Weak Insider Trading and Behavioral Finance

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Abstract

In this paper, we study the optimal portfolio selection problem for weakly informed traders in the sense of Baudoin [1]. Apart from expected utility maximizers, we consider investors with other preference paradigms. In particular, we consider agents following cumulative prospect theory as developed by Tversky and Kahneman [12] as well as Yaari's dual theory of choice [13]. We solve the corresponding optimization problems, in both non-informed and informed case, i.e. when the agent has an additional weak information. Finally, comparison results among investors with different preferences and information sets are given, together with explicit examples. In particular, the insider's gain, i.e. the difference between the optimal values of an informed and a non informed investor, is explicitly computed.

Keywords: weak information, insider trading, behavioral finance, loss aversion, probability distorsion, minimal probability measure, Yaari's dual theory of choice.
AMS classification (2000): 91B28, 60G44
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1 Introduction

In a paper by Baudoin [1] the notions of weak information and weakly informed agent are introduced, where the latter is an agent having an additional information about the law (under the objective probability measure) of a functional of the price process. In contrast to the well-known strong information approach (initiated in [10]), we follow Baudoin's approach and assume that there is an extra-informed investor acting in the market, who knows the law (under the historical probability $\mathbb{P}$) of a functional Y related to the asset prices. In this approach the historical probability $\mathbb{P}$ is assumed to be unknown to every agent, whereas everyone knows the equivalent martingale measure $\mathbb{Q}$, which is assumed to be unique, i.e. the market is complete (see Assumption 2.1). Therefore, knowing the $\mathbb{P}$-law of Y translates to an informational advantage. The assumption that nobody in the market observe the prices under $\mathbb{P}$ is justified by the reasonable fact that the model for the prices can be calibrated on observed data under $\mathbb{Q}$, while all the agents ignore the *effective* drifts in the price dynamics (see Remark 1 in [2]).

In [1], the author studies a portfolio optimization problem for a non-informed agent and an insider respectively. He is then able to characterize the optimal terminal wealth and the corresponding optimal value. Moreover, he finds an explicit formula for a particular choice of the utility function. It is important to note that only Expected Utility maximizers (EU or classical, henceforth) are considered in [1]. A natural question is : What happens if one considers different preference paradigms than EU's? More specifically, we think of an investor whose goal is not necessarily to maximize the expected utility from terminal wealth. In the utility maximization literature, the EU case developed by Von Neumann and Morgenstern in the early 30's is the most treated thanks to its relative simplicity and the possibility of using a dual theory allowing to solve a wide range of problems. However, it is empirically observed that real world people systematically violate the hypotheses standing behind EU (this lead to a number of so-called paradoxes and puzzles).

In this paper, we consider two alternative models.

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- Cumulative Prospect Theory (CPT thereafter): this paradigm is fully described in [12] and it is a further development of the original Prospect Theory by Kahneman and Tversky (see [9]). Briefly, according to CPT an economic agent evaluates her payoff with respect to a reference level B: if the payoff is greater than B, then it is considered as a gain. On the other side, a payoff lower than B becomes a loss for a CPT agent, and a loss hurts more than an equivalent gain (loss aversion). This type of investor does not use a utility function. More precisely, she has two *value* functions, a concave one for the gains and a convex one for the losses. Hence, the overall form of her "utility" function is so-called S-shaped and she is risk-averse w.r.t. gains while risk-lover w.r.t. losses. Finally, laboratory evidence shows that people tend to overweight relatively large gains and losses of small probabilities. This feature is captured via two *reversed S-shaped* functions (one for the gains and one for the losses) describing probability distortions. Loosely speaking, the shape of such a weighting function looks like a reversed S, i.e. it is monotone increasing, greater than the identity for small probabilities and lower than the identity for probabilities near 1. A general mathematical treatment in continuous time for CPT can be found in [6], where it is necessary to use Choquet capacities instead of classical expectations and to split the objective function into two parts, one for the gains and one for losses.

- Yaari's Dual Theory of Choice: in 1987, Yaari proposed in [13] a different set of axioms than Von Neumann and Morgenstern's. The result was a *dual* representation of the expected utility criterion, where in the preference value functional the distortion applies to decumulative probabilities instead of payoffs (recall that a utility function $u(\cdot)$ can be viewed as a distortion on payoffs). A mathematical formulation of Yaari's model in continuous time can be found in [4], where $w(\cdot)$ is used as a probability distortion function. In [13] it has been shown that the risk-aversion is characterized by a convex $w(\cdot)$, i.e. by an overweighting of relatively small payoffs and underweighting of relatively large payoffs, whereas the opposite case of a risk-loving agent is described by a concave $w(\cdot)$.

For any of the previous two paradigms, we will solve the optimization problem for a non-informed investor and for an insider. We stress that, in this paper, agents will always be small traders, in the sense that their investment choices do not affect the asset prices. Our study is strongly motivated by the contributions in [4] and [6]. We will use the same mathematical framework and keep their notation to get a more transparent comparison with their results.

An important issue in this family of non-classical problem is well-posedness. Indeed, it is shown in [6] that ill-posed problems, i.e. having infinite optimal value, can quite easily arise if one does not make the right assumptions on the *value* functions and/or the probability distortions. We will give sufficient conditions for well-posedness during our analysis.

At last, we recall that the existing literature lacks of explicit examples and explicit computations of the optimal value for both CPT and Yaari's models. This is why we focus on examples which, to the best of our knowledge, are new.

The paper is organized as follows. In Section 2, we recall the weak information setting as developed in [1] and in Section 2.1 we consider the maximization problems of an EU agent, whose results are already proved in [1]. Then, Section 3 deals with the problem in the CPT case and Section 4 is devoted to comparison results between differently informed CPT agents. Section 5 concerns a Yaari-type investor and Section 6 concludes. Some proofs are presented in the Appendix.

2 The weak information approach

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ be an atomless probability space, where $\mathbb{F} := \{\mathscr{F}_t\}_{0 \le t \le T}$ is a completed and right-continuous filtration with $\mathscr{F}_0$ being the trivial σ-algebra and $T > 0$ a constant time horizon.

We consider a continuous-time market model with one riskless asset whose price is $S_0 \equiv 1$ and m traded risky assets, whose evolution is described by the process $S(t) = (S_1(t), \dots, S_m(t))$. Notice that we do not specify any particular dynamics for our price processes. The main assumption we make is the following.

Assumption 2.1. *The price process $(S_1(t), \dots, S_m(t))$ is a continuous and adapted square integrable martingale on $(\Omega, \mathbb{F}, \mathbb{Q})$. Moreover, $\mathbb{Q}$ is the unique probability measure under which $S(t)$ is a local martingale, i.e. the market is complete.*

We consider two types of agents acting in this market, a non informed agent (or N-agent) and an informed agent (I-agent). An N-agent relies on $\mathbb{Q}$ as well as on the observable past and present prices when taking her investment decisions. On the other hand, an I-agent has also some privileged information concerning the law of a functional Y of the stock prices. Specifically, I-agent knows the distribution of Y under the so-called "historical" measure $\mathbb{P}$ governing market prices.

From now on, we assume that Y is a scalar random variable (everything shown below can be easily generalized to a vector valued random variable or to more general functionals Y taking values in a Polish space $\mathcal{P}$). We will denote by $\mathbb{Q}_Y$ the law of Y under $\mathbb{Q}$ and by ν the *effective* law of Y known by I-agent. Therefore we have

$$\mathbb{Q}_Y(B) = \mathbb{Q}\{Y \in B\}, \quad \forall B \in \mathscr{B}(\mathbb{R}).$$

Assumption 2.2. *ν is equivalent to $\mathbb{Q}_Y$, the (real) density is $\xi := \frac{d\nu}{d\mathbb{Q}_Y}$ and $\xi(Y)$ is $\mathbb{Q}$-a.s. bounded.*

The privileged information (Y, ν) can be naturally associated to a new measure called by Baudoin the *minimal* probability.

Definition 2.1. *The probability measure $\mathbb{Q}^\nu$ defined on $(\Omega, \mathscr{F}_T)$ by:*

$$\mathbb{Q}^\nu(A) := \int_{\mathbb{R}} \mathbb{Q}(A|Y = y)\, \nu(\mathrm{d}y), \quad A \in \mathscr{F}_T \tag{2.1}$$

is called the minimal *probability associated to the weak information (Y, ν).*

The expression *minimal probability* used by Baudoin is justified in [1, Proposition 6], showing that $\mathbb{Q}^\nu$ fulfills some class of minimization problems. Moreover, observe that $\mathbb{Q}^\nu$ does not depend on the choice of the utility function in a standard portfolio selection model, thus in a behavioral setting this amounts to say that the *minimal* probability is unaffected by the probability distortions and the value functions (see Remark 3.3 later in this paper).

We now turn to utility maximization problems for non-informed and informed investors under the weak information approach.

2.1 The classical agents' models and their solutions

In a classical portfolio selection model, i.e. when N-agent's objective is to maximize her expected utility from terminal wealth, all the results have already been derived in [1]. For reader's convenience, we recall here the solution of this problem assuming that the considered investor is endowed with a positive initial wealth x_0 and with a utility function satisfying the following standard assumption.

Assumption 2.3. *The utility function $U : (0, +\infty) \to \mathbb{R}$ is strictly increasing, strictly concave and twice continuously differentiable and satisfies the Inada conditions $U'(+\infty) = 0$, $U'(0+) = +\infty$.*

Before formulating the optimization problems, we need to define a suitable class of portfolio processes. We are going to use a slight modification of the definition of tame portfolios given in [6], which is well adapted to solve the optimal investment problems of both EU and CPT agents, in the informed as well as in the non-informed case. Let's denote by $\Pi_i(t)$ the number of shares of the i-th risky asset held by our trader at time t.

Definition 2.2. *An admissible portfolio is a couple $(x_0, \Pi(\cdot))$, where x_0 is an initial wealth and $\Pi(\cdot)$ is a $\mathbb{F}$-predictable process, $(S(t))$-integrable and such that the corresponding wealth process*

$$x(t) := x_0 + \int_0^t \Pi(u)\, \mathrm{d}S(u), \quad 0 \le t \le T, \tag{2.2}$$

is a $(\mathbb{F}, \mathbb{Q})$-martingale. Moreover, we say that an admissible portfolio $\Pi(\cdot)$ is $\mathbb{Q}$-tame if the corresponding wealth $x(\cdot)$ is $\mathbb{Q}$-a.s. bounded from below, where the bound may depend on $\Pi(\cdot)$.

Remark 2.1. Notice that the terminal wealth $x(T) \equiv X$ of any *tame* portfolio $(x_0, \Pi(\cdot))$ is a $\mathscr{F}_T$-measurable random variable, $\mathbb{Q}$-a.s. bounded from below and such that $\mathbb{E}^{\mathbb{Q}}[X] = x_0$. Conversely, thanks to Assumption 2.1 a standard completeness argument can be applied, so that any bounded from below contingent claim X with $\mathbb{E}^{\mathbb{Q}}[X] = x_0$ can be replicated by a $\mathbb{Q}$-*tame* portfolio $\Pi(\cdot)$ with the initial wealth x_0.[1] Hence, in the formulation of the optimization problems of both EU and CPT agents, we can replace the (dynamic) constraints on strategies $\Pi(\cdot)$ with (static) constraints on contingent claims X as in [6]. This is usual in the martingale approach.

Let us recall what are the optimal strategies for non-informed and informed agents.

- For a N-agent, the most natural way to evaluate her own utility from terminal wealth X is to choose the martingale measure $\mathbb{Q}$ when computing the expectation (in fact she does not know the historical measure $\mathbb{P}$, so she can not use it! See [2]). Therefore, the EU non informed agent's problem is

$$\begin{aligned} &\text{Maximize} \quad \mathbb{E}^{\mathbb{Q}}[U(X)] \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}}[X] = x_0,\ X \text{ is } \mathscr{F}_T\text{-measurable and } \mathbb{Q}\text{-a.s. bounded from below.} \end{aligned} \tag{EU-N}$$

  The solution to Problem (EU-N) is the trivial null portfolio, $\Pi \equiv 0$, thanks to a simple application of Jensen's inequality and the concavity assumption on $U(\cdot)$.

  In the CPT case things will be different even for non-informed "risk-neutral" agents, i.e. agents evaluating their gains/losses under the risk-neutral measure $\mathbb{Q}$.

- According to Baudoin and Nguyen-Ngoc [1, 2] in a classical portfolio optimization problem for an informed agent who has the weak information (Y, ν) and a utility function $U(\cdot)$ satisfying Assumption 2.3, we can define the financial value of the weak information (Y, ν) for an insider with initial endowment $x_0 > 0$ as follows (X will denote any terminal payoff which can be attained using admissible strategies):

$$u(x_0, \nu) := \inf_{\mu \in \mathcal{E}^\nu} \sup_{\Pi \text{ admissible}} \mathbb{E}^\mu[U(X)], \tag{2.3}$$

  where $\mathcal{E}^\nu$ is the set of probability measures on $\mathscr{F}_T$ which are equivalent to $\mathbb{Q}$ and such that the law of Y under those measures is ν. Using convex duality and the martingale dual approach in complete markets, one has the following result easily adapted from [2, Theorem 1]: Assume that the expectations below are finite. Then for each initial endowment $x_0 > 0$,

$$u(x_0, \nu) = \sup_{\Pi \text{ admissible}} \mathbb{E}^\nu[U(X)] = \mathbb{E}^\nu\left[U\left((U')^{-1}\left(\tfrac{\Lambda(x_0)}{\xi(Y)}\right)\right)\right], \tag{2.4}$$

  where $\Lambda(x_0)$ is defined by

$$\mathbb{E}^\nu\left[\tfrac{1}{\xi(Y)}\,(U')^{-1}\left(\tfrac{\Lambda(x_0)}{\xi(Y)}\right)\right] = x_0.$$

  Moreover, under $\mathbb{Q}^\nu$ the optimal terminal wealth is given by

$$X^* = (U')^{-1}\left(\tfrac{\Lambda(x_0)}{\xi(Y)}\right). \tag{2.5}$$

Remark 2.2. Note that if the insider has no additional information, i.e. $\nu = \mathbb{Q}_Y$, then we have $\xi(Y) = 1$ $\mathbb{Q}$-a.s.. Therefore, we deduce $u(x_0, \nu) = U(x_0)$ and $X = x_0$ $\mathbb{Q}$-a.s., which is nothing but the N-agent's solution. Finally, as a corollary one can even show that $u(x_0, \nu) \geq U(x_0)$ where the equality holds for $\nu = \mathbb{Q}_Y$.

Turning back to the portfolio optimization problem of a weakly informed classical insider, we see that it can be equivalently defined as

$$\begin{aligned} &\text{Maximize} \quad \mathbb{E}^\nu[U(X)] \\ &\text{subject to} \quad \mathbb{E}^\nu\left[\tfrac{1}{\xi(Y)}X\right] = x_0 > 0,\ X \text{ is } \mathscr{F}_T\text{-measurable and } \mathbb{Q}^\nu\text{-a.s. bounded from below,} \end{aligned} \tag{EU-I}$$

thanks to the first equality in equation (2.4).

We also remark that the constraint in Problem (EU-I) is a direct consequence of the relation $d\mathbb{Q}^\nu = \frac{d\nu}{d\mathbb{Q}_Y}(Y)$ (see [1, Remark 4]). Indeed, we can write $1/\xi$ as a density of $\mathbb{Q}_Y$ w.r.t. ν and $1/\xi(Y)$ is $\mathbb{Q}$-a.s. bounded. Now it immediately follows $x_0 = \mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^\nu[(1/\xi(Y))X]$, as it appears in (EU-I).

[1] See for example [11], Definition 6.1 and Theorem 6.6 or [6], Proposition 2.1. We also remark that in [6], absolute value portfolio strategies were used instead of our "number of shares" strategies.

Example 2.1 ([1], Proposition 67)**.** Let $W^{\mathbb{Q}}$ be a $(\Omega, \mathbb{F}, \mathbb{Q})$-Brownian motion and consider a market with only one risky asset whose price dynamics is

$$dS(t) = \sigma S(t) dW^{\mathbb{Q}}(t), \quad t \in [0,T], \quad S(0) = s_0 > 0,$$

for some constant $\sigma > 0$, or equivalently

$$S(t) = s_0 \exp\left(\sigma W_t^{\mathbb{Q}} - \frac{\sigma^2}{2} t\right).$$

Hence, by a change of variable, a weak information on the final price $S(T)$ is equivalent to a weak information on the Gaussian random variable $W_T^{\mathbb{Q}}$. Suppose I-agent has the privileged information $(W_T^{\mathbb{Q}}, \nu)$, where

$$\nu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-m)^2}{2s^2}\right) \mathrm{d}x$$

is Gaussian with mean $m \in \mathbb{R}$ and variance $s^2 \leq T$, with $0 < s \leq \sqrt{T}$ (in what follows, we will write $\nu \sim \mathcal{N}(m, s^2)$). Note that Assumption 2.2 is fulfilled and we can also explicitly compute

$$\xi(Y) = \xi(W_T^{\mathbb{Q}}) = \frac{\sqrt{T}}{s} \exp\left(-\frac{(W_T^{\mathbb{Q}} - m)^2}{2s^2} + \frac{(W_T^{\mathbb{Q}})^2}{2T}\right). \tag{2.6}$$

Therefore, if we set $\delta = \frac{s^2 - T}{T}$, then for a power utility function $U(x) = x^\alpha$, $\alpha \in (0,1)$, one can compute

$$u(x_0, \nu) = x_0^\alpha \frac{1}{\sqrt{1+\delta}} \left(\frac{1-\alpha}{\frac{1}{1+\delta} - \alpha}\right)^{\frac{1-\alpha}{2}} \exp\left(\frac{\alpha m^2}{2[T(1-\alpha) - \alpha\delta T]}\right).$$

Specifically, if $m = 0$ and $s^2 = T$ (i.e. $\delta = 0$) then we recover the no additional information case, as $\nu = \mathbb{Q}_Y$. If $m \neq 0$ and $s^2 = T$ then I-agent has some additional information regarding the drift but not the variance of the Brownian motion; in this case we have $u(x_0, \nu) = x_0^\alpha \exp\left(\frac{\alpha m^2}{2[T(1-\alpha)]}\right)$ and the bigger m, the more valuable the information. Viceversa, if $m = 0$ and $s^2 < T$ then we obtain $u(x_0, \nu) = x_0^\alpha \frac{1}{\sqrt{1+\delta}} \left(\frac{1-\alpha}{\frac{1}{1+\delta} - \alpha}\right)^{\frac{1-\alpha}{2}}$, which tends to infinity as $\delta \downarrow -1$ or, equivalently, as $s \downarrow 0$. Thus a more precise knowledge on the final price leads to a higher value of the weak information, as naturally expected. This example will be studied in full details in the CPT case as well.

