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► **To cite this version:**

Luciano Campi, Matteo del Vigna. Weak Insider Trading and Behavioral Finance. 2011. hal-00566185v1

HAL Id: hal-00566185

<https://hal.science/hal-00566185v1>

Preprint submitted on 15 Feb 2011 (v1), last revised 5 Dec 2011 (v3)

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Weak Insider Trading and Behavioral Finance*

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February 11, 2011

Abstract

In this paper, we study the optimal portfolio selection problem of weakly informed traders in the sense of Baudoin [1]. Instead of considering only expected utility maximizers, we also take into consideration different preference paradigms. In particular, we analyze a representative agent who follows the tenets of cumulative prospect theory as developed by Kahneman and Tversky [15], together with an investor acting as in Yaari's dual theory of choice [16] and a trader who faces the so-called goal reaching maximizer. For everyone of these different maximizers, we frame the corresponding optimization problems, one in the non-informed case and the other one when the agent possesses a weak information. At last, comparison results among different types of investor and differently informed investor are given, together with explicit examples. Specifically, the insider's gain, or the difference between the optimal values of an informed and a non informed investor, is explicitly evaluated.

Keywords: weak information, insider trading, behavioral finance, loss aversion, probability distortion, Yaari's dual theory of choice, goal reaching maximizer.

AMS classification (2000): 91B28, 60G44

JEL classification: D82, D81, G11

1 Introduction

We start from an idea of Baudoin, [1], when modeling the existence of insiders in financial markets. In contrast to the well-known strong information approach (see for example [13]), we assume that the extra-informed investor is endowed with the knowledge of a functional Y (or more simply a random variable) related to the asset prices and its distribution with respect to the historical probability measure that underlies the market. In fact, this historical probability \mathbb{P} is unknown to every agent, whereas everyone knows the so-called equivalent martingale measure, namely \mathbb{Q} . Therefore, having a \mathbb{P} -knowledge of Y translates in an informational advantage. We also note that in this setting, the discounted prices of the assets are martingales. This could be well realistic, since a model for the prices under \mathbb{Q} can be calibrated by observed data, while the insider ignores the *effective* drifts of the prices.

In [1], the author studies a portfolio optimization problem for a non-informed agent and an insider respectively. He is then able to characterize the optimal terminal wealth and the corresponding optimal value; moreover, he finds explicit formula for particular choice of the utility function. It is important to note that he considers Expected Utility maximizers (EU or classical, in what follows). Our question is: what happens if we apply this way of modeling insider to different preference paradigms? More specifically, we now think of an investor whose objective function is not to maximize the expected utility of the terminal wealth. In the literature, the EU case developed by Von Neumann and Morgenstern in the early 30's is the most treated thanks to its relative simplicity and the existence of a dual theory which permits to elegantly solve rather difficult problems. However, it is not the only existing paradigm and, more importantly, it is empirically proved that real world people systematically violate the hypothesis which stands behind EU (this lead to a number of so-called paradoxes and puzzles). In this paper, we consider three alternative models.

*This research benefited from the support of the "Chair Les Particuliers Face aux Risques", Fondation du Risque (Groupama-ENSAE-Dauphine) as well as from the GIP-ANR "Croyances" project of the Europlace Institute of Finance.

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- Cumulative Prospect Theory (CPT thereafter): this paradigm is fully described in [15] and it is a further development of the original Prospect Theory by Kahneman and Tversky; see [12]. Briefly, according to CPT an economic agent evaluates its payoff with respect to a reference level B ; if the payoff is greater than B , then it has to be considered as a gain. On the contrary, a payoff lower than B becomes a loss for a CPT agent, and a loss hurts more than an equivalent gain (loss aversion). Secondly, this type of investor does not use a utility value; more properly, she has two *value* functions, a concave one for the gains and a convex one for the losses. Hence, the overall form of her “utility” function is so-called S-shaped and she exhibits risk-aversion in gains while she is risk-lover w.r.t. losses. Finally, actual probabilities are perceived as distorted by CPT agents; mathematically speaking, there exists two increasing functions, one for the gains and one for the losses, that describe this distortion. From laboratory evidence, it is generally assessed that these weighting functions are *reversed S-shaped*, i.e. they are greater than the identity for small probabilities and lower than the identity for probabilities near to 1. Intuitively, this means that people tend to overestimate small probabilities and vice versa. A general mathematical model in continuous time for CPT can be found in [8], where it is necessary to use Choquet capacities instead of classical expectations and to split the objective function in the gains and losses parts.
- Goal Reaching Model: in this case, the objective function of the considered investor consists in maximizing the probability of having a terminal payoff greater than a specified level. This model has been extensively treated by Browne in [4] and it is usually referred to a fund manager that has to beat the benchmark. The strangeness of this model is condensed in the objective function, where the probability of an event should be maximized.
- Yaari’s Dual Theory of Choice: in 1989, Yaari proposed in [16] a different set of hypothesis w.r.t. those of Von Neumann and Morgenstern. The result was a *dual* representation of the expected utility criterion, where the decumulative probabilities were distorted instead of payoffs (recall that a utility function $u(\cdot)$ is nothing but a distortion over the payoffs). A mathematical formulation for Yaari’s model in continuous time can be found in [5], where $w(\cdot)$ is used as the probability distortion function. In [16] it has been shown that risk-aversion is characterized by a convex $w(\cdot)$, i.e. by a global underestimation of probabilities, whereas risk-loving is identified by a concave $w(\cdot)$. We note that this is no longer true in the CPT model, as we have two probability distortions. However, the economic intuition is the same.

In each of the previous cases, we will frame and solve the optimization problem for a non-informed investor and for an insider. We always have in mind an hypothetical situation of a small trader, in the sense that her investment choices do not affect the asset prices; otherwise stated, we do not allow for price impacts.

An important issue in this family of non-classical problem is the well-posedness of the model. As is shown in [8] and [5], ill-posedness can quite easily arise if we do not make proper assumption over the *value* functions and/or the probability distortions. We will give sufficient conditions for well-posedness during our analysis.

At last, we recall that in the existing literature there is lack of explicit examples and explicit evaluations of the optimal value for CPT and Yaari’s models. This is why we focused on examples and, to the best of our knowledge, they are completely new.

The paper is organized as follows. In Section 2, we recover the weak information setting as developed in [1] and in Section 3 we consider the problems for an EU agent, whose results were already proved in [1]. Then, Section 4 analyzes the problem in the CPT case and Section 5 is devoted to comparison results between EU and CPT agents, which are the two most common classes considered in the literature. Section 6 is about the goal reaching problem and Section 7 concerns a Yaari-type investor. Finally, Section 8 concludes. Involved proofs are presented in the Appendix.

2 The weak information approach

Fix an atom-less probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is a completed and right-continuous filtration with \mathcal{F}_0 being the trivial σ -algebra and $T > 0$ a constant terminal time.

We consider a market model with $m + 1$ traded assets in a continuous time setting, where the first asset $S_0(t)$ is the risk-free asset, or bank account, and the other m assets $S(t) = (S_1(t), \dots, S_m(t))$

are the risky ones, or stocks. It is important to highlight the fact that we do not specify any particular dynamics for our price processes; on the contrary, the main hypothesis concerning them is the following one, which retains a no-arbitrage and completeness condition over the market.

Assumption 2.1. *The price process $(S_0(t), S_1(t), \dots, S_m(t))$ is a continuous and adapted square integrable martingale over $(\Omega, \mathbb{F}, \mathbb{Q})$. Moreover, \mathbb{Q} is the unique such measure with this property.*

From an economic point of view, Assumption 2.1 is equivalent to say that \mathbb{Q} is the so-called *discounting* or *spot* martingale measure; its existence implies working in a standard no-arbitrage financial market, whereas its uniqueness provides completeness.

Having said this, a non informed agent (or N-agent in what follows) is only able to afford on \mathbb{Q} and the observable past and present prices when taking her investment decisions. On the other hand, an informed agent (briefly, I-agent) also possesses privileged information concerning the law of a functional Y of the stock prices. Specifically, I-agent knows the distribution of Y under the so-called “historical” measure \mathbb{P} governing market prices.

From now on, we assume that Y is a scalar random variable (everything shown below can be easily generalized to a vector valued random variable or to more complicated functional Y taking values in a Polish space \mathcal{P}). We will denote with \mathbb{Q}_Y the law of Y under \mathbb{Q} and with ν the *effective* law of Y known by I-agent; therefore we have

$$\mathbb{Q}_Y(B) = \mathbb{Q}\{Y \in B\}, \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

Assumption 2.2. *ν is equivalent to \mathbb{Q}_Y and the density $\xi := \frac{d\nu}{d\mathbb{Q}_Y}$ is \mathbb{Q} -a.s. bounded.*

Note that Assumption 2.2 rules out some interesting cases of information owned by the insider. Now, the privileged information (Y, ν) can be naturally associated with a new measure called by Baudoin the *minimal* probability.

Definition 2.1. *The probability measure \mathbb{Q}^ν defined on (Ω, \mathcal{F}_T) by:*

$$\mathbb{Q}^\nu(A) := \int_{\mathbb{R}} \mathbb{Q}(A|Y=y) \nu(dy), \quad A \in \mathcal{F}_T \quad (2.1)$$

is called the minimal probability associated with the weak information (Y, ν) .

