT. Cubic sevenfolds are examples of Fano manifolds of Calabi-Yau type. We study them in relation with the Cartan cubic, the E_6 -invariant cubic in \mathbb{P}^{26} . We show that a generic cubic sevenfold X can be described as a linear section of the Cartan cubic, in finitely many ways. To each such “Cartan representation” we associate a rank nine vector bundle on X with very special cohomological properties. In particular it allows to define auto-equivalences of the non-commutative Calabi-Yau threefold associated to X by Kuznetsov. Finally we show that the generic eight dimensional section of the Cartan cubic is rational.

1. INTRODUCTION

Manifolds of Calabi-Yau type were defined in [IMa2] as compact complex manifolds of odd dimension whose middle dimensional Hodge structure is similar to that of a Calabi-Yau threefold. Certain manifolds of Calabi-Yau type were used in order to construct mirrors of rigid Calabi-Yau threefolds, and in several respects they behave very much like Calabi-Yau threefolds. This applies in particular to the cubic sevenfold, which happens to be the mirror of the so-called Z -variety – a rigid Calabi-Yau threefold obtained as a desingularization of the quotient of a product of three Fermat cubic curves by the group $\mathbb{Z}_3 \times \mathbb{Z}_3$, see e.g. [Sch], [CDP], [BBVW].

In some way, manifolds of Calabi-Yau type come just after manifolds of K3 type: manifolds of even dimension $2n$, whose middle dimensional Hodge structure is similar to that of a K3 surface. For $n \geq 2$, projective $2n$ -folds of K3 type seem to be scarce, but they are specially interesting in the fact that if one has a well-behaved family of $(n - 1)$ -cycles on the variety, then the Abel-Jacobi map produces a closed holomorphic two-form on the base of the family. This could be a method to construct new complex symplectic manifolds. Indeed, examples suggest that these holomorphic forms tend to be as non-degenerate as they can, see e.g. [KMM]. This is what we will check for the cubic sevenfold and its family of planes: since we start with a manifold of Calabi-Yau type, the Abel-Jacobi map will produce a holomorphic three-form on the corresponding Fano variety, and we show that it is generically non-degenerate (Proposition 2.1), in a sense that we will make precise.

Then we will concentrate, for the remaining of the paper, on a phenomenon which was first observed in [IMa2], and that we will study here in

much more depth: one can construct manifolds of Calabi-Yau type as linear sections of the Cayley plane, and this turns out to be again related to cubic sevenfolds. The *Cayley plane*, which is homogeneous under E_6 , is the simplest of the homogeneous spaces of exceptional type. It has many beautiful geometric properties, in particular it supports a plane projective geometry, whose “lines” are eight dimensional quadrics. Its minimal equivariant embedding is inside a projective space of dimension 26, in which its secant variety is an E_6 -invariant cubic hypersurface that we call the *Cartan cubic*. The Cartan cubic is singular exactly along the Cayley plane, so that its general linear section of dimension at most eight is smooth.

We prove in Proposition 3.1 that a general cubic sevenfold X can always be described as a linear section of the Cartan cubic. Moreover, there is, up to isomorphism, only finitely many such *Cartan representations*. Using the plane projective geometry supported by the Cayley plane we show that such a representation induces on X a rank nine vector bundle \mathcal{E}_X with very nice properties: in particular, it is simple, infinitesimally rigid (Theorem 3.4) and arithmetically Cohen-Macaulay (Proposition 3.3). Kuznetsov observed in [Ku1] that the derived category of coherent sheaves on a smooth hypersurface contains, under some hypothesis on the dimension and the degree, a full subcategory which is a Calabi-Yau category, of smaller dimension than the hypersurface. In particular the derived category $D^b(X)$ of our smooth cubic sevenfold contains a full subcategory \mathcal{A}_X which is Calabi-Yau of dimension three (a “*non-commutative Calabi-Yau threefold*”). It turns out that $\mathcal{E}_X(-1)$ and $\mathcal{E}_X(-2)$ both define objects in \mathcal{A}_X , and these objects are spherical. As shown by Seidel and Thomas, the corresponding spherical twists define auto-equivalences of \mathcal{A}_X (Corollary 5.3). Recall that $D^b(X)$ itself is poor in symmetries, X being Fano. Since \mathcal{A}_X is Calabi-Yau, its structure and its symmetries are potentially much richer.

Cartan representations of cubic sevenfolds are very similar to Pfaffian representations of cubic fourfolds, which are linear sections of the secant cubic to the Grassmannian $G(2,6)$. (There are nevertheless important differences, in particular a general cubic fourfold is *not* Pfaffian.) Starting from a Pfaffian representation, a classical construction consists in considering the orthogonal linear section of the dual Grassmannian: this is a K3 surface of genus eight, whose derived category can be embedded inside the derived category of the fourfold [Ku2]. We expect something similar for a Cartan representation of a general cubic sevenfold X . The orthogonal linear section of the dual Cayley plane is a Fano sevenfold Y of index three. Using the plane projective geometry supported by $\mathbb{O}\mathbb{P}^2$, we show that Y is birationally equivalent (but not isomorphic) to X (Proposition 4.2). This is an analogue of the Tregub-Takeuchi birational equivalence between the general prime Fano threefold of degree fourteen and the orthogonal cubic threefold, as we explain in a short Appendix. We conjecture that an explicit subcategory \mathcal{A}_Y of $D^b(Y)$ should be Calabi-Yau of dimension three, and should even be equivalent to \mathcal{A}_X . This would give rise to an isomorphism, very

similar to the Pfaffian-Calabi-Yau derived equivalence of [BC], between two non-commutative Calabi-Yau threefolds, associated to two quite different sevenfolds of Calabi-Yau type.

Pfaffian fourfolds are famous examples of *rational* cubic fourfolds. We conclude the paper by showing (Theorem 6.1) that eight dimensional linear sections of the Cartan cubic give rise to rational cubic eightfolds (but very non generic). The main ingredient of the proof is the close connection between E_6 and $Spin_{10}$, and the fact that the spinor varieties of $Spin_{10}$ have four dimensional linear sections which are varieties with one apparent double point. The very nice geometric and homological properties of Pfaffian fourfolds thus seem to propagate in rather different ways to cubics of dimension seven or eight.

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2. PLANES ON THE CUBIC SEVENFOLD

Here we shall see that on the 8-dimensional family $F = F(X)$ of planes on the cubic sevenfold X there exists a generically non-degenerate holomorphic 3-form ω . The existence of the form $\omega \in H^{3,0}(F(X))$ is based on the fact that the space $H^{5,2}(X)$ is one-dimensional, and ω can be constructed for example by following the techniques from [KMM]. This is an analog of the well-known observation about the cubic fourfold Y and its family of lines $F(Y)$, due to Beauville and Donagi: the unique (3,1)-form (up to scalars) on the cubic fourfold Y yields a non-degenerate holomorphic 2-form on $F(X)$, thus establishing that $F(X)$ is a holomorphic symplectic fourfold, see [BD].

2.1. Basics. The cubic sevenfold and its Abel-Jacobi mapping have been studied by Albano and Collino [AC]. It also appeared in the physics literature, since it was observed in [CDP] that it could be considered as the mirror of a rigid Calabi-Yau threefold, obtained as the quotient of a product of three elliptic curves by some threefold symmetry group.

In [IMa2] we defined *manifolds of Calabi-Yau type* as compact complex manifolds X of odd dimension $2n + 1$ such that:

- (1) The middle dimensional Hodge structure is numerically similar to that of a Calabi-Yau threefold, that is

$$h^{n+2,n-1}(X) = 1, \quad \text{and} \quad h^{n+p+1,n-p}(X) = 0 \text{ for } p \geq 2.$$

- (2) For any generator $\omega \in H^{n+2,n-1}(X)$, the contraction map

$$H^1(X, TX) \xrightarrow{\omega} H^{n-1}(X, \Omega_X^{n+1})$$

is an isomorphism.

- (3) The Hodge numbers $h^{k,0}(X) = 0$ for $1 \leq k \leq 2n$.

It is a straightforward consequence of the description of the Hodge structure of a smooth projective hypersurface in terms of its Jacobian ring, that

a smooth cubic sevenfold X is a Fano manifold of Calabi-Yau type. The relevant Hodge numbers are

$$h^{5,2}(X) = 1, \quad h^{4,3}(X) = 84.$$

2.2. A 3-form on the family of planes in the cubic sevenfold. Our first interest in cubic sevenfolds stemmed from the fact that the equality $h^{5,2}(X) = 1$ looks very similar to the equality $h^{3,1}(Y) = 1$ that holds for a smooth cubic fourfold Y . As we have already mentioned, it is a famous result of Beauville and Donagi [BD] that the Fano fourfold $F(Y)$ of lines on the general cubic fourfold Y is a symplectic fourfold: it is endowed with a closed holomorphic two-form $\omega \in H^{2,0}(F(Y))$ which is everywhere non-degenerate. Moreover, if $\mathcal{L} \subset F(Y) \times Y$ is the family of lines on Y , with its projections p and q to $F(Y)$ and Y , then the natural map

$$p_*q^* : H^{3,1}(Y) \rightarrow H^{2,0}(F(Y))$$

is an isomorphism.

If now X is a general cubic sevenfold in \mathbb{P}^8 , then the Fano variety $F = F(X)$ of *planes* on X is a smooth connected eightfold, and with the same notations as above, the map

$$p_*q^* : H^{5,2}(X) \rightarrow H^{3,0}(F)$$

is an isomorphism [AC]. In particular $H^{3,0}(F) = \mathbb{C}$, so that F is endowed with a holomorphic three-form ω , which is closed by the results of [KMM]. Although F is not a symplectic variety (its canonical bundle is non-trivial), we have a partially similar result to the case of the cubic fourfold.

Proposition 2.1. *Let X be a general cubic sevenfold, F the eightfold of planes on X , and ω a generator of $H^{3,0}(F)$. Then ω is generically non-degenerate.*

Before proving this statement, a few words of explanation are in order. Indeed, if the concept of non-degeneracy is completely clear for two-forms, it is not for three-forms in general. The point here is that F has dimension eight, so that ω is at each point of F a three-form in eight variables. Now, it is known that the space of skew-symmetric three forms in eight variables has a finite number of GL_8 -orbits, which were first classified by Gurevich [Gu]. In particular there is an open orbit $\mathcal{O}_{gen} \subset \wedge^3 \mathbb{C}^8$, and ω can be defined as being non-degenerate at some point P if

$$\omega \in \mathcal{O}_{gen} \subset \wedge^3 (T_P F)^\vee.$$

Proof. Following [KMM], we can describe the three-form in terms of the extension class $\nu \in Ext^1(\Omega_X^1, \mathcal{O}_X(-3))$ defining the conormal exact sequence. Its square is an element $S^2\nu \in Ext^2(\Omega_X^2, \mathcal{O}_X(-6))$. Let P be a general plane in X . The normal bundle $N_{P/X}$ has rank five and determinant $\wedge^5 N_{P/X} = \mathcal{O}_P(3)$. Moreover, the quotient $TX|_P \rightarrow N_{P/X}$ can be

interpreted as a global section t of $\Omega_{X|P}^1 \otimes N_{P/X}$. We can thus define the composition

$$\begin{aligned} \wedge^3 H^0(N_{P/X}) &\rightarrow H^0(\wedge^3 N_{P/X}) \xrightarrow{\wedge^2 t} H^0(\wedge^5 N_{P/X} \otimes \Omega_{X|P}^2) = \\ &= H^0(\Omega_{X|P}^2(3)) \xrightarrow{S^2 \nu} H^2(\omega_P) = \mathbb{C}. \end{aligned}$$

Once we identify $T_P F$ with $H^0(N_{P/X})$, this is precisely the three-form ω (up to scalar) and we can make an explicit local computation.

So let $\mathbb{C}^9 = A \oplus B$ be a direct sum decomposition such that the plane P is the projectivization of A . Let x_1, x_2, x_3 be coordinates on A and y_1, \dots, y_6 be coordinates on B . Let $c(x, y)$ be a cubic polynomial defining the hypersurface X . The hypothesis that X contains P implies that c can be written as

$$c(x, y) = \sum_{i=1}^6 y_i q_i(x, y)$$

for some quadratic forms q_1, \dots, q_6 . The normal bundle $N_{P/X} \subset B \otimes \mathcal{O}_P(1)$ can be described as the set of sextuples $\ell_1(x), \dots, \ell_6(x)$ such that

$$\sum_{i=1}^6 \ell_i(x) q_i(x, 0) = 0.$$

The space of global sections $H^0(N_{P/X})$ has then dimension eight if and only if any cubic form on A can be written in the previous form, that is, $\sum_{i=1}^6 \ell_i(x) q_i(x, 0)$ for some linear forms $\ell_i(x)$. In the sequel we make a stronger assumption: we suppose that the quadrics $q_1(x, 0), \dots, q_6(x, 0)$ are linearly independent. Otherwise said, they define an isomorphism between B and $S^2 A$.