3 The CPT agents' models and their solutions

In this section we will give the solution of portfolio selection problems of CPT non-informed and informed agents. We will keep as much as possible the setting and the notation used by Jin and Zhou [6] to describe the preferences and the objective function of a CPT investor. We point out that our results are linked to those in [6]. However, they need a complete proof as we are working in a slight different setting. Loosely speaking, in [6] the investor knows the "historical" probability $\mathbb{P}$ and she performs a standard change of measure based on a pricing kernel (or state price density) ρ, thus obtaining martingale processes for the prices under an equivalent probability. After that, a complete solution based on ρ is derived under some technical assumptions.

In the present framework we start from the very beginning with martingale prices. Therefore, no change of measure is needed and $\rho \equiv 1$ a.s., i.e. it is totally concentrated. As a consequence, we will see that the structure of the solution for a N-agent will be *law dependent*, in the sense that only the distribution of a random variable will affect her optimal value.

Now, we briefly recall the cornerstones of the CPT preferences behind the formulation above and its assumptions. In CPT, the trader's goal is to select the portfolio that will produce a terminal wealth X maximizing her "utility". Such a "utility" (also called prospect value, in Kahneman and Tversky's terminology) will come up from the algebraic sum of some *expected distorted values* of gains and losses w.r.t. a reference wealth that we set once for all at the value 0. Mathematically speaking, we will make the following assumptions, corresponding to Assumptions 2.3 and 2.4 in [6]. If the random variable X represents a final wealth at time T and our CPT agent uses μ as a reference measure, then she will assign to X the prospect value $V(X)$, which is defined by

$$V(X) := V_+(X^+) - V_-(X^-) \tag{3.1}$$

with its components $V_+(\cdot)$ and $V_-(\cdot)$ given by

$$V_+(Y) := \int_0^{+\infty} T_+(\mu\{u_+(Y) > y\})\,\mathrm{d}y, \qquad V_-(Y) := \int_0^{+\infty} T_-(\mu\{u_-(Y) > y\})\,\mathrm{d}y \tag{3.2}$$

for any random variable $Y \geq 0$ μ-a.s.. Here, X^+ and X^- denote the positive and the negative part of X respectively. The functions $u_+(\cdot), u_-(\cdot)$ and $T_+(\cdot), T_-(\cdot)$ appearing above are assumed to satisfy the following conditions.

Assumption 3.1. *$u_+(\cdot)$ and $u_-(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$, are strictly increasing, concave, with $u_+(0) = u_-(0) = 0$. Moreover, $u_+(\cdot)$ is strictly concave and twice differentiable, satisfying the Inada conditions $u'_+(0+) = +\infty$ and $u'_+(+\infty) = 0$.*

Assumption 3.2. *$T_+(\cdot)$ and $T_-(\cdot) : [0,1] \mapsto [0,1]$, are differentiable and strictly increasing, with $T_+(0) = T_-(0) = 0$ and $T_+(1) = T_-(1) = 1$.*

Our CPT agent will look for a terminal wealth X, which is $\mathscr{F}_T$-measurable and a.s. bounded from below w.r.t. her reference probability. However, her initial endowment can be any amount $x_0 \in \mathbb{R}$ and not necessarily non-negative. In this paper, the reference measure μ will alternatively be the risk neutral measure $\mathbb{Q}$ for the non-informed investor and the minimal probability measure $\mathbb{Q}^\nu$ for the informed one.

3.1 The non informed agent's problem

We now consider a non-informed agent who evaluates her total utility distinguishing gains from losses w.r.t. to the reference level 0. For the moment, probability distortions are not allowed, i.e. $T_\pm(\cdot) = id(\cdot)$. Such an investor represents an intermediate case between a classical agent and a *behavioral* agent *à la* Kahneman and Tversky.
Within this framework, it seems reasonable to define the problem of a non informed agent as

$$\begin{aligned} &\text{Maximize} \quad V(X) = \mathbb{E}^{\mathbb{Q}}[u_+(X^+)] - \mathbb{E}^{\mathbb{Q}}[u_-(X^-)] \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}}[X] = x_0,\ X \text{ is } \mathscr{F}_T\text{-measurable and } \mathbb{Q}\text{-a.s. bounded from below.} \end{aligned} \tag{3.3}$$

Unfortunately there are bad news about Problem (3.3) because under Assumptions 3.1 and 3.2 it can easily be ill-posed. Before giving a more precise statement (and its proof), we note that an investor with the previous objective function would better choose a fixed reward x_+ whenever X is positive, thanks to Jensen's inequality and the concavity of $u_+(\cdot)$. Otherwise, conditioned to $X \leq 0$, she will try to minimize the expected loss. The following ill-posedness result depends substantially on a comparison between the magnitude of the utility from large gains and that of disutility from large losses.

Proposition 3.1. *Assume $\lim_{x\to+\infty} u_+(x) = +\infty$ and $\lim_{x\to+\infty} \frac{u_+(x)}{u_-(x)} \in (1, +\infty]$, where the previous limit exists. Then Problem (3.3) is ill-posed.*

Proof. Consider the sequence of admissible terminal wealths (X_n), where

$$X_n = \begin{cases} n(x_0^+ + 1) & \text{with } \mathbb{Q}\text{-probability } 1/2, \\ 2x_0 - n(x_0^+ + 1) & \text{with } \mathbb{Q}\text{-probability } 1/2, \end{cases}$$

for n sufficiently large. Then we have

$$V(X_n) = \frac{1}{2}\left[u_+\left(n(x_0^+ + 1)\right) - u_-\left(n(x_0^+ + 1) - 2x_0\right)\right] \to +\infty,$$

as $n \to +\infty$, thanks to our assumptions $\lim_{x\to+\infty} u_+(x) = +\infty$ and $\lim_{x\to+\infty} \frac{u_+(x)}{u_-(x)} \in (1, +\infty]$. $\square$

There are different ways out of this drawback. Obviously, we could choose suitable value functions $u_{\pm}(\cdot)$, e.g. imposing $\lim_{x\to+\infty} u_+(x) < +\infty$. We could alternatively introduce probability distortions, especially on the loss part as explained in [6]. Finally, we could impose a loss control, i.e. a lower bound L on the maximal loss which can be suffered by the investor (for more details on the subject, see [8]). One could also use a combination of the previous modifications.

Let us consider the case where the probability distortions satisfy Assumption 3.2. Thus the problem for a CPT N-agent will be

$$\begin{aligned} &\text{Maximize} \quad V(X) = V_+(X^+) - V_-(X^-) \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}}[X] = x_0,\ X \text{ is } \mathscr{F}_T\text{-measurable and } \mathbb{Q}\text{-a.s. bounded from below,} \end{aligned} \tag{CPT-N}$$

where we set

$$V_+(X^+) := \int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X^+) > y\})\,\mathrm{d}y, \qquad V_-(X^-) := \int_0^{+\infty} T_-(\mathbb{Q}\{u_-(X^-) > y\})\,\mathrm{d}y.$$

The main difference between our Problem (CPT-N) and the optimization problem in [6] concerns the constraint on the expected value of the terminal wealth X. More specifically, in [6], equation (2.6), the budget constraint was $\mathbb{E}^{\mathbb{P}}[\rho X] = x_0$, where the law of the state price density ρ was assumed to be atom-less w.r.t. $\mathbb{P}$. Now, we do not have that atom-less density as we are already working under the martingale measure $\mathbb{Q}$. We also recall that the assumption on ρ being atom-less w.r.t. $\mathbb{P}$ was imposed in [6] just to avoid technical difficulties. In our case, the absence of a weighting random variable (this was actually the role played by ρ) will change the structure of the solution to Problem (CPT-N) as well as its economical interpretation.
For reader's convenience, we will report below only the main results, while the proofs are postponed in the Appendix.

For any fixed random variable Z uniformly distributed over $(0,1)$ w.r.t. $\mathbb{Q}$ (i.e. $Z \sim U(0,1)$, for short) and given a pair (p, x_+) with $p \in [0,1]$ and $x_+ \geq x_0^+$, define $v_+(p, x_+)$ as the optimal value of the following problem:

$$\begin{aligned} &\text{Maximize} \quad V_+(X) = \textstyle\int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X) > y\})\,\mathrm{d}y \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}}[X] = x_+, \quad X \geq 0 \text{ on } \{Z \leq p\}, \quad X = 0 \text{ on } \{Z > p\}. \end{aligned} \tag{3.4}$$

Next, we set up the optimization problem

$$\begin{aligned} &\text{Maximize} \quad v_+(p, x_+) - u_-\left(\tfrac{x_+ - x_0}{1-p}\right) T_-(1-p) \\ &\text{subject to} \quad \begin{cases} p \in [0,1], \quad x_+ \geq x_0^+, \\ x_+ = 0 \text{ if } p = 1, \quad x_+ = x_0 \text{ if } p = 0, \end{cases} \end{aligned} \tag{3.5}$$

where we conventionally define $u_-\left(\frac{x_+ - x_0}{1-p}\right) T_-(1-p) := 0$ if $p = 1$ and $x_+ = x_0$. Finally, we denote by X^* the optimal solution to Problem (CPT-N) and we make the following hypothesis.

Assumption 3.3. *$T_+'(z)$ is non-increasing for $z \in (0,1]$, $\liminf_{x\to+\infty} -\frac{x u_+''(x)}{u_+'(x)} > 0$ and for any $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$ we have $\mathbb{E}^{\mathbb{Q}}\left[u_+\left((u_+')^{-1}(\frac{1}{T'(Z)})\right) T'(Z)\right] < +\infty$.*

Under Assumption 3.3, for *any* $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$ we have

$$X^* = (u_+')^{-1}\left(\tfrac{\lambda}{T_+'(Z)}\right) I_{Z\leq p^*} - \tfrac{x_+^* - x_0}{1-p^*} I_{Z>p^*}, \tag{3.6}$$

$$V(X^*) = \mathbb{E}^{\mathbb{Q}}\left[u_+\left((u_+')^{-1}\left(\tfrac{\lambda}{T_+'(Z)}\right)\right) T_+'(Z) I_{Z\leq p^*}\right] - u_-\left(\tfrac{x_+^* - x_0}{1-p^*}\right) T_-(1-p^*), \tag{3.7}$$

where the pair (p^*, x_+^*) is optimal for Problem (3.5) and the Lagrange multiplier λ satisfies

$$\mathbb{E}^{\mathbb{Q}}\left[(u_+')^{-1}\left(\tfrac{\lambda}{T_+'(Z)}\right) I_{Z\leq p^*}\right] = x_+^*. \tag{3.8}$$

Remark 3.1. Firstly, our result shows that a CPT non-informed investor is interested in probabilities (and not in events). This is a by-product of the law-invariance property of the CPT preferences and the fact that she observes the evolution of the price process under the martingale measure $\mathbb{Q}$. These facts are eventually reflected by the indifference in the choice of Z. For instance, in a Brownian motion driven market as in Example 2.1, the agent can choose $Z = F_W(W_T^{\mathbb{Q}})$, where $F_W(\cdot)$ is the distribution function of $W_T^{\mathbb{Q}}$. In this way she will obtain a gain when the price of the risky stock is lower than a certain threshold. However, she could also choose $Z = 1 - F_W(W_T^{\mathbb{Q}})$, representing the opposite situation.

Secondly, we highlight that the explicit solution given by (3.6) is available only when $T'_+(\cdot)$ is non-increasing over $(0, 1]$. Combining this observation with Assumption 3.2, a necessary condition to get (3.6) is $T_+(\cdot)$ to be concave. Notice that a reversed S-shaped $T_+(\cdot)$ does not fulfill this condition.

Before going further, we consider the case of power utilities. In [6], the authors were able to find a much more explicit solution assuming generic probability weighting functions $T_\pm(\cdot)$ and $u_+(x) = x^\alpha$, $u_-(x) = k_- x^\alpha$ with $\alpha \in (0, 1)$, $k_- \geq 1$.[2] We now adapt their reasoning and we choose the special distortion on gains $T_+(p) = p^\gamma$, $\gamma \in (0, 1]$, as suggested by the previous Remark 3.1. Intuitively, this concave function should reflect an overweighting of relatively large gains w.r.t. smaller payoffs. With straightforward computations, for $\alpha < \gamma$ we find

$$\varphi_N(p) := \mathbb{E}^{\mathbb{Q}}\left[T'_+(Z)^{1/(1-\alpha)} I_{Z\leq p}\right] = \gamma^{1/(1-\alpha)} \left(\frac{1-\alpha}{\gamma-\alpha}\right) p^{\frac{\gamma-\alpha}{1-\alpha}}, \qquad p \in [0, 1], \tag{3.9}$$

$$k_N(p) := \frac{k_- T_-(1-p)}{(1-p)^\alpha \varphi_N(p)^{1-\alpha}} = \frac{k_-}{\gamma}\left(\frac{\gamma-\alpha}{1-\alpha}\right)^{1-\alpha} \frac{T_-(1-p)}{(1-p)^\alpha p^{\gamma-\alpha}}, \qquad p \in (0, 1], \tag{3.10}$$

and following the same lines as in [6], Theorem 9.1, we have

Proposition 3.2. *In the CRRA case with $x_0 \geq 0$ and $T_+(p) = p^\gamma$, $\gamma \in (0, 1]$:*

(i) if $0 < \alpha < \gamma \leq 1$ and $\inf_{p\in(0,1]} k_N(p) \geq 1$, then Problem (CPT-N) *is well-posed and*

$$X^* = x_0 \left(\frac{\gamma-\alpha}{1-\alpha}\right) Z^{\frac{\gamma-1}{1-\alpha}}, \qquad Z \sim U(0, 1); \tag{3.11}$$

$$V(X^*) = x_0^\alpha \gamma \left(\frac{1-\alpha}{\gamma-\alpha}\right)^{1-\alpha}. \tag{3.12}$$

(ii) if $\alpha \geq \gamma$ or $\inf_{p\in(0,1]} k_N(p) < 1$, then Problem (CPT-N) *is ill-posed.*

It is clear by the parameters' condition in *(i)* that the curvature of the value function on gains must be greater than that of the distortion $T_+(\cdot)$ if we hope to find a financially meaningful solution. Moreover, the well-posedness of this model strongly depends on the shape of $T_-(\cdot)$. We also note that the optimal value $V(X^*)$ is decreasing in γ, whereas it does not exhibit a clear dependence in α. As a particular case, we have the following corollary.

Corollary 3.1. *With the same assumptions of Proposition 3.2 and $T_-(p) = p^\delta$, $\delta \in (0, 1)$:*

(i) if $0 < \delta \leq \alpha < \gamma < 1$ and $k_- \geq f(\alpha, \gamma, \delta)$, where

$$f(\alpha, \gamma, \delta) := \gamma \frac{(1-\alpha)^{1-\alpha}}{(\gamma-\alpha)^{1-\gamma}} \frac{(\alpha-\delta)^{\alpha-\delta}}{(\gamma-\delta)^{\gamma-\delta}}, \tag{3.13}$$

then Problem (CPT-N) *is well-posed;*

(ii) if $0 < \delta \leq \alpha < \gamma = 1$, then Problem (CPT-N) *is well-posed;*

(iii) if $\delta > \alpha$ then Problem (CPT-N) *is ill-posed.*

[2]We recall that the parameter k_- is usually called the *loss aversion coefficient*, as in this framework it reflects the idea that "losses loom larger than gains". In what follows, we will refer to this case as to CRRA, due to the Constant Relative Risk Aversion coefficient exhibited by the value functions.

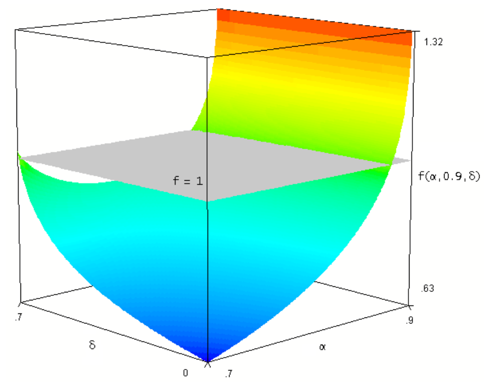

Figure 1: a comparison between the well-posedness threshold and the level of k_-.

Proof. Using (3.10) and the special form of $T_-(\cdot)$, it is immediate to compute the infimum of $k_N(p)$ over $(0,1]$ via first order conditions. Now, case *(iii)* follows if we let p tend to 1. In the other cases, we have that the infimum is reached for $\hat{p} = \frac{\gamma-\alpha}{\gamma-\delta} \leq 1$. Hence, we find $k_N(\hat{p}) = f(\alpha,\gamma,\delta)$ with the subsequent well-posedness condition $k_- \geq f(\alpha,\gamma,\delta)$. If $\gamma = 1$, equation (3.13) reduces to

$$f(\alpha,1,\delta) = \frac{(1-\alpha)^{1-\alpha}(\alpha-\delta)^{\alpha-\delta}}{(1-\delta)^{1-\delta}} \leq 1.$$

To see this, note that we have $1-\delta$, $1-\alpha$, $\alpha-\delta \in (0,1)$. Moreover, the function $g(x) := x^x \equiv e^{x\ln x}$ is well defined for $x \in (0,1)$. To prove the previous relation, we only have to show that for every $0 < y < x < 1$ we have

$$y\ln y + (x-y)\ln(x-y) - x\ln x \leq 0.$$

But this is true because

$$\sup_{0<y<x<1} y\ln y + (x-y)\ln(x-y) - x\ln x = 0,$$

as it is easily seen using standard minimization techniques. □

Remark 3.2. We stress that the *ad hoc* choice of concave $T_\pm(\cdot)$ corresponds to an investor who underweights relatively small gains and losses and overweights relatively large gains and losses.[3] Lengthy but not difficult computations show that $f(\cdot,\cdot,\cdot)$ is decreasing in γ and increasing in δ, confirming the economic intuition. In fact, the lower the overestimation of gains is, the higher the loss aversion coefficient has to be in order to compensate its effect and for the problem to reach well-posedness. However, the dependence on α is not monotonic.