The meaning of the word *minimal* used by Baudoin is easily understood referring to the following proposition, which shows that \mathbb{Q}^ν fulfills a minimum criterion. Specifically, let \mathcal{E}^ν be the set of probability measures on Ω which are equivalent to \mathbb{Q} and such that the law of Y under those measures is ν . Then we have

Theorem 2.1 ([1], Proposition 6). *For every convex function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$,*

$$\min_{\mu \in \mathcal{E}^\nu} \mathbb{E} \left[\varphi \left(\frac{d\mu}{d\mathbb{Q}} \right) \right] = \mathbb{E} \left[\varphi \left(\frac{d\mathbb{Q}^\nu}{d\mathbb{Q}} \right) \right].$$

If $\frac{d\mathbb{Q}^\nu}{d\mathbb{Q}}$ is bounded, then the preceding values are finite.

In particular, we will see that employing the minimal probability means that the additional value for the insider is the lowest increase in utility that she can obtain given her extra information. Moreover, we can observe that \mathbb{Q}^ν does not depend on the choice of the utility function in a standard portfolio selection model, thus in a behavioral setting this amounts to say that the *minimal* probability is unaffected by the probability distortions and the value functions.

It is not difficult to show the following useful properties:

1. If $X : \Omega \rightarrow \mathbb{R}$ is a bounded random variable, then $\mathbb{E}^\nu[X|Y] = \mathbb{E}^\mathbb{Q}[X|Y]$ \mathbb{Q} -a.s., where \mathbb{E}^ν denotes the expectation w.r.t. the measure \mathbb{Q}^ν .
2. The law of Y under \mathbb{Q}^ν is ν , i.e. $\nu(B) = \mathbb{Q}^\nu\{Y \in B\}$, $\forall B \in \mathcal{B}(\mathbb{R})$.
3. $\mathbb{Q}^\nu = \mathbb{Q} \iff \nu = \mathbb{Q}_Y$.

4. We have the following equivalence relationship:

$$d\mathbb{Q}^\nu = \frac{d\nu}{d\mathbb{Q}_Y}(Y) d\mathbb{Q}. \quad (2.2)$$

To avoid ambiguity, this means that $\forall A \in \mathcal{F}_T$ we have $\mathbb{E}^\nu[I_A] = \mathbb{E} \left[\frac{d\nu}{d\mathbb{Q}_Y}(Y) I_A \right]$.

5. If $A \in \mathcal{F}_T$ is \mathbb{Q} -independent of Y , then it is also \mathbb{Q}^ν -independent of Y .

In particular, (2.2), Property 3 and 5 follow immediately by Definition 2.1; these in turn imply Property 1 thanks to the Bayes' rule for conditional expectations; finally Property 2 can be easily checked for every Borel set $(-\infty, b]$, $b \in \mathbb{R}$.

3 The classical agents' models and their solutions

In a classical portfolio selection model, i.e. when N-agent's objective is to maximize her expected utility from terminal wealth, all the results have already been derived in [1]. For the sake of completeness and to perform comparisons, we report here the solution of this problem assuming that the considered investor is endowed with a positive initial wealth x_0 and a utility function which satisfies the following hypothesis.

Assumption 3.1. *The utility function $U : (0, +\infty) \rightarrow \mathbb{R}$ is strictly increasing, strictly concave and twice continuously differentiable and satisfy the Inada conditions $U'(+\infty) = 0$, $U'(0+) = +\infty$.*

In [1] it is used the convention $U(x) = -\infty$ for $x \leq 0$. This is because the initial wealth x_0 is assumed to be positive; however, in the case of a power utility function one could take $U(0) = 0$.

At this point, we have to define a suitable class of *admissible* portfolios or strategies. Let's denote with $\Pi_i(t)$ the number of shares of the i -th risky asset held by our trader at time t .

Assumption 3.2. *$\Pi(\cdot)$ is a \mathbb{Q} -a.s \mathbb{F} -predictable process, square integrable w.r.t. the price process $S(\cdot)$ over $[0, T]$ and bounded from below. Moreover, $\Pi(\cdot)$ is \mathbb{Q} -tame, i.e. the corresponding discounted wealth is \mathbb{Q} -a.s. bounded from below, where the bound may depend on $\Pi(\cdot)$.*

Under the previous assumption, the value of any portfolio is a \mathbb{Q} -martingale whose \mathbb{Q} -dynamics are

$$dx(t) = \Pi(t)'dS(t), \quad t \in [0, T]; \quad x(0) = x_0 > 0. \quad (3.1)$$

Furthermore, for every *admissible* portfolio $\Pi(\cdot)$ we will find as terminal wealth $x(T) \equiv X$ a \mathcal{F}_T -measurable random variable, \mathbb{Q} -a.s. lower bounded and such that $\mathbb{E}^\mathbb{Q}[X] = x_0$. Moreover, thanks to Assumption 2.1 a standard completeness argument can be applied, so that X can be replicated by a \mathbb{Q} -tame portfolio $\Pi(\cdot)$.¹ Hence, when formulating the optimization problems of our agents, both EU and CPT, we will impose a set of constraints related to X and not to $\Pi(\cdot)$ as is usual in the martingale approach (see e.g. [13]).

3.1 The non informed agent's problem

For a N-agent, the most natural way to evaluate her own utility from terminal wealth X is to choose the martingale measure \mathbb{Q} when computing the expectation (in fact she does not know the historical measure \mathbb{P} , so she can not use it!). Therefore, the EU non informed agent's problem is

$$\begin{aligned} & \text{Maximize} \quad \mathbb{E}^\mathbb{Q}[U(X)] \\ & \text{subject to} \quad \mathbb{E}^\mathbb{Q}[X] = x_0, \quad X \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{Q}\text{-a.s. lower bounded.} \end{aligned} \quad (\text{EU-N})$$

The solution to Problem (EU-N) is straightforward thanks to Jensen's inequality and the concavity assumption on $U(\cdot)$. In fact, for every feasible solution X we have

$$\mathbb{E}^\mathbb{Q}[U(X)] \leq U(\mathbb{E}^\mathbb{Q}[X]) = U(x_0),$$

¹See for example [14], Definition 6.1 and Theorem 6.6 or [8], Proposition 2.1. We also remark that in [8], absolute value portfolio strategies were used instead of our "number of shares" strategies.

so the best choice for N-agent is $X^* \equiv x_0$, with optimal value $U(x_0)$. In other words, $\Pi \equiv 0$ will be the selected portfolio, which implies null risky investment. The explanation of this behavior is obvious if one thinks to the *risk neutrality* which stands behind the Black-Scholes model and is perfectly reflected by the measure \mathbb{Q} . We will see that in the non-informed CPT case, the optimal strategy is not as trivial as is here.

3.2 The insider's problem

Let's have a look at the results obtained by Baudoin in [1] for a classical portfolio optimization problem with the presence of an extra-informed agent who possesses the weak information (Y, ν) and an utility function $U(\cdot)$ satisfying Assumption 3.1. If we still denote with X the final wealth, her portfolio optimization problem will be defined as

$$\begin{aligned} & \text{Maximize} && \mathbb{E}^\nu[U(X)] \\ & \text{subject to} && \mathbb{E}^\nu \left[\frac{1}{\xi(Y)} X \right] = x_0 > 0, X \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{Q}^\nu\text{-a.s. lower bounded.} \end{aligned} \tag{EU-I}$$

Recalling Theorem 2.1, we observe that our definition is strongly connected to Definition 4 in [2], which we report below.

Definition 3.1 ([2], Definition 4). *The financial value of the weak information (Y, ν) for an insider with initial endowment $x_0 > 0$ is*

$$u(x_0, \nu) := \inf_{\mu \in \mathcal{E}^\nu} \sup_{\Pi \text{ admissible}} \mathbb{E}^\mu[U(X)]. \tag{3.2}$$

The link between our definition and (3.2) is easily deduced from the *convexity* of the sup function. Moreover, it would be more natural to perform a shift over $u(x_0, \nu)$ and define the value of the privileged information (or *insider's gain*) as $u(x_0, \nu) - U(x_0)$, i.e. the difference between the final utility of the insider and the optimal value of a non informed investor.

We finally remark that the constraint in Problem (EU-I) is a direct consequence of the relation (2.2); indeed, we see that

$$\nu \{ \xi^{-1}(0) \} = \int_{\{ \xi^{-1}(0) \}} d\nu = \int_{\{ \xi^{-1}(0) \}} \xi d\mathbb{Q}_Y = 0.$$

Moreover, thanks to Assumption 2.2, we have

$$\mathbb{Q} \{ \xi(Y) = 0 \} = \mathbb{Q} \{ Y \in \xi^{-1}(0) \} = \mathbb{Q}_Y \{ \xi^{-1}(0) \} = \nu \{ \xi^{-1}(0) \} = 0, \tag{3.3}$$

so that we can write $\frac{1}{\xi}$ as a density of \mathbb{Q}_Y w.r.t. ν , which is \mathbb{Q} -a.s. finite and non-zero. Now it immediately follows $x_0 = \mathbb{E}^\mathbb{Q}[X] = \mathbb{E}^\nu \left[\frac{1}{\xi(Y)} X \right]$, as it appears in (EU-I).