This suggests to change our notation by identifying $i = 1 \dots 6$ with pairs of integers (j, k) from 1 to 3, and letting $q_{jk}(x, 0) = x_j x_k$. Correspondingly, B has a basis e_{jk} and $N_{P/X} \subset B \otimes \mathcal{O}_P(1)$ is the set of sextuples $(\ell_{jk}(x))$ such that $\sum_{jk} \ell_{jk}(x) x_j x_k = 0$. An easy computation shows that $\wedge^5 N_{P/X} \subset \wedge^5 B \otimes \mathcal{O}_P(5) \simeq B^\vee(5)$ is the copy of $\mathcal{O}_P(3)$ generated by the global section $n(x) = \sum_{jk} x_j x_k e_{jk}^\vee$ of $B^\vee(2)$, which never vanishes.

Note that the map $TX|_P \rightarrow N_{P/X}$ induces a map from $Ext^1(\Omega_X^1, \mathcal{O}_X(-3))$ to $Ext^1(N_{P/X}^\vee, \mathcal{O}_X(-3))$, mapping the class ν to the class ν_P of the extension

$$(1) \quad 0 \rightarrow N_{P/X} \rightarrow B \otimes \mathcal{O}_P(1) \rightarrow \mathcal{O}_P(3) \rightarrow 0.$$

This allows to compute our three-form on $H^0(N_{P/X})$ as the composition

$$\begin{aligned} \wedge^3 H^0(N_{P/X}) &\rightarrow H^0(\wedge^3 N_{P/X}) \\ &\simeq H^0(\wedge^2 N_{P/X}^\vee \otimes \mathcal{O}_P(3)) \xrightarrow{S^2 \nu_P} H^2(\omega_P) = \mathbb{C}. \end{aligned}$$

Let us find a Čech representative of ν_P relative to the canonical affine covering of P . On $x_i \neq 0$, the section $x_i \otimes e_i^2$ yields a splitting of (1), which implies that ν_P is defined by the Čech cocycle

$$g_{ij} = \frac{x_i \otimes e_{ii}}{x_i^3} - \frac{x_j \otimes e_{jj}}{x_j^3}.$$

The symmetric square of ν_P in $Ext^2(\wedge^2 N_{P/X}^\vee, \mathcal{O}_X(-6))$ is defined by the Čech cocycle

$$g_{ijk} = \frac{e_{ii} \wedge e_{jj}}{x_i^2 x_j^2} + \frac{e_{jj} \wedge e_{kk}}{x_j^2 x_k^2} + \frac{e_{kk} \wedge e_{ii}}{x_k^2 x_i^2}.$$

Let now a, b, c be three global sections of $N_{P/X}$. For example, we can identify a with a traceless matrix (a_{jk}) of order three, in such a way that, as a global section of $B(1)$,

$$a = \sum_{\sigma \in \mathcal{S}_3} \sum_{k=1}^3 \epsilon(\sigma) a_{\sigma(1)k} x_{\sigma(2)} e_{\sigma(3)k}.$$

Then our three form ψ on $H^0(N_{P/X})$ maps $a \wedge b \wedge c$ to the Čech cocycle of ω_P defined by $a \wedge b \wedge c \wedge g_{123}$, which can be written, when considered as a rational section of $\wedge^5 N_{P/X}$, as a multiple of $n(x)$ by some rational function having poles at most along the three coordinate hyperplanes. Since $1/x_1 x_2 x_3$ defines a cocycle generating $H^2(\omega_P)$, while any other monomial (with exponents in \mathbb{Z}) is a coboundary, we deduce the identity

$$a \wedge b \wedge c = \psi(a, b, c) x_1 x_2 x_3 e_{12} \wedge e_{23} \wedge e_{13} + \text{other terms}.$$

At that point the simplest thing to do is an explicit computation: we find, in terms of the linear forms α_{jk} on \mathfrak{sl}_3 taking the values of the (jk) -entries (so that $\alpha_{11} + \alpha_{22} + \alpha_{33} = 0$):

$$\begin{aligned} \psi = & \alpha_{12} \wedge \alpha_{31} \wedge \alpha_{23} + \alpha_{13} \wedge \alpha_{21} \wedge \alpha_{32} + \alpha_{12} \wedge \alpha_{21} \wedge (\alpha_{22} - \alpha_{11}) + \\ & + \alpha_{23} \wedge \alpha_{32} \wedge (\alpha_{33} - \alpha_{22}) + \alpha_{13} \wedge \alpha_{31} \wedge (\alpha_{11} - \alpha_{33}), \end{aligned}$$

in which we easily recognize the unique invariant three-form on \mathfrak{sl}_3 :

$$\psi(a, b, c) = \text{trace}([a, b]c).$$

Any element in the stabilizer of ψ clearly induces a Lie algebra isomorphism of \mathfrak{sl}_3 . This stabilizer is thus locally isomorphic to SL_3 . In particular it has dimension 8, hence codimension 56 in $GL(\mathfrak{sl}_3)$. But 56 is also the dimension of $\wedge^3 \mathfrak{sl}_3$, which shows that the $GL(\mathfrak{sl}_3)$ -orbit of the determinant is the open orbit \mathcal{O}_{gen} of non-degenerate three-forms. \square

Remark. Following [KS] (Proposition 10 page 90), the complement of \mathcal{O}_{gen} is defined by a covariant of degree 16 on $\wedge^3 \mathbb{C}^3$. This covariant defines a map $S^{16} \Omega_{F(X)}^3 \rightarrow \omega_{F(X)}^6$. Since $\omega_{F(X)} = \mathcal{O}_{F(X)}(1)$, the degeneracy locus of the three-form ω is a sextic hypersurface in $F(X)$.

3. CARTAN REPRESENTATIONS OF CUBIC SEVENFOLDS

By a result of L. Hesse from 1844 the homogeneous form defining a smooth plane cubic can be represented in three non-equivalent ways as a determinant of a symmetric 3×3 matrix with linear entries, see [He]. By a later construction of A. Dixon (1902), the representations of the equation of a plane curve as a symmetric determinant correspond to its non-effective theta-characteristics, see [Di] or §4 in [Be]. By other classical results originating from A. Clebsch (1866), a smooth cubic surface $S \subset \mathbb{P}^3$ has exactly 72 determinantal representations, i.e. the cubic form $F(x)$ of S has 72 non-equivalent representations as a determinant of a 3×3 matrix M with linear entries, see [Cl] or Cor.6.4 in [Be].

What about cubics of higher dimensions? The determinantal cubics of dimensions ≥ 3 are singular, but one can nevertheless proceed by replacing determinantal by Pfaffian representations, i.e. representations as Pfaffians of skew-symmetric matrices of order 6 with linear entries. It has been found essentially in [MT], [IMr] and [Dr], and restated in this form in [Be], that the general cubic threefold Y has a family of Pfaffian representations which is 5-dimensional and birational to the intermediate Jacobian of Y . As for cubic fourfolds, it is well-known that the general cubic fourfold is not Pfaffian. For cubics of higher dimensions the situation is even worse – while the general Pfaffian cubic fourfold is still smooth (and rational!) the Pfaffian cubics of dimension greater than five are singular, and the question is which generalization of the Pfaffian, if any, can represent cubics of higher dimensions.

Looking back, we see that the spaces of symmetric 3×3 matrices, matrices of order 3, and skew-symmetric matrices of order 6 are in fact the first three of a series of four Jordan algebras – they are the algebras of Hermitian matrices over the composition algebras \mathbf{R} , \mathbf{C} and \mathbf{H} . Then, writing-down a cubic form F as a symmetric determinant, as a determinant, or as a Pfaffian is the same as giving a presentation of the cubic $F = 0$ as a linear section of the determinant in the Jordan algebra over \mathbf{R} , \mathbf{C} or \mathbf{H} .

What is left is the fourth Jordan algebra – the space of Hermitian 3×3 matrices over the Cayley algebra \mathbf{O} of octonions. The octonionic determinant is a special E_6 -invariant cubic in \mathbb{P}^{26} – the Cartan cubic which is singular in codimension 9. Therefore we can ask about representations of the smooth cubics X as linear sections of the Cartan cubic upto dimension 8. We call them the *Cartan* representations of X .

As we show below, the general cubic sevenfold is a section of the Cartan cubic in a finite number of ways, and by this property it looks similar to the cubic curve and the cubic surface. But while the numbers of determinantal representations as above of cubic curves and of cubic surfaces are known since Hesse and Clebsch, the number of Cartan representations of the general cubic sevenfold remains an unsolved question.

3.1. The Cayley plane and the Cartan cubic. Let \mathbf{O} denote the normed algebra of real octonions (see e.g. [Ba]), and let \mathbb{O} be its complexification. The space

$$J_3(\mathbb{O}) = \left\{ x = \begin{pmatrix} r & w & \bar{v} \\ \bar{w} & s & u \\ v & \bar{u} & t \end{pmatrix} : r, s, t \in \mathbf{C}, u, v, w \in \mathbb{O} \right\} \cong \mathbf{C}^{27}$$

of \mathbb{O} -Hermitian matrices of order 3, is traditionally known as the *exceptional simple Jordan algebra*, for the Jordan multiplication $A \circ B = \frac{1}{2}(AB + BA)$. The automorphism group of this Jordan algebra is an algebraic group of type F_4 , and preserves the determinant

$$\text{Det}(x) = rst - r|u|^2 - s|v|^2 - t|w|^2 + 2\text{Re}(uvw),$$

where the term $\text{Re}(uvw)$ makes sense uniquely, despite the lack of associativity of \mathbb{O} . The subgroup of $GL(J_3(\mathbb{O}))$ consisting of automorphisms preserving this cubic polynomial is the adjoint group of type E_6 . The Jordan algebra $J_3(\mathbb{O})$ and its dual are the minimal (non-trivial and irreducible) representations of this group. An equation of the invariant cubic hypersurface was once written down by Elie Cartan in terms of the configuration of the 27 lines on a smooth cubic surface. We will call it the *Cartan cubic*.

The action of E_6 on the projectivization $\mathbb{P}J_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface ($\text{Det} = 0$), the regular part of this hypersurface, and its singular part which is the closed E_6 -orbit. These three orbits can be considered as the (projectivizations of the) sets of matrices of rank three, two, and one respectively.

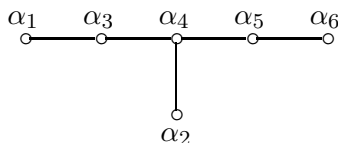
The closed orbit, corresponding to rank one matrices, is called the *Cayley plane* and denoted $\mathbb{O}\mathbb{P}^2$ (the reasons for this notation and terminology will soon be explained). It can be defined by the quadratic equation

$$x^2 = \text{trace}(x)x, \quad x \in J_3(\mathbb{O}).$$

It will be useful to notice that a dense open subset can be parametrized explicitly as the set of matrices of the form

$$(2) \quad \begin{pmatrix} 1 & w & \bar{v} \\ \bar{w} & |w|^2 & \bar{w}\bar{v} \\ v & vw & |v|^2 \end{pmatrix}, \quad v, w \in \mathbb{O}.$$

The Cayley plane can also be identified with the quotient of E_6 by the maximal parabolic subgroup P_1 defined by the simple root α_1 corresponding to the end of one the long arms of the Dynkin diagram (we follow the notations of [Bou]). The semi-simple part of this maximal parabolic is isomorphic to Spin_{10} .



The symmetric node α_6 of the Dynkin diagram corresponds to the dual representation $J_3(\mathbb{O})^\vee$, which is exchanged with $J_3(\mathbb{O})$ by an outer automorphism of the group. In particular the orbits inside $\mathbb{P}J_3(\mathbb{O})$ and $\mathbb{P}J_3(\mathbb{O})^\vee$ are isomorphic (although non-equivariantly), and the closed orbit E_6/P_6 inside $\mathbb{P}J_3(\mathbb{O})^\vee$ is again a copy of the Cayley plane. We denote it by $\bar{\mathbb{O}}\mathbb{P}^2$ and call it the *dual Cayley plane*.

The Cartan cubic inside $\mathbb{P}J_3(\mathbb{O})^\vee$ can be described either as the projective dual variety to the Cayley plane, or as the secant variety to the dual Cayley plane. As a secant variety, it is degenerate, and therefore any point p of the open orbit in $\mathbb{P}J_3(\mathbb{O})^\vee$ defines a positive dimensional *entry locus* in the dual Cayley plane (the trace on $\bar{\mathbb{O}}\mathbb{P}^2$ of the secants through p , see e.g. [Z] for background and more details on Severi varieties). One can check that these entry-loci are eight dimensional quadrics, and that their family is parametrized by the Cayley plane itself. Explicitly, if x is in $\mathbb{O}\mathbb{P}^2$, the orthogonal to its tangent space cuts the dual Cayley plane along one of these quadrics Q_x .

Quite remarkably, these quadrics must be considered as \mathbb{O} -lines inside the dual Cayley plane, and we get the familiar picture of a plane projective geometry: the two Cayley planes parametrize points and lines (respectively lines and points) on them, and the basic axioms of a plane geometry hold generically [Ti, Fr]:

- (1) Let $x, x' \in \mathbb{O}\mathbb{P}^2$ be two distinct points, such that the line xx' is not contained in $\mathbb{O}\mathbb{P}^2$. Then $Q_x \cap Q_{x'}$ is a unique reduced point.
- (2) Let $y, y' \in \bar{\mathbb{O}}\mathbb{P}^2$ such that the line yy' is not contained in $\bar{\mathbb{O}}\mathbb{P}^2$. Then there is a unique x in $\mathbb{O}\mathbb{P}^2$ such that the \mathbb{O} -line Q_x passes through y and y' .