For a better understanding of the previous corollary, in Figure 1 we provide a plot representing a 3D surface of the well-posedness threshold $f(\cdots)$ in case *(i)*, where we arbitrarily fix $\gamma = 0.9$ and we take $\alpha \in [0.7, 0.9)$ and $\delta \in (0, 0.7]$. An horizontal plane at the level $f = 1$ is drawn to facilitate the distinction between a surely well-posed case, i.e. when the surface stands below the plane, or a probable ill-posed case, i.e. when the loss-aversion coefficient has to be sufficiently high to ensure condition (3.13). More generally, we can also note that for the reversed S-shaped $T_-(\cdot)$ used in [12], namely $T_-(p) = \frac{p^\delta}{(p^\delta+(1-p)^\delta)^{1/\delta}}$ with $\delta \in (0.28, 1)$, we have

$$k_N(p) = const \times \frac{(1-p)^{\delta-\alpha}p^{\alpha-\gamma}}{[(1-p)^\delta + p^\delta]^{1/\delta}}.$$

In this case, if $\delta \geq \alpha$ or $\delta < \alpha < \gamma$ we have a systematic ill-posedness because $\lim_{p\to 1^-} k_N(p) = 0$.

[3] This is no longer true if we assume $T_+(\cdot) = id(\cdot)$, as our trader will not weight gains while she would exhibit some distortion on the loss side.

3.2 The insider's problem

In this section we will solve the portfolio optimization problem for an informed agent with CPT preferences. We keep Assumption 3.1 on the value functions $u_\pm(\cdot)$ and Assumption 3.2 on the probability distortions $T_\pm(\cdot)$. Furthermore, thanks to the equivalence between $\mathbb{Q}$ and $\mathbb{Q}^\nu$ stated in Assumption 2.2, the CPT I-agent can still rely on the admissible portfolios described in Definition 2.2. We remark that the dynamics of the wealth process $x(\cdot)$ under $\mathbb{Q}$ remains the same as in equation (2.2), whereas it drastically changes under $\mathbb{Q}^\nu$.[4]

We now define the value of the weak information for the I-agent analogously to (EU-I). The optimization problem for a CPT insider with the weak information (Y,ν) and the initial endowment $x_0 \in \mathbb{R}$ is

$$\begin{aligned} &\text{Maximize} \quad V^\nu(X) := V_+^\nu(X^+) - V_-^\nu(X^-) \\ &\text{subject to} \quad \mathbb{E}^\nu\left[\tfrac{1}{\xi(Y)}X\right] = x_0,\ X \text{ is } \mathscr{F}_T\text{-measurable and } \mathbb{Q}^\nu\ a.s. \text{ bounded from below,} \end{aligned} \tag{CPT-I}$$

where

$$V_+^\nu(X^+) := \int_0^{+\infty} T_+(\mathbb{Q}^\nu\left\{u_+(X^+) > y\right\})\,\mathrm{d}y, \quad V_-^\nu(X^-) := \int_0^{+\infty} T_-(\mathbb{Q}^\nu\left\{u_-(X^-) > y\right\})\,\mathrm{d}y. \tag{3.14}$$

Here X represents the terminal payoff obtained via the initial wealth x_0 and the dynamics (2.2). The optimal value of problem (CPT-I) will be denoted $V(x_0,\nu)$.

Remark 3.3. Notice that the maximization problem (CPT-I) is formulated under the minimal probability $\mathbb{Q}^\nu$. To see why this makes sense, we recall that the historical probability $\mathbb{P}$ is unknown to I-agent and thus can not be used. Moreover, using the martingale measure $\mathbb{Q}$ (as in N-agent's optimization problem) doesn't make sense since it does not exploit the information advantage of I-agent, hence it must be replaced by a different measure reflecting the extra knowledge. Thus, the insider chooses a probability belonging to the set $\mathcal{E}^\nu$ and it seems natural for her to select a measure which is not influenced by $u_\pm(\cdot)$ and $T_\pm(\cdot)$. As a matter of fact, a CPT trader is able to correctly assess probabilities of events. Therefore, those functions shouldn't affect probabilities because they are used in the CPT paradigm only to describe risk attitudes.

Another reason behind the choice of $\mathbb{Q}^\nu$ relies on the fact that it reflects a Bayesian updating, in the sense that it involves the conditional probabilities $\mathbb{Q}(\cdot|Y=y)$. Moreover, the properties of the minimal probability in [1, Remark 4], especially properties 1. and 3., make it desirable to use $\mathbb{Q}^\nu$ in Problem (CPT-I).[5] Furthermore, dropping the distinction between losses and gains and probability distortions as well, we recover Problem (EU-I). On the other hand, in the extreme case of no additional information $\nu = \mathbb{Q}_Y$ we have $V_\pm^\nu(X) = V_\pm(X)$ thanks to *minimal* probability's property 3 in [1, Remark 4], so we turn back to Problem (CPT-N).

To conclude this remark, we observe that we could define the *financial value of the weak information* (Y,ν) analogously to Baudoin [1] as

$$\inf_{\mu\in\mathcal{E}^\nu} \sup_{\Pi\ admissible} V_+^\mu(X^+) - V_-^\mu(X^-),$$

where $V_\pm^\mu(\cdot)$ are defined similarly as in equation (3.14). Unfortunately, due to the complexity of the preferences of the CPT I-agent, it is not clear whether the previous problem is equivalent to Problem (CPT-I).

It is important to note that Problem (CPT-I) is nothing but a special case of the problem originally studied by Jin and Zhou [6] (see their equation (2.7)), where

1. the measure $\mathbb{P}$ is replaced by $\mathbb{Q}^\nu$ in both the objective function and the constraints;
2. the random variable ρ is replaced by the new random variable $\frac{1}{\xi(Y)}$.

Therefore, we have to check that all the assumptions imposed in [6] on ρ are now fulfilled by $\frac{1}{\xi(Y)}$ and then we will be able to use all the results found in [6] with the obvious modifications, i.e. substitute for $\frac{1}{\xi(Y)}$ and $\mathbb{Q}^\nu$ in every explicit expression. First of all, to avoid undue technicalities, the assumption of ρ having no atoms w.r.t. $\mathbb{P}$ ([6], Assumption 2.2) is now translated to the following

[4] For more details, see [1] where the theory of Conditioned Stochastic Differential Equations (CSDEs) is developed.

[5] In [2], the authors confirm that $\mathbb{Q}^\nu$ has been built with the purpose of keeping the independence property. Then, it seems reasonable for us to keep such a feature in the evaluation function of a CPT agent.

Assumption 3.4. *The random variable* $\frac{1}{\xi(Y)}$ *has no atoms under* $\mathbb{Q}^\nu$*, i.e.*

$$\mathbb{Q}^\nu\left\{\tfrac{1}{\xi(Y)} = a\right\} = 0 \quad \forall\, a \geq 0.$$

Other technical conditions on $\xi(Y)$ are straightforward to check; in fact we have $\frac{1}{\xi(Y)} \in (0, +\infty)$ $\mathbb{Q}^\nu$-a.s. thanks to our Assumption 2.2. Moreover, $\mathbb{E}^\nu[\frac{1}{\xi(Y)}] = 1$ follows directly from the definition of ξ.

Remark 3.4. It is worth noticing that

$$\mathbb{Q}^\nu\left\{\tfrac{1}{\xi(Y)} = a\right\} = \mathbb{E}^{\mathbb{Q}}\left[\xi(Y) I_{\left\{\frac{1}{\xi(Y)}=a\right\}}\right] = \tfrac{1}{a}\mathbb{Q}\left\{\xi(Y) = \tfrac{1}{a}\right\},$$

so that the $\mathbb{Q}^\nu$-law of $1/\xi(Y)$ is atomless if and only if the $\mathbb{Q}$-law of $\xi(Y)$ is. The latter condition is satisfied in many common situations (see our examples at the end of this section).

We now adapt the analysis made in [6] with the necessary modifications for a CPT informed investor. In what follows, we will need a new set of variables for I-agent, that will be equipped with the superscript $^\nu$, to distinguish them from the same variables for N-agents. We define

$$\overline{\tfrac{1}{\xi(Y)}} \equiv \operatorname{esssup}_{\mathbb{Q}^\nu} \tfrac{1}{\xi(Y)} := \sup\{a \in \mathbb{R} : \mathbb{Q}^\nu\{\tfrac{1}{\xi(Y)} > a\} > 0\},$$

$$\underline{\tfrac{1}{\xi(Y)}} \equiv \operatorname{essinf}_{\mathbb{Q}^\nu} \tfrac{1}{\xi(Y)} := \inf\{a \in \mathbb{R} : \mathbb{Q}^\nu\{\tfrac{1}{\xi(Y)} < a\} > 0\}.$$

Once again well-posedness is an important issue as in the case of N-agent's problem. With some slight adjustments to Theorems 3.1 and 3.2 in [6] we have:

Proposition 3.3. *Problem* (CPT-I) *is ill-posed if there exists a nonnegative* $\mathscr{F}_T$*-measurable random variable* X *such that* $\mathbb{E}^\nu\left[\frac{1}{\xi(Y)}X\right] < +\infty$ *and* $V_+^\nu(X) = +\infty$.

Proposition 3.4. *If* $u_+(+\infty) = +\infty$, $\overline{\frac{1}{\xi(Y)}} = +\infty$, *and* $T_-(\cdot) = id(\cdot)$, *then Problem* (CPT-I) *is ill-posed.*

Thus, to avoid systematic ill-posedness, we will impose:

Assumption 3.5 (see [6], Assumption 3.1)**.** $V_+^\nu(X) < +\infty$ *for any nonnegative,* $\mathscr{F}_T$*-measurable random variable* X *satisfying* $\mathbb{E}^\nu\left[\frac{1}{\xi(Y)}X\right] < +\infty$.

Remark 3.5. Note that we do not have yet a comparison result between the optimal value of a CPT I-agent, $V(x_0, \nu)$, and the optimal value for a CPT N-agent's problem, so for the moment we can not conclude that an insider always gets more than a non informed agent in this behavioral context, neither we can say that ill-posedness for N-agent implies ill-posedness for I-agent.

We recall the main steps to get to the solution to Problem (CPT-I) (for more details see [6]). First, for a given pair (A, x_+), with $A \in \mathscr{F}_T$ and $x_+ \geq x_0^+$, define the problem

$$\begin{array}{ll} \text{Maximize} & V_+^\nu(X) = \int_0^{+\infty} T_+(\mathbb{Q}^\nu\{u_+(X) > y\})\, \mathrm{d}y \\ \text{subject to} & \mathbb{E}^\nu[\frac{1}{\xi(Y)}X] = x_+, \quad X \geq 0 \ \ \mathbb{Q}^\nu\text{-a.s.}, \quad X = 0 \ \ \mathbb{Q}^\nu\text{-a.s. on } A^C. \end{array} \tag{3.15}$$

Note that Assumption 3.5 implies that $V_+^\nu(X)$ is a finite nonnegative number for any feasible X. We now define $v_+^\nu(A, x_+)$, the optimal value of problem (3.15), in this way:

- if $\mathbb{Q}^\nu(A) > 0$ then the feasible region of (3.15) is non-empty and $v_+^\nu(A, x_+)$ is defined as the supremum of (3.15);

- if $\mathbb{Q}^\nu(A) = 0$ and $x_+ = 0$, then the only feasible solution for (3.15) is $X = 0$ $\mathbb{Q}^\nu$-a.s., so $v_+^\nu(A, x_+) := 0$;

- if $\mathbb{Q}^\nu(A) = 0$ and $x_+ > 0$, then (3.15) has an empty feasible region, therefore $v_+^\nu(A, x_+) := -\infty$.

For any $c \geq 0$, set $v_+^\nu(c, x_+) := v_+^\nu(\{\omega \in \Omega : \frac{1}{\xi(Y(\omega))} \leq c\}, x_+)$; moreover, define $F^\nu(\cdot)$ and $F^{\mathbb{Q}}(\cdot)$ the distribution functions of $\frac{1}{\xi(Y)}$ w.r.t. $\mathbb{Q}^\nu$ and $\mathbb{Q}$ respectively. Following the guidelines of [6], equation (4.4), we set up the "simpler" problem

$$\begin{aligned} &\text{Maximize} \quad v_+^\nu(c, x_+) - u_-\left(\tfrac{x_+ - x_0}{1 - F^{\mathbb{Q}}(c)}\right) T_-(1 - F^\nu(c)) \\ &\text{subject to} \quad \begin{cases} \underline{\frac{1}{\xi(Y)}} \leq c \leq \overline{\frac{1}{\xi(Y)}}, \quad x_+ \geq x_0^+, \\ x_+ = 0 \text{ if } c = \underline{\frac{1}{\xi(Y)}}, \quad x_+ = x_0 \text{ if } c = \overline{\frac{1}{\xi(Y)}}, \end{cases} \end{aligned} \tag{3.16}$$

where we use the convention

$$u_-\left(\tfrac{x_+ - x_0}{1 - F^{\mathbb{Q}}(c)}\right) T_-(1 - F^\nu(c)) := 0 \quad \text{if} \quad c = \overline{\frac{1}{\xi(Y)}} \quad \text{and} \quad x_+ = x_0. \tag{3.17}$$

We are now ready to state the results for a CPT agent who has the weak information (Y, ν):

Proposition 3.5. *Assume that $u_-(\cdot)$ is strictly concave at 0. We have the following conclusions:*

(i) If $X^{\nu *}$ is optimal for Problem (CPT-I), *then*

$$c^{\nu *} := (F^\nu)^{-1}\left(\mathbb{Q}^\nu\{X^{\nu *} \geq 0\}\right), \quad x_+^{\nu *} := \mathbb{E}^\nu\left[\tfrac{1}{\xi(Y)}(X^{\nu *})^+\right]$$

are optimal for Problem (3.16). *Moreover, $\{\omega : X^{\nu *} \geq 0\}$ and $\{\omega : \frac{1}{\xi(Y)} \leq c^{\nu *}\}$ are identical up to a $\mathbb{Q}^\nu$-null probability set, and*

$$(X^{\nu *})^- = \frac{x_+^{\nu *} - x_0}{1 - F^{\mathbb{Q}}(c^{\nu *})} I_{\frac{1}{\xi(Y)} > c^{\nu *}} \quad \mathbb{Q}^\nu\text{-a.s..}$$

(ii) If $(c^{\nu *}, x_+^{\nu *})$ is optimal for Problem (3.16) *and $X_+^{\nu *}$ is optimal for Problem* (3.15) *with parameters $(\{\frac{1}{\xi(Y)} \leq c^{\nu *}\}, x_+^{\nu *})$, then*

$$X^{\nu *} := X_+^{\nu *} I_{\frac{1}{\xi(Y)} \leq c^{\nu *}} - \frac{x_+^{\nu *} - x_0}{1 - F^{\mathbb{Q}}(c^{\nu *})} I_{\frac{1}{\xi(Y)} > c^{\nu *}}$$

is optimal for Problem (CPT-I).

Therefore, in order to solve Problem (CPT-I) we can exploit the following algorithm:

Step 1 Solve Problem (3.15) with given parameters $(\{\frac{1}{\xi(Y)} \leq c\}, x_+)$, where $\underline{\frac{1}{\xi(Y)}} \leq c \leq \overline{\frac{1}{\xi(Y)}}$ and $x_+ \geq x_0^+$, in order to obtain $v_+^\nu(c, x_+)$ and the optimal solution $X_+^{\nu *}(c, x_+)$.

Step 2 Solve Problem (3.16) to get $(c^{\nu *}, x_+^{\nu *})$.

Step 3 (i) If $(c^{\nu *}, x_+^{\nu *}) = (\overline{\frac{1}{\xi(Y)}}, x_0)$, then $X_+^{\nu *}(\overline{\frac{1}{\xi(Y)}}, x_0)$ solves Problem (CPT-I).

(ii) Else $X_+^{\nu *}(c^{\nu *}, x_+^{\nu *}) I_{\frac{1}{\xi(Y)} \leq c^{\nu *}} - \frac{x_+^{\nu *} - x_0}{1 - F^{\mathbb{Q}}(c^{\nu *})} I_{\frac{1}{\xi(Y)} > c^{\nu *}}$ solves Problem (CPT-I).

To get an explicit solution we now have to impose conditions similar to that in Assumption 3.3. In particular, we set

Assumption 3.6. *(i) $\frac{(F^\nu)^{-1}(z)}{T_+'(z)}$ is non-decreasing in $z \in (0, 1]$;*

(ii) $\liminf_{x \to +\infty} \frac{-x u_+''(x)}{u_+'(x)} > 0$;

(iii) $\mathbb{E}^\nu\left[u_+\left((u_+')^{-1}\left(\frac{1}{\xi(Y) T_+'(F^\nu(\frac{1}{\xi(Y)}))}\right)\right) T_+'(F^\nu(\frac{1}{\xi(Y)}))\right] < +\infty$.