Thanks to convex duality and the martingale dual approach in a complete market framework, we obtain the following result adapted from [2].

Theorem 3.1 ([2], Theorem 1). *Assume that the expectations below are finite. Then for each initial endowment $x_0 > 0$,*

$$u(x_0, \nu) = \sup_{\Pi \text{ admissible}} \mathbb{E}^\nu[U(X)] = \mathbb{E}^\nu \left[U \left((U')^{-1} \left(\frac{\Lambda(x_0)}{\xi(Y)} \right) \right) \right], \tag{3.4}$$

where $\Lambda(x_0)$ is defined by

$$\mathbb{E}^\nu \left[\frac{1}{\xi(Y)} (U')^{-1} \left(\frac{\Lambda(x_0)}{\xi(Y)} \right) \right] = x_0.$$

Moreover, under \mathbb{Q}^ν the optimal terminal wealth is given by

$$X^* = (U')^{-1} \left(\frac{\Lambda(x_0)}{\xi(Y)} \right). \tag{3.5}$$

Note that if the insider only has a *minimal* information, i.e. $\nu = \mathbb{Q}_Y$, then we have $\xi(Y) = 1$ \mathbb{Q} -a.s.. Therefore, we deduce $u(x_0, \nu) = U(x_0)$ and $X = x_0$ \mathbb{Q} -a.s., which is nothing but the N-agent's solution. Finally, as a corollary one can even show that we always have $u(x_0, \nu) \geq U(x_0)$ and the equality takes place for $\nu = \mathbb{Q}_Y$.

Example 3.1. If $\alpha \in (0, 1)$ and $U(x) = x^\alpha$ then by straightforward calculations

$$u(x_0, \nu) = x_0^\alpha \left(\mathbb{E}^{\mathbb{Q}} \left[\xi(Y)^{\frac{1}{1-\alpha}} \right] \right)^{1-\alpha},$$

$$X^* = x_0 \frac{\xi(Y)^{\frac{1}{1-\alpha}}}{\mathbb{E}^{\mathbb{Q}} \left[\xi(Y)^{\frac{1}{1-\alpha}} \right]}.$$

Example 3.2 ([1], Proposition 67). Let's fix a $(\Omega, \mathbb{F}, \mathbb{Q})$ -Brownian motion $W^{\mathbb{Q}}$ and assume a market with only one risky asset whose price dynamics is

$$dS(t) = \sigma S(t) dW^{\mathbb{Q}}(t), \quad t \in [0, T], \quad S(0) = s_0 > 0,$$

for some constant $\sigma > 0$. It is widely known that the prices can also be expressed as

$$S(t) = s_0 \exp \left(\sigma W_t^{\mathbb{Q}} - \frac{\sigma^2}{2} t \right).$$

Hence, by a change of variable, a weak information on the final price $S(T)$ is equivalent to a weak information on the Gaussian random variable $W_T^{\mathbb{Q}}$. Suppose I-agent has the privileged information $(W_T^{\mathbb{Q}}, \nu)$, where

$$\nu(dx) = \frac{1}{\sqrt{2\pi}s} \exp \left(-\frac{(x-m)^2}{2s^2} \right) dx$$

is Gaussian with mean $m \in \mathbb{R}$ and variance $s^2 \leq T$, with $0 < s \leq \sqrt{T}$ (in what follows, we will write $\nu \sim \mathcal{N}(m, s^2)$). Note that Assumption 2.2 is fulfilled and we can also explicitly compute

$$\xi(Y) = \xi(W_T^{\mathbb{Q}}) = \frac{\sqrt{T}}{s} \exp \left(-\frac{(W_T^{\mathbb{Q}} - m)^2}{2s^2} + \frac{(W_T^{\mathbb{Q}})^2}{2T} \right) \quad (3.6)$$

Therefore, if we set $\delta = \frac{s^2 - T}{T}$, then for a power utility function $U(x) = x^\alpha$, $\alpha \in (0, 1)$, one can compute

$$u(x_0, \nu) = x_0^\alpha \frac{1}{\sqrt{1+\delta}} \left(\frac{1-\alpha}{\frac{1}{1+\delta} - \alpha} \right)^{\frac{1-\alpha}{2}} \exp \left(\frac{\alpha m^2}{2[T(1-\alpha) - \alpha\delta T]} \right).$$

Specifically, if $m = 0$ and $s^2 = T$ (i.e. $\delta = 0$) then we recover the *minimal* information case, as $\nu = \mathbb{Q}_Y$. If $m \neq 0$ and $s^2 = T$ then I-agent has some additional information regarding the drift but not the variance of the Brownian motion; in this case we have $u(x_0, \nu) = x_0^\alpha \exp \left(\frac{\alpha m^2}{2[T(1-\alpha)]} \right)$ and the information is more valuable as m becomes greater. Vice versa, if $m = 0$ and $s^2 < T$ then we obtain $u(x_0, \nu) = x_0^\alpha \frac{1}{\sqrt{1+\delta}} \left(\frac{1-\alpha}{\frac{1}{1+\delta} - \alpha} \right)^{\frac{1-\alpha}{2}}$, which tends to infinity as $\delta \downarrow -1$, or equivalently as $s \downarrow 0$. Thus a more precise knowledge on the terminal value of the underlying Brownian motion, which is the same as on the final price, leads to a higher value of the weak information, as naturally expected.

4 The CPT agents' models and their solutions

In this section we are going to frame and solve the portfolio selection problems relative to a non-informed and an insider CPT agent respectively. In particular, we will recover as much as possible the structure and the notation implemented in [8] by Jin and Zhou to describe the preferences and the corresponding objective functions of an investor who takes her decisions affording on the CPT tenets. We immediately point out that our results have a direct link with those in [8]; however, they need a complete proof as we are working in a slight different setting. Loosely speaking, in [8] the investor knows the "historical" probability \mathbb{P} and she performs a standard change of measure affording on the so-called pricing kernel (or state price density) ρ , thus obtaining martingale processes for the prices under an equivalent probability; after that, technical assumptions are imposed and a complete

solution based on ρ is derived. More specifically, using (4.2) and (4.3) below to define the objective function $V(\cdot)$, in [8] the optimization problem for a CPT agent is framed in their equation (2.7) as

$$\begin{aligned} & \text{Maximize} && V(X) \\ & \text{subject to} && \mathbb{E}^{\mathbb{P}}[\rho X] = x_0, X \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{P}\text{-a.s. lower bounded.} \end{aligned} \tag{4.1}$$

However, in the present framework we start from the very beginning with martingale prices; therefore, no change of measure is needed as ρ is totally concentrated. Thus, the structure of our solution for a N-agent will only be *law dependent*, in the sense that only the distribution of a random variable will affect her optimal value.

Now, we recall the cornerstones of the CPT preferences and their mathematical formulations. In this *behavioral* scenario, the trader's goal is to select the portfolio that will engender a terminal wealth X maximizing her "utility". This "utility" (or prospect value, in Kahneman and Tversky terminology) will come up from the algebraic sum of some *expected distorted values* of gains and losses w.r.t. to a reference wealth that we set once for all at the value 0. Mathematically speaking, we will make the following assumptions, corresponding to Assumptions 2.3 and 2.4 in [8].

Assumption 4.1. $u_+(\cdot)$ and $u_-(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$, are strictly increasing, concave, with $u_+(0) = u_-(0) = 0$. Moreover, $u_+(\cdot)$ is strictly concave and twice differentiable, satisfying the Inada conditions $u'_+(0+) = +\infty$ and $u'_+(+\infty) = 0$.

Assumption 4.2. $T_+(\cdot)$ and $T_-(\cdot) : [0, 1] \mapsto [0, 1]$, are differentiable and strictly increasing, with $T_+(0) = T_-(0) = 0$ and $T_+(1) = T_-(1) = 1$.

If the random variable X represents a final wealth at time T and our CPT agent takes μ as her reference measure, then she will assign to X the prospect value $V(X)$, defined by

$$V(X) := V_+(X^+) - V_-(X^-) \tag{4.2}$$

where

$$V_+(Y) := \int_0^{+\infty} T_+(\mu\{u_+(Y) > y\}) dy, \quad V_-(Y) := \int_0^{+\infty} T_-(\mu\{u_-(Y) > y\}) dy \tag{4.3}$$

for any random variable $Y \geq 0$ μ -a.s.; here, X^+ and X^- denote the positive and the negative part of X respectively.

Regarding the market, we maintain Assumption 3.2 over the class of admissible portfolios; hence our CPT agent will search a terminal wealth X , \mathcal{F}_T -measurable and a.s. lower bounded w.r.t. her reference probability. However, her initial endowment can be any amount $x_0 \in \mathbb{R}$ and not necessarily non-negative. Thus, the initial budget constraint will be simply adapted to this generalization.