Of course the same properties hold in the dual Cayley plane. We can encode our incidence geometry in the following diagram:

$$\begin{array}{ccc}
 & E_6/P_{1,6} & \\
 \mu \swarrow & & \searrow \nu \\
 x \in \mathbb{O}\mathbb{P}^2 & & \bar{\mathbb{O}}\mathbb{P}^2 \supset Q_x
 \end{array}$$

where $P_{1,6} \subset E_6$ is the parabolic subgroup defined as the intersection of P_1 and P_6 . In particular $Q_x = \nu(\mu^{-1}(x))$.

Another useful property that we want to mention is that the Cartan cubic is a homoloïdal polynomial, in the sense that its derivatives define a birational transformation of the projective space. More intrinsically, we

have a birational quadratic map

$$\begin{aligned} PDet : \mathbb{P}J_3(\mathbb{O}) &\dashrightarrow \mathbb{P}J_3(\mathbb{O})^\vee \\ x &\mapsto Det(x, x, *), \end{aligned}$$

compatible with the E_6 -action. To be precise, the polarized determinant $PDet$ defines an isomorphism between the open orbits on both sides, but it contracts the Cartan cubic to the dual Cayley plane. Moreover it blows-up the Cayley plane itself, which is its indeterminacy locus, to the dual Cartan cubic, in such a way that a point $x \in \mathbb{O}\mathbb{P}^2$ is blown-up into the linear span of the corresponding quadric Q_x . Of course, the picture being symmetric, the inverse of $PDet$ must be the quadratic map defined by the derivatives of the dual Cartan cubic.

In order to make explicit computations, it is convenient to identify $J_3(\mathbb{O})$ and $J_3(\mathbb{O})^\vee$ (as vector spaces) through the following scalar product:

$$\left\langle \begin{pmatrix} r & w & \bar{v} \\ \bar{w} & s & u \\ v & \bar{u} & t \end{pmatrix}, \begin{pmatrix} r' & w' & \bar{v}' \\ \bar{w}' & s' & u' \\ v' & \bar{u}' & t' \end{pmatrix} \right\rangle = rr' + ss' + tt' - 2\operatorname{Re}(\bar{u}u' + \bar{v}v' + \bar{w}w').$$

With this convention, the map $PDet$ is given explicitly by

$$(3) \quad x = \begin{pmatrix} r & w & \bar{v} \\ \bar{w} & s & u \\ v & \bar{u} & t \end{pmatrix} \mapsto \begin{pmatrix} st - |u|^2 & tw - \bar{v}u & wu - s\bar{v} \\ t\bar{w} - uv & rt - |v|^2 & ru - \bar{w}v \\ \bar{u}\bar{w} - sv & r\bar{u} - vw & rs - |w|^2 \end{pmatrix}.$$

The main interest of this presentation is that the equation of the dual Cartan cubic is given exactly by the same formula as Det . In particular the above expression of $PDet$, which is a kind of *comatrix* map, defines a birational involution of \mathbb{P}^{26} .

3.2. Cartan representations. Given a cubic hypersurface X of dimension d , we define a *Cartan representation* of X as a linear section of the Cartan cubic isomorphic to X :

$$X \simeq \mathcal{C} \cap L \quad \text{for} \quad \mathbb{P}^{d+1} \simeq L \subset \mathbb{P}J_3(\mathbb{O}).$$

Note that the Cartan cubic being singular in codimension nine, a *smooth* cubic hypersurface of dimension bigger than eight cannot have any Cartan representation. The main result of this section is that, on the contrary, a general cubic hypersurface of dimension smaller than eight always admits a Cartan representation. (The boundary case of eight dimensional sections will be considered later). A more precise statement is the following.

Proposition 3.1. *A generic cubic sevenfold is, up to isomorphism, a linear section of the Cartan cubic, in a finite, non zero, number of different ways.*

Proof. Let \mathcal{M}_3^7 denote the moduli space of cubic sevenfolds. What we need to prove is that the rational map

$$\Psi : G(9, J_3(\mathbb{O})) // E_6 \dashrightarrow \mathcal{M}_3^7,$$

obtained by sending $L \in G(9, J_3(\mathbb{O}))$ to the isomorphism class of $X_L = \mathcal{C} \cap L$, is generically finite. The two moduli spaces have the same dimension so we just need to check that Ψ is generically étale, hence to find an L at which its differential is surjective. This will follow from the surjectivity of the map

$$\varphi \in \text{Hom}(L, J_3(\mathbb{O})) \mapsto P_\varphi(x) = \text{Det}(x, x, \varphi(x)) \in S^3 L^\vee.$$

We will choose our L as the image of a map

$$(r, y) \mapsto \begin{pmatrix} r & \alpha(y) & \bar{y} \\ \bar{\alpha}(y) & r & \beta(y) \\ y & \bar{\beta}(y) & r \end{pmatrix},$$

where $\alpha, \beta : \mathbb{O} \rightarrow \mathbb{O}$ are linear endomorphisms (of \mathbb{O} considered as an eight-dimensional vector space).

Consider the morphism φ from L to $J_3(\mathbb{O})$ defined by

$$(r, y) \mapsto \begin{pmatrix} a & w & \bar{v} \\ \bar{w} & b & u \\ v & \bar{u} & c \end{pmatrix},$$

for some linear forms a, b, c and linear functional u, v, w . The corresponding polynomial $P_\varphi(x)$ is

$$\begin{aligned} P_\varphi(x) &= r^2(a + b + c) - a|\alpha(y)|^2 - b|y|^2 - c|\beta(y)|^2 \\ &\quad - 2r\text{Re}(u\beta(y) + vy + w\alpha(y)) + 2\text{Re}(w\beta(y)y + \beta(y)v\alpha(y) + uy\alpha(y)). \end{aligned}$$

So what we need to prove is that the 27 following quadrics in r, y generate the whole space of cubics

$$\begin{aligned} r^2 - |\alpha(y)|^2, & \quad y\alpha(y) - r\beta(y), \\ r^2 - |y|^2, & \quad \alpha(y)\beta(y) - ry, \\ r^2 - |\beta(y)|^2, & \quad \beta(y)y - r\alpha(y). \end{aligned}$$

The following easy lemma is left to the reader.

Lemma 3.2. *A sufficient condition for this to be true is that the sixteen quadrics $y\alpha(y)$ and $\beta(y)y$ generate the whole space of cubics in $y \in \mathbb{O}$.*

To finish the proof, we thus only need to exhibit $\alpha(y)$ and $\beta(y)$ for which the lemma does hold. Let us choose them of the form

$$\alpha(y) = \sum_i s_i y_{\sigma(i)} e_i, \quad \beta(y) = \sum_i t_i y_{\tau(i)} e_i,$$

for some coefficients s_i, t_i and permutations σ, τ . More precisely, let $\sigma(i) = i + 1$ and $\beta(i) = i + 3$ (using the cyclic order on $0, \dots, 7$), and $s = t = (1, 2, -1, 3, 1, -1, 2, 1)$. A computation with MACAULAY2 shows that the lemma does indeed hold in that case. \square

Question. What is the number of Cartan representations of a generic cubic sevenfold? (Otherwise said, what is the degree of Ψ ?) Is it equal to one, in which case we would get a *canonical form* for cubic sevenfolds? Or bigger,

in which case the derived category would have many interesting symmetries, as we will see later on?

Remark. In [IMa2] we have shown that the Cayley plane belongs to a series of four homogeneous spaces for which we expect very similar phenomena. One of these is the Grassmannian $G(2, 10)$, and the statement analogous to Proposition 3.1 in that case is the fact that a general quintic threefold (the archetypal Calabi-Yau) admits a finite number of Pfaffian representations. This was proved by Beauville and Schreyer [Be] but the precise number of such representations is not known (although it may be computable, being a Donaldson-Thomas invariant). The two other cases are spinor varieties, for which the corresponding statements have not been checked yet.

3.3. A special vector bundle. The next question to ask is: what kind of additional data do we need on a cubic sevenfold, in order to define it as a linear section of the Cartan cubic?

In the case of quintic threefolds, it is easy to see that Pfaffian representations are in correspondence with certain special rank two vector bundles, obtained as the restriction of the kernel bundle on the regular part of the Pfaffian cubic, which parametrizes skew-symmetric forms of corank two.

On the regular part of the Cartan cubic, there is a natural quadric bundle \mathcal{Q} of relative dimension eight, whose fiber at p is just the corresponding entry locus Q_p . The linear span $\langle Q_p \rangle$ contains p , and since Q_p is always smooth, the polar to p with respect to Q_p , is a hyperplane $\mathbb{P}(\mathcal{E}_p^\vee) \subset \langle Q_p \rangle$. This defines a vector bundle \mathcal{E}^\vee of rank nine, which is a sub-bundle of the trivial vector bundle with fiber $J_3(\mathbb{O})^\vee$. We will denote by \mathcal{K} the quotient bundle, whose rank is eighteen.

Our next aim is to understand the special properties of these vector bundles, when restricted to a cubic sevenfold X defined as a general linear section $X = \mathcal{C} \cap L$ of the Cartan cubic.

Proposition 3.3. *Let \mathcal{E}_X and \mathcal{K}_X denote the restrictions of \mathcal{E} and \mathcal{K} to X .*

- (1) \mathcal{E}_X is arithmetically Cohen-Macaulay;
- (2) its Hilbert polynomial is $P_{\mathcal{E}_X}(k) = 27 \binom{k+7}{7}$;
- (3) the space of its global sections is $H^0(X, \mathcal{E}_X) = J_3(\mathbb{O})$;
- (4) $\mathcal{E}_X^\vee = \mathcal{E}_X(-2)$ and $\mathcal{K}_X^\vee = \mathcal{K}_X(-1)$;
- (5) $\chi(\text{End}(\mathcal{E}_X)) = 0$.

Proof. The polarization of Det defines over $\mathbb{P}J_3(\mathbb{O})^\vee$ a map

$$J_3(\mathbb{O})^\vee \otimes \mathcal{O}_{\mathbb{P}J_3(\mathbb{O})^\vee}(-1) \xrightarrow{\delta} J_3(\mathbb{O}) \otimes \mathcal{O}_{\mathbb{P}J_3(\mathbb{O})^\vee}.$$

This map is invertible outside \mathcal{C} , and we claim that its cokernel \mathcal{F} has constant rank 9 over the smooth part \mathcal{C}_0 of \mathcal{C} (that is, outside the Cayley plane). To check this, by homogeneity we just need to check this claim at

one point of \mathcal{C}_0 ; we choose

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the entry locus of that point is the quadric

$$Q_p = \left\{ \begin{pmatrix} r & z & 0 \\ z & s & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad rs - |z|^2 = 0 \right\}.$$

An easy computation shows that the image of δ_p is generated by the linear forms $r + s, t, x, y$. Its dual can then be identified with the kernel of this linear form, that is

$$\mathcal{F}_p^\vee = \left\{ \begin{pmatrix} r & z & 0 \\ z & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r \in \mathbb{C}, z \in \mathbb{O} \right\}.$$

But this is clearly the linear space polar to p with respect to the quadric Q_p . Hence an identification $\mathcal{F}^\vee \simeq \mathcal{E}^\vee$ over \mathcal{C}_0 . In particular, we deduce over the linear span \mathbb{P}_X of X an exact sequence of sheaves

$$0 \rightarrow J_3(\mathbb{O})^\vee \otimes \mathcal{O}_{\mathbb{P}_X}(-1) \rightarrow J_3(\mathbb{O}) \otimes \mathcal{O}_{\mathbb{P}_X} \rightarrow \mathcal{E}_X \rightarrow 0.$$

This immediately implies that \mathcal{E}_X is arithmetically Cohen-Macaulay. That $\mathcal{E}_X^\vee = \mathcal{E}_X(-2)$ follows from the fact that the quadric bundle Q_p is everywhere non degenerate over \mathcal{C}_0 . Similarly we get $\mathcal{K}_X^\vee = \mathcal{K}_X(1)$. Also we directly deduce the Hilbert polynomial of \mathcal{E}_X .