Briefly, condition *(i)* is related to the fact that the distortion on gains should not be too extreme.[6] Then, hypothesis *(ii)* on the RRA coefficient on gains is the same as for N-agent (recall that the value functions, as well as the probability distortions, are assumed to be the same for the two types of agent in order to facilitate a comparative analysis).
Under Assumption 3.6, both $v_+^\nu(c, x_+)$ and the corresponding optimal solution $X_+^{\nu *}$ to Problem (3.15) can be expressed more explicitly, together with the optimal solution $X^{\nu *}$ of (CPT-I):

$$v_+^\nu(c, x_+) = \mathbb{E}^\nu \left[u_+ \left((u'_+)^{-1} \left(\frac{\lambda^\nu(c, x_+)}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) \right) T'_+(F^\nu(\tfrac{1}{\xi(Y)})) I_{\frac{1}{\xi(Y)} \le c} \right], \tag{3.18}$$

$$X_+^{\nu *} = (u'_+)^{-1} \left(\frac{\lambda^\nu(c, x_+)}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) I_{\frac{1}{\xi(Y)} \le c}, \tag{3.19}$$

$$X^{\nu *} = (u'_+)^{-1} \left(\frac{\lambda^\nu(c^{\nu *}, x_+^{\nu *})}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) I_{\frac{1}{\xi(Y)} \le c^{\nu *}} - \frac{x_+^{\nu *} - x_0}{1 - F^{\mathbb{Q}}(c^{\nu *})} I_{\frac{1}{\xi(Y)} > c^{\nu *}}, \tag{3.20}$$

where $\lambda^\nu(c, x_+)$ satisfies $\mathbb{E}^\nu \left[(u'_+)^{-1} \left(\frac{\lambda^\nu(c,x_+)}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) \frac{1}{\xi(Y)} I_{\frac{1}{\xi(Y)} \le c} \right] = x_+$.

Remark 3.6. Before going further, let us explore in details what are the implications of the optimal policy adopted by a weakly informed CPT investor. As Jin and Zhou noticed in [6], Footnote 7, a non-informed agent selects a final payoff which looks like a gamble on a good state of the world[7]. In fact, in their framework a trader obtains a final wealth greater than her reference point if and only if the event $\{\rho \le c^*\}$ happens. In a market with one risky asset, constant coefficients and null interest rate, this amounts to say that the final price of the stock, namely $S(T)$, must be greater than a certain threshold depending on c^*. This can be easily shown by noting that

$$S(T) = s_0 \exp \left(\left(b - \tfrac{\sigma^2}{2} \right) T + \sigma W_T^{\mathbb{P}} \right),$$

where $W^{\mathbb{P}}$ is a $(\mathbb{F}, \mathbb{P})$-Brownian motion over $[0, T]$. This in turn implies

$$\{\rho \le c^*\} = \left\{ \exp \left(-\tfrac{b^2}{2\sigma^2} T - \tfrac{b}{\sigma} W_T^{\mathbb{P}} \right) \le c^* \right\} = \left\{ S(T) \ge s_0 \exp \left(\tfrac{b - \sigma^2}{2} T - \tfrac{\sigma^2}{b} \ln c^* \right) \right\}.$$

Obviously one can see that the greater c^*, the higher is the $\mathbb{P}$-probability to reach a final gain. Can we find a similar explanation for a weakly informed CPT investor? A good state of the world for I-agent is the event

$$\left\{ \tfrac{1}{\xi(Y)} \le c^{\nu *} \right\} = \left\{ \xi(Y) \ge \tfrac{1}{c^{\nu *}} \right\}.$$

Again it is clear that the greater $c^{\nu *}$, the higher is the $\mathbb{Q}^\nu$-probability of a terminal gain. Moreover, note that the optimal threshold $c^{\nu *}$ varies with the weak information (Y, ν)! (Recall that $c^{\nu *}$ is obtained in Step 2 of the previous algorithm, where one has to solve Problem (3.16)). It would be interesting to analyze how much $c^{\nu *}$ or the probability of a terminal (positive) gain $\mathbb{Q}^\nu\{\xi(Y) \ge 1/c^{\nu *}\}$ vary depending on $\xi(Y)$, and this is in general not an easy task. However we are able to provide an interesting example where this dependence can be estimated (see Example 3.1 below).

Analogously to the non-informed agent case, let's now assume that the I-agent has CRRA value functions. We follow again the argument described in [6], Section 9, but now using the following functions

$$\varphi^\nu(c) := \mathbb{E}^\nu \left[\left(\xi(Y) T'_+(F^\nu(\tfrac{1}{\xi(Y)})) \right)^{1/(1-\alpha)} \frac{1}{\xi(Y)} I_{\frac{1}{\xi(Y)} \le c} \right] > 0, \quad \underline{\frac{1}{\xi(Y)}} < c \le \overline{\frac{1}{\xi(Y)}}, \tag{3.21}$$

$$k^\nu(c) := \frac{k_- T_-(1 - F^\nu(c))}{\varphi^\nu(c)^{1-\alpha} \left(1 - F^{\mathbb{Q}}(c)\right)^\alpha} > 0, \quad \underline{\frac{1}{\xi(Y)}} < c \le \overline{\frac{1}{\xi(Y)}}. \tag{3.22}$$

Note that the case $c = \underline{\frac{1}{\xi(Y)}}$ is trivial and that once again the sign of the initial wealth x_0 is crucial.

[6] We observe that our Assumption 3.6 is nothing but Assumption 4.1 in [6]. However, in their context condition *(i)* concerns the distribution function of the state price density ρ, thus it involves the market parameters. On the contrary, in our case *(i)* imposes a link between the distortion $T_+(\cdot)$ and $F^\nu(\cdot)$, therefore it is a condition on I-agent's weak information. A similar remark holds for *(iii)*.

[7] Remember that in the original framework the agent knows the historical measure $\mathbb{P}$ but this is by no means helpful, i.e. it does not give any advantage because $\mathbb{P}$ is common knowledge.

Proposition 3.6. *Assume that $x_0 \geq 0$ and Assumption 3.6 holds.*

(i) If $\inf_{c > \underline{\frac{1}{\xi(Y)}}} k^\nu(c) \geq 1$, *then Problem* (CPT-I) *is well-posed and*

$$X^{\nu *} = \frac{x_0}{\varphi^\nu(\overline{\frac{1}{\xi(Y)}})} \left(\xi(Y) T'_+ (F^\nu(\tfrac{1}{\xi(Y)})) \right)^{1/(1-\alpha)}, \tag{3.23}$$

$$V^\nu(x_0, \nu) = x_0^\alpha \varphi^\nu(\overline{\tfrac{1}{\xi(Y)}})^{1-\alpha}. \tag{3.24}$$

(ii) If $\inf_{c > \underline{\frac{1}{\xi(Y)}}} k^\nu(c) < 1$, *then Problem* (CPT-I) *is ill-posed.*

Note that a null initial wealth corresponds to a null risky investment and a null financial value. Finally, if $x_0 < 0$ it is sufficient to adapt the results of [6], Theorem 9.2, to the present case.

Example 3.1 (*Computation of $V(x_0, \nu)$ with $T_+(\cdot)$ convex*)**.** We provide an example where the optimal value of a CPT I-agent with weak information (Y, ν) can be explicitly computed. We assume CRRA value functions with $x_0 \geq 0$ for the informed agent and a single risky asset market analogous to that of Example 2.1, with weak information given by $Y = W_T^{\mathbb{Q}}$ and $\nu \sim \mathcal{N}(0, s^2)$ with $0 < s \leq \sqrt{T}$. It is easy to compute

$$\frac{1}{\xi(Y)} = \frac{s}{\sqrt{T}} \exp\left\{ \frac{T - s^2}{2Ts^2} (W_T^{\mathbb{Q}})^2 \right\}, \tag{3.25}$$

which immediately gives $\underline{\frac{1}{\xi(Y)}} = \frac{s}{\sqrt{T}}$ and $\overline{\frac{1}{\xi(Y)}} = +\infty$. Next, Assumption 3.4 is clearly satisfied, i.e. $\frac{1}{\xi(Y)}$ has no atoms under $\mathbb{Q}^\nu$, as $\frac{1}{\xi(Y)}$ does not have atoms under $\mathbb{Q}$ and the two measures are equivalent. Moreover $\frac{1}{\xi(Y)} \in (0, +\infty)$ $\mathbb{Q}^\nu$-a.s. and its $\mathbb{Q}^\nu$-expected value is 1. Thus every technical condition is fulfilled.
The next step consists in verifying the three conditions in Assumption 3.6. *(ii)* follows immediately by the CRRA hypothesis and *(iii)* will be checked *a posteriori* once we have performed the necessary computations. For condition *(i)*, we observe that the law of Y under $\mathbb{Q}^\nu$ is exactly ν. Hence, with some tedious but elementary computations one can check that the distribution function of $\frac{1}{\xi(Y)}$ under $\mathbb{Q}^\nu$ is given by

$$F^\nu(c) = \mathbb{Q}^\nu\left\{ \tfrac{1}{\xi(Y)} \leq c \right\} = \begin{cases} 0 & \text{if } c \leq \frac{s}{\sqrt{T}}, \\ 2\mathcal{N}\left(\sqrt{\dfrac{2T}{T - s^2} \ln\left(c\frac{\sqrt{T}}{s} \right)} \right) - 1 & \text{if } c > \frac{s}{\sqrt{T}}, \end{cases} \tag{3.26}$$

where $\mathcal{N}(\cdot)$ is the distribution function of a standard Gaussian random variable. The left inverse of $F^\nu(\cdot)$ is given by

$$(F^\nu)^{-1}(z) = \frac{s}{\sqrt{T}} \exp\left(\frac{T - s^2}{2T} \left[\mathcal{N}^{-1}\left(\frac{z+1}{2} \right) \right]^2 \right), \qquad z \in [0, 1). \tag{3.27}$$

Now, condition *(i)* requires the ratio $\frac{(F^\nu)^{-1}}{T'_+}(z)$ to be non decreasing over $(0, 1]$. If the distortion $T_+(\cdot)$ is assumed to be twice continuously differentiable, we see that this is indeed the case whenever the derivative of that ratio is non-negative. Note that a sufficient condition for this to happen is $T''_+(\cdot) \leq 0$ over $[0, 1]$, as it ensures

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{(F^\nu)^{-1}}{T'_+}(z) = \frac{\left[(F^\nu)^{-1} \right]' T'_+ - (F^\nu)^{-1} T''_+}{(T'_+)^2}(z) \geq 0, \qquad z \in (0, 1], \tag{3.28}$$

thanks to the fact that $\left[(F^\nu)^{-1} \right]'(\cdot)$, $T'_+(\cdot)$ and $(F^\nu)^{-1}(\cdot)$ are non-negative functions. By the way, $T''_+(\cdot) \leq 0$ is only a sufficient condition and not necessary. Therefore, we can try to use a non concave $T_+(\cdot)$ and check the validity of *(i)*.
It turns out that a class of weighting functions that fulfills both Assumption 3.2 and the previous condition *(i)* is given by[8]

$$T_+(p) = 2\mathcal{N}\left(\sqrt{1 - 2a}\, \mathcal{N}^{-1}\left(\frac{p+1}{2} \right) \right) - 1, \qquad a \in \left(0, \tfrac{1}{2}\right). \tag{3.29}$$

[8] To define $T_+(1)$ we use the convention $\mathcal{N}(+\infty) = 1$.

It is not difficult to check that such distortions are globally convex over $(0,1)$, thus implying a prudential criterion when evaluating gains. The lack of concavity restricts our attention to Problem (CPT-I), as Assumption 3.3 for Problem (CPT-N) is not fulfilled. A closer look at equation (3.29) shows that those weighting functions are nothing but the primitives of

$$T'_+(p) = \sqrt{1-2a}\ \exp\left(a\left[\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\right]^2\right), \qquad a \in \left(0, \tfrac{1}{2}\right). \tag{3.30}$$

By using (3.27) and (3.30), rather long calculations show that condition *(i)* is indeed fulfilled if and only if $a \le \frac{T-s^2}{2T} < \frac{1}{2}$ (however, we will choose $a < \frac{T-s^2}{2T}$ as the equality leads to integrability problems, so that condition *(iii)* in Assumption 3.6 might not hold). Forgetting for a while the ill-posedness issue, we now apply Proposition 3.6. After cumbersome (but not difficult) computations, we find

$$\begin{aligned} X^{\nu *} &= x_0 \frac{\sqrt{T-s^2-2aT}}{s\sqrt{1-\alpha}} \exp\left(-\frac{T-s^2-2aT}{2Ts^2(1-\alpha)}(W_T^{\mathbb{Q}})^2\right), \\ V(x_0,\nu) &= x_0^{\alpha} \frac{\sqrt{T(1-2a)}}{s}\left(\frac{s\sqrt{1-\alpha}}{\sqrt{T-s^2-2aT}}\right)^{1-\alpha}. \end{aligned}$$

Performing first order derivatives, it is immediate to see that $V(x_0,\nu)$ is increasing in T, whereas it is decreasing in a and in s. This is perfectly coherent with intuition, as the more accurate the information, the greater should be its value. It is interesting to note that the magnitude of the parameter a determines the "degree" of convexity of $T_+(\cdot)$ and as $a \to 0$, $T_+(\cdot)$ tends to the identity function. As expected, we obtain $\lim_{a\to 0^+} V(x_0,\nu) = u(x_0,\nu)$ as in Example 2.1, because CPT and EU preferences coincide.

According to the last observation in Remark 3.6, we can provide in this example some information on how the threshold $c^{\nu *}$ varies with the information (Y,ν). A closer look at the shape of the distribution function $F^{\nu}(\cdot)$ in (3.26) shows that if $s \to 0$ (which corresponds to a more and more accurate information), then the random variable $1/\xi(Y)$ tends to be more and more concentrated around 0, i.e. (3.26) tends to $I_{c>0}$. In the other extreme case, as $s \to \sqrt{T}$ (which corresponds to no additional information case) $1/\xi(Y)$ tends to be more and more concentrated around 1, i.e. (3.26) tends to $I_{c>1}$.

We now come back to the ill-posedness issue. In order to ensure well-posedness, one has to compute $\varphi^{\nu}(c)$ and specify a particular form for $T_-(\cdot)$ and check whether $\inf_{c>\frac{1}{\xi(Y)}} k^{\nu}(c) \ge 1$ as we did in Corollary 3.1. Nonetheless, we observe that we can also find an estimate of the value $V(x_0,\nu)$ in the well-posed case. If $\inf_{c>0} k^{\nu}(c) \ge 1$, then we know that $V(x_0,\nu) = x_0^{\alpha}\varphi^{\nu}(+\infty)^{1-\alpha}$ and we can compute

$$\varphi^{\nu}(+\infty) = \mathbb{E}^{\mathbb{Q}}\left[\xi(Y)^{\frac{1}{1-\alpha}} T'_+\left(F^{\nu}\left(\tfrac{1}{\xi(Y)}\right)\right)^{\frac{1}{1-\alpha}}\right] \ge \inf_{p\in[0,1]} T'_+(p)\, \mathbb{E}^{\mathbb{Q}}\left[\xi(Y)^{\frac{1}{1-\alpha}}\right] \longrightarrow +\infty \tag{3.31}$$

as $s \to 0^+$ whenever the infimum appearing in (3.31) above is strictly positive. On one hand, this fact suggests that well-posedness can only be assessed for weak information that are not too accurate, i.e. when s^2 is close to T. On the other hand, the condition on $\inf_{p\in[0,1]} T'_+(p) > 0$ is fulfilled by our particular choice in (3.30). Moreover, this intuition is implicit in the empirical estimation in [12] where the suggested distortion $T_+(p) = \frac{p^{\delta}}{(p^{\delta}+(1-p)^{\delta})^{1/\delta}}$ automatically satisfies $\inf_{p\in[0,1]} T'_+(p) > 0$ for sufficiently high δ (approximatively $\delta > 0.28$, whereas in [12] it was estimated $\delta = 0.69$).

Example 3.2 (*Reversed S-shaped probability distortion* $T_+(\cdot)$). We are aware of the fact that empirical observations suggest probability weighting not to be globally convex neither globally concave. While in the previous example an *ad hoc* construction has been performed in order to obtain explicit (and sensible!) expressions, we now suggest a particular reversed S-shaped $T_+(\cdot)$ which may look like an observable one. Following the same lines as in [6], Example 6.1, and using the framework in Example 3.1, it is not difficult to exhibit such a distortion. Setting $\delta = \frac{s^2-T}{T}$, for a given set of parameters

$a < 0$, $0 < b < -\frac{1}{\delta}$, $c_0 > \frac{s}{\sqrt{T}}$ we obtained

$$T_+(p) = \begin{cases} 4k\left(\frac{s}{\sqrt{T}}\right)^a \delta_a \left[\mathcal{N}\left(\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\delta_a\right) - \frac{1}{2}\right] & p \in [0, p_0), \\ 4k\left(\frac{s}{\sqrt{T}}\right)^a \delta_a \left[\mathcal{N}\left(\mathcal{N}^{-1}\left(\frac{p_0+1}{2}\right)\delta_a\right) - \frac{1}{2}\right] & \\ \quad + 4\tilde{k}\left(\frac{s}{\sqrt{T}}\right)^b \delta_b \left[\mathcal{N}\left(\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\delta_b\right) - \mathcal{N}\left(\mathcal{N}^{-1}\left(\frac{p_0+1}{2}\right)\delta_b\right)\right] & p \in (p_0, 1), \end{cases} \tag{3.32}$$

where $\delta_a := \sqrt{\frac{1}{\delta}\left(a + \frac{1}{\delta}\right)}$, $\delta_b := \sqrt{\frac{1}{\delta}\left(b + \frac{1}{\delta}\right)}$, $\tilde{k} := kc_0^{a-b}$, $p_0 := F^\nu(c_0)$ and the real number k is uniquely determined by the terminal condition $T_+(1) = 1$. Note that such a $T_+(\cdot)$ is a non decreasing function over $[0, 1]$ and $T'_+(p) \to +\infty$ as $p \to 0$ or $p \to 1$, which is consistent with the empirical estimates. However, it is important to note that the overall construction depends on the weak information (Y, ν), thus it seems to be completely unrealistic. This flaw was still present in Example 6.1 of [6], where the *ad hoc* distortion depends on the market parameters! To conclude, we note that the condition $\inf_{p\in[0,1]} T'_+(p) > 0$ which ensured (3.31) is satisfied for the $T_+(\cdot)$ that we exhibit in (3.32). In fact we have

$$T'_+(p) = T'_+(F^\nu(x)) = \begin{cases} kx^a & \text{if } 0 < x \le c_0, \\ \tilde{k}x^b & \text{if } x > c_0, \end{cases}$$

which is always greater than or equal to $kc_0^a > 0$.