4.1 The non informed agent's problem

Before formulating the most general version of the model, we proceed step by step and for the moment we consider a non-informed agent who evaluates her total utility distinguishing gains from losses w.r.t. to the reference level set to 0; no probability distortion is allowed, i.e. $T_{\pm}(\cdot) = id(\cdot)$. Such an investor stands in the middle between a classical agent and a *behavioral* agent *à la* Kahneman and Tversky. Within this framework, it seems reasonable to define the problem of a non informed agent as

$$\begin{aligned} & \text{Maximize} && V(X) = \mathbb{E}^{\mathbb{Q}}[u_+(X^+)] - \mathbb{E}^{\mathbb{Q}}[u_-(X^-)] \\ & \text{subject to} && \mathbb{E}^{\mathbb{Q}}[X] = x_0, X \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{Q}\text{-a.s. lower bounded.} \end{aligned} \tag{4.4}$$

Unfortunately there are bad news about Problem (4.4) because with Assumptions 4.1 and 4.2 in force it can easily be ill-posed. Before proceeding to the proof, we note that for an investor with the previous objective function it is better to choose a fixed reward x_+ whenever X is positive, thanks to Jensen's inequality and the concavity of $u_+(\cdot)$. Otherwise, conditioned to $X \leq 0$, she will choose a *lottery* with a \mathbb{Q} -a.s. null payoff accompanied by a huge negative payoff with an infinitesimal probability. This policy will produce an infinite optimal value for the agent; see also Theorem 3.2 in [8] and the subsequent remark.

Proposition 4.1. *If $u_+(+\infty) = +\infty$ then Problem (4.4) is ill-posed.*

Proof. Consider the sequence of terminal wealths $(X_n)_n$, where

$$X_n = \begin{cases} n(x_0^+ + 1) & \text{with probability } p, \\ 0 & \text{with probability } 1 - p - \frac{1}{n^2}, \\ n^2[x_0 - np(x_0^+ + 1)] & \text{with probability } \frac{1}{n^2}, \end{cases}$$

$p \in (0, 1 - \frac{1}{n^2})$ and n is sufficiently big. Note that for every $n \in \mathbb{N}$, $\mathbb{E}^{\mathbb{Q}}[X_n] = x_0$; moreover, for X_n to be \mathcal{F}_T -measurable it is sufficient to choose a pair of disjoint events $A, B_n \in \mathcal{F}_T$ with probability p and $\frac{1}{n^2}$ respectively, no matter what these events are; e.g. one can choose $A = \{S_1(T) \leq \alpha\}$, where α is such that $p = \mathbb{E}^{\mathbb{Q}}[I_A]$ and for each n , $B_n = \{S_1(T) \geq \beta_n\}$, where $\beta_n > \alpha$ is such that $\frac{1}{n^2} = \mathbb{E}^{\mathbb{Q}}[I_{B_n}]$. Then we can compute

$$\mathbb{E}^{\mathbb{Q}}[u_+(X_n^+)] = pu_+(n(x_0^+ + 1)) \longrightarrow +\infty$$

as $n \rightarrow +\infty$. Moreover

$$\mathbb{E}^{\mathbb{Q}}[u_-(X_n^-)] = \frac{1}{n^2}u_-(n^2[x_0 - np(x_0^+ + 1)]^+) \leq u_-([x_0 - np(x_0^+ + 1)]^+) \longrightarrow 0$$

as $n \rightarrow +\infty$, where the last inequality follows by the concavity of $u_-(\cdot)$ and $u_-(0) = 0$. \square

At this point there are two ways out of this unpleasant situation; the first one is to introduce probability distortions, specially on the loss part as explained in [8]. The second one is to impose a loss control, i.e. a lower bound L on the maximal loss which can be suffered by the investor; obviously, one can consider both modifications. For more details on this subject, see [11].

Let's consider the case with probability distortions satisfying Assumption 4.2. Thus the problem for a CPT N-agent will be

$$\begin{aligned} & \text{Maximize} && V(X) = V_+(X^+) - V_-(X^-) \\ & \text{subject to} && \mathbb{E}^{\mathbb{Q}}[X] = x_0, X \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{Q}\text{-a.s. lower bounded,} \end{aligned} \tag{CPT-N}$$

where we set

$$V_+(X^+) := \int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X^+) > y\}) dy, \quad V_-(X^-) := \int_0^{+\infty} T_-(\mathbb{Q}\{u_-(X^-) > y\}) dy.$$

Now, the big difference between our Problem (CPT-N) and the relative optimization problem which has to be solved by a *behavioral* agent in [8] concerns the constraint on the expected value of the terminal wealth X . More specifically, in [8], equation (2.6), the budget constraint was $\mathbb{E}^{\mathbb{P}}[\rho X] = x_0$, where the state price density ρ was supposed to be atom-less w.r.t. to \mathbb{P} . Now, we do not have that atom-less density as we are already working under the martingale measure \mathbb{Q} . We also recall that the assumption on ρ being atom-less w.r.t. \mathbb{P} was imposed in [8] just to avoid technical difficulties. In our case, the absence of a weighting random variable (this was actually the role played by ρ) will change the structure of the solution to Problem (CPT-N) and its economical interpretation.

For a better readability, we will report below only the main results, referring the interested reader to the Appendix for a detailed proof. Now, for any fixed random variable Z uniformly distributed over $(0, 1)$ w.r.t. \mathbb{Q} ($Z \sim U(0, 1)$ for short) and given the pair (p, x_+) with $p \in [0, 1]$ and $x_+ \geq x_0^+$, define as $v_+(p, x_+)$ the optimal value of the following problem:

$$\begin{aligned} & \text{Maximize} && V_+(X) = \int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X) > y\}) dy \\ & \text{subject to} && \mathbb{E}^{\mathbb{Q}}[X] = x_+, \quad X \geq 0 \text{ on } \{Z \leq p\}, \quad X = 0 \text{ on } \{Z > p\}. \end{aligned} \tag{4.5}$$

Next, we set up the optimization problem

$$\begin{aligned} & \text{Maximize} && v_+(p, x_+) - u_-\left(\frac{x_+ - x_0}{1-p}\right) T_-(1-p) \\ & \text{subject to} && \begin{cases} p \in [0, 1], & x_+ \geq x_0^+, \\ x_+ = 0 \text{ if } p = 1, & x_+ = x_0 \text{ if } p = 0, \end{cases} \end{aligned} \tag{4.6}$$

where we conventionally define $u_-\left(\frac{x_+ - x_0}{1-p}\right) T_-(1-p) := 0$ if $p = 1$ and $x_+ = x_0$. Finally, we denote with X^* the optimal solution to Problem (CPT-N) and we make the following hypothesis.

Assumption 4.3. $T'_+(z)$ is non-increasing for $z \in (0, 1]$, $\liminf_{x \rightarrow +\infty} -\frac{xu'_+(x)}{u'_+(x)} > 0$ and for any $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} we have $\mathbb{E}^{\mathbb{Q}} \left[u_+ \left((u'_+)^{-1} \left(\frac{1}{T'_+(Z)} \right) \right) T'(Z) \right] < +\infty$.

Now, with Assumption 4.3 in force, for any $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} we have

$$X^* = (u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) I_{Z \leq p^*} - \frac{x_+^* - x_0}{1 - p^*} I_{Z > p^*}, \quad (4.7)$$

$$V(X^*) = \mathbb{E}^{\mathbb{Q}} \left[u_+ \left((u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) \right) T'_+(Z) I_{Z \leq p^*} \right] - u_- \left(\frac{x_+^* - x_0}{1 - p^*} \right) T_-(1 - p^*), \quad (4.8)$$

where (p^*, x_+^*) are optimal for Problem (4.6) and the Lagrange multiplier λ satisfies

$$\mathbb{E}^{\mathbb{Q}} \left[(u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) I_{Z \leq p^*} \right] = x_+^*. \quad (4.9)$$

Remark 4.1. Firstly, our result shows that a CPT non-informed investor is only interested in probabilities (and not in events). This is a by-product of her knowledge of the martingale measure \mathbb{Q} ; the risk neutrality arising from \mathbb{Q} is reflected by the indifference in the choice of Z . For instance, in a Brownian motion driven market as in Example 3.2, she can choose $Z = F_W(W_T^{\mathbb{Q}})$, where $F_W(\cdot)$ is the distribution function of $W_T^{\mathbb{Q}}$; in this way she will obtain a gain when the price of the risky stock is lower than a certain threshold; however, she is indifferent in choosing $Z = 1 - F_W(W_T^{\mathbb{Q}})$, representing the perfect opposite situation.

Secondly, we highlight the fact that the explicit solution given by (4.7) is only available when $T'_+(\cdot)$ is non-increasing over $(0, 1]$. Combining this observation with Assumption 4.2, a necessary condition to get (4.7) is $T_+(\cdot)$ to be concave. In this way, a reversed S-shaped $T_+(\cdot)$, as it is empirically observed, does not fulfill this condition.