Now we can restrict the previous sequence to X , which gives

$$(4) \quad 0 \rightarrow \mathcal{E}_X(-3) \rightarrow J_3(\mathbb{O})^\vee \otimes \mathcal{O}_X(-1) \rightarrow J_3(\mathbb{O}) \otimes \mathcal{O}_X \rightarrow \mathcal{E}_X \rightarrow 0.$$

This exact sequence is self-dual, up to a twist. We can tensor it by \mathcal{E}_X^\vee to deduce the Hilbert polynomial of $\text{End}(\mathcal{E}_X)$. Indeed, this polynomial verifies the two equations

$$\begin{aligned} P_{\text{End}(\mathcal{E}_X)}(k) - P_{\text{End}(\mathcal{E}_X)}(k-3) &= 27(P_{\mathcal{E}_X}(k-2) - P_{\mathcal{E}_X}(k-3)), \\ P_{\text{End}(\mathcal{E}_X)}(k) &= -P_{\text{End}(\mathcal{E}_X)}(-k-6). \end{aligned}$$

The first equation follows from the exact sequence above, and the second one from Serre duality, since $\omega_X = \mathcal{O}_X(-6)$. There is a unique solution to these two equations, given by the following formula:

$$P_{\text{End}(\mathcal{E}_X)}(k) = 3^5 \binom{k+6}{7} - 3^4 \binom{k+5}{5} + 3^3 \binom{k+4}{3} - 3^2 \binom{k+3}{1}.$$

In particular $\chi_{\text{End}(\mathcal{E}_X)} = P_{\text{End}(\mathcal{E}_X)}(0) = 0$. \square

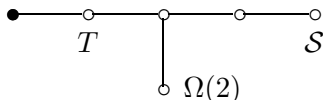
Remark. The characteristic classes of \mathcal{E}_X can easily be computed from the sequence (4), for example the Chern character

$$ch(\mathcal{E}_X) = 27 \frac{1 - \exp(-h)}{1 - \exp(-3h)},$$

where h is the hyperplane class on X . One easily deduces the Chern classes of the self-dual bundle $\mathcal{E}_X(-1)$:

$$c(\mathcal{E}_X(-1)) = 1 - 3h^2 + 9h^4 - 39h^6.$$

3.4. Relation with the normal bundle. Recall that with our conventions, the Cayley plane $\mathbb{O}\mathbb{P}^2 = E_6/P_1$. In particular the category of E_6 -homogeneous vector bundles on $\mathbb{O}\mathbb{P}^2$ is equivalent to the category of P_1 -modules. But recall that the semi-simple part of P_1 is a copy of $Spin_{10}$, whose minimal non trivial representations are the ten-dimensional vector representation, and the two sixteen-dimensional half-spin representations. The corresponding homogeneous vector bundles are, up to twists, the tangent bundle T and cotangent bundle Ω for the latter two, and the normal bundle for the former. To fix notations, we will denote by \mathcal{S} the irreducible homogeneous vector bundle on $\mathbb{O}\mathbb{P}^2$ such that $H^0(\mathbb{O}\mathbb{P}^2, \mathcal{S}) = J_3(\mathbb{O})$.



The normal bundle is then $\mathcal{N} = \mathcal{S}(1)$. This implies that the projectivization of the bundle \mathcal{S}^\vee over $\mathbb{O}\mathbb{P}^2$ is a desingularization of the dual Cayley cubic: the line bundle $\mathcal{O}_{\mathcal{S}}(1)$ is generated by its global sections, whose space is isomorphic with $H^0(\mathbb{O}\mathbb{P}^2, \mathcal{S}) = J_3(\mathbb{O})$. We have a diagram

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{S}^\vee) & \\ \pi \swarrow & & \searrow \gamma \\ \mathbb{O}\mathbb{P}^2 & & \mathcal{C} \supset \bar{\mathbb{O}}\mathbb{P}^2 \end{array}$$

where the morphism γ , which is defined by the linear system $|\mathcal{O}_{\mathcal{S}}(1)|$, is an isomorphism outside the dual Cayley plane $\bar{\mathbb{O}}\mathbb{P}^2$. (We use the same notation for the Cartan cubic in the dual space.) Moreover the pre-image of $\bar{\mathbb{O}}\mathbb{P}^2$ is the incidence variety $E_6/P_{1,6}$, embedded as a divisor in $\mathbb{P}(\mathcal{S}^\vee)$. Note that the rational map $\pi \circ \gamma^{-1} : \mathcal{C} \rightarrow \mathbb{O}\mathbb{P}^2$ is the restriction of the comatrix map $PDet$.

Since X does not meet $\bar{\mathbb{O}}\mathbb{P}^2$, γ is an isomorphism over X , and $\gamma^{-1}(X)$ is the complete intersection, in $\mathbb{P}(\mathcal{S}^\vee)$, of 18 general sections of $\mathcal{O}_{\mathcal{S}}(1)$. By construction, we have

$$(5) \quad \pi^* \mathcal{S}_X^\vee = \mathcal{E}_X^\vee \oplus \mathcal{O}_X(-1),$$

if we denote by $\pi^* \mathcal{S}_X$ the restriction of $\pi^* \mathcal{S}$ to $\gamma^{-1}(X) \simeq X$; note that the restriction of $\mathcal{O}_{\mathcal{S}}(1)$ coincides with the pull-back of $\mathcal{O}_X(1)$ by γ . If we let H_X denote the restriction to $\gamma^{-1}(X)$ of $\pi^* \mathcal{O}_{\mathbb{O}\mathbb{P}^2}(1)$, we have $H_X \simeq \mathcal{O}_X(2)$,

and the map $\theta|_X : X \rightarrow \mathbb{O}\mathbb{P}^2$ is defined by a sub-linear system of $|\mathcal{O}_X(2)|$. (One can check that for X generic, $\theta|_X$ is an embedding.)

Theorem 3.4. *The vector bundle \mathcal{E}_X is simple and infinitesimally rigid.*

More precisely,

$$h^i(X, \text{End}(\mathcal{E}_X)) = \delta_{i,0} + \delta_{i,3}.$$

Proof. We start with a simple observation showing that the higher cohomology groups of $\text{End}(\mathcal{E}_X)$ cannot all be trivial.

Lemma 3.5. *One has a natural duality*

$$H^i(X, \text{End}(\mathcal{E}_X)) \simeq H^{3-i}(X, \text{End}(\mathcal{E}_X))^\vee.$$

In particular $H^i(X, \text{End}(\mathcal{E}_X)) = 0$ for $i \geq 4$.

Proof. Consider the exact sequence (4). Observe that $\mathcal{E}_X(-3)$ is acyclic. Indeed, \mathcal{E}_X is aCM, $H^0(\mathcal{E}_X^\vee(-3)) = 0$, and by Serre duality, $H^7(\mathcal{E}_X(-3)) = H^0(\mathcal{E}_X^\vee(-3)) = H^0(\mathcal{E}_X(-5)) = 0$. We deduce that for any i ,

$$H^i(\text{End}(\mathcal{E}_X)) = H^{i+2}(\text{End}(\mathcal{E}_X)(-3)).$$

But the latter is Serre dual to $H^{5-i}(\text{End}(\mathcal{E}_X)(-3))$ which, by the previous assertion, is isomorphic to $H^{3-i}(\text{End}(\mathcal{E}_X))$. \square

Now we use the fact that $\mathcal{E}_X^\vee = \mathcal{E}_X(2)$ to decompose $\text{End}(\mathcal{E}_X)$ as the sum of its symmetric and skew-symmetric parts $S^2\mathcal{E}_X^\vee(-2)$ and $\wedge^2\mathcal{E}_X^\vee(-2)$, which we treat separately.

A. We start with the latter. To determine its cohomology, we will use the fact that $\wedge^2(\pi^*\mathcal{S}_X) = \wedge^2\mathcal{E}_X \oplus \mathcal{E}_X(1)$. Moreover, we observe that we can use the Koszul complex

$$(6) \quad 0 \rightarrow \wedge^{18}L \otimes \mathcal{O}_{\mathcal{S}}(-18) \rightarrow \cdots \rightarrow L \otimes \mathcal{O}_{\mathcal{S}}(-1) \rightarrow \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{O}_X \rightarrow 0$$

to deduce the cohomology of $\wedge^2(\pi^*\mathcal{S}_X)$ from that of $\wedge^2(\pi^*\mathcal{S})$ and its twists, which we can derive from Bott's theorem.

Since $\mathcal{O}_X(-2) = H_X^{-1}$, we will twist the Koszul complex by $\pi^*(\wedge^2\mathcal{S} \otimes H^{-1})$. This bundle is acyclic since $\wedge^2\mathcal{S} \otimes H^{-1}$ is acyclic on $\mathbb{O}\mathbb{P}^2$. Its twists by $\mathcal{O}_{\mathcal{S}}(-k)$ are also acyclic for $k \leq 9$, since they are acyclic on the fibers of π . For $k \geq 10$ and any bundle F on $\mathbb{O}\mathbb{P}^2$, we have an isomorphism

$$\begin{aligned} H^q(\mathbb{P}(\mathcal{S}^\vee), \pi^*F \otimes \mathcal{O}_{\mathcal{S}}(-k)) &= H^{q-9}(\mathbb{O}\mathbb{P}^2, F \otimes \text{Sym}^{k-10}\mathcal{S}^\vee \otimes \det(\mathcal{S}^\vee)) \\ &= H^{q-9}(\mathbb{O}\mathbb{P}^2, F \otimes \text{Sym}^{k-10}\mathcal{S} \otimes H^{5-k}). \end{aligned}$$

We will need a few decomposition formulas. Denote by \mathcal{S}_m the homogeneous vector bundle on $\mathbb{O}\mathbb{P}^2$ defined by the weight $m\omega_6$. In particular $\mathcal{S}_1 = \mathcal{S}$.

Lemma 3.6. *For any positive integer m ,*

$$\text{Sym}^m \mathcal{S} = \bigoplus_{\ell \geq 0} \mathcal{S}_{m-2\ell} \otimes H^\ell.$$

Proof. This follows from the decomposition formulas of the symmetric powers of the natural representations of special orthogonal groups, which are classical. \square

Applying the formula above to $F = \wedge^2 \mathcal{S} \otimes H^{-1}$, we thus get

$$H^q(\mathbb{P}(\mathcal{S}^\vee), \pi^*(\wedge^2 \mathcal{S} \otimes H^{-1}) \otimes \mathcal{O}_{\mathcal{S}}(-k)) = \bigoplus_{\ell \geq 0} H^{q-9}(\mathbb{O}\mathbb{P}^2, \wedge^2 \mathcal{S} \otimes \mathcal{S}_{k-2\ell-10} \otimes H^{4-k+\ell}).$$

Then we need to decompose the tensor products $\wedge^2 \mathcal{S} \otimes \mathcal{S}_m$ into irreducible components. This decomposition is given by the next Lemma, where for future use we include a few more similar formulas. We shall denote by $\mathcal{S}_{p,q,r}$ the homogeneous vector bundle on $\mathbb{O}\mathbb{P}^2$ defined by the weight $p\omega_6 + q\omega_5 + r\omega_4$, and we let $\mathcal{S}_{p,q} = \mathcal{S}_{p,q,0}$.

- Lemma 3.7.** (1) $\mathcal{S}_m \otimes \mathcal{S} = \mathcal{S}_{m+1} \oplus \mathcal{S}_{m-1,1} \oplus \mathcal{S}_{m-1} \otimes H$.
(2) $\mathcal{S}_m \otimes \wedge^2 \mathcal{S} = \mathcal{S}_{m,1} \oplus \mathcal{S}_{m-1,0,1} \oplus \mathcal{S}_m \otimes H \oplus \mathcal{S}_{m-2,1} \otimes H$.
(3) $\mathcal{S}_m \otimes \mathcal{S}_2 = \mathcal{S}_{m+2} \oplus \mathcal{S}_{m,1} \oplus \mathcal{S}_{m-2,2} \oplus \mathcal{S}_m \otimes H \oplus \mathcal{S}_{m-2,1} \otimes H \oplus \mathcal{S}_{m-2} \otimes H^2$.

Proof. Routine representation theory. \square

Now we can decompose each term $\wedge^2 \mathcal{S} \otimes \mathcal{S}_{k-2\ell-10}$ into irreducible components, and compute the cohomology of each term. The special cases of Bott's theorem we will need are covered by the following lemma:

Lemma 3.8. *Let $p \geq 0$ and $m \geq 1$. Then*

- (1) $H^i(\mathbb{O}\mathbb{P}^2, \mathcal{S}_p \otimes H^{-m}) = 0$ for $i < 16$, except
 $H^8(\mathbb{O}\mathbb{P}^2, \mathcal{S}_p \otimes H^{-m}) = V_{(p+4-m)\omega_1 + (m-8)\omega_6}$ if $8 \leq m \leq p+4$.
(2) $H^i(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,1} \otimes H^{-m}) = 0$ for $i < 16$, except
 $H^8(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,1} \otimes H^{-m}) = V_{(p+5-m)\omega_1 + (m-9)\omega_6}$ if $9 \leq m \leq p+5$.
(3) $H^i(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,2} \otimes H^{-m}) = 0$ for $i < 16$, except
 $H^4(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,2} \otimes H^{-5}) = V_{p\omega_1}$,
 $H^8(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,2} \otimes H^{-m}) = V_{(p+6-m)\omega_1 + (m-10)\omega_6}$ if $10 \leq m \leq p+6$,
 $H^{12}(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,2} \otimes H^{-p-11}) = V_{p\omega_6}$.
(4) $H^i(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,0,1} \otimes H^{-m}) = 0$ for $i < 16$, except
 $H^2(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,0,1} \otimes H^{-3}) = V_{p\omega_1}$,
 $H^6(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,0,1} \otimes H^{-7}) = V_{(p-2)\omega_1}$,
 $H^8(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,0,1} \otimes H^{-m}) = V_{(p+5-m)\omega_1 + (m-10)\omega_6}$ if $10 \leq m \leq p+5$,
 $H^{10}(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,0,1} \otimes H^{-p-8}) = V_{(p-2)\omega_6}$,
 $H^{14}(\mathbb{O}\mathbb{P}^2, \mathcal{S}_{p,0,1} \otimes H^{-p-12}) = V_{p\omega_6}$.