4 Comparison between differently informed CPT agents

Our analysis distinguished four different types of investors, depending on their information (N-agents versus I-agents) and on their valuation criteria or preferences paradigms (classical EU maximizers versus CPT investors *à la* Kahneman and Tversky).

In [2], the authors already compared an EU N-agent with an EU I-agent; the main result is the fact that the insider always gets more than a non informed agent (see also the estimate in Example 2.1). Furthermore, the differences between an EU N-agent and a CPT N-agent are easy to analyze: on one hand, we have seen that the optimal policy for a classical N-agent is to choose a constant wealth, i.e. $X^* = x_0$ $\mathbb{Q}$ a.s., leading to the optimal value $U(x_0)$, whereas the CPT N-agent's strategy is characterized by her substantial indifference between events with the same probability. This phenomenon produces structurally different optimal final wealths, as the *behavioral* agent can even exploit a leverage effect by choosing a negative final wealth with positive probability. Moreover, this kind of investor can obviously select $(p^*, x_+^*) = (1, x_0)$, thus obtaining $V(X^*) = u_+(x_0)$. However this strategy is not necessarily the best one, as she has to face Problem (3.5).

The comparison between EU and CPT agents sharing the same extra information is interesting only from a qualitative point of view. This is because decision criteria are extremely far from each other. For an additional insight, we refer the interested reader to [4] and [6] where the optimal strategies as well as the optimal values of the problems are compared. Here, we only notice that one can have $u(x_0, \nu) < +\infty$ and $V(x_0, \nu) = +\infty$ even if the two insiders share a common extra information (Y, ν). To see this, assume the same market setting as in Example 3.1. Now, for any fixed initial endowment $x_0 > 0$, Example 2.1 shows that *for every* $s > 0$ we have $u(x_0, \nu) < +\infty$, whereas it tends to infinity only when $s \downarrow 0$. However, if we assume $T_-(\cdot) = id(\cdot)$, then *for every* $s > 0$ Proposition 3.4 implies ill-posedness for the I-agent, i.e. $V(x_0, \nu) = +\infty$.

From now on, we compare the solutions and the optimal values of the problems faced by a CPT N-agent and a weakly informed CPT I-agent. We assume that they share the same initial endowment $x_0 \in \mathbb{R}$, the same utility functions $u_\pm(\cdot)$ and the same probability weightings $T_\pm(\cdot)$. We suppose that Assumption 3.3 and 3.6 are in force and I-agent has a weak information given by (Y, ν). In asymmetric information models (see e.g. [10]), a fundamental issue consists in proving that an insider always "gets" more than a non-informed trader and the difference between the optimal values of these two investors is usually called *insider's gain*. Now, we are going to prove that this fact still holds in such a *behavioral* setting. In other words, we show the inequality $V(x_0, \nu) \ge V(X^*)$, whose intuitive meaning is that any additional information is an advantage for the investor, even if she is a *behavioral* one.

Before stating the main result, we need the following preliminary lemma which compares $F^\nu(\cdot)$ and $F^{\mathbb{Q}}(\cdot)$, the distribution functions of the random variable $1/\xi(Y)$ w.r.t. $\mathbb{Q}^\nu$ and $\mathbb{Q}$ respectively.

Lemma 4.1. *The following inequality holds:*

$$F^\nu(c) \geq F^{\mathbb{Q}}(c), \quad \forall\, c \in \left[\underline{\tfrac{1}{\xi(Y)}}, \overline{\tfrac{1}{\xi(Y)}}\right]. \tag{4.1}$$

Moreover, if $c \in \left(\underline{\frac{1}{\xi(Y)}}, \overline{\frac{1}{\xi(Y)}}\right)$ *and* $\xi \not\equiv 1$ *then* (4.1) *holds strictly.*

Proof. If $c \geq 1$, then it is sufficient to use the estimation

$$\begin{aligned}
F^\nu(c) &= 1 - \mathbb{E}^{\mathbb{Q}}\left[\xi(Y) I_{\frac{1}{\xi(Y)}>c}\right] \\
&= F^{\mathbb{Q}}(c) + \mathbb{E}^{\mathbb{Q}}\left[(1-\xi(Y)) I_{\frac{1}{\xi(Y)}>c}\right] \\
&\geq F^{\mathbb{Q}}(c) + \left(1 - \tfrac{1}{c}\right)(1 - F^{\mathbb{Q}}(c)) \\
&\geq F^{\mathbb{Q}}(c).
\end{aligned}$$

Otherwise, if $c < 1$ we observe that the function $f(c) := \mathbb{E}^{\mathbb{Q}}\left[(1-\xi(Y))\, I_{\frac{1}{\xi(Y)}>c}\right]$ is increasing in $\left(\underline{\frac{1}{\xi(Y)}}, 1\right)$ and $\lim_{c \downarrow \underline{\frac{1}{\xi(Y)}}} f(c) = 0$. Finally, the strict inequality is a consequence of Assumption 2.2, i.e. the equivalence of ν and $\mathbb{Q}_Y$. □

We are now ready to prove the existence of the *insider's gain.* Note that it suffices to find a particular feasible solution to Problem (CPT-I) whose prospect value for the informed trader is greater than or equal to $V(X^*)$. A quick look at Problems (CPT-N) and (CPT-I) shows that they share the same feasible set. Hence, we could even choose X^* as the insider's terminal wealth. This random variable will not be the optimal solution to Problem (CPT-I). However, we will be able to prove that $V^\nu(X^*) \geq V(X^*)$ and this will in turn imply $V(x_0, \nu) \geq V(X^*)$.

Theorem 4.1. *Let Assumption 2.2 hold. Then*

$$V(x_0, \nu) \geq V(X^*).$$

Moreover, if $\xi \not\equiv 1$, $V(X^*) < +\infty$ *and the optimal solution* (p^*, x_+^*) *to Problem* (3.5) *is such that* $p^* \in (0,1)$*, then the inequality is strict.*

Proof. First of all, we recall that a *behavioral* N-agent endowed with a concave $T_-(\cdot)$ is indifferent in choosing (3.6) as optimal solution for any given $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$. Now we distinguish between two cases, namely when the optimal value for N-agent is finite or not. In the first case, the non-informed trader can select $\tilde{Z} = F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)$. Using (3.6), (3.7) and setting $c^* := (F^{\mathbb{Q}})^{-1}(p^*)$, we find

$$X^* = (u_+')^{-1}\left(\tfrac{\lambda(c^*, x_+^*)}{T_+'(Z)}\right) I_{\frac{1}{\xi(Y)} \leq c^*} - \frac{x_+^* - x_0}{1 - F^{\mathbb{Q}}(c^*)} I_{\frac{1}{\xi(Y)} < c^*}, \tag{4.2}$$

$$V(X^*) = V_+(X^{*+}) - u_-\left(\frac{x_+^* - x_0}{1 - F^{\mathbb{Q}}(c^*)}\right) T_-(1 - F^{\mathbb{Q}}(c^*)), \tag{4.3}$$

where (p^*, x_+^*) are optimal for Problem (3.5) and $\lambda(c^*, x_+^*)$ is determined by the budget constraint. On the other hand, if the informed investor chooses X^* as her terminal wealth, then she obtains the prospect value

$$V^\nu(X^*) = V^\nu(X^{*+}) - u_-\left(\frac{x_+^* - x_0}{1 - F^{\mathbb{Q}}(c^*)}\right) T_-(1 - F^\nu(c^*)). \tag{4.4}$$

Now, using Lemma 4.1, it is immediate to see that the negative part of the prospect value for N-agent is greater (in absolute value) than that of I-agent. Moreover, using the fact that $u_+\left((u_+')^{-1}\right)(\cdot)$ is strictly decreasing, we can explicitly write

$$= \int_0^{+\infty} T_+\left(\mathbb{Q}\left(\left\{T_+'\left(F^{\mathbb{Q}}(\tfrac{1}{\xi(Y)})\right) > \tfrac{\lambda(c^*, x_+^*)}{u_+'\left(u_+^{-1}(y)\right)}\right\} \cap \left\{\tfrac{1}{\xi(Y)} \leq c^*\right\}\right)\right) \mathrm{d}y. \tag{4.5}$$

Now it suffices to note that $T'_+(\cdot)$ is monotone decreasing, whereas $F^{\mathbb{Q}}(\cdot)$ is monotone increasing; furthermore, $V_+^{\nu}(X^{*+})$ can be written analogously to $V_+(X^{*+})$ just by replacing $\mathbb{Q}$ with $\mathbb{Q}^{\nu}$. Therefore, applying again Lemma 4.1 we have $V_+^{\nu}(X^{*+}) \geq V_+(X^{*+})$, which in turn implies the desired inequality. In the ill-posed case for N-agent, note that we can find a sequence of feasible solutions to Problem (3.5), namely $\{(p^n, x_+^n)\}_{n\in\mathbb{N}}$, where $p^n = (F^{\mathbb{Q}})^{-1}(c^n)$ for some $c^n \in \left[\frac{1}{\underline{\xi(Y)}}, \frac{1}{\overline{\xi(Y)}}\right]$, for every $n \in \mathbb{N}$. This sequence will in turn induce a sequence of feasible terminal wealths $\{X^{\nu,n}\}_{n\in\mathbb{N}}$. Now, using the previous argument, it is easily seen that if I-agent chooses that sequence of terminal wealths, then her optimal value will diverge to $+\infty$ too. Finally, the strict version is a consequence of (4.1) holding strictly. $\square$

We remark that in general the optimal pair (c^*, x_+^*) for N-agent will be different than $(c^{\nu*}, x_+^{\nu*})$. Moreover, we have seen that ill-posedness for (CPT-N) implies ill-posedness for (CPT-I). A natural question arising from this observation is whether it is possible to find an example where (CPT-N) is well-posed and (CPT-I) is ill-posed. The answer is positive and we are going to exploit some results previously obtained in a single risky asset market driven by an $(\Omega, \mathbb{F}, \mathbb{Q})$-Brownian motion (see Example 3.1 for the notation).

Proposition 4.1. *Assume CRRA preferences with $x_0 \geq 0$, $T_+(p) = p$ and $T_-(p) = p^{\delta}$, $0 < \delta < \alpha < 1$. If the weak information of CPT I-agent is given by $Y = W_T^{\mathbb{Q}}$ and $\nu \sim \mathcal{N}(0, s^2)$ with $0 < s \leq \sqrt{T}$, then for sufficiently small s Problem* (CPT-I) *is ill-posed.*

Proof. To start, recall that with these assumptions on the agents' preferences, Corollary 3.1 ensures well-posedness for Problem (CPT-N). Then, ill-posedness for Problem (CPT-I) follows from Proposition 3.6 if we are able to show that

$$\inf_{c > \frac{1}{\underline{\xi(Y)}}} k^{\nu}(c) \equiv \inf_{c > \frac{1}{\underline{\xi(Y)}}} \frac{k_-(1 - F^{\nu}(c))^{\delta}}{(1 - F^{\mathbb{Q}}(c))^{\alpha} \left(\mathbb{E}^{\mathbb{Q}}\left[\xi(Y)^{\frac{1}{1-\alpha}} I_{\frac{1}{\xi(Y)} \leq c}\right]\right)^{1-\alpha}} < 1. \tag{4.6}$$

Now, we apply Jensen's inequality to the convex function $f(x) = x^{\frac{1}{1-\alpha}}$, $\alpha \in (0,1)$, and we estimate the infimum choosing $\hat{c} = (F^{\mathbb{Q}})^{-1}\left(\frac{1-\alpha}{1-\delta}\right)$. Hence we obtain

$$\inf_{c > \frac{1}{\underline{\xi(Y)}}} k^{\nu}(c) \leq \inf_{c > \frac{1}{\underline{\xi(Y)}}} \frac{k_-(1 - F^{\nu}(c))^{\delta}}{(1 - F^{\mathbb{Q}}(c))^{\alpha} F^{\nu}(c)} \leq k_- \left(\frac{1-\delta}{\alpha - \delta}\right)^{\alpha} \frac{(1 - F^{\nu}(\hat{c}))^{\delta}}{F^{\nu}(\hat{c})},$$

where it is important to note that $\hat{c}$ depends both on the preference parameters and on the weak information. At this point it is not difficult to compute

$$F^{\mathbb{Q}}(c) \equiv \mathbb{Q}\left\{\frac{1}{\xi(Y)} \leq c\right\} = \begin{cases} 0 & \text{if } c \leq \frac{s}{\sqrt{T}}, \\ 2\mathcal{N}\left(\sqrt{\frac{2s^2}{T-s^2} \ln\left(c\frac{\sqrt{T}}{s}\right)}\right) - 1 & \text{if } c > \frac{s}{\sqrt{T}}, \end{cases} \tag{4.7}$$

where as usual $\mathcal{N}(\cdot)$ is the standard Gaussian distribution function. Next, we find

$$(F^{\mathbb{Q}})^{-1}(z) = \frac{s}{\sqrt{T}} \exp\left(\frac{T - s^2}{2s^2}\left[\mathcal{N}^{-1}\left(\frac{z+1}{2}\right)\right]^2\right), \quad z \in [0, 1). \tag{4.8}$$

Using the explicit expression of $F^{\nu}(\cdot)$ in equation (3.26), we can compute

$$F^{\nu}(\hat{c}) = 2\mathcal{N}\left(\frac{\sqrt{T}}{s}\mathcal{N}^{-1}\left(\frac{2 - \alpha - \delta}{2(1-\delta)}\right)\right) - 1.$$

Now we see that for every choice of $k_- \geq 1$ and $0 < \delta < \alpha < 1$, there exists a $\tilde{s} > 0$ such that for every $s < \tilde{s}$ the inequality in (4.6) is fulfilled. $\square$

Economically speaking, the meaning of the previous proposition is that there can always exist a particular weak information which ensures well-posedness for N-agent's problems and ill-posedness for the informed investor. Obviously, this extra information must be sufficiently accurate (in our case $s < \tilde{s}$) in order to provide an infinite optimal value for I-agent. We recognize that our estimation

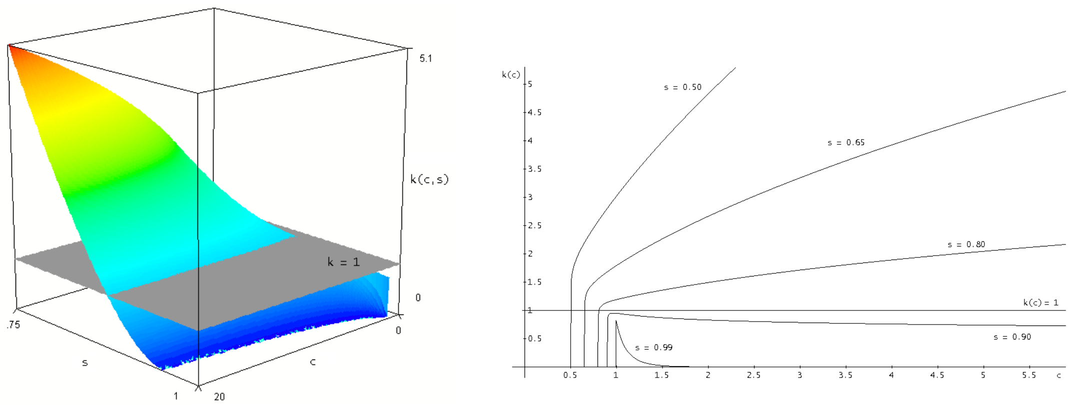

Figure 2: a graphical analysis for the ill-posedness of the CPT insider's problem.

for (4.6) is effectively rough. For a more detailed analysis, we note that an explicit expression for $k^\nu(c)$ can be provided, even if it is quite cumbersome. However, it is not difficult to perform a graphical analysis whose results are shown in Figure 2. Fixing $\alpha = 0.88$, $\delta = 0.7$ and $T = 1$, k^ν reduces to a function of k_-, s and c; isolating the loss aversion coefficient k_-, we can now see whether $k_- \leq \sup_{c>s} k(c,s)$ which in turn implies ill-posedness for the CPT insider's problem. In the left-side plot, the 3D surface of $k(c,s)$ shows that even for a quite elevated k_- we still have ill-posedness. On the contrary, if s is sufficiently close to 1, then every $k_- \geq 1$ leads to well-posedness, as the surface lies below the horizontal plane at level $k \equiv 1$ and $k(\cdot, s)$ becomes monotonically decreasing. Finally, for particular values of s, i.e. for specific types of weak information, we have drawn in the right-side plot the corresponding curves $k(c)$ which confirm what previously stated.

Remark 4.1. It is worth noting that with the same hypotheses as in Proposition 4.1, the analogous problem for a *classical* informed trader has a completely different solution. Indeed, Problem (EU-I) is well-posed *for every* $s > 0$ and its optimal value tends to diverge only if $s \downarrow 0$. On the other hand, if we assume $T_-(\cdot) = id$, then Problem (CPT-I) too becomes ill-posed *for every* $s > 0$, thus showing a substantial lack of robustness.

To conclude this section, we now provide an example where the insider's gain can be explicitly computed and whose results have a clear and intuitive economic explanation.