Before proceeding further, we provide a result concerning power utility functions. In [8], the authors were able to find a much more explicit solution assuming generic probability weighting functions $T_{\pm}(\cdot)$ and $u_+(x) = x^\alpha$, $u_-(x) = k_- x^\alpha$ with $\alpha \in (0, 1)$, $k_- \geq 1$.² We now adapt their reasoning and we choose the special distortion on gains $T_+(p) = p^\gamma$, $\gamma \in (0, 1]$, as suggested by the previous Remark 4.1. Intuitively, this concave function should reflect an overestimation of every probability of a winning event. With straightforward computations, for $\alpha < \gamma$ we find

$$\varphi_N(p) := \mathbb{E}^{\mathbb{Q}} \left[T'_+(Z)^{1/(1-\alpha)} I_{Z \leq p} \right] = \gamma^{1/(1-\alpha)} \left(\frac{1-\alpha}{\gamma-\alpha} \right) p^{\frac{\gamma-\alpha}{1-\alpha}}, \quad p \in [0, 1], \quad (4.10)$$

$$k_N(p) := \frac{k_- T_-(1-p)}{(1-p)^\alpha \varphi_N(p)^{1-\alpha}} = \frac{k_-}{\gamma} \left(\frac{\gamma-\alpha}{1-\alpha} \right)^{1-\alpha} \frac{T_-(1-p)}{(1-p)^\alpha p^{\gamma-\alpha}}, \quad p \in (0, 1], \quad (4.11)$$

and following the same lines as in [8], Theorem 9.1, we have

Proposition 4.2. *In the CRRA case with $x_0 \geq 0$ and $T_+(p) = p^\gamma$, $\gamma \in (0, 1]$:*

(i) *if $0 < \alpha < \gamma \leq 1$ and $\inf_{p \in (0, 1]} k_N(p) \geq 1$, then Problem (CPT-N) is well-posed and*

$$X^* = x_0 \left(\frac{\gamma-\alpha}{1-\alpha} \right) Z^{\frac{\gamma-1}{1-\alpha}}, \quad Z \sim U(0, 1); \quad (4.12)$$

$$V(X^*) = x_0^\alpha \gamma \left(\frac{1-\alpha}{\gamma-\alpha} \right)^{1-\alpha}. \quad (4.13)$$

(ii) *if $\alpha \geq \gamma$ or $\inf_{p \in (0, 1]} k_N(p) < 1$, then Problem (CPT-N) is ill-posed.*

It is clear by the parameters' condition in (i) that the curvature of the value function on gains must be greater than that of the distortion $T_+(\cdot)$ if we hope to find a financially meaningful solution; not only, the well-posedness of this model strongly depends on the shape of $T_-(\cdot)$. We also note that the optimal value $V(X^*)$ is decreasing in γ , whereas it does not exhibit a clear dependence in α . As a particular case, we have the following corollary.

²We recall that the parameter k_- is usually called the *loss aversion coefficient*, as in this framework it reflects the concept that "losses loom larger than gains". In what follows, we will refer to this case as to CRRA, due to the Constant Relative Risk Aversion coefficient exhibited by the value functions.

Corollary 4.1. *With the same assumptions of Proposition 4.2 and $T_-(p) = p^\delta$, $\delta \in (0, 1)$:*

(i) *if $0 < \delta \leq \alpha < \gamma < 1$ and $k_- \geq f(\alpha, \gamma, \delta)$, where*

$$f(\alpha, \gamma, \delta) := \gamma \frac{(1-\alpha)^{1-\alpha} (\alpha-\delta)^{\alpha-\delta}}{(\gamma-\alpha)^{\gamma-1} (\gamma-\delta)^{\gamma-\delta}}, \quad (4.14)$$

then Problem (CPT-N) is well-posed;

(ii) *if $0 < \delta \leq \alpha < \gamma = 1$, then Problem (CPT-N) is well-posed;*

(iii) *if $\delta > \alpha$ then Problem (CPT-N) is ill-posed.*

Proof. Using (4.11) and the special form of $T_-(\cdot)$, it is immediate to compute the infimum of $k_N(p)$ over $(0, 1]$ via first order conditions. Now, case (iii) follows if we let p tend to 1; in the other cases, we have that the infimum is reached for $\hat{p} = \frac{\gamma-\alpha}{\gamma-\delta} \leq 1$. Hence, we find $k_N(\hat{p}) = f(\alpha, \gamma, \delta)$ with the subsequent well-posedness condition $k_- \geq f(\alpha, \gamma, \delta)$. If $\gamma = 1$, equation (4.14) reduces to

$$f(\alpha, 1, \delta) = \frac{(1-\alpha)^{1-\alpha} (\alpha-\delta)^{\alpha-\delta}}{(1-\delta)^{1-\delta}} \leq 1.$$

To see this, note that we have $1-\delta, 1-\alpha, \alpha-\delta \in (0, 1)$; moreover, the function $g(x) := x^x \equiv e^{x \ln x}$ is well defined for $x \in (0, 1)$. To prove the previous relation, we only have to show that for every $0 < y < x < 1$ we have

$$x \ln x - y \ln y \leq (x-y) \ln(y-x).$$

But this is true because

$$\sup_{0 < x < 1, 0 < y < 1} x \ln x - y \ln y - (y-x) \ln(y-x) = 0,$$

as it is easily seen using standard minimization techniques. \square

We stress that the *ad hoc* choice of concave $T_\pm(\cdot)$ stands for an investor who overestimates winning and losses chances, thus being somewhat coherent when taking her decisions.³ Furthermore, lengthy but not difficult computations show that $f(\cdot, \cdot, \cdot)$ is decreasing in γ and increasing in δ , as economic intuition suggests. In fact, the lower is the overestimation of gains, the higher has the loss aversion coefficient to be in order to compensate its effect and for the problem to reach well-posedness. However, the dependence on α is not monotonic. For a better understanding of the preceding corollary, in Figure 1 we provide a plot representing a 3D surface of the well-posedness threshold $f(\cdot \cdot \cdot)$ in case (i), where we arbitrarily fix $\gamma = 0.9$ and we take $\alpha \in [0.7, 0.9]$ and $\delta \in (0, 0.7]$. An horizontal plane at the level $f = 1$ is depicted to facilitate the distinction between a surely well-posed case, i.e. when the surface stands below the plane, or a probable ill-posed case, i.e. when the loss-aversion coefficient has to be sufficiently high to ensure condition (4.14). More generally, we can also note that for the reversed S-shaped $T_-(\cdot)$ conjectured by Kahneman and Tversky in [15], namely $T_-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}$ with $\delta \in (0, 1)$, if $\delta > \alpha$ then we still have a systematic ill-posedness, whereas in the opposite case we have to rely on numerical and graphical analysis.

4.2 The insider's problem

In this section we are going to solve the portfolio optimization problem for an informed agent who possesses CPT preferences; in particular, this type of investor has to choose her best portfolio among a suitable class of *admissible* ones in order to maximize her balanced expected utility from gains and losses; moreover, she will be able to use the *minimal* probability \mathbb{Q}' in order to exploit her weak information (Y, ν) but at the same time she weights those probabilities with T_\pm due the her *distorting* evaluation criteria. Thus, we retain Assumption 4.1 on the value functions $u_\pm(\cdot)$ and Assumption 4.2

³This is no longer true if we assume $T_+(\cdot) = id(\cdot)$, as our trader would be perfectly capable to judge gain chances whereas it distorts losing probabilities. Nonetheless, the case $\gamma = 1$ would have a strong link with the theory of coherent risk measures if we further assume a linear utility function. But in our setting, this possibility is obviously ruled out as we require a strictly concave $u_-(\cdot)$ at the origin; hence $V_-(\cdot)$ and similarly $V_+(\cdot)$ can not be risk measures. For more information, see [3].

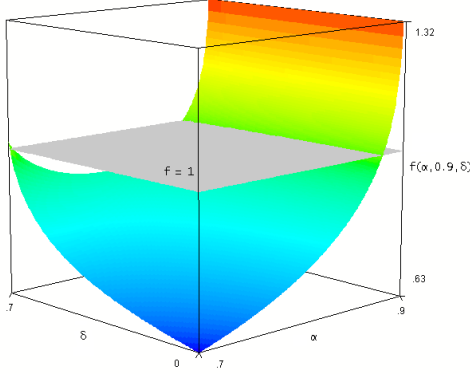


Figure 1: a comparison between the well-posedness threshold and the level of k_- .

on the probability distortions $T_{\pm}(\cdot)$; furthermore, thanks to the equivalence between the measures \mathbb{Q} and \mathbb{Q}^{ν} stated in Assumption 2.2, this CPT I-agent can still afford on the admissible portfolios described in Definition 3.2. We remark that the dynamics of the wealth process $x(\cdot)$ under \mathbb{Q} remain the same as in equation (3.1), whereas they drastically change w.r.t. \mathbb{Q}^{ν} .⁴ However, when solving this problem we look for the optimal terminal wealth X ; the issue of its replication comes up subsequently.

Now we can define the value of the weak information for I-agent in a way analogous to Definition 3.1:

Definition 4.1. *The financial value of the weak information (Y, ν) for a CPT agent with initial endowment $x_0 \in \mathbb{R}$ is*

$$V(x_0, \nu) := \sup_{\Pi \text{ admissible}} V_+^{\nu}(X^+) - V_-^{\nu}(X^-)$$

where

$$V_+^{\nu}(X^+) := \int_0^{+\infty} T_+(\mathbb{Q}^{\nu} \{u_+(X^+) > y\}) dy, \quad V_-^{\nu}(X^-) := \int_0^{+\infty} T_-(\mathbb{Q}^{\nu} \{u_-(X^-) > y\}) dy. \quad (4.15)$$

Here X represents the terminal payoff obtained via the initial wealth x_0 and the dynamics (3.1).