Proof. Apply Bott's theorem as in [Ma]. \square

We deduce the following statement: for $q < 25$,

$$H^q(\mathbb{P}(\mathcal{S}^\vee), \pi^*(\wedge^2 \mathcal{S} \otimes H^{-1}) \otimes \mathcal{O}_{\mathcal{S}}(-k)) = \delta_{q,19} V_{(k-15)\omega_6}.$$

This implies that for $q < 25 - 18 = 7$, $H^q(X, \wedge^2 \mathcal{S}_X^\vee(-2))$ is the q -th cohomology group of the complex

$$0 \rightarrow V_{3\omega_6} \rightarrow L \otimes V_{2\omega_6} \rightarrow \wedge^2 L \otimes V_{\omega_6} \rightarrow \wedge^3 L \rightarrow 0,$$

going from degree zero to degree five. In particular, $H^0(X, \wedge^2 \mathcal{S}_X^\vee(-2)) = 0$.

Proposition 3.9. *For L generic, the map $L \otimes V_{2\omega_1} \rightarrow V_{3\omega_1}$ is surjective. Therefore $H^1(X, \wedge^2 \mathcal{S}_X^\vee(-2)) = 0$.*

Proof. We start with the commutative diagram

$$\begin{array}{ccccc} & & V_{\omega_1} \otimes V_{\omega_6} & & \\ & & \downarrow & & \\ \wedge^2 V_{\omega_1} \otimes V_{\omega_1} & \rightarrow & V_{\omega_1} \otimes S^2 V_{\omega_1} & \rightarrow & S^3 V_{\omega_1} \\ \parallel & & \downarrow & & \downarrow \\ \wedge^2 V_{\omega_1} \otimes V_{\omega_1} & \rightarrow & V_{\omega_1} \otimes V_{2\omega_1} & \rightarrow & V_{3\omega_1} \end{array}$$

The horizontal complex in the middle is exact, being part of a Koszul complex. It surjects to the horizontal complex of the bottom, which is again exact, as one can check from the decompositions into irreducible components, as given by LiE [LiE]. The vertical complex in the middle is also exact, the map $V_{\omega_6} = V_{\omega_1}^\vee \rightarrow S^2 V_{\omega_1}$ being given by the differential of the invariant cubic Det .

We deduce from this diagram that the kernel \bar{K} of the map $L \otimes V_{2\omega_1} \rightarrow V_{3\omega_1}$ is simply the image of the kernel K of the map $L \otimes S^2 V_{\omega_1} \rightarrow S^3 V_{\omega_1}$.

Lemma 3.10. *Let $L \subset V$ be any subspace of dimension ℓ and codimension c . Then the kernel K of the map $L \otimes S^2 V \rightarrow S^3 V$ is the image of $\wedge^2 L \otimes V$. In particular*

$$\dim(K) = \frac{\ell(\ell^2 - 1)}{3} + c \frac{\ell(\ell - 1)}{2}.$$

Proof. Straightforward. \square

In our case, letting $\ell = 18$ and $c = 9$ we get $\dim(K) = 3315$.

Lemma 3.11. *The map $K \rightarrow \bar{K}$ is an isomorphism.*

Proof. We need to check that K does not meet the kernel $L \otimes V_{\omega_6}$ of the projection to $L \otimes V_{2\omega_1}$. Otherwise said, we need to check that the map to $Sym^3 V_{\omega_1}$ is injective on $L \otimes V_{\omega_6} \subset V_{\omega_1} \otimes S^2 V_{\omega_1}$. Let ℓ_i be a basis of L , that we complete into a basis of V_{ω_1} by some vectors m_j generating a supplement M to L . Let $(y_k) = (\ell_i^\vee, m_j^\vee)$ denote the dual basis. Any element of $L \otimes V_{\omega_6}$ can be written as $\sum_i \ell_i \otimes x_i$ for some vectors $x_i \in V_{\omega_6} = V_{\omega_1}^\vee$. Its image in $L \otimes S^2 V_{\omega_1}$ is

$$\sum_{i,j,k} \ell_i \otimes Det(x_i, y_j, y_k) y_j^\vee y_k^\vee \mapsto \sum_{i,j,k} Det(x_i, y_j, y_k) \ell_i y_j^\vee y_k^\vee \in S^3 V_{\omega_1}.$$

The decomposition $V_{\omega_1} = L \oplus M$ induces a direct sum decomposition $S^3V_{\omega_1} = S^3L \oplus S^2LM \oplus LS^2M \oplus S^3M$. The component of the previous tensor on LS^2M is

$$\sum_{i,j,k} \text{Det}(x_i, m_j^\vee, m_k^\vee) \ell_i m_j m_k.$$

For it to vanish, we need that $\text{Det}(x_i, m_j^\vee, m_k^\vee) = 0$ for any i, j, k . But this implies that $x_i = 0$ for all i since for a generic L , one can check that the map

$$S^2L^\perp \hookrightarrow S^2V_{\omega_6} \rightarrow V_{\omega_6}^\vee$$

is surjective. \square

We can now complete the proof of the surjectivity of $\phi : L \otimes V_{2\omega_1} \rightarrow V_{3\omega_1}$. We know that $\dim V_{2\omega_1} = 351$ and $\dim V_{3\omega_1} = 3003$. By the two previous lemmas the kernel of ϕ has dimension 3315, and therefore the dimension of its image is $18 \times 351 - 3315 = 3003$. Hence the surjectivity. This completes the proof of the Proposition. \square

B. Now we treat $\text{Sym}^2\mathcal{E}_X(-2)$, essentially in the same way. We first observe that $\text{Sym}^2(\pi^*\mathcal{S}_X) = \text{Sym}^2\mathcal{E}_X \oplus \pi^*\mathcal{S}_X(1)$. Recall that $\text{Sym}^2(\mathcal{S}) = \mathcal{S}_2 \oplus H$, where the second factor will contribute to the homotheties in $\text{End}(\mathcal{E}_X)$. Therefore what we need to compute is the cohomology of $\pi^*(\mathcal{S}_2 \otimes H^{-1})$ restricted to X . Proceeding as before, we check that for $q < 25$,

$$H^q(\mathbb{P}(\mathcal{S}^\vee), \pi^*(\mathcal{S}_2 \otimes H^{-1}) \otimes \mathcal{O}_{\mathcal{S}}(-k)) = \delta_{q,17}V_{(k-12)\omega_6} \oplus \delta_{q,21}V_{(k-18)\omega_6}.$$

This implies that for $q < 21 - 18 = 3$, the cohomology group $H^q(X, \pi^*(\mathcal{S}_2 \otimes H^{-1})|_X)$ can be computed as the q -th cohomology group of the complex

$$(7) \quad 0 \rightarrow V_{6\omega_6} \rightarrow L \otimes V_{5\omega_6} \rightarrow \wedge^2 L \otimes V_{4\omega_6} \rightarrow \wedge^2 L \otimes V_{3\omega_6} \rightarrow \wedge^3 L,$$

where the rightmost term is in degree two. On the other hand, recall that $\text{Sym}^2\mathcal{E}_X(-2)$ is only a direct factor of $\pi^*(\mathcal{S}_2 \otimes H^{-1})|_X = \pi^*(\mathcal{S}_2)|_X(-2)$, with complementary factor $\mathcal{S}_X(-1) = \mathcal{S}_X \otimes H_X^{-1}(1)$. After a computation similar to (but easier than) the previous one, we get that for $q < 25$,

$$H^q(\mathbb{P}(\mathcal{S}^\vee), \pi^*(\mathcal{S} \otimes H^{-1}) \otimes \mathcal{O}_{\mathcal{S}}(-1-k)) = \delta_{q,17}V_{(k-12)\omega_6},$$

and we can conclude that for $q < 25 - 18 = 7$, $H^q(X, \mathcal{S}_X(-1))$ is the q -th cohomology group of the complex

$$(8) \quad 0 \rightarrow V_{6\omega_6} \rightarrow L \otimes V_{5\omega_6} \rightarrow \wedge^2 L \otimes V_{4\omega_6} \rightarrow \wedge^2 L \otimes V_{3\omega_6} \rightarrow \wedge^3 L \rightarrow \dots,$$

with the same grading as in the complex (7). In particular, $H^q(X, \mathcal{S}_X(-1))$ coincides with $H^q(X, \pi^*(\mathcal{S}_2 \otimes H^{-1})|_X)$ for $q < 3$, and therefore

$$H^q(X, \text{Sym}^2\mathcal{E}_X^\vee(-2)) = \delta_{q,0}\mathbb{C} \quad \text{if } q < 3.$$

The proof of the theorem is now complete. Indeed, we have proved that $H^0(X, \text{End}(\mathcal{E}_X)) = H^0(X, \wedge^2\mathcal{E}_X(-2)) \oplus H^0(X, \text{Sym}^2\mathcal{E}_X(-2)) = 0 \oplus \mathbb{C} = \mathbb{C}$,

so that \mathcal{E}_X is simple, and moreover that $H^1(X, \text{End}(\mathcal{E}_X)) = 0$. By Lemma 3.5 we deduce that $H^2(X, \text{End}(\mathcal{E}_X)) = 0$ and $H^3(X, \text{End}(\mathcal{E}_X)) = \mathbb{C}$, and we already know that $H^q(X, \text{End}(\mathcal{E}_X)) = 0$ for $q > 3$. \square

Remark. Observe that the term

$$H^{21}(\mathbb{P}(\mathcal{S}^\vee), \pi^*(\mathcal{S}_2 \otimes H^{-1})) \otimes \mathcal{O}_S(-18) = \mathbb{C}$$

does contribute to $H^3(X, \text{Sym}^2 \mathcal{E}_X^\vee(-2))$, in agreement with the fact that $H^3(X, \text{End}(\mathcal{E}_X))$ is one-dimensional.

Remark. It would be interesting to solve the following reconstruction problem: being given a smooth cubic sevenfold X , and a rank nine vector bundle \mathcal{E} on X , having all the properties we have just established for our bundle \mathcal{E}_X (and maybe some others), does it necessarily come from a Cartan representation of X ? And if this the case, how can we reconstruct effectively this representation? The global sections of \mathcal{E} provide what should be a copy of $J_3(\mathbb{O})$, and one would like to reconstruct the cubic determinant just from the bundle. Unfortunately we have not been able to do that for \mathcal{E}_X .

4. THE TREGUB-TAKEUCHI BIRATIONALITY FOR CUBIC SEVENFOLDS

By the results of the previous section, the general cubic sevenfold X has a finite number of Cartan representations, where a Cartan representation is the same as a presentation of X as a linear section of the Cartan cubic $\mathcal{C} \subset \mathbb{P}J_3(\mathbb{O}) = \mathbb{P}^{26}$ with a subspace \mathbb{P}^8 . In the dual space $\hat{\mathbb{P}}^{26} = \mathbb{P}J_3(\mathbb{O})^\vee$, the 18 linear forms defining $\mathbb{P}^8 \subset \mathbb{P}^{26}$ span a subspace \mathbb{P}^{17} which intersects the Cayley plane $\mathbb{O}\mathbb{P}^2 \subset \hat{\mathbb{P}}^{26}$ along a sevenfold Y , the orthogonal linear section of X with respect to its Cartan representation. In this section we prove that X and Y are birational one to each other by constructing a birationality which is an analog of the Tregub-Takeuchi birationality between the cubic threefold and its general orthogonal linear section in $G(2, 6)$.

The analogy is the following. The general Pfaffian representation of the cubic threefold X' is the same as the general representation of X' as a linear section of the Pfaffian cubic $Pf \subset \mathbb{P}J_3(\mathbb{H})$ with a subspace \mathbb{P}^4 , where the projective Jordan algebra $\mathbb{P}J_3(\mathbb{H})$ is identified with the projective space $\mathbb{P}(\wedge^2 V)$, where $V = \mathbb{C}^6$. The 10 linear forms defining the subspace $\mathbb{P}^4 \subset \mathbb{P}^{14}$ span a subspace \mathbb{P}^9 in $\hat{\mathbb{P}}^{14} = \mathbb{P}(\wedge^2 V^\vee)$ which intersects the quaternionic plane $\mathbb{H}\mathbb{P}^2 = G(2, 6) \subset \hat{\mathbb{P}}^{14}$ along a Fano threefold Y' of degree 14, the orthogonal section of X' . It is known since G. Fano that Y' and X' are birational. The two types of birationalities between X' and Y' – of Fano-Iskovskikh and of Tregub-Takeuchi are related to curves on the cubic threefold X' which are hyperplane sections of surfaces with one apparent double point (OADP), correspondingly del Pezzo surfaces of degree 5 and rational quartic scrolls, see [Tr], [Ta], [CMR], [AR]. Especially the Tregub-Takeuchi birationality, or more precisely its inverse $Y' \rightarrow X'$ starts from a point $p \in Y'$, and after a blow-up of p and a flop, ends with a contraction of a divisor to a rational normal quartic in curve Y' .