Example 4.1 (*Explicit evaluation of the insider's gain*)**.** We use exactly the same setting of Example 3.1 changing only the probability distortion on gains of the informed investors. Precisely, this time we assume

$$T_+(p) = 2\mathcal{N}\left(\sqrt{1+2b}\,\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\right) - 1, \qquad b > 0. \tag{4.9}$$

These weighting functions are globally concave over $(0,1)$ [9] and are the primitives of

$$T_+'(p) = \sqrt{1+2b}\,\exp\left(-b\left[\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\right]^2\right), \qquad b > 0. \tag{4.10}$$

As we did in Example 3.1, we check condition *(i)* of Assumption 3.3, as *(ii)* is clearly true and *(iii)* will be controlled *ex post*. Using equation (3.27) and performing the first order derivative, it is immediate to see that $\frac{(F^\nu)^{-1}}{T_+'}(\cdot)$ is non decreasing over $(0,1]$ for every $b > 0$. Hence, we can make our computations assuming well-posedness for Problem (CPT-I), which implies that of Problem (CPT-N) as well (thanks to Theorem 4.1). For the non-informed investor, we exploit the results of Proposition 3.2, which give us

$$X^* = x_0\sqrt{\tfrac{1-\alpha+2b}{(1-\alpha)(1+2b)^{1/(1-\alpha)}}}\,T_+'(Z)^{1/(1-\alpha)}, \tag{4.11}$$

$$V(X^*) = x_0^\alpha\sqrt{1+2b}\left(\sqrt{\tfrac{1-\alpha}{1-\alpha+2b}}\right)^{1-\alpha}. \tag{4.12}$$

[9] Thus, we observe an overestimation of relatively large gains and an underestimation of small gains.

On the other hand, for the CPT insider we apply Proposition 3.6 which yields

$$X^{\nu *} = x_0 \tfrac{T-s^2\alpha+2bT}{s^2(1-\alpha)} \exp\left\{-\tfrac{T-s^2+2bT}{2Ts^2(1-\alpha)}(W_T^{\mathbb{Q}})^2\right\}, \tag{4.13}$$

$$V(x_0,\nu) = x_0^\alpha \sqrt{\tfrac{T(1+2b)}{s^2}} \left(\sqrt{\tfrac{s^2(1-\alpha)}{T-s^2\alpha+2bT}}\right)^{1-\alpha}. \tag{4.14}$$

The insider's gain is thus given by $V(x_0,\nu) - V(X^*)$. For our purposes, it is more convenient to compute the ratio

$$\frac{V(x_0,\nu)}{V(X^*)} = \frac{\sqrt{T}}{s}\left(\sqrt{\frac{s^2(1-\alpha)}{T-s^2\alpha+2bT}}\right)^{1-\alpha} \geq 1, \tag{4.15}$$

which is increasing in both b, T and decreasing in s, whereas the dependence on α is not monotone. Note that this makes perfectly sense since a greater T (or a lower s) improves the accuracy of insider's information. Moreover, if $s \uparrow \sqrt{T}$, the ratio (4.15) decreases to 1. On the other hand, as $b \downarrow 0$ we see that $T_+(\cdot)$ converges uniformly to the identity function and, in case of well-posedness, we recover the same results of Example 2.1, where the agent was a *classical* insider. Finally, if $\alpha \uparrow 1$, then (4.15) tends to $\sqrt{T}/s$, which is equivalent to say that if the trader becomes risk neutral, then the ratio between the optimal values is nothing but an index of the "goodness" of the extra information.

The comparison between the optimal terminal wealths X^* and $X^{\nu *}$ exhibits the already known flaw of being dependent on the choice of Z; in particular, if $Z = F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)$ then straightforward computations show that $X^{\nu *} \geq X^*$ if and only if the terminal price of the stock lies in a certain range, whereas if $Z = 1 - F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)$ we obtain the opposite result.

5 The Yaari's models and their solutions

In this section we are going to look at the model proposed by Yaari in [13]. That model is somewhat linked to CPT model since a probability distortion $w(\cdot)$ is applied as well. However, in that model gains are not distinct from losses, so what is important for the trader is the level of terminal wealth X. Moreover, the value function is simply the identity, hence distortions on payments are not allowed. We will solve the problems relative to a non informed investor and an insider respectively, following the approach developed in [4]. At last, we will provide an example where the insider's gain can be explicitly computed. From now on, the following assumptions on the distortion $w(\cdot)$ will be in force.

Assumption 5.1 (see [4], Assumption 3.3)**.** $w(\cdot) : [0,1] \to [0,1]$ *is continuous and strictly increasing with* $w(0) = 0$*,* $w(1) = 1$*. Furthermore,* $w(\cdot)$ *is continuously differentiable on* $(0,1)$.

5.1 The non informed agent's problem

For our N-agent, we adapt the solution scheme proposed in [4], Section 3.2. Assuming an initial endowment $x_0 > 0$, a standard formulation of this model would be

$$\begin{aligned} &\text{Maximize} \quad V_Y(X) := \textstyle\int_0^{+\infty} w\left(\mathbb{Q}\{X > x\}\right) \mathrm{d}x \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}}[X] = x_0, \quad X \geq 0, \quad X \text{ is } \mathscr{F}_T\text{-measurable.} \end{aligned} \tag{YA-N}$$

Once again, we note that the objective function is law-invariant, in the sense that if X is a feasible solution to (YA-N) with distribution function $F_X(\cdot)$, then for every $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$ we have $V_Y(X) = V_Y\left((F_X)^{-1}(Z)\right)$ (see e.g. Lemma A.1). Moreover, the structure of the objective function may be a source of ill-posedness, similarly to what happened for the CPT model. A straightforward adaptation of the proof in [4], Theorem 3.4, shows the next result.

Proposition 5.1. *Under Assumption 5.1, Problem* (YA-N) *is ill-posed if* $\liminf_{z\downarrow 0} w'(z) = +\infty$*, and well-posed if* $\limsup_{z\downarrow 0} w'(z) < +\infty$.

In particular, $w(z) = z^\gamma$ leads to well-posedness if $\gamma > 1$, whereas $\gamma < 1$ implies ill-posedness. We also recall that in Yaari's model, a convex distortion is equivalent to risk aversion. Moreover, using Jensen's inequality we see that if $w(\cdot)$ is convex, then Problem (YA-N) has the trivial solution $X_Y^* = x_0$ $\mathbb{Q}$ a.s. with $V_Y(X_Y^*) = x_0$ (we will use this fact in Example 5.1). Therefore, it remains some other interesting shapes of $w(\cdot)$ to analyze. Following [4], we impose this technical condition.

Assumption 5.2 (see [4], Assumption 3.5). *$M(z) := w'(1-z)$ is continuous on $(0,1)$, and there exists $z_0 \in (0,1)$ such that $M(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$.*

In other words, the previous assumption describes an S-shaped distortion function which is useful for the subsequent mathematical analysis whereas it is not properly suitable in an economic sense from a descriptive or a normative point of view. In fact, such an S-shaped $w(\cdot)$ implies underweighting of both relatively large and small payoffs. Using exactly the same argument as in the proofs of Proposition 3.6 and Theorem 3.7 in [4], we obtain the main result of this section.

Proposition 5.2. *Suppose Assumption 5.2 holds. Define*

$$z(\lambda) := \inf\left\{z \in (0, z_0] : M(z) = \lambda\right\}, \tag{5.1}$$

$$h(\lambda) := w\left(1 - z(\lambda)\right) - \lambda\left(1 - z(\lambda)\right), \tag{5.2}$$

and let λ^* be the unique positive root of $h(\cdot)$. Then, for every $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$, we have $X_Y^* = b^* I_{z(\lambda^*) < Z \le 1}$, where $b^* = \frac{x_0}{1 - z(\lambda^*)}$ is determined by the budget constraint. Moreover, $V_Y(X_Y^*) = b^* w\left(1 - z(\lambda^*)\right)$.

5.2 The insider's problem

For a weakly informed trader who follows the tenets of Yaari's dual theory of choice, the optimization problem can be naturally set as follows

$$\begin{aligned} &\text{Maximize} \quad V_Y^\nu(X) := \textstyle\int_0^{+\infty} w\left(\mathbb{Q}^\nu\left\{X > x\right\}\right) \mathrm{d}x \\ &\text{subject to} \quad \mathbb{E}^\nu\left[\tfrac{1}{\xi(Y)} X\right] = x_0, \quad X \ge 0, \quad X \text{ is } \mathscr{F}_T\text{-measurable.} \end{aligned} \tag{YA-I}$$

We will call $X_Y^{\nu *}$ its optimal solution, with optimal value $V_Y(x_0, \nu)$. Once again, we recover the same structure as in [4], Problem (2.11), and we replace ρ with $\frac{1}{\xi(Y)}$ and $\mathbb{P}$ with $\mathbb{Q}^\nu$. Before giving the solution, we impose the technical hypothesis

Assumption 5.3 (see [4], Assumption 5.2). *$M^\nu(z) := \frac{w'(1-z)}{(F^\nu)^{-1}(1-z)}$ is continuous on $(0,1)$, and there exists $z_0 \in (0,1)$ such that $M(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$.*

The solution to Problem (YA-I) is completely described in the next proposition.

Proposition 5.3. *Suppose Assumption 5.3 holds. Define*

$$z^\nu(\lambda^\nu) := \inf\left\{z \in (0, z_0] : M^\nu(z) = \lambda^\nu\right\}, \tag{5.3}$$

$$h^\nu(\lambda) := \int_{z^\nu(\lambda)}^1 \left[w'(1-z) - \lambda (F^\nu)^{-1}(1-z)\right] \mathrm{d}z. \tag{5.4}$$

Let $\lambda^{\nu *}$ be the unique positive root of $h^\nu(\cdot)$. Then, $X_Y^{\nu *} = b^\nu I_{\frac{1}{\xi(Y)} \le c^\nu}$, where c^ν is the unique root of

$$\varphi^\nu(c) := x w\left(F^\nu(x)\right) - w'\left(F^\nu(x)\right) \int_0^x s \, \mathrm{d}F^\nu(s) \tag{5.5}$$

over $\left((F^\nu)^{-1}(1-z_0), \overline{\frac{1}{\xi(Y)}}\right)$ and b^ν is implicitly defined by the budget constraint $\mathbb{E}^\nu\left[\frac{1}{\xi(Y)} X_Y^{\nu *}\right] = x_0$. Moreover, $V_Y(x_0, \nu) = \lambda^{\nu *} x_0$.

Proof. Use the same arguments as in [4], Proposition 3.6 and Theorem 3.7. □

Before giving an explicit example, we show that an I-agent always gets a higher optimal value than an N-agent. In fact, Problems (YA-N) and (YA-I) share the same feasible set; therefore, the non-informed agent can choose $Z = F^{\mathbb{Q}}(\frac{1}{\xi(Y)})$ and the insider can select the corresponding X_Y^* as her terminal wealth. Hence, using Lemma 4.1, we can compute

$$\begin{aligned}
V_Y^\nu(X_Y^*) &= \int_0^{+\infty} w\left(\mathbb{Q}^\nu\left\{b^* I_{z(\lambda^*)<F^{\mathbb{Q}}(\frac{1}{\xi(Y)})} > x\right\}\right) \mathrm{d}x \\
&= \int_0^{b^*} w\left(F^\nu\left((F^{\mathbb{Q}})^{-1}(1-z(\lambda^*))\right)\right) \mathrm{d}x \\
&\geq \int_0^{b^*} w\left(1-z(\lambda^*)\right) \mathrm{d}x \\
&= V_Y(X_Y^*),
\end{aligned}$$

which obviously implies $V_Y(x_0,\nu) \geq V_Y(X^*)$. This time too the comparison between the optimal terminal wealths is not very sensible, as it strongly depends on the choice of Z.

Example 5.1 (*Evaluation of the insider's gain in Yaari's model*)**.** Consider the single risky asset setting as in Example 2.1. We assume that the weak information of I-agent is given by $Y = F_W(W_T^{\mathbb{Q}})$ and $\nu(\mathrm{d}x) = [(2-2a)x + a]\,\mathrm{d}x$, $a \in (0,1)$, where $F_W(\cdot)$ is the cdf of the random variable $W_T^{\mathbb{Q}}$. Note that $Y \sim U(0,1)$ w.r.t. $\mathbb{Q}$ and the economic intuition behind this example is that the insider has a weak knowledge about the terminal price, as the distortion applied by $F_W(\cdot)$ is irrelevant due to its strict monotonicity. Furthermore, the parameter a is an index of the goodness of the extra information: (Y,ν) becomes in particular more and more valuable as $a \to 0^+$, whereas if $a \to 1^-$ we recover the no additional information case. At this point, we can immediately compute

$$\frac{1}{\xi(Y)} = \frac{1}{(2-2a)F_W(W_T^{\mathbb{Q}})+a}, \qquad \underline{\frac{1}{\xi(Y)}} = \frac{1}{2-a}, \qquad \overline{\frac{1}{\xi(Y)}} = \frac{1}{a}. \tag{5.6}$$

Now, we assume a risk averse investor endowed with probability distortion $w(z) = z^\gamma$, $\gamma > 1$. As noticed in Section 5.1, for such a convex $w(\cdot)$ we already know that $X_Y^* = x_0$ $\mathbb{Q}$ a.s. and $V_Y(X_Y^*) = x_0$. Next, we check the validity of Assumption 5.3. Using equation (5.6) together with the uniform distribution of Y, we find

$$M(z) = \gamma(1-z)^{\gamma-1}\sqrt{4z(a-1)+(a-2)^2}, \qquad z_0 = \frac{a^2(1-\gamma)+2(1-a)}{2(1-a)(2\gamma-1)}. \tag{5.7}$$

Then, we look for a root of $\varphi^\nu(\cdot)$ as defined in Proposition 5.3. It turns out that an admissible $c^\nu \in \left((F^\nu)^{-1}(1-z_0), \overline{\frac{1}{\xi(Y)}}\right)$ is obtained only under an additional condition over the parameters γ and a. More precisely, we have

$$c^\nu = \frac{2\gamma-1}{2-a} \quad \text{if} \quad \gamma < \frac{1}{a}. \tag{5.8}$$

Observe that whenever $\gamma < 1/a$, the quantity z_0 in (5.7) belongs to $(0,1)$. The final step is to find the optimal solution $X_Y^{\nu *}$ together with its optimal value. Using the budget constraint we have

$$X_Y^{\nu *} = b^{\nu *} I_{W_T^{\mathbb{Q}} \geq (F_W)^{-1}\left(\frac{a\gamma-1}{(a-1)(2\gamma-1)}\right)}, \tag{5.9}$$

$$V_Y(x_0,\nu) = x_0\,\frac{\gamma^\gamma(\gamma-1)^{\gamma-1}(2-a)^{2\gamma-1}}{(1-a)^{\gamma-1}(2\gamma-1)^{2\gamma-1}}, \tag{5.10}$$

where $b^{\nu *} = x_0\,\frac{(1-a)(2\gamma-1)}{(2-a)(\gamma-1)}$. We remark that our insider will obtain $b^{\nu *}$ if the terminal prices are higher than a certain threshold which is decreasing in both a and γ as economic intuition suggests. Furthermore, $b^{\nu *}$ is decreasing in both parameters and the $\mathbb{Q}$-probability of obtaining $b^{\nu *}$ is nothing but $\frac{(\gamma-1)(2-a)}{(2\gamma-1)(1-a)}$, which is increasing in both γ and a. Finally, we note that $V_Y(x_0,\nu) \geq x_0$ obviously holds. However, there is no clear dependence of $V_Y(x_0,\nu)$ in the parameters.

6 Conclusions

In this paper we considered portfolio optimization problems for investors following different preference paradigms. Classical expected utility, CPT and Yaari's dual theory maximizers have been studied under both (weakly) informed and non-informed case. The informed case is easy to handle, since the techniques developed by, e.g., [6] in both CPT and Yaari-type cases, can be applied with basically no changes. On the contrary, for the non informed investor, those results cannot be directly applied. Nonetheless, the corresponding optimization problems can be solved using similar techniques, leading to a family of optimal solutions, for which uniqueness *in distribution* of the solution replaces the uniqueness almost surely. In particular, a non expected utility trader obtains an optimal terminal payoff which looks like a gamble on the final price, where this payoff can even be negative in the CPT case. We proved the intuitive fact that the optimal value of a CPT informed agent is always bigger than that of a CPT non-informed agent. In other terms, the value of the (weak) information is always positive. Moreover, in CPT I-agent case, ill-posedness is an even more delicate issue than in the non-informed case. In some involved examples, we performed some graphical analysis which helped us to understand well-posedness as a function of model's parameters.

Another contribution of this paper are explicit computations of the optimal terminal wealths of a CPT and a Yaari-type insider. In particular, we proposed two new classes of probability distortions, a convex and a concave one and a new example of weak information which turns out to be economically meaningful (see Examples 3.1, 4.1 and 5.1).

The partial and the strong information cases are left for future research.

A A Choquet maximization problem

Our aim is to solve a general utility maximization problem which includes a Choquet capacity:

$$\begin{array}{ll} \text{Maximize} & V_1(X) = \int_0^{+\infty} T(\mathbb{P}\{u(X) > y\})\,\mathrm{d}y \\ \text{subject to} & \mathbb{E}^{\mathbb{P}}[X] = a, \ X \geq 0, \end{array} \tag{A.1}$$

where $a \geq 0$, $T : [0,1] \to [0,1]$ is a strictly increasing, differentiable function with $T(0) = 0$, $T(1) = 1$, and $u(\cdot)$ is a strictly concave, strictly increasing, twice differentiable function with $u(0) = 0$, $u'(0) = +\infty$, $u'(+\infty) = 0$. Note that the only difference with the Choquet maximization problem solved in [6], Appendix C, is that their weighting function ξ in the constraint is not atomless, being here a Dirac mass. This makes impossible to use directly their results.
We will denote by X^* the optimal solution to Problem (A.1). The case $a = 0$ is trivial, as it implies $X^* = 0$ with optimal value $V_1(X^*) = 0$; therefore assume $a > 0$. First of all we have the following result, which states the *law-invariance* property of the problem.