We can now consider any real initial wealth x_0 , not only the positive ones. In what follows we will recover some notation we used in Section 3.1 for the classical portfolio selection model for a weak informed agent. We can now define the problem of a CPT I-agent as

$$\begin{aligned} & \text{Maximize} && V^{\nu}(X) := V_+^{\nu}(X^+) - V_-^{\nu}(X^-) \\ & \text{subject to} && \mathbb{E}^{\nu} \left[\frac{1}{\xi(Y)} X \right] = x_0, X \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{Q}^{\nu} \text{ a.s. lower bounded.} \end{aligned} \quad (\text{CPT-I})$$

Note that this definition is in some sense half-way between that of Jin and Zhou in [8] and that of Baudoin in [1]; in fact, if we suppress the distinction between losses and gains and we do not admit probability distortions, then we recover Problem (EU-I). On the other hand, in the extreme case of *minimal* information $\nu = \mathbb{Q}_Y$ we have $V_{\pm}^{\nu}(X) = V_{\pm}(X)$ thanks to the properties of the *minimal* probability, so we turn back to Problem (CPT-N).

At this point it is sufficient to note that Problem (CPT-I) is nothing but a specialization of Problem (4.1) with only two changes:

1. we use the measure \mathbb{Q}^{ν} instead of \mathbb{P} both in the objective function and in the constraints;
2. the random variable ρ is replaced by the new random variable $\frac{1}{\xi(Y)}$.

Economically speaking, the historical measure \mathbb{P} which is unknown to I-agent and thus can not be used, is replaced by the *minimal* probability \mathbb{Q}^{ν} , which now represents the probability valuation criteria of the insider (and as usual they are perceived as distorted by $T_{\pm}(\cdot)$). This fact should reflect her privileged information and one can wonder if it produces a greater optimal value w.r.t. a non

⁴For more details, see [1] where the theory of Conditioned Stochastic Differential Equations (CSDEs) is developed.

informed investor (we will prove in Theorem 5.1 that this fact always hold).

Secondly, the state price density ρ is also unknown; however I-agent can rely on her own *density*. An interpretation of this phenomenon is that the typical *law dependence* which is revealed when $\xi(Y) = 1$ \mathbb{Q} -a.s. (as it happened in Problem (CPT-N)) is now distorted by a generic random variable $\xi(Y)$. This reflects an Ω -dependence of the insider's preferences once she obtains some extra information, as now I-agent also cares about events.

Turning back to the mathematical counterpart of Problem (CPT-I), we see that we only have to check that all the assumptions imposed in [8] on ρ are now fulfilled by $\frac{1}{\xi(Y)}$ and then we will be able to recover all the results found in [8] with the obvious modifications, i.e. substitute for $\frac{1}{\xi(Y)}$ and \mathbb{Q}^ν in every explicit expression.

First of all, to avoid undue technicalities, the assumption of ρ having no atoms w.r.t. \mathbb{P} ([8], Assumption 2.2) is now translated in our

Assumption 4.4. *The random variable $\frac{1}{\xi(Y)}$ has no atoms w.r.t. \mathbb{Q}^ν , i.e.*

$$\mathbb{Q}^\nu \left\{ \frac{1}{\xi(Y)} = a \right\} = 0 \quad \forall a \geq 0.$$

But it is easily seen that

$$\mathbb{Q}^\nu \left\{ \frac{1}{\xi(Y)} = a \right\} = \mathbb{E}^\mathbb{Q} \left[\xi(Y) I_{\left\{ \frac{1}{\xi(Y)} = a \right\}} \right] = \frac{1}{a} \mathbb{Q} \left\{ \xi(Y) = \frac{1}{a} \right\};$$

hence we could impose this last condition which is indeed satisfied in most common situations, e.g. when Y is an absolutely continuous random variable and the law ν preserves its absolute continuity on the same support. Other technical conditions on $\xi(Y)$ are straightforward to check; in fact we have $\frac{1}{\xi(Y)} \in (0, +\infty)$ \mathbb{Q}^ν -a.s. thanks to our Assumption 2.2. Moreover, $\mathbb{E}^\nu \left[\frac{1}{\xi(Y)} \right] = 1$ follows directly from the definition of ξ .

We now modify the analysis made in [8] with the necessary modifications for a CPT informed investor. In what follows, we will need a new set of variables for I-agent; they will be accompanied with a ν , in contrast to the variables for N-agent which will be recovered by Section 4.1 and do not have any upper symbol. Unfortunately we will obtain quite cumbersome explicit expressions; however we decide to retain $\frac{1}{\xi(Y)}$ instead of its reciprocal because it will simplify some comparison results in the subsequent sections. Now we define

$$\overline{\frac{1}{\xi(Y)}} \equiv \text{esssup}_{\mathbb{Q}^\nu} \frac{1}{\xi(Y)} := \sup \{ a \in \mathbb{R} : \mathbb{Q}^\nu \left\{ \frac{1}{\xi(Y)} > a \right\} > 0 \},$$

$$\underline{\frac{1}{\xi(Y)}} \equiv \text{essinf}_{\mathbb{Q}^\nu} \frac{1}{\xi(Y)} := \inf \{ a \in \mathbb{R} : \mathbb{Q}^\nu \left\{ \frac{1}{\xi(Y)} < a \right\} > 0 \}.$$

Once again well-posedness is an important issue as in the case of N-agent's problem; with some slight adjustments to Theorems 3.1 and 3.2 in [8] we can formulate:

Proposition 4.3. *Problem (CPT-I) is ill-posed if there exists a nonnegative \mathcal{F}_T -measurable random variable X such that $\mathbb{E}^\nu \left[\frac{1}{\xi(Y)} X \right] < +\infty$ and $V_+^\nu(X) = +\infty$.*

Proposition 4.4. *If $u_+(+\infty) = +\infty$, $\overline{\frac{1}{\xi(Y)}} = +\infty$, and $T_-(\cdot) = \text{id}(\cdot)$, then Problem (CPT-I) is ill-posed.*

Thus, to avoid systematic ill-posedness, we will impose:

Assumption 4.5 (see [8], Assumption 3.1). *$V_+^\nu(X) < +\infty$ for any nonnegative, \mathcal{F}_T -measurable random variable X satisfying $\mathbb{E}^\nu \left[\frac{1}{\xi(Y)} X \right] < +\infty$.*

Remark 4.2. Note that we do not yet have a comparison result between the financial value of the weak information for a CPT I-agent, $V(x_0, \nu)$, and the optimal value for a CPT N-agent's problem, so for the moment we can not conclude that an insider always gets more than a non informed agent in this behavioral context, neither we can say that ill-posedness for N-agent implies ill-posedness for I-agent.

For reasons of space, we only report the main steps to get the solution to Problem (CPT-I). At first, for a given pair (A, x_+) , with $A \in \mathcal{F}_T$ and $x_+ \geq x_0^+$, define the problem

$$\begin{aligned} & \text{Maximize} && V_+^\nu(X) = \int_0^{+\infty} T_+(\mathbb{Q}^\nu\{u_+(X) > y\}) dy \\ & \text{subject to} && \mathbb{E}^\nu[\frac{1}{\xi(Y)}X] = x_+, \quad X \geq 0 \quad \mathbb{Q}^\nu\text{-a.s.}, \quad X = 0 \quad \mathbb{Q}^\nu\text{-a.s. on } A^C. \end{aligned} \quad (4.16)$$

Note that Assumption 4.5 implies that $V_+^\nu(X)$ is a finite nonnegative number for any feasible X . We now define $v_+^\nu(A, x_+)$, the optimal value of problem (4.16), in this way:

- if $\mathbb{Q}^\nu(A) > 0$ then the feasible region of (4.16) is non-empty and $v_+^\nu(A, x_+)$ is defined as the supremum of (4.16);
- if $\mathbb{Q}^\nu(A) = 0$ and $x_+ = 0$, then the only feasible solution for (4.16) is $X = 0$ \mathbb{Q}^ν -a.s., so $v_+^\nu(A, x_+) := 0$;
- if $\mathbb{Q}^\nu(A) = 0$ and $x_+ > 0$, then (4.16) has an empty feasible region, therefore $v_+^\nu(A, x_+) := -\infty$.