Here we describe the analog of the inverse Tregub-Takeuchi birationality between the cubic sevenfold X and its dual sevenfold Y defined by a Cartan representation of X . It is an analog at least because the construction described in the proof of Proposition 4.2 applied to the cubic threefold X' and its dual Y' as above, and after replacing $J_3(\mathbb{O})$ by $J_3(\mathbb{H})$ gives the Tregub-Takeuchi birationality as described e.g. in [Tr] and [Ta] (see the Appendix). Just as in the 3-dimensional case, the inverse birationality $Y \rightarrow X$ starts with a blow-up of a point $p \in Y$. In the seven dimensional case the birationality ends with a contraction of a divisor onto a prime Fano threefold $Z \subset X$ of degree 12, which is a hyperplane section of a fourfold with OADP. The connection with varieties with OADP inside the Pfaffian cubic fourfolds and Cartan cubic eightfolds is commented in Section 6. One geometric explanation why the rational quartic curve $C \subset X$ and the Fano threefold $Z \subset X$ of degree 12 appear as indeterminacy loci of the birationalities related to points on their dual varieties is that the projective tangent space to a point p on $\mathbb{H}\mathbb{P}^2 = G(2, 6)$ intersects on $G(2, 6)$ a cone over $\mathbb{P}^1 \times \mathbb{P}^3$ (whose general linear section of dimension is a rational normal quartic), while the projective tangent space to $\mathbb{O}\mathbb{P}^2$ at p intersects on $\mathbb{O}\mathbb{P}^2$ a cone over the spinor variety $S_{10} = OG(5, 10)$ (whose general 3-dimensional linear section is a prime Fano threefold of degree 12), see [LM1].

4.1. The orthogonal linear sections of the cubic sevenfold on the Cayley plane. In [IMa2] we encountered the Cayley plane and its linear sections as candidates for being Fano manifolds of Calabi-Yau type. Starting from a Cartan representation of a general cubic sevenfold, $X = \mathcal{C} \cap L$, we consider the orthogonal section $Y = \mathbb{O}\mathbb{P}^2 \cap L^\perp$.

Proposition 4.1. *The variety Y is a Fano manifold of Calabi-Yau type, of dimension seven and index three.*

Proof. The Cayley plane has dimension sixteen and index twelve, hence a general section of codimension nine has dimension seven and index three. The fact that it is of Calabi-Yau type follows from [IMa2, Proposition 4.5] and Proposition 3.1. \square

Remark. The situation here is very similar to what happens for cubic fourfolds. Recall that the Pfaffian cubic fourfolds are those that can be obtained as linear sections of the Pfaffian cubic in \mathbb{P}^{14} . The Pfaffian cubic can be defined as the secant variety to $G(2, 6)$ (or the projective dual variety to the dual Grassmannian $G(4, 6)$), and that $G(2, 6)$ is nothing else than the projective plane $\mathbb{H}\mathbb{P}^2$ over the quaternions.

Given such a smooth four dimensional section of the Pfaffian cubic, the orthogonal section of the dual Grassmannian is now a smooth K3 surface, and the Hodge structure of the cubic fourfold is indeed “of K3 type”. The main difference with cubic sevenfolds is that the general cubic fourfold is not Pfaffian, the Pfaffian ones form a codimension one family in the moduli space. Nevertheless, the Hodge structure of a cubic fourfold is always of K3

type, and this allows in particular to associate to them complete families of symplectic fourfolds [BD, IRa].

4.2. The Tregub-Takeuchi birationalities for the cubic sevenfold.

We will use the basic properties of the incidence geometry on the Cayley plane in order to prove that:

Proposition 4.2. *The general cubic sevenfold $X = \mathcal{C} \cap L$ and the dual section $Y = \mathbb{O}\mathbb{P}^2 \cap L^\perp$ are birationally equivalent.*

Remark. Since X has index six but Y has index three, they are not isomorphic. Their Hodge numbers coincide in middle dimension, and in fact they do coincide in general because the Cayley plane and the projective space have the same Betti (and Hodge) numbers up to degree six.

Proof. We start by fixing a general point $y_0 \in Y$. There is a corresponding quadric Q_{y_0} on $\mathbb{O}\mathbb{P}^2$.

Let $y \in Y$ be general. The corresponding quadric Q_y meets Q_{y_0} along a unique point of $\mathbb{O}\mathbb{P}^2$, say z . The tangent space T_z to the dual Cayley plane at this point is contained in the Cartan cubic, and we claim that it meets L at a unique point. Indeed, T_z is a projective space of dimension 16, and L has codimension 18. But y and y_0 belong to L^\perp and they contain T_z (recall that T_z is orthogonal to the linear span of \bar{Q}_z , and that the fact that z belongs to Q_y is equivalent to the condition that y belongs to \bar{Q}_z). So the intersection of T_z with L is in fact given by sixteen general conditions, and we end up with a unique point $\psi(y) \in X$.

Conversely, let $x \in X$ be a general point, then the entry-locus Q_x meets Q_{y_0} at a unique point of $\mathbb{O}\mathbb{P}^2$, that we again call z . In turn z defines a quadric \bar{Q}_z on the Cayley plane $\mathbb{O}\mathbb{P}^2$ that we will cut out with L^\perp . The latter has codimension nine but it contains x , and we claim that the orthogonal hyperplane to x contains \bar{Q}_z . In order to check this we have to make a little computation. We may suppose that y_0 and $PDet(x) = y_1$ are two general points on $\mathbb{O}\mathbb{P}^2$. The group E_6 acts transitively on pairs of points in the Cayley plane which are not joined by a line contained in $\mathbb{O}\mathbb{P}^2$, we can therefore suppose that

$$y_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This implies that

$$Q_{y_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad Q_{y_1} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(We have omitted the quadratic conditions expressing that the determinants of the non-trivial 2×2 blocs in the matrices above, must vanish.) Hence the

intersection point

$$z = Q_{y_0} \cap Q_{y_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad Q_z = \begin{pmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{pmatrix}.$$

Finally, recall that a point x such that $PDet(x) = y_1$ must be contained in the linear span of Q_{y_1} . If we take into account the condition that x is orthogonal to y_0 , we deduce that it must be of the form

$$x = \begin{pmatrix} 0 & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore $Q_z \subset x^\perp$, as claimed.

We conclude that the intersection of \bar{Q}_z with L^\perp is in fact given by eight general linear conditions, and we end up with two points: one must be y_0 , and we denote the other one $\phi(x)$.

It is then straightforward to check that the rational maps ψ and ϕ between X and Y are inverse one of the other. This concludes the proof. \square

Question. Note that ψ and ϕ factorize through Q_{y_0} , which implies that the general cubic sevenfold is birational to a complete intersection of bidegree $(2, d)$ in \mathbb{P}^9 . What is d ?

Analyzing in detail the structure of the birational maps ϕ and ψ seems to be rather complicated. The very first steps in this direction are the following two statements.

Proposition 4.3. *The rational map from $\mathbb{O}\mathbb{P}^2$ to $\bar{\mathbb{O}}\mathbb{P}^2$, mapping a general point y to $Q_y \cap Q_{y_0}$, coincides with the double projection p from y_0 .*

Proof. We may suppose that y_0 is the point that we have chosen in the proof of Proposition 4.2. Then we can use the local parametrization of $\mathbb{O}\mathbb{P}^2$ around y_0 given by 2,

$$y = \begin{pmatrix} 1 & w & \bar{v} \\ \bar{w} & |w|^2 & \bar{w}\bar{v} \\ v & vw & |v|^2 \end{pmatrix} \mapsto p(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |w|^2 & \bar{w}\bar{v} \\ 0 & vw & |v|^2 \end{pmatrix}.$$

The point $p(y)$ certainly belongs to Q_{y_0} . There remains to prove that it also belongs to Q_y , hence that it is orthogonal to the tangent space to the Cayley plane at y . This is a straightforward explicit computation. \square

Proposition 4.4. *Suppose X, Y and y_0 to be general. The indeterminacy locus of ψ is a cone over a smooth canonical curve of genus seven. The indeterminacy locus of ϕ is a smooth prime Fano threefold of degree twelve.*

Proof. There are two possible accidents that could prevent $\psi(y)$ from being defined. The first one is that Q_y meets Q_{y_0} in more than one point. The second one is that $Q_y \cap Q_{y_0}$ is a single point z but such that the tangent space T_z meets L in more than one point. A dimension count shows that

generically, the second situation cannot happen. The first one happens when y belongs to the cone C_{y_0} spanned by the lines through y_0 in the Cayley plane. By [LM2], this is a cone over the spinor variety $S_{10} = OG(5, 10)$, inside a sixteen dimensional projectivized half-spin representation. Cutting this cone by L^\perp , we get a cone over a codimension nine generic linear section of S_{10} , which is a canonical curve of genus seven.

Let now x be a point in X . To prevent $\phi(x)$ from being defined, the only possible accident is that Q_x meets Q_{y_0} in more than one point. If we let $y = PDet(x)$, so that $Q_x = Q_y$, this means that y belongs to the cone C_{y_0} . Let us determine $PDet^{-1}(C_{y_0})$. We may suppose that

$$y_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{hence } C_{y_0} = \left\{ \begin{pmatrix} 1 & a & \bar{b} \\ \bar{a} & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, |a|^2 = |b|^2 = ab = 0 \right\},$$

as we can see from the explicit parametrization of a neighbourhood of y_0 given in (2). This implies in particular that the conditions $|a|^2 = |b|^2 = ab = 0$ on the pair $(a, b) \in \mathbb{O} \oplus \mathbb{O}$ define a copy of the spinor variety S_{10} in a half-spin representation. Now, (3) shows that x belongs to $PDet^{-1}(C_{y_0})$ if and only if $Det(x) = 0$ and

$$|v|^2 = rt, \quad |w|^2 = rs, \quad vw = r\bar{u}.$$

If $r \neq 0$, these conditions imply that $wu = s\bar{v}$ and $uv = t\bar{w}$, so that in fact $PDet(x) = y_0$. Generically, this cannot happen in our situation since this would imply that x belongs to the linear span of Q_{y_0} , which is too small to meet L . So we must let $r = 0$, in which case we are left with the conditions

$$|v|^2 = 0, \quad |w|^2 = 0, \quad vw = 0.$$

As we have just seen, this defines a copy of S_{10} . Since the parameters s, t, u remain free, we conclude that x must belong to the join of S_{10} with a \mathbb{P}^9 . Cutting this join with L amounts to cut S_{10} along a generic three dimensional linear section, which is a prime Fano threefold of degree 12. \square

5. DERIVED CATEGORIES

5.1. The Calabi-Yau subcategory. Let again X be a smooth cubic sevenfold. Since it is Fano of index 6, the collection $\mathcal{O}_X, \dots, \mathcal{O}_X(5)$ is exceptional. Denote by \mathcal{A}_X the full subcategory of $D^b(X)$, the derived category of coherent sheaves on X , defined as the left semi-orthogonal to this exceptional collection. The following statement is a special case of [Ku1, Corollary 4.3].

Proposition 5.1. *\mathcal{A}_X is a three-dimensional Calabi-Yau category.*

The terminology *non-commutative Calabi-Yau* is sometimes used. Kuznetsov has given in [Ku2] interesting examples of non-commutative K3's, which are deformations of commutative ones (i.e., of derived categories of coherent sheaves on genuine K3 surfaces). Here the situation is a bit different, since our non-commutative Calabi-Yau cannot be the derived category of any

Calabi-Yau threefold Z , or even a deformation of such a derived category. Indeed, computing the Hochschild cohomology of \mathcal{A}_X with the help of [Ku3, Corollary 7.5], we would deduce from the HKR isomorphism theorem that the Hodge numbers of Z should be such that

$$\sum_p h^{p,p}(Z) = \sum_p h^{p,p}(X) - 6 = 2 !$$

5.2. Spherical twists. Suppose that X is given with a Cartan representation $X = \mathcal{C} \cap L$. Recall that this induces a vector bundle \mathcal{E}_X of rank nine on X , such that $\mathcal{E}_X(-1)$ is self-dual.

Proposition 5.2. $\mathcal{E}_X(-1)$ and $\mathcal{E}_X(-2)$ are two objects of \mathcal{A}_X .