Lemma A.1. *Suppose that Problem* (A.1) *admits a feasible solution X whose distribution function is $G(\cdot)$; then for every random variable $Z \sim U(0,1)$ w.r.t. $\mathbb{P}$ we have $V_1(X) = V_1\left(G^{-1}(Z)\right)$.*

Proof. One can easily guess from the structure of Problem (A.1) that the only relevant feature of the optimal solution is its distribution. Formally, for any such Z we can compute

$$\mathbb{E}^{\mathbb{P}}[G^{-1}(Z)] = \int_0^{+\infty} \mathbb{P}\{G^{-1}(Z) > y\}\,\mathrm{d}y = \int_0^{+\infty} \mathbb{P}\{X > y\}\,\mathrm{d}y = \mathbb{E}^{\mathbb{P}}[X] = a,$$

thus the random variable $G^{-1}(Z)$ is feasible and we have

$$\begin{aligned} V_1(X) &= \int_0^{+\infty} T(\mathbb{P}\{u(X) > y\})\,\mathrm{d}y = \int_0^{+\infty} T(\mathbb{P}\{X > u^{-1}(y)\})\,\mathrm{d}y \\ &= \int_0^{+\infty} T(1 - \mathbb{P}\{X \leq u^{-1}(y)\})\,\mathrm{d}y = \int_0^{+\infty} T(1 - G(u^{-1}(y)))\,\mathrm{d}y \\ &= \int_0^{+\infty} T(\mathbb{P}\{Z > G(u^{-1}(y))\})\,\mathrm{d}y = \int_0^{+\infty} T(\mathbb{P}\{u(G^{-1}(Z)) > y\})\,\mathrm{d}y \\ &= V_1(G^{-1}(Z)) \end{aligned}$$

as claimed. □

We notice at once the difference between our Lemma A.1 and Lemma C.1 in [6]: we do not have an *almost sure* result. However we proved that for *any* such Z the previous equivalence holds, thus it is clearly true even for an optimal X^*. Thus we are free to choose any Z uniformly distributed. This is a general feature of our results, i.e. replacing the *almost sureness* with a weaker condition on the distribution functions which gives us an additional degree of freedom. From now on, we follow [6] with some slight modifications. Let's introduce the problem

$$\begin{aligned} &\text{Maximize} \quad v_1(G) := \int_0^{+\infty} T(\mathbb{P}\{u(G^{-1}(Z)) > y\})\,\mathrm{d}y \\ &\text{subject to} \quad \begin{cases} \mathbb{E}^{\mathbb{P}}[G^{-1}(Z)] = a, \\ G \text{ is the distribution function of a non negative random variable,} \end{cases} \end{aligned} \tag{A.2}$$

which changes the domain of our problem from a set of random variables to a set of functions. The functions $G(\cdot)$ appearing in the constraints must be non-decreasing, càdlàg and satisfy $G(0-) = 0$, $G(+\infty) = 1$. From Lemma A.1 we deduce the equivalence between the two previous Problems (A.1) and (A.2).

Proposition A.1. *If G^* is optimal for Problem* (A.2)*, then for any $Z \sim U(0,1)$ w.r.t. $\mathbb{P}$ the random variable $X^* := (G^*)^{-1}(Z)$ is optimal for Problem* (A.1)*. Conversely, if X^* is optimal for* (A.1)*, then its distribution function G^* is optimal for* (A.2)*.*

Performing the same calculations as in [6] and setting

$$\Gamma := \left\{ g : [0,1) \to \mathbb{R}^+, g \text{ is non-decreasing, left continuous, with } g(0) = 0 \right\},$$

we can rewrite Problem (A.2) as

$$\begin{aligned} &\text{Maximize} \quad \overline{v}_1(g) := \mathbb{E}[u(g(Z))T'(1-Z)] \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{P}}[g(Z)] = a, \ g \in \Gamma. \end{aligned} \tag{A.3}$$

Thanks to the assumptions on $T(\cdot)$ and $u(\cdot)$ we now have a concave optimization problem in $g(\cdot)$ and we can use Lagrange method. Thus, for a given $\lambda \in \mathbb{R}$, we can solve

$$\begin{aligned} &\text{Maximize} \quad \overline{v}_1^{\lambda}(g) := \mathbb{E}[u(g(Z))T'(1-Z) - \lambda g(Z)] \\ &\text{subject to} \quad g \in \Gamma, \end{aligned} \tag{A.4}$$

and then determine λ via the original constraint. As noticed in [6], if we ignore the constraint and apply standard maximization techniques we find $g(z) = (u')^{-1}(\lambda/T'(1-z))$. Moreover, if $T'(z)$ is non-increasing in $z \in (0,1]$, then $g(z)$ is non-decreasing in $z \in [0,1)$ and therefore it solves Problem (A.4). However, if $T'(z)$ is not non-increasing then we are not able to find an explicit solution[10]. We remark that if $T(z)$ is twice continuously differentiable, then $T'(z)$ non-increasing amounts to require a concave $T(\cdot)$. In particular $T(\cdot) = id$ satisfies this condition.

Denote $R_u(x) := -\frac{xu''(x)}{u'(x)}$, $x > 0$, the index of Relative Risk Aversion (RRA for short) of the function $u(\cdot)$. We have

Proposition A.2. *Assume that $T'(z)$ is non-increasing in $z \in (0,1]$ and $\liminf_{x\to+\infty} R_u(x) > 0$. Then for any $Z \sim U(0,1)$ w.r.t. $\mathbb{P}$, the following claims are equivalent:*

(i) Problem (A.3) *is well-posed for any $a > 0$.*

(ii) Problem (A.3) *admits a unique optimal solution for any $a > 0$.*

(iii) $\mathbb{E}^{\mathbb{P}}\left[u\left((u')^{-1}(\frac{1}{T'(1-Z)})\right)T'(1-Z)\right] < +\infty.$

(vi) $\mathbb{E}^{\mathbb{P}}\left[u\left((u')^{-1}(\frac{\lambda}{T'(1-Z)})\right)T'(1-Z)\right] < +\infty \ \forall\, \lambda > 0.$

Furthermore, when one of the above (i)-(iv) holds, the optimal solution to Problem (A.3) *is*

$$g^*(x) \equiv (G^*)^{-1}(x) = (u')^{-1}\left(\frac{\lambda}{T'(1-x)}\right), \ x \in [0,1),$$

where $\lambda > 0$ is the one satisfying $\mathbb{E}^{\mathbb{P}}[(G^*)^{-1}(1-Z)] = a$.

[10] If one restricts the domain to the set of step functions $g \in \Gamma$, then solving (A.4) is equivalent to solving a non-linear programming problem in $\mathbb{R}^n$, which once again does not have an easy explicit solution.

Proof. As in the proof of Proposition C.2 in [6], we can define a new probability measure $\tilde{\mathbb{P}}$ such that $\mathrm{d}\tilde{\mathbb{P}} = T'(1-Z)\,\mathrm{d}\mathbb{P}$ and a random variable $\zeta := \frac{1}{T'(1-Z)}$ which is positive $\mathbb{P}$-a.s.. We can now rewrite Problem (A.3) as follows.

$$\begin{aligned} &\text{Maximize} && \overline{v}_1(g) := \mathbb{E}^{\tilde{\mathbb{P}}}[u(g(Z))] \\ &\text{subject to} && \mathbb{E}^{\tilde{\mathbb{P}}}[\zeta g(Z)] = a,\ g \in \Gamma. \end{aligned}$$

By [5], Theorem 5.4, we get the result. □

We remark that the claim *(ii)* (as it appears in Proposition C.2 of [6]) still holds true because the optimal solution $g^*(\cdot)$ to Problem (A.3) determines the inverse of a distribution function whereas the optimal solution X^* to Problem (A.1) is not unique $\mathbb{P}$-a.s. as it depends on the choice of Z. However, X^* is unique *in law*. Moreover we could also replace $1-Z$ with Z in every explicit expression containing an expected value, since $Z \sim U(0,1)$ as well. Now we can state the main result of this section.

Theorem A.1. *Assume that $T'(z)$ is non-increasing in $z \in (0,1]$ and $\liminf_{x\to+\infty} R_u(x) > 0$; for any fixed $Z \sim U(0,1)$ w.r.t. $\mathbb{P}$ define $X(\lambda) := (u')^{-1}\left(\frac{\lambda}{T'(1-Z)}\right)$ for $\lambda > 0$. If $V_1(X(1)) < +\infty$, then $X(\lambda)$ is an optimal solution of Problem (A.1), where λ is the one satisfying $\mathbb{E}^{\mathbb{P}}[X(\lambda)] = a$. If $V_1(X(1)) = +\infty$, then Problem (A.1) is ill-posed.*

With the obvious changes in the proofs, we can also state a necessary condition for optimality as in [6].

Lemma A.2. *If $g(\cdot)$ is optimal for Problem (A.4), then either $g \equiv 0$ or $g(x) > 0\ \forall\, x > 0$.*

Theorem A.2. *If X^* is an optimal solution for Problem (A.1) with some $a > 0$, then $\mathbb{P}\{X^* = 0\} = 0$.*

Note that these last results do not depend on the choice of Z. They will be useful in order to state monotonicity properties of the value function of a CPT non-informed agent.

B A Choquet minimization problem

In this section we solve a general utility minimization problem including a Choquet capacity:

$$\begin{aligned} &\text{Minimize} && V_2(X) := \int_0^{+\infty} T(\mathbb{P}\{u(X) > y\})\,\mathrm{d}y \\ &\text{subject to} && \mathbb{E}^{\mathbb{P}}[X] = a,\ X \geq 0, \end{aligned} \tag{B.1}$$

where a, $T(\cdot)$ satisfy the same hypothesis employed in Problem (A.1) and $u(\cdot)$ is strictly increasing, concave and $u(0) = 0$. Once again the only difference with respect to the Choquet minimization problem solved in [6], Appendix D, is the absence of the atom-less weighting function ξ.
We will denote as usual by X^* the optimal solution to Problem (B.1). Note that there is always a feasible solution, namely $X = a$ $\mathbb{P}$-a.s.; hence the optimal value of Problem (B.1) is a finite non-negative number. Proceeding as in Appendix A we can show the following *law-invariance* lemma.

Lemma B.1. *Suppose Problem (B.1) admits a feasible solution X whose distribution function is $G(\cdot)$; then for every random variable $Z \sim U(0,1)$ w.r.t. $\mathbb{P}$ we have $V_1(X) = V_1\left(G^{-1}(Z)\right)$.*

Thus, we can look for a solution to the following problem:

$$\begin{aligned} &\text{Minimize} && \overline{v}_2(g) := \mathbb{E}[u(g(Z))T'(1-Z)] \\ &\text{subject to} && \mathbb{E}^{\mathbb{P}}[g(Z)] = a,\ g \in \Gamma, \end{aligned} \tag{B.2}$$

where $g(\cdot)$ represents the inverse of a distribution function $G(\cdot)$, i.e. $g(\cdot) = G^{-1}(\cdot)$. As already pointed out in [6], Problem (B.2) is a difficult one since we have to minimize a concave objective function in a function space. Again we can seek among *corner point solutions* and by straightforward modifications of the proof of Proposition D.2 in [6], we can prove the next result.

Proposition B.1. *Assume that $u(\cdot)$ is strictly concave at 0. Then the optimal solution for Problem (B.2), if it exists, must be in the form $g(t) = \frac{a}{1-b} I_{(b,1)}(t)$, $t \in [0,1)$.*

Obviously by left continuity of $g(\cdot)$ we can extend the optimal $g(\cdot)$ over $[0,1]$ by setting $g(1) := \frac{a}{1-b}$. Moreover, $g(\cdot)$ is uniformly bounded in $t \in [0,1]$, so it follows by the previous lemma that an X^* optimal for Problem (B.1) is uniformly bounded from above. Thanks to the previous proposition, we can reduce our problem to finding an optimal real number $b \in [0,1)$. Therefore we introduce the following minimization problem:

$$\begin{aligned} &\text{Minimize} \quad \overline{v}_2(b) := \mathbb{E}[u(g(Z))T'(1-Z)] \\ &\text{subject to} \quad g(\cdot) = \tfrac{a}{1-b} I_{(b,1]}(\cdot), \quad 0 \le b < 1. \end{aligned} \tag{B.3}$$

Adapting the proofs of Proposition D.3 and Theorem D.1 in [6], we can obtain the following result.

Proposition B.2. *Problems* (B.2) *and* (B.3) *have the same infimum values.*

Theorem B.1. *Problems* (B.1) *and* (B.3) *have the same infimum values. If, in addition, $u(\cdot)$ is strictly concave at* 0*, then Problem* (B.1) *admits an optimal solution if and only if the following problem*

$$\min_{0 \le b < 1} u\left(\frac{a}{1-b}\right) T(1-b)$$

admits an optimal solution b^*, in which case the optimal solution to Problem (B.1) *is of the form $X^* = \frac{a}{1-b^*} I_{(b^*,1]}(Z)$ for any choice of $Z \sim U(0,1)$ w.r.t. $\mathbb{P}$.*

C The solution of a CPT non-informed agent's problem

We will now proceed to solve Problem (CPT-N). The scheme of the solution is nothing but the one already showed in [6]. Some results will be just restated without proofs as they need only slight and straightforward adaptations. As already noted, the main changes are due to the constraint $\mathbb{E}^{\mathbb{Q}}[X] = x_0$. Recall Problem (CPT-N):

$$\begin{aligned} &\text{Maximize} \quad V(X) = V_+(X^+) - V_-(X^-) \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}}[X] = x_0,\ X \text{ is } \mathscr{F}_T\text{-measurable and } \mathbb{Q}\ a.s. \text{ bounded from below,} \end{aligned}$$

where

$$V_+(X^+) := \int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X^+) > y\})\,\mathrm{d}y, \qquad V_-(X^-) := \int_0^{+\infty} T_-(\mathbb{Q}\{u_-(X^-) > y\})\,\mathrm{d}y.$$

As noticed in [6], Proposition 3.1, to avoid systematic ill-posedness we will impose

Assumption C.1. $V_+(X) < +\infty$ *for any nonnegative, $\mathscr{F}_T$-measurable random variable X satisfying* $\mathbb{E}^{\mathbb{Q}}[X] < +\infty$.

We now split Problem (CPT-N) into its *positive* and *negative* part, also defining their respective optimal values $v_+(A, x_+)$ and $v_-(A, x_+)$ as usual; after that we merge them back:

- *Positive Part Problem*: given the pair (A, x_+), with $A \in \mathscr{F}_T$ and $x_+ \ge x_0^+$,

$$\begin{aligned} &\text{Maximize} \quad V_+(X) = \textstyle\int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X) > y\})\,\mathrm{d}y \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}}[X] = x_+, \quad X \ge 0 \ \ \mathbb{Q}\,a.s., \quad X = 0 \ \ \mathbb{Q}\,a.s. \text{ on } A^C. \end{aligned} \tag{C.1}$$

- *Negative Part Problem*: given the pair (A, x_+), with $A \in \mathscr{F}_T$ and $x_+ \ge x_0^+$,

$$\begin{aligned} &\text{Minimize} \quad V_-(X) = \textstyle\int_0^{+\infty} T_-(\mathbb{Q}\{u_-(X) > y\})\,\mathrm{d}y \\ &\text{subject to} \quad \begin{cases} \mathbb{E}^{\mathbb{Q}}[X] = x_+ - x_0, \quad X \ge 0 \ \ \mathbb{Q}\ a.s., \quad X = 0 \ \ \mathbb{Q}\ a.s. \text{ on } A, \\ X \text{ is upper bounded } \mathbb{Q}\ a.s.. \end{cases} \end{aligned} \tag{C.2}$$

- *Merged Problem*:

$$\begin{aligned} &\text{Maximize} \quad v_+(A, x_+) - v_-(A, x_+) \\ &\text{subject to} \quad \begin{cases} A \in \mathscr{F}_T, \quad x_+ \ge x_0^+, \\ x_+ = 0 \ \text{ if } \ \mathbb{Q}(A) = 0, \quad x_+ = x_0 \ \text{ if } \ \mathbb{Q}(A) = 1. \end{cases} \end{aligned} \tag{C.3}$$

With only a few and simple adaptations, we can prove the following two results.

Proposition C.1 ([6], Proposition 5.1)**.** *Problem* (CPT-N) *is ill-posed if and only if Problem* (C.3) *is ill-posed.*

Proposition C.2 ([6], Proposition 5.2)**.** *Given X^*, define $A^* := \{\omega : X^* \geq 0\}$ and $x_+^* := \mathbb{E}^{\mathbb{Q}}[(X^*)^+]$. Then X^* is optimal for Problem* (CPT-N) *if and only if (A^*, x_+^*) are optimal for Problem* (C.3) *and $(X^*)^+$ and $(X^*)^-$ are respectively optimal for Problems* (C.1) *and* (C.2) *with parameters (A^*, x_+^*).*

Therefore, the original Problem (CPT-N) for N-agent is equivalent once more to the set of Problems (C.1)-(C.3). The next step is the crucial one, as it completely changes the structure of the solution of our problem. We will not be able to obtain the *almost sure* characterization results obtained in [6]. On the other hand, we can avoid the technical details related to the *comonotonicity* and *anti-comonotonicity* of the random variables employed in the solution (see [6], Appendix B, where a series of so-called *quantile* problems is solved).

The fact is that the density ρ allowed for a huge simplification of the overall procedure, since it made possible to look for a solution where the set A was of the form $\{\rho \leq c\}$ for some real number $c \in [\underline{\rho}, \overline{\rho}]$. Now we can find a quite similar result adapting the proof of Theorem 5.1 in [6]; this will substantially reduce the complexity of Problem (C.3).

Theorem C.1. *For any feasible (A, x_+) of Problem* (C.3) *such that $\mathbb{Q}(A) = p$ and for every $(\Omega, \mathscr{F})$ random variable $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$, we have*

$$v_+(\overline{A}, x_+) - v_-(\overline{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+), \tag{C.4}$$

where $\overline{A} := \{Z \leq p\}$.