For any $c \geq 0$, set $v_+^\nu(c, x_+) := v_+^\nu(\{\omega \in \Omega : \frac{1}{\xi(Y(\omega))} \leq c\}, x_+)$; moreover, define $F^\nu(\cdot)$ and $F^\mathbb{Q}(\cdot)$ the distribution functions of $\frac{1}{\xi(Y)}$ w.r.t. \mathbb{Q}^ν and \mathbb{Q} respectively. Following the guidelines of [8], equation (4.4), we set up the "simpler" problem

$$\begin{aligned} & \text{Maximize} && v_+^\nu(c, x_+) - u_- \left(\frac{x_+ - x_0}{1 - F^\mathbb{Q}(c)} \right) T_-(1 - F^\nu(c)) \\ & \text{subject to} && \begin{cases} \frac{1}{\xi(Y)} \leq c \leq \overline{\frac{1}{\xi(Y)}}, & x_+ \geq x_0^+, \\ x_+ = 0 \text{ if } c = \underline{\frac{1}{\xi(Y)}}, & x_+ = x_0 \text{ if } c = \overline{\frac{1}{\xi(Y)}}, \end{cases} \end{aligned} \quad (4.17)$$

where we use the convention

$$u_- \left(\frac{x_+ - x_0}{1 - F^\mathbb{Q}(c)} \right) T_-(1 - F^\nu(c)) := 0 \quad \text{if } c = \overline{\frac{1}{\xi(Y)}} \quad \text{and} \quad x_+ = x_0. \quad (4.18)$$

Now we are ready to state the results for a CPT agent who has the weak information (Y, ν) :

Proposition 4.5. *Assume that $u_-(\cdot)$ is strictly concave at 0. We have the following conclusions:*

(i) *If $X^{\nu*}$ is optimal for Problem (CPT-I), then*

$$c^{\nu*} := (F^\nu)^{-1}(\mathbb{Q}^\nu\{X^{\nu*} \geq 0\}), \quad x_+^{\nu*} := \mathbb{E}^\nu \left[\frac{1}{\xi(Y)} (X^{\nu*})^+ \right]$$

are optimal for Problem (4.17). Moreover, $\{\omega : X^{\nu} \geq 0\}$ and $\{\omega : \frac{1}{\xi(Y)} \leq c^{\nu*}\}$ are identical up to a \mathbb{Q}^ν -null probability set, and*

$$(X^{\nu*})^- = \frac{x_+^{\nu*} - x_0}{1 - F^\mathbb{Q}(c^{\nu*})} I_{\frac{1}{\xi(Y)} > c^{\nu*}} \quad \mathbb{Q}^\nu\text{-a.s.}$$

(ii) *If $(c^{\nu*}, x_+^{\nu*})$ is optimal for Problem (4.17) and $X_+^{\nu*}$ is optimal for Problem (4.16) with parameters $(\{\frac{1}{\xi(Y)} \leq c^{\nu*}\}, x_+^{\nu*})$, then*

$$X^{\nu*} := X_+^{\nu*} I_{\frac{1}{\xi(Y)} \leq c^{\nu*}} - \frac{x_+^{\nu*} - x_0}{1 - F^\mathbb{Q}(c^{\nu*})} I_{\frac{1}{\xi(Y)} > c^{\nu*}}$$

is optimal for Problem (CPT-I).

Therefore, in order to solve Problem (CPT-I) we can exploit the following algorithm:

Step 1 Solve Problem (4.16) with given parameters $(\{\frac{1}{\xi(Y)} \leq c\}, x_+)$, where $\frac{1}{\xi(Y)} \leq c \leq \overline{\frac{1}{\xi(Y)}}$ and $x_+ \geq x_0^+$, in order to obtain $v_+^\nu(c, x_+)$ and the optimal solution $X_+^{\nu*}(c, x_+)$.

Step 2 Solve Problem (4.17) to get $(c^{\nu*}, x_+^{\nu*})$.

Step 3 (i) If $(c^{\nu^*}, x_+^{\nu^*}) = (\frac{1}{\xi(Y)}, x_0)$, then $X_+^{\nu^*}(\frac{1}{\xi(Y)}, x_0)$ solves Problem (CPT-I).

(ii) Else $X_+^{\nu^*}(c^{\nu^*}, x_+^{\nu^*}) I_{\frac{1}{\xi(Y)} \leq c^{\nu^*}} - \frac{x_+^{\nu^*} - x_0}{1 - F^{\mathbb{Q}}(c^{\nu^*})} I_{\frac{1}{\xi(Y)} > c^{\nu^*}}$ solves Problem (CPT-I).

To get an explicit solution we now have to impose conditions similar to that in Assumption 4.3. In particular, we set

Assumption 4.6.

(i) $\frac{(F^\nu)^{-1}(z)}{T'_+(z)}$ is non-decreasing in $z \in (0, 1]$;

(ii) $\liminf_{x \rightarrow +\infty} \frac{-x u'_+(x)}{u'_+(x)} > 0$;

(iii) $\mathbb{E}^\nu \left[u_+ \left((u'_+)^{-1} \left(\frac{1}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)})}) \right) \right) T'_+(F^\nu(\frac{1}{\xi(Y)})) \right] < +\infty$.

Briefly, condition (i) is related to the fact that the distortion on gains should not be too extreme⁵; then, hypothesis (ii) on the RRA coefficient on gains is the same as for N-agent (recall that the value functions, as well as the probability distortions, are supposed to be the same for the two types of agent in order to facilitate a comparative analysis).

Now, with Assumption 4.6 in force, $v'_+(c, x_+)$ and the corresponding optimal solution $X_+^{\nu^*}$ to Problem (4.16) can be expressed more explicitly, together with the optimal solution X^{ν^*} of (CPT-I):

$$v'_+(c, x_+) = \mathbb{E}^\nu \left[u_+ \left((u'_+)^{-1} \left(\frac{\lambda^\nu(c, x_+)}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) \right) T'_+(F^\nu(\frac{1}{\xi(Y)})) I_{\frac{1}{\xi(Y)} \leq c} \right], \quad (4.19)$$

$$X_+^{\nu^*} = (u'_+)^{-1} \left(\frac{\lambda^\nu(c, x_+)}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) I_{\frac{1}{\xi(Y)} \leq c}, \quad (4.20)$$

$$X^{\nu^*} = (u'_+)^{-1} \left(\frac{\lambda^\nu(c^{\nu^*}, x_+^{\nu^*})}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) I_{\frac{1}{\xi(Y)} \leq c^{\nu^*}} - \frac{x_+^{\nu^*} - x_0}{1 - F^{\mathbb{Q}}(c^{\nu^*})} I_{\frac{1}{\xi(Y)} > c^{\nu^*}}, \quad (4.21)$$

where $\lambda^\nu(c, x_+)$ satisfies $\mathbb{E}^\nu \left[(u'_+)^{-1} \left(\frac{\lambda^\nu(c, x_+)}{\xi(Y) T'_+(F^\nu(\frac{1}{\xi(Y)}))} \right) \frac{1}{\xi(Y)} I_{\frac{1}{\xi(Y)} \leq c} \right] = x_+$.

Before proceeding further, let's explore in detail what are the implications of such a policy adopted by a CPT weak informed investor. As Jin and Zhou noticed in [8], Footnote 7, a non-informed agent selects a final payoff which resembles a gamble on a good state of the world⁶. In fact, in their framework a trader obtained a final wealth greater than her reference point if and only if the event $\{\rho \leq c^*\}$ happened. But in a one risky asset market with constant coefficients and null interest rate, this amounts to say that the final price of the stock, namely $S(T)$, must be greater than a certain threshold which depends on c^* ; this can be easily shown by noting that

$$S(T) = s_0 \exp \left(\left(b - \frac{\sigma^2}{2} \right) T + \sigma W_T^{\mathbb{P}} \right),$$

where $W^{\mathbb{P}}$ is a (\mathbb{F}, \mathbb{P}) -Brownian motion over $[0, T]$. This in turn implies

$$\{\rho \leq c^*\} = \left\{ \exp \left(-\frac{b^2}{2\sigma^2} T - \frac{b}{\sigma} W_T^{\mathbb{P}} \right) \leq c^* \right\} = \left\{ S(T) \geq s_0 \exp \left(\frac{b - \sigma^2}{2} T - \frac{\sigma^2}{b} \ln c^* \right) \right\}.$$

Obviously one can see that the greater is c^* , the higher is the \mathbb{P} -probability to reach a final gain. Can we find a similar explanation for a weak informed CPT investor? It is straightforward from (4.21) that also this time the final payoff shows a betting strategy adopted by the insider. However, now a good state of the world for I-agent is the event

$$\left\{ \frac{1}{\xi(Y)} \leq c^{\nu^*} \right\} = \left\{ \xi(Y) \geq \frac{1}{c^{\nu^*}} \right\}.$$

⁵We observe that our Assumption 4.6 is nothing but Assumption 4.1 in [8]. However, in their context conditions (i) concerned the distribution function of the state prices ρ , thus it involved the market parameters. On the contrary, in our case (i) imposes a link between the distortion $T_+(\cdot)$ and $F^\nu(\cdot)$, therefore it is a condition about the kind of weak information that I-agent possesses; a similar remark holds for (iii).

⁶Remember that in the original framework the agent knows the historical measure \mathbb{P} but this is by no means helpful, i.e. it does not give any advantage because \mathbb{P} is common knowledge.

Again it is clear that the greater is c^{ν^*} , the higher is the \mathbb{Q}^ν -probability of a terminal gain. Otherwise stated, with c^{ν^*} held fixed, the greater is the random variable $\xi(Y)$, the higher is that probability. Intuitively, a greater value of $\xi(Y)$ means a more accurate information; just think of Example 3.2, where the bell-shaped Gaussian density function becomes more and more concentrated as $s \downarrow 0$. In the limit this will produce a certain information about the final price, which in turn should imply an infinite optimal value for the insider, i.e. some kind of arbitrage on the market.