Proof. Immediate consequence of Proposition 3.3 and Serre duality. \square

A nice consequence is that Theorem 3.4 can now be interpreted as the assertion that $\mathcal{E}_X(-1)$ and $\mathcal{E}_X(-2)$ are *spherical objects* in \mathcal{A}_X , in the sense of Seidel and Thomas [ST]. In particular, with their notations and terminology:

Corollary 5.3. *The spherical twists $T_{\mathcal{E}_X(-1)}$ and $T_{\mathcal{E}_X(-2)}$ are auto-equivalences of \mathcal{A}_X .*

These auto-equivalences have infinite order. This can be seen at the level of K-theory: the K-theory of \mathcal{A}_X is free of rank two, hence has a basis given by the classes of $\mathcal{E}_X(-1)$ and $\mathcal{E}_X(-2)$. The matrices $\Phi_{\mathcal{E}_X(-1)}$ and $\Phi_{\mathcal{E}_X(-2)}$ of the induced automorphisms of $K(\mathcal{A}_X)$, expressed in these basis, are easy to compute. Since $P_{\text{End}(\mathcal{E}_X)}(-1) = 9$, they are given by

$$\Phi_{\mathcal{E}_X(-1)} = \begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix}, \quad \Phi_{\mathcal{E}_X(-2)} = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}.$$

Remark. The category \mathcal{A}_X should have a very interesting group of auto-equivalences, all the bigger than the number of Cartan representations of the cubic is large. This is of course related to the fact that \mathcal{A}_X is Calabi-Yau, contrary to $D^b(X)$ which, X being Fano, has no interesting auto-equivalence.

5.3. Homological projective duality. We expect for cubic sevenfolds certain phenomena that would illustrate the principles of *homological projective duality* [Ku1]. Starting once again from a Cartan representation $X = \mathcal{C} \cap L$ of a smooth cubic sevenfold, we would like to compare the derived category of X with that of the orthogonal section $Y = \mathbb{O}\mathbb{P}^2 \cap L^\perp$, that we suppose to be smooth as well.

In order to describe $D^b(Y)$ we will start from the Cayley plane. Recall that on $\mathbb{O}\mathbb{P}^2$ we have denoted by \mathcal{S} the rank ten spin bundle. It is endowed with a non degenerate quadratic form

$$\text{Sym}^2 \mathcal{S} \rightarrow \mathcal{O}_{\mathbb{O}\mathbb{P}^2}(1),$$

(which define the quadrics on the dual Cayley plane parametrized by $\mathbb{O}\mathbb{P}^2$), whose kernel is an irreducible homogeneous bundle that we denoted \mathcal{S}_2 .

Rephrasing the results of Lemmas 1, 2, 3, 4, 6 in [Ma] (which also imply that \mathcal{S} and \mathcal{S}_2 are exceptional bundles), we can state the following result:

Theorem 5.4. *In the derived category of coherent sheaves on $\mathbb{O}\mathbb{P}^2$, let*

$$\begin{aligned}\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 &= \langle \mathcal{S}_2^\vee, \mathcal{S}^\vee, \mathcal{O}_{\mathbb{O}\mathbb{P}^2} \rangle, \\ \mathcal{A}_3 = \cdots = \mathcal{A}_{11} &= \langle \mathcal{S}^\vee, \mathcal{O}_{\mathbb{O}\mathbb{P}^2} \rangle.\end{aligned}$$

Then $\mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{11}(11)$ is a semi-orthogonal collection in $\mathcal{D}^b(\mathbb{O}\mathbb{P}^2)$.

This semi-orthogonal collection is certainly complete (otherwise said, it should define a *Lefschetz decomposition* of $\mathcal{D}^b(\mathbb{O}\mathbb{P}^2)$), but we have not been able to prove it.

If we denote by \mathcal{S}_Y the restriction of \mathcal{S} to the linear section Y , we deduce:

Proposition 5.5. *The sequence $\mathcal{O}_Y, \mathcal{S}_Y, \mathcal{O}_Y(1), \mathcal{S}_Y(1), \mathcal{O}_Y(2), \mathcal{S}_Y(2)$ is exceptional.*

Proof. Use the Koszul complex describing \mathcal{O}_Y and apply Bott's theorem on the Cayley plane. More precisely, use Lemma 1 and Lemma 3 in [Ma]. \square

It is then tempting to consider the right orthogonal \mathcal{A}_Y in $\mathcal{D}^b(Y)$ to this exceptional collection and ask:

- Is \mathcal{A}_Y a three-dimensional Calabi-Yau category?
- Is \mathcal{A}_Y equivalent to \mathcal{A}_X ?

Note that if X has $d > 1$ Cartan representations, the corresponding orthogonal sections Y_1, \dots, Y_d would define non-commutative Fourier partners to the non-commutative Calabi-Yau threefold \mathcal{A}_X .

Remark. It follows from the results of [Ma, Lemma 4] that $\mathcal{S}_{2,Y}(-1)$ belongs to \mathcal{A}_Y . Moreover, completing the computations made for proving this Lemma, one gets the following statement: if $0 \leq i \leq 13$,

$$h^q(\mathbb{O}\mathbb{P}^2, \text{End}(\mathcal{S}_2)(-i)) = \delta_{i,0}\delta_{q,0} + \delta_{i,3}\delta_{q,4} + \delta_{i,6}\delta_{q,8} + \delta_{i,9}\delta_{q,12}.$$

One can easily deduce that the values of $h^q(Y, \text{End}(\mathcal{S}_{2,Y}))$ are equal to 1, 84, 84, 1, 0, 0, 0, 0. This is remarkably coherent with the expected Calabi-Yau property of \mathcal{A}_Y .

6. THE RATIONALITY OF THE CARTAN CUBIC EIGHTFOLDS

Contrary to cubic sevenfolds, an eight dimensional linear section of the Cartan cubic cannot be a general cubic eightfold. Indeed, the dimension of this family of cubics is $\dim G(10, 27) - \dim E_6 = 170 - 78 = 92$, while the moduli space of cubic eightfolds has dimension $\dim |\mathcal{O}_{\mathbb{P}^9}(3)| - \dim PGL_{10} = 219 - 99 = 120$.

In the spirit of the study in Section 3, the Cartan cubic eightfold is the octonionic analog of the Pfaffian cubic fourfold, the last corresponding to the quaternions. The most important property of the Pfaffian cubic fourfold, known since G. Fano is that it is rational, see [Fa]. In this section we prove that the Cartan cubic eightfold is also rational.

By definition, an n -dimensional variety $V \subset \mathbb{P}^{2n+1}$ is a *variety with one apparent double point* (VOADP) if through the general point $x \in \mathbb{P}^{2n+1}$ passes a unique secant line to V . For example the union of two non-intersecting n -dimensional linear spaces in \mathbb{P}^{2n+1} is a VOADP.

A cubic $2n$ -fold $X \subset \mathbb{P}^{2n+1}$ containing a VOADP V is rational. In fact, since through the general point of X passes a unique secant line to V , taking the intersection of this secant line with a general hyperplane H yields a birationality between X and H . For example, a smooth cubic surface S is rational because it contains space cubic curves, or pairs of non intersecting lines, which are examples of VOADP.

By [CG] any smooth cubic threefold is nonrational, but it is unknown whether the general cubic hypersurfaces of dimension ≥ 4 are rational or not. The singular cubic hypersurfaces in any dimension are evidently rational. The general Pfaffian cubic 4-fold X is smooth and rational since X contains quintic del Pezzo surfaces and rational quartic scrolls, which are other examples of VOADP (see e.g. [AR]).

Other interesting series of examples of smooth rational cubic fourfolds have been found by Hassett [Ha], but as far as we know, until now the only examples of smooth rational cubic hypersurfaces of dimension ≥ 5 are of even dimension $2n$ and contain a VOADP. One delicate part in these examples is the proof that a general cubic $2n$ -fold through a particular kind of VOADP is indeed smooth. Moreover the examples of n -dimensional VOADP are not too many (see [CMR]), which seems to be one more reason why there are very few known examples of rational smooth cubic $2n$ -folds of dimension bigger than four (see e.g. [AR] for a modern treatment of Fano's results on Pfaffian cubic fourfolds and for examples of smooth cubic sixfolds containing a 3-dimensional VOADP).

We shall see below that the general Cartan cubic 8-fold X contains a 4-dimensional VOADP Y , which implies that X is rational. This special VOADP Y is a transversal section of the spinor variety S_{10} (Ex. 2.8 in [CMR]). By the comments to Section 4, the rationality construction coming from the degree 12 VOADP Y on the Cartan cubic 8-fold X is the analog of the rationality construction coming from the quartic rational normal scroll on the Pfaffian cubic fourfold, as described in [Fa] and Thm. 4.3 in [AR].

Theorem 6.1. *The general Cartan cubic eightfold is rational.*

Proof. The key observation is the following.

Lemma 6.2. *Any general $\mathbb{P}^{15} \subset \mathbb{P}(H)$ is the linear span of a copy of the spinor variety S_{10} , contained in the Cayley cubic.*

As we have already mentioned, this implies the theorem as follows. Let X be the cubic eightfold defined as the linear section of the Cayley cubic by a general $\mathbb{P}_X^9 \subset \mathbb{P}(H)$. By the Lemma, there exists a copy S_{10} of the spinor variety, contained in the Cayley cubic, whose linear span contains \mathbb{P}_X^9 as a

general linear subspace. Then the intersection $Y = S_{10} \cap \mathbb{P}_X^9$ is smooth of dimension four, and is a VOADP.

Now, as we already mentioned, any cubic X of even dimension $2m$ containing a m -dimensional VOADP Y is rational. Indeed, consider a general hyperplane H in \mathbb{P}^{2m+1} and x a general point of X . There is a unique secant line ℓ_x to Y passing through x and we can let $\pi(x) = \ell_x \cap H$. Then π is a birational isomorphism whose inverse is described in much the same way: if h is a general point of H , there is a unique secant line d_h to Y passing through h ; this line d_h cuts X in two points of Y , and a third point which is $\pi^{-1}(h)$. \square

Proof of the Lemma. Recall that the spinor variety S_{10} parametrizes the space of lines in the Cayley plane passing through a given point. We have deduced that

$$S_{10} \simeq \mathbb{P}\{u + v \in \mathbb{O} \oplus \mathbb{O}, |u|^2 = |v|^2 = u\bar{v} = 0\} \subset \mathbb{P}(\mathbb{O} \oplus \mathbb{O}) \simeq \mathbb{P}^{15}.$$

Let ρ, τ be linear forms on $\mathbb{O} \oplus \mathbb{O}$, and let α, β be endomorphisms of \mathbb{O} . Then, letting $y = \alpha(u) + \tau(v)$, the set of matrices of the form

$$\begin{pmatrix} \rho(u, v) & u & \bar{y} \\ \bar{u} & 0 & \bar{v} \\ y & v & \sigma(u, v) \end{pmatrix}, \quad |u|^2 = |v|^2 = u\bar{v} = 0,$$

defines a copy of S_{10} contained in the Cayley cubic. The corresponding family F_0 of linear spaces is a smooth subset of $G(16, H)$; to be precise, consider the decomposition $H = L_0 \oplus L_1$, where

$$L_0 = \left\{ \begin{pmatrix} 0 & u & 0 \\ \bar{u} & 0 & \bar{v} \\ 0 & v & 0 \end{pmatrix} \right\}, \quad L_1 = \left\{ \begin{pmatrix} r & 0 & y \\ 0 & s & 0 \\ \bar{y} & 0 & t \end{pmatrix} \right\}.$$

Then the set of points in $G(16, H)$ defined by subspaces transverse to y_1 form an affine neighbourhood of L_0 isomorphic to $\text{Hom}(L_0, L_1)$, inside which F_0 is the linear subspace $\text{Hom}(L_0, L'_1)$, if $L'_1 \subset L_1$ denotes the hyperplane defined by $s = 0$. Consider the map

$$\begin{aligned} \psi : E_6 \times F_0 &\rightarrow G(16, H) \\ (g, L) &\mapsto g(L). \end{aligned}$$

The image of ψ consists in linear spaces spanned by a copy of the spinor variety contained in the Cayley cubic, since this is the case for F_0 and this property is preserved by the action of E_6 . We claim that ψ is dominant, which implies the Lemma. More precisely, we will check that the differential of ψ at $(1, L_0)$ is surjective. To see this first observe that $\text{Im}(\psi_*) \subset T_{L_0}G(16, H) \simeq \text{Hom}(L_0, L_1)$ contains $T_{L_0}F_0 \simeq \text{Hom}(L_0, L'_1)$. It also contains the image $\psi_*(\mathfrak{e}_6)$ of the Lie algebra \mathfrak{e}_6 , obtained by differentiating the restriction of ψ to $E_6 \times \{L_0\}$. So we just need to prove that $\text{Hom}(L_0, L'_1)$ and $\psi_*(\mathfrak{e}_6)$ do span $\text{Hom}(L_0, L_1)$, which is equivalent to the fact that the quotient map $\mathfrak{e}_6 \rightarrow \text{Hom}(L_0, L_1/L'_1) \simeq L_0^\vee$ is surjective.

To check this, remember that $\mathfrak{e}_6 \simeq \mathfrak{f}_4 \oplus H_0$, where $H_0 \subset H$ is the hyperplane of traceless matrices, action on H by Jordan multiplication:

$$M \in H_0 \mapsto T_M \in \text{End}(H), \quad T_M(X) = \frac{1}{2}(XM + MX).$$

In particular, for a traceless matrix

$$M = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix},$$

a short computation yields

$$T_M(L_0) = \left\{ \begin{pmatrix} * & * & * \\ * & \langle a, u \rangle + \langle c, v \rangle & * \\ * & * & * \end{pmatrix}, \quad u, v \in \mathbb{O} \right\}.$$

The central coefficient defines the restriction to $H_0 \subset \mathfrak{e}_6$ of the map to L_0^\vee we are interested in. It is obviously surjective, hence we are done. \square

7. APPENDIX

We have shown in Proposition 4.2 that the general cubic sevenfold, considered as a linear section of the Cartan cubic, is birational to the orthogonal section of the Cayley plane. In this appendix, we show that the same construction, when octonions are replaced by quaternions, allows to recover the Tregub-Takeuchi birationality between a Pfaffian cubic threefold and the orthogonal section of the Grassmannian $G(2, 6)$.