Proof. Fix a random variable $Z \sim U(0,1)$. The cases $x_+ = x_0^+$ and $p = 0$ or $p = 1$ are trivial, so we assume that $x_+ > x_0^+$ and $p \in (0,1)$. Define $B := A^C$ and $\overline{A} := \{Z \leq p\}$ and set

$$\begin{aligned} A_1 &= A \cap \{Z \leq p\}, \qquad & A_2 &= A \cap \{Z > p\}, \\ B_1 &= B \cap \{Z \leq p\}, \qquad & B_2 &= B \cap \{Z > p\}. \end{aligned}$$

Note that $\mathbb{Q}(A_1 \cup A_2) = \mathbb{Q}(A_1 \cup B_1) = p$, so that $\mathbb{Q}(A_2) = \mathbb{Q}(B_1)$. If $\mathbb{Q}(A_2) = 0$ then the result is trivial, so suppose $\mathbb{Q}(A_2) > 0$. Choose a feasible solution X_1 for Problem (C.1) with parameters (A, x_+); we will prove that $V_+(X_1) \leq v_+(\overline{A}, x_+)$ (the proof for a feasible solution X_2 for Problem (C.2) is analogous). To this end, define $f_1(t) := \mathbb{Q}\{X_1 \leq t | A_2\}$, $g_1(t) := \mathbb{Q}\{Z \leq t | B_1\}$, $t \in [0,1]$, $Z_1 := g_1(Z)$ and $Y_1 := f_1^{-1}(Z_1)$. Note that Z has no atoms w.r.t. $\mathbb{Q}$, which in turn implies that it has no atoms w.r.t. $\mathbb{Q}(\cdot | B_1)$. Moreover one can show that $Z_1 \sim U(0,1)$ w.r.t. $\mathbb{Q}(\cdot | B_1)$, implying $\mathbb{Q}\{Y_1 \leq t | B_1\} = \mathbb{Q}\{Z_1 \leq f_1(t) | B_1\} = f_1(t)$. To see this note that

$$g_1(t) = \frac{\mathbb{Q}\{A^C \cap (Z \leq t) \cap (Z \leq p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}} = \frac{\mathbb{Q}\{A^C \cap (Z \leq t \wedge p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}},$$

so we can compute

$$\mathbb{Q}\{Z_1 \leq t | B_1\} = \frac{\mathbb{Q}\{A^C \cap (Z_1 \leq t) \cap (Z \leq p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}} = \frac{\mathbb{Q}\{A^C \cap (Z \leq g_1^{-1}(t) \wedge p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}} = g_1\left(g_1^{-1}(t)\right) = t.$$

Consequently, $\mathbb{E}^{\mathbb{Q}}[X_1 I_{A_2}] = \mathbb{Q}(A_2)\mathbb{E}^{\mathbb{Q}}[X_1 | A_2] = \mathbb{E}^{\mathbb{Q}}[Y_1 I_{B_1}]$. Now set $\overline{X}_1 := X_1 I_{A_1} + Y_1 I_{B_1}$. Then $\mathbb{E}^{\mathbb{Q}}[X_1] = \mathbb{E}^{\mathbb{Q}}[\overline{X}_1]$, so $\overline{X}_1$ is feasible for Problem (C.1) with parameters $(\overline{A}, x_+)$. Finally it is obviously seen that $\mathbb{Q}\{\overline{X}_1 > t\} = \mathbb{Q}\{X_1 > t\}$, therefore by the definition of $V_+(\cdot)$ it follows that $V_+(\overline{X}_1) \geq V_+(X_1)$. Combining this with the similar result for the *Negative Part Problem* we get the desired inequality (C.4). $\square$

The meaning of Theorem C.1 is that a non-informed agent cares only about the probability of events, no matter what structure they have or what economic phenomenon they represent. In what follows, it will be clear that for such an agent investing in a risky asset is not so different than tossing a coin or betting on horses![11]

[11] Recall that also in the original framework in [6], the optimal policy for an investor was to behave like a gambler, but she would choose a terminal gain accompanied with a high price of the underlying stock, opposite to a final loss if the terminal price would have fallen below a certain threshold.

We can now proceed similarly to Jin and Zhou [6], using $v_+(p, x_+)$ and $v_-(p, x_+)$ to denote $v_+(\{\omega : Z \le p\}, x_+)$ and $v_-(\{\omega : Z \le p\}, x_+)$ respectively. Note that we can freely choose Z, and the previous definition is in some sense *independent* of Z thanks to previous Theorem C.1. Accordingly, we replace Problem (C.3) by the easier constrained optimization problem in $\mathbb{R}^2$:

$$\begin{aligned} &\text{Maximize} \quad v_+(p, x_+) - v_-(p, x_+) \\ &\text{subject to} \quad \begin{cases} p \in [0, 1], \quad x_+ \ge x_0^+, \\ x_+ = 0 \ \text{ if } \ p = 1, \quad x_+ = x_0 \ \text{ if } \ p = 0. \end{cases} \end{aligned} \tag{C.5}$$

Using Theorem C.1 we obtain the general structure of the solution to Problem (CPT-N), which is indeed similar to the in [6]. In what follows we will consider such a Z fixed and denote with X^* the optimal solution depending on Z.

Theorem C.2. *Given X^* and Z, define $p^* := \mathbb{Q}\{X^* \ge 0\}$, $x_+^* := \mathbb{E}^{\mathbb{Q}}[(X^*)^+]$. Then X^* is optimal for Problem* (CPT-N) *if and only if (p^*, x_+^*) is optimal for Problem* (C.5) *and $(X^*)^+ I_{Z \le p^*}$ and $(X^*)^- I_{Z > p^*}$ are respectively optimal for Problems* (C.1) *and* (C.2) *with parameters $(\{\omega : Z \le p^*\}, x_+^*)$.*

The next step consists in solving the *positive* and the *negative* part Problems (C.1) and (C.2) using the results obtained in Appendix A and B respectively. In order to obtain a more explicit result, we impose the following conditions.

Assumption C.2. *$T_+'(z)$ is non-increasing for $z \in (0, 1]$, $\liminf_{x \to +\infty} -\frac{x u_+''(x)}{u_+'(x)} > 0$ and for any $Z \sim U(0, 1)$ w.r.t. $\mathbb{Q}$ we have $\mathbb{E}^{\mathbb{Q}}\left[u_+\left((u_+')^{-1}(\frac{1}{T'(Z)})\right) T'(Z)\right] < +\infty$.*

At this point we can perform the same procedure used in [6], Section 6.1, to obtain

Theorem C.3. *Let Assumption C.2 hold. For any $Z \sim U(0, 1)$ w.r.t. $\mathbb{Q}$ and for a given $p \in [0, 1]$, set $A := \{\omega : Z \le p\}$; let $x_+ \ge x_0^+$ be given. Then:*

(i) if $x_+ = 0$, then the optimal solution of Problem (C.1) *is $X^* = 0$ and $v_+(p, x_+) = 0$;*

(ii) if $x_+ > 0$, $p = 0$ then there is no feasible solution to Problem (C.1) *and $v_+(p, x_+) = -\infty$;*

(iii) if $x_+ > 0$, $p \in (0, 1]$ then the optimal solution to (C.1) *is $X^*(\lambda) = (u_+')^{-1}\left(\frac{\lambda}{T_+'(Z)}\right) I_{Z \le p}$ with the optimal value $v_+(p, x_+) = \mathbb{E}^{\mathbb{Q}}\left[u_+\left((u_+')^{-1}\left(\frac{\lambda}{T_+'(Z)}\right)\right) T_+'(Z) I_{Z \le p}\right]$, where $\lambda > 0$ is the unique real number satisfying $\mathbb{E}^{\mathbb{Q}}[X^*(\lambda)] = x_+$.*

Proof. Cases *(i)* and *(ii)* are trivial; to prove *(iii)* we follow an argument similar to that in the proof of Theorem 6.1 in [6]. Define $T_A(x) := \frac{T_+(x\mathbb{Q}(A))}{T_+(\mathbb{Q}(A))} = \frac{T_+(xp)}{T_+(p)}$, $x \in [0, 1]$ and the conditional probability measure $\mathbb{Q}_A := \mathbb{Q}(\cdot|A)$. Now consider Problem (C.1) in the conditional probability space $(\Omega \cap A, \mathscr{F} \cap A, \mathbb{Q}_A)$, i.e.

$$\begin{aligned} &\text{Maximize} \quad V_+(Y) = T_+(p) \textstyle\int_0^{+\infty} T_A(\mathbb{Q}_A\{u_+(Y) > y\})\, \mathrm{d}y \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{Q}_A}[Y] = \frac{x_+}{p}, \quad Y \ge 0. \end{aligned} \tag{C.6}$$

We can apply Theorem A.1 to Problem (C.6) choosing *any* random variable $\tilde{Z} \sim U(0, 1)$ w.r.t. $\mathbb{Q}_A$; note that every required assumption for Theorem A.1 is still fulfilled. At this point, in order to simplify calculations as much as possible, we see that once Z is chosen there is a *canonical* choice of $\tilde{Z}$: $\tilde{Z} = 1 - g(Z)$, where $g(z) := \mathbb{Q}\{Z \le t|A\}$. In fact we can show that if $Z \sim U(0, 1)$ w.r.t. $\mathbb{Q}$, then $\tilde{Z}$ has the same distribution w.r.t. $\mathbb{Q}_A$. To see this, note that

$$\mathbb{Q}_A\{\tilde{Z} \le t\} = \frac{\mathbb{Q}\{\tilde{Z} \le t, A\}}{p} = \frac{\mathbb{Q}\{1 - g(Z) \le t, Z \le p\}}{p} = t, \quad t \in (0, 1),$$

where we used

$$g(t) = \frac{\mathbb{Q}\{Z \le t \wedge p\}}{p} = \frac{t \wedge p}{p}.$$

Using such a choice of $\tilde{Z}$ we can find that an optimal solution to Problem (C.1) is given by $X^* = (u'_+)^{-1}\left(\frac{\overline{\lambda} T_+(p)}{pT'_+(pg(Z))}\right) I_{Z\leq p}$, where $\overline{\lambda}$ is uniquely determined by the constraint. We now observe that on the set $\{Z \leq p\}$ we have $g(Z) = Z/p$; finally we set $\lambda := \frac{\overline{\lambda} T_+(p)}{p}$ to find our results. □

Comparing this result to the analogous in [6], we see that the link between the two solutions is substantially made by the replacement of the set $\{\rho \leq c\}$ with $\{Z \leq p\}$. In particular $c = \underline{\rho}$ corresponds to $p = 0$ and $c = \overline{\rho}$ corresponds to $p = 1$. Thanks to the free choice of Z, we see once more that a non-informed agent is only interested in probabilities, not in events.

With a simple modification in the proof of [6], Proposition 6.2, we can also state the strict monotonicity of the optimal value $v_+(\cdot, x_+)$ w.r.t. p.

Proposition C.3. *If $x_+ > 0$ and $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$, then Problem* (C.1) *admits an optimal solution with parameters $(\{Z \leq p\}, x_+)$ only if $v_+(\overline{p}, x_+) > v_+(p, x_+)$ for any $\overline{p} > p$.*

We now proceed to solve the *negative* part Problem (C.2). We follow again the arguments applied in [6], Section 7, combining them with our results in Appendix B.

Theorem C.4. *Assume that $u_-(\cdot)$ is strictly concave at 0. For any $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$ and for a given $p \in [0,1]$ set $A := \{\omega : Z \leq p\}$. Let $x_+ \geq x_0^+$ be given. Then:*

(i) if $p = 1$, $x_+ = x_0$ then the optimal solution of Problem (C.2) *is $X^* = 0$ and $v_-(p, x_+) = 0$;*

(ii) if $p = 1$, $x_+ \neq x_0$ then there is no feasible solution to Problem (C.2) *and $v_-(p, x_+) = +\infty$;*

(iii) if $p \in [0,1)$ then $v_-(p, x_+) = \inf_{0\leq b<1} u_-\left(\frac{x_+ - x_0}{(1-p)(1-b)}\right) T_-\left((1-p)(1-b)\right)$. Moreover, Problem (C.2) *with parameters (A, x_+) admits an optimal solution X^* if and only if the minimization problem*

$$\min_{0\leq b<1} u_-\left(\frac{x_+ - x_0}{(1-p)(1-b)}\right) T_-\left((1-p)(1-b)\right) \tag{C.7}$$

admits an optimal solution b^*, in which case $X^* = \frac{x_+ - x_0}{(1-p)(1-b^*)} I_{Z>(1-p)b^*+p}$.

Proof. Cases *(i)* and *(ii)* are trivial; to prove *(iii)* we define $T_{A^C}(x) := \frac{T_-(x\mathbb{Q}(A^C)}{T_-(\mathbb{Q}(A^C))} = \frac{T_-(x(1-p))}{T_-(1-p)}$, $x \in [0,1]$ and the conditional probability measure $\mathbb{Q}_{A^C} := \mathbb{Q}(\cdot|A^C)$. Let's consider Problem (C.2) in the conditional probability space $(\Omega \cap A^C, \mathscr{F} \cap A^C, \mathbb{Q}_{A^C})$:

$$\begin{array}{ll} \text{Minimize} & V_-(Y) = T_-(1-p)\int_0^{+\infty} T_{A^C}(\mathbb{Q}_{A^C}\{u_-(Y) > y\})\,\mathrm{d}y \\ \text{subject to} & \mathbb{E}^{\mathbb{Q}_{A^C}}[Y] = \frac{x_+ - x_0}{1-p}, \quad Y \geq 0, \quad Y\ \mathbb{Q}_{A^C}\ a.s.\ \text{bounded.} \end{array} \tag{C.8}$$

Now we apply Theorem B.1 to Problem (C.8), choosing *any* random variable $\tilde{Z} \sim U(0,1)$ w.r.t. $\mathbb{Q}_{A^C}$. Once again, when Z is chosen there is a *canonical* choice of $\tilde{Z}$: $\tilde{Z} = g(Z)$, where $g(t) := \mathbb{Q}\{Z \leq t|A^C\}$. Indeed, if $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$ then $\tilde{Z}$ has the same distribution w.r.t. $\mathbb{Q}_{A^C}$. To see this, observe that

$$\mathbb{Q}_{A^C}\{\tilde{Z} \leq t\} = \frac{\mathbb{Q}\{\tilde{Z} \leq t, A^C\}}{1-p} = \frac{\mathbb{Q}\{g(Z) \leq t, Z > p\}}{1-p} = \frac{\mathbb{Q}\{Z \leq g^{-1}(t), Z > p\}}{1-p},$$

but we can compute

$$g(t) = \frac{\mathbb{Q}\{Z \leq t, Z > p\}}{1-p} = \frac{t-p}{1-p} \wedge 0,$$

therefore we obtain $\mathbb{Q}_{A^C}\{\tilde{Z} \leq t\} = t$, $t \in (0,1)$. Using such a choice of $\tilde{Z}$ and recalling that an optimal solution to Problem (C.8) is automatically bounded (if it exists), we can find that an optimal solution to Problem (C.2) is

$$X^* = \tfrac{x_+ - x_0}{(1-p)(1-b^*)} I_{Z>p} I_{g(Z)\in(b^*,1]} = \tfrac{x_+ - x_0}{(1-p)(1-b^*)} I_{Z>(1-p)b^*+p},$$

thanks to the fact that on the set $\{Z > p\}$ we have $g(Z) = \frac{Z-p}{1-p} \geq b^*$ if and only $Z \geq (1-p)b^* + p$. □

At last we have to merge these results to obtain the overall solution to Problem (CPT-N). As in [6] we take an intermediate step using the following problem

$$\begin{array}{ll} \text{Maximize} & v_+(p, x_+) - u_-\left(\frac{x_+ - x_0}{1-p}\right) T_-(1-p) \\ \text{subject to} & \begin{cases} p \in [0,1], \quad x_+ \geq x_0^+, \\ x_+ = 0 \ \text{ if } \ p = 1, \quad x_+ = x_0 \ \text{ if } \ p = 0, \end{cases} \end{array} \tag{C.9}$$

where we set $u_-\left(\frac{x_+ - x_0}{1-p}\right) T_-(1-p) := 0$ if $p = 1$ and $x_+ = x_0$. By simply adapting the proofs in [6], Lemma 8.1 and Proposition 8.1, we claim:

Lemma C.1. *For any feasible pair* (p, x_+) *for Problem* (C.5), $u_-\left(\frac{x_+ - x_0}{1-p}\right) T_-(1-p) \geq v_-(p, x_+)$.

Proposition C.4. *Problems* (C.5) *and* (C.9) *have the same supremum values.*

Finally we state the main result of this section:

Theorem C.5. *Assume that* $u_-(\cdot)$ *is strictly concave at* 0. *We have the following results:*

(i) if X^* *is optimal for Problem* (CPT-N), *then* $p^* := \mathbb{Q}\{X^* \geq 0\}$, $x_+^* := \mathbb{E}^{\mathbb{Q}}[(X^*)^+]$ *are optimal for Problem* (C.9);

(ii) if (p^*, x_+^*) *is optimal for Problem* (C.9) *and* X_+^* *is optimal for Problem* (C.1) *with parameters* $(\{Z \leq p^*\}, x_+^*)$, *where* $Z \sim U(0,1)$ *w.r.t.* $\mathbb{Q}$, *then* $X^* := (X^*)^+ I_{Z \leq p^*} - \frac{x_+^* - x_0}{1-p^*} I_{Z > p^*}$ *is optimal for Problem* (CPT-N).

To conclude, if Assumption C.2 is in force, then for any $Z \sim U(0,1)$ w.r.t. $\mathbb{Q}$ we have

$$\begin{aligned} X^* &= (u_+')^{-1}\left(\frac{\lambda}{T_+'(Z)}\right) I_{Z \leq p^*} - \frac{x_+^* - x_0}{1-p^*} I_{Z > p^*}, \\ V(X^*) &= \mathbb{E}^{\mathbb{Q}}\left[u_+\left((u_+')^{-1}\left(\frac{\lambda}{T_+'(Z)}\right)\right) T_+'(Z) I_{Z \leq p^*}\right] - u_-\left(\frac{x_+^* - x_0}{1-p^*}\right) T_-(1-p^*), \end{aligned}$$

where (p^*, x_+^*) are optimal for Problem (C.9) and λ satisfies $\mathbb{E}^{\mathbb{Q}}\left[(u_+')^{-1}\left(\frac{\lambda}{T_+'(Z)}\right) I_{Z \leq p^*}\right] = x_+^*$. We finally notice that this construction could be considered as an adaptation of the model set up in [6] if we had started with prices following a geometric or an arithmetic Brownian motion, as it is often assumed in the finance literature.

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