First we recall the Tregub-Takeuchi birationality, see [Ta]. Let $Y = Y_{14}$ be a smooth prime Fano threefold of degree 14, and let $y_0 \in Y$ be a point through which do not pass any of the one-dimensional family of lines on Y . The linear system $|2 - 5y_0|$ defines a birational isomorphism $\psi_T : Y \rightarrow X$ with a cubic threefold X , the *Tregub-Takeuchi* birationality. It can be decomposed as

$$\begin{array}{ccccc} & & Y' & \dashrightarrow & Y^+ \\ & \swarrow \sigma & & \searrow & \\ Y & & & & X \\ & \swarrow p & & \searrow & \\ & & \bar{Y} & & \end{array}$$

where $p : Y \dashrightarrow \bar{Y}$ is the double projection through y_0 , $\sigma : Y' \rightarrow Y$ is the blow-up of y_0 , $Y' \dashrightarrow Y^+$ is a flop over \bar{Y} , and $Y^+ \rightarrow X$ is the contraction of an irreducible divisor N^+ of Y^+ onto a rational normal quartic curve Γ on X (the blow-up of Γ). There is on Y a \mathbb{P}^1 -family of 1-cycles whose general member is a rational quintic curve C_t with a double point at y_0 , and the proper image on Y^+ of the general C_t is the general fiber of the contraction $N^+ \rightarrow \Gamma$.

We have already mentioned the fact that $G(2, 6) = \mathbb{H}\mathbb{P}^2$, with its “plane projective geometry” defined by the family of copies of $G(2, 4) \simeq \mathbb{Q}^4$ parametrized by the dual Grassmannian $G(4, 6)$. As Mukai has shown, the general

prime Fano threefold Y can be realized as a linear section $Y = G(2, 6) \cap L^\perp$, with $L^\perp = \mathbb{P}(V_{10}) \simeq \mathbb{P}^9$. The intersection of the Pfaffian cubic in the dual projective space, with $L \simeq \mathbb{P}^4$, is a smooth cubic threefold, and it follows from the results of [IMr] that it must coincide with the target X of the Tregub-Takeuchi birationality.

On the other hand, we have the birational map $\psi : Y \dashrightarrow X$ similar to the one we constructed in the similar situation over the octonions. Recall the construction: take a general point y on Y , then $Q_y \cap Q_{y_0}$ is a single point z , and the tangent space T_z to the Grassmannian $G(4, 6)$ at z meets L in a unique point $\psi(y)$.

Proposition 7.1. *The birational maps ψ and ψ_T are the same.*

Proof. First observe that y and y_0 define two planes P and P_0 in \mathbb{C}^6 , and the point $z = Q_y \cap Q_{y_0}$ in $G(4, 6)$ is nothing else than $P + P_0$ when these two planes are transverse. Exactly as in the octonionic case, the rational map $y \mapsto z$ coincides with the double projection from y_0 .

Now we study the \mathbb{P}^1 -family of quintic rational curves $C_t \subset Y$, $t \in \mathbb{P}^1$, passing doubly through y_0 . Let $\mathbb{P}_{y_0}^8 = \mathbb{P}(P_0 \wedge \mathbb{C}^6)$ be the projective tangent space to $G(2, 6)$ at y_0 , and let $\mathbb{P}_{y_0}^3 = \mathbb{P}(U_4) = \mathbb{P}((P_0 \wedge \mathbb{C}^6) \cap V_{10})$ be the projective tangent space to Y at y_0 . These quintics are obtained as follows. For any hyperplane $U_5 \subset \mathbb{C}^6$, the intersection $C = Y \cap G(2, U_5)$ is a quintic curve of arithmetic genus $p_a(C) = 1$, since $G(2, U_5)$ has degree five and index five. In this \mathbb{P}^5 -family, those having a double point at y_0 are the rational curves of our \mathbb{P}^1 -family C_t . They correspond to those 5-spaces U_t such that the projective tangent spaces to $G(2, U_t)$ and to Y at y_0 intersect each other along a plane (the tangent space to C_t at the node y_0), that is

$$\dim(U_4 \cap \wedge^2 U_t) = 3.$$

Lemma 7.2. *Let C_t be a rational quintic on Y passing doubly through y_0 . Then the birationality $\psi : Y \dashrightarrow X$ contracts C_t .*

Proof. Since the quintic curve C_t has a double point at y_0 , the double projection \bar{p} sends C_t to a line $\ell_t \subset \bar{Y}$. A point $y_s \in C_t$, $y_s \neq y_0$ is mapped by \bar{p} to a point $z_s = z(y_s, y_0)$ on the line $\ell_t \subset \bar{Y} \subset Q_{y_0} \subset G(4, 6)$. We will then obtain $\psi(y_s)$ as the intersection point of the tangent space T_{z_s} to $G(4, 6)$ at z_s , with $L \simeq \mathbb{P}^4$, and we must check that this point does not depend on s . For this we observe that

$$\bigcap_{z \in \ell_t} T_z := T_{\ell_t}$$

is five dimensional. Moreover, we claim that T_{ℓ_t} is orthogonal to the linear span of C_t . This will imply the claim, since $\langle C_t \rangle^\perp$ is a \mathbb{P}^9 inside which $L \simeq \mathbb{P}^4$ and $T_{\ell_t} \simeq \mathbb{P}^5$ will meet, generically, at a unique point x_t . But then $\{\psi(y_s)\} = T_{z_s} \cap L \subset T_{\ell_t} \cap L = \{x_t\}$, hence ψ contracts C_t to the point x_t .

There remains to prove the previous claim. It is enough to check that any $y_s \in C_t$ is orthogonal to T_{z_s} . Recall that if y_s represents a plane $P_s \subset \mathbb{C}^6$, transverse to P_0 , then z_s represents the four-space $P_s + P_0$. But then $\hat{T}_{z_s} =$

$\wedge^3(P_s + P_0) \wedge \mathbb{C}^6 \subset P_s \wedge (\wedge^3 \mathbb{C}^6)$, and therefore $\wedge^2 P_s \wedge \hat{T}_{z_s} = 0$, which is the required orthogonality condition. \square

We can finally conclude the proof of the proposition. Our two birational maps ψ and ψ_T induce two birational morphisms ψ^+ and ψ_T^+ from Y^+ to X . These two morphisms contract the divisor N^+ , and the fibers of their restrictions to N^+ are the same. Since the relative Picard number is one, ψ^+ and ψ_T^+ cannot contract any other divisor. Then $\psi^+ \circ (\psi_T^+)^{-1}$ is an isomorphism between $X - \psi_T^+(N^+)$ and $X - \psi^+(N^+)$, and extends continuously to X . Hence it must extend to an automorphism of X . \square

REFERENCES

- [AC] Albano A., Collino A., *On the Griffiths group of the cubic sevenfold*, Math. Ann. **299** (1994), no. 4, 715–726.
- [AR] Alzati A., Russo F., *Some extremal contractions between smooth varieties arising from projective geometry*, Proc. London Math. Soc. **89** (2004), 2553.
- [Ba] Baez J. *The octonions*, Bull. Amer. Math. Soc. **39** (2002), 145–205.
- [BBVW] Becker K., Becker M., Vafa C., Walcher J., *Moduli stabilization in non-geometric backgrounds*, Nucl. Phys. B **770** (2007), 1–46
- [Be] Beauville A., *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64.
- [BD] Beauville A., Donagi R., *La variété des droites d’une hypersurface cubique de dimension 4*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706.
- [BK] Buckley A., Kosir T., *Determinantal representations of smooth cubic surfaces*, Geom. Dedicata **125** (2007), 115–140.
- [BC] Borisov L., Caldararu A., *The Pfaffian-Grassmannian derived equivalence*, J. Algebraic Geom. **18** (2009), no. 2, 201–222.
- [Bou] Bourbaki N., Groupes et algèbres de Lie, Hermann, Paris 1968.
- [CDP] Candelas P., Derrick E., Parkes L., *Generalized Calabi-Yau Manifolds and the Mirror of a Rigid Manifold*, Nucl.Phys. B **407** (1993), 115–154.
- [CG] Clemens C. H., Griffiths Ph. A., *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356.
- [Cl] Clebsch A., *Die Geometrie auf den Flächen dritter Ordnung*, J. reine angew. Math **65** (1866), 359–380.
- [CMR] Ciliberto C., Mella M., Russo F., *Varieties with one apparent double point*, J. Algebraic Geom. **13** (2004), 475512.
- [Di] Dixon A.C., *Note on the reduction of a ternary quantic to a symmetric determinant*, Proc. Cambridge Phil. Soc. **11** (1902), 350–351.
- [Dr] Druel S., *Espace des modules des faisceaux semi-stables de rang 2 et de classes de Chern $c_1 = 0$, $c_2 = 2$ et $c_3 = 0$ sur une hypersurface cubique lisse de P^4* , Int. Math. Res. Not. **19** (2000), 985–1004.
- [Fa] Fano G., *Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4° ordine*, Comm. Math. Helv. **15** (1942), 71–80.
- [Fr] Freudenthal H., *Lie groups in the foundations of geometry*, Advances in Math. **1** (1964), 145–190.
- [Gu] Gurevich G.B., *Foundations of the theory of algebraic invariants*, Noordhoff 1964.
- [Ha] Hassett B., *Some rational cubic fourfolds*, J. Algebraic Geom. **8** (1999), 103–114
- [He] Hesse L., *Über die Elimination der Variablen aus drei algebraischen Gleichungen von zweitem Grad mit zwei Variablen*, J. reine angew. Math. **28** (1844), 68–96.
- [IMr] Iliev A., Markushevich D., *The Abel-Jacobi Map for a cubic threefold and periods of Fano threefolds of degree 14*, Documenta Math. **5** (2000), 23–47.

- [IMa1] Iliev A., Manivel L., *The Chow ring of the Cayley plane*, Compos. Math. **141** (2005), 146–160.
- [IMa2] Iliev A., Manivel L., *Fano manifolds of Calabi-Yau type*, preprint 2011.
- [IRa] Iliev A., Ranestad K., *K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, Trans. Amer. Math. Soc. **353** (2001), 1455–1468.
- [KS] Kimura T., Sato M. *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.
- [Ku1] Kuznetsov A., *Derived categories of cubic and V_{14} threefolds*, Proc. Steklov Inst. Math. **246** (2004), 171–194.
- [Ku2] Kuznetsov A., *Derived categories of cubic fourfolds*, in Cohomological and geometric approaches to rationality problems, 219–243, Progr. Math. **282**, Birkhäuser 2010.
- [Ku3] Kuznetsov A., *Hochschild homology and semiorthogonal decompositions*, arXiv:0904.4330.
- [KMM] Kuznetsov A., Markushevich D., *Symplectic structures on moduli spaces of sheaves via the Atiyah class*, J. Geom. Phys. **59** (2009), 843–860.
- [KMM] Kuznetsov A., Manivel L., Markushevich D., *Abel-Jacobi maps for hypersurfaces and noncommutative Calabi-Yau's*, Commun. Contemp. Math. **12** (2010), no. 3, 373–416.
- [LM1] Landsberg, J. M., Manivel, L., *The projective geometry of Freudenthal's magic square*, J. Algebra **239** (2001), 447–512.
- [LM2] Landsberg, J. M., Manivel, L., *On the projective geometry of rational homogeneous varieties*, Comment. Math. Helv. **78** (2003), 65100.
- [LiE] Cohen A.M., van Leeuwen M.A., Lisser B., LiE, a package for Lie group computations, available at <http://young.sp2mi.univ-poitiers.fr/marc/LiE/>
- [Ma] Manivel, L., *On the derived category of the Cayley plane*, arXiv:0907.2784, to appear in Journal of Algebra.
- [MT] Markushevich D., Tikhomirov A., *The Abel-Jacobi map of a moduli component of vector bundles on the cubic threefold*, J. Algebraic Geom. **10** (2001), 37–62.
- [Sch] Schimrnigk R., *Mirror Symmetry and String Vacua from a Special Class of Fano Varieties*, Int. J. Mod. Phys. A **11** (1996), 3049–3096.
- [ST] Seidel P., Thomas R., *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108.
- [Ta] Takeuchi K., *Some birational maps of Fano 3-folds*, Compositio Math. **71** (1989), 265–283.
- [Ti] Tits J., *Les groupes de Lie exceptionnels et leur interprétation gomtrique*, Bull. Soc. Math. Belg. **8** (1956), 4881.
- [Tr] Tregub S., *Construction of a birational isomorphism of a cubic threefold and Fano variety of the first kind with $g = 8$, associated with a normal rational curve of degree 4*, Moscow Univ. Math. Bull. **40** (1985), 78–80.
- [Z] Zak F.L., *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs **127**, AMS 1993.

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