Remark 4.3. The problem here is that this reasoning is more subtle than it could seem; in fact the optimal threshold c^{ν^*} varies with the weak information (Y, ν) ! (Recall that c^{ν^*} is obtained in Step 2 of the preceding algorithm, where one has to solve Problem (4.17)). A correct argument should analyze how much c^{ν^*} varies depending on $\xi(Y)$, and this in general is not an easy task. However we are able to provide an interesting example where this dependence can be estimated, see Example 4.1 below.

As we did for a non-informed agent, let's now suppose that I-agent has CRRA value functions; no additional assumption is made on the probability distortions and we also remark that nothing can be said in general on the distribution of $\frac{1}{\xi(Y)}$, as it strongly depends both on the random variable Y and on its effective law ν . We follow again the argument described in [8], Section 9, but now we have to rely on the functions

$$\varphi^\nu(c) := \mathbb{E}^\nu \left[\left(\xi(Y) T'_+ \left(F^\nu \left(\frac{1}{\xi(Y)} \right) \right) \right)^{1/(1-\alpha)} \frac{1}{\xi(Y)} I_{\frac{1}{\xi(Y)} \leq c} \right] > 0, \quad \frac{1}{\xi(Y)} < c \leq \frac{1}{\xi(Y)}, \quad (4.22)$$

$$k^\nu(c) := \frac{k_- T_- (1 - F^\nu(c))}{\varphi^\nu(c)^{1-\alpha} (1 - F^\mathbb{Q}(c))^\alpha} > 0, \quad \frac{1}{\xi(Y)} < c \leq \frac{1}{\xi(Y)}. \quad (4.23)$$

Note that the case $c = \frac{1}{\xi(Y)}$ is trivial and once again the sign of the initial wealth x_0 is crucial.

Proposition 4.6. *Assume that $x_0 \geq 0$ and Assumption 4.6 holds.*

(i) *If $\inf_{c > \frac{1}{\xi(Y)}} k^\nu(c) \geq 1$, then Problem (CPT-I) is well-posed and*

$$X^{\nu^*} = \frac{x_0}{\varphi^\nu \left(\frac{1}{\xi(Y)} \right)} \left(\xi(Y) T'_+ \left(F^\nu \left(\frac{1}{\xi(Y)} \right) \right) \right)^{1/(1-\alpha)}, \quad (4.24)$$

$$V^\nu(x_0, \nu) = x_0^\alpha \varphi^\nu \left(\frac{1}{\xi(Y)} \right)^{1-\alpha}. \quad (4.25)$$

(ii) *If $\inf_{c > \frac{1}{\xi(Y)}} k^\nu(c) < 1$, then Problem (CPT-I) is ill-posed.*

Note that a null initial wealth is still accompanied by a null risky investment and a null financial value. Finally, if $x_0 < 0$ it is sufficient to adapt the results of [8], Theorem 9.2, to the present case.

Example 4.1 (*Evaluation of $V(x_0, \nu)$ with $T_+(\cdot)$ convex*). We are going to provide a concrete example where the financial value of the weak information (Y, ν) can be explicitly computed. We assume CRRA value functions with $x_0 \geq 0$ for our informed agent and a single risky asset market analogous to that of Example 3.2, with weak information given by $Y = W_T^\mathbb{Q}$ and $\nu \sim \mathcal{N}(0, s^2)$ with $0 < s \leq \sqrt{T}$. Hence, it is easy to compute

$$\frac{1}{\xi(Y)} = \frac{s}{\sqrt{T}} \exp \left\{ \frac{T - s^2}{2Ts^2} (W_T^\mathbb{Q})^2 \right\}, \quad (4.26)$$

which immediately gives $\frac{1}{\xi(Y)} = \frac{s}{\sqrt{T}}$ and $\frac{1}{\xi(Y)} = +\infty$. Next we are able to check Assumption 4.4, i.e. that $\frac{1}{\xi(Y)}$ has not atom w.r.t. \mathbb{Q}^ν , as $\frac{1}{\xi(Y)}$ does not have atoms w.r.t. \mathbb{Q} and the two measures are equivalent. Moreover $\frac{1}{\xi(Y)} \in (0, +\infty)$ \mathbb{Q}^ν -a.s. and its expected value w.r.t. \mathbb{Q}^ν is 1. Thus every technical condition is fulfilled and we are allowed to proceed in our analysis.

The next step consists in verifying the three conditions in Assumption 4.6; however, (ii) follows immediately by the CRRA hypothesis and (iii) will be checked *a posteriori* once we have performed the necessary computations. Regarding condition (i), we observe that the law of Y w.r.t. \mathbb{Q}^ν is exactly ν . Hence, with some tedious but elementary computations one can check that the distribution function of $\frac{1}{\xi(Y)}$ w.r.t. to \mathbb{Q}^ν is given by

$$F^\nu(c) = \mathbb{Q}^\nu \left\{ \frac{1}{\xi(Y)} \leq c \right\} = \begin{cases} 0 & \text{if } c \leq \frac{s}{\sqrt{T}}, \\ 2\mathcal{N} \left(\sqrt{\frac{2T}{T - s^2}} \ln \left(c \frac{\sqrt{T}}{s} \right) \right) - 1 & \text{if } c > \frac{s}{\sqrt{T}}, \end{cases} \quad (4.27)$$

where $\mathcal{N}(\cdot)$ is the distribution function of a standard Gaussian random variable. By this last expression we can obtain the left inverse of $F^\nu(\cdot)$ as

$$(F^\nu)^{-1}(z) = \frac{s}{\sqrt{T}} \exp\left(\frac{T-s^2}{2T} \left[\mathcal{N}^{-1}\left(\frac{z+1}{2}\right)\right]^2\right), \quad z \in [0, 1]. \quad (4.28)$$

Now, condition (i) requires the ratio $\frac{(F^\nu)^{-1}}{T'_+}(z)$ to be non decreasing over $(0, 1]$; if the distortion $T_+(\cdot)$ is assumed to be twice continuously differentiable, we see that this is indeed the case whenever the derivative of that ratio is non-negative. Note that a sufficient condition for this to happen is $T''_+(\cdot) \leq 0$ over $[0, 1]$, as it ensures

$$\frac{d}{dz} \frac{(F^\nu)^{-1}}{T'_+}(z) = \frac{[(F^\nu)^{-1}]' T'_+ - (F^\nu)^{-1} T''_+}{(T'_+)^2}(z) \geq 0, \quad z \in (0, 1], \quad (4.29)$$

thanks to the fact that $[(F^\nu)^{-1}]'(\cdot)$, $T'_+(\cdot)$ and $(F^\nu)^{-1}(\cdot)$ are non-negative functions. By the way, $T''_+(\cdot) \leq 0$ is only a sufficient condition, not a necessary one; therefore, we can try to use a non concave $T_+(\cdot)$ and check the validity of (i).

It turns out that a class of weighting functions that fulfills both Assumption 4.2 and the previous condition (i) is given by⁷

$$T_+(p) = 2\mathcal{N}\left(\sqrt{1-2a} \mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\right) - 1, \quad a \in (0, \frac{1}{2}). \quad (4.30)$$

Despite this unusual expression, it is not difficult to check that such distortions are globally convex over $(0, 1)$, thus implying a prudential criterion when evaluating gains. The lack of concavity restricts our attention to Problem (CPT-I), as Assumption 4.3 for Problem (CPT-N) is not fulfilled. A closer look at equation (4.30) shows that they are nothing but the primitives of

$$T'_+(p) = \sqrt{1-2a} \exp\left(a \left[\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\right]^2\right), \quad a \in (0, \frac{1}{2}). \quad (4.31)$$

By using (4.28) and (4.31), tedious calculations shows that condition (i) is indeed fulfilled if and only if $a \leq \frac{T-s^2}{2T} < \frac{1}{2}$ (however, we will choose $a < \frac{T-s^2}{2T}$ as the equality causes integrability troubles, leading to violate condition (iii) of Assumption 4.6). Leaving aside the ill-posedness issue, we now apply Proposition 4.6. After cumbersome (but not difficult) computations, we find

$$X^{\nu*} = x_0 \frac{\sqrt{T-s^2-2aT}}{s\sqrt{1-\alpha}} \exp\left(-\frac{T-s^2-2aT}{2Ts^2(1-\alpha)} (W_T^\mathbb{Q})^2\right),$$

$$V(x_0, \nu) = x_0^\alpha \frac{\sqrt{T(1-2a)}}{s} \left(\frac{s\sqrt{1-\alpha}}{\sqrt{T-s^2-2aT}}\right)^{1-\alpha}.$$

Performing first order derivatives, it is immediate to see that $V(x_0, \nu)$ is increasing in the terminal time T , whereas it is decreasing in the parameter a and in s . This is perfectly coherent with intuition, as the more accurate information is, the greater should be its value. It is interesting to note that the magnitude of the parameter a determines the “degree” of convexity of $T_+(\cdot)$ and as $a \rightarrow 0$, $T_+(\cdot)$ tends to the identity function. As expected, we obtain $\lim_{a \rightarrow 0^+} V(x_0, \nu) = u(x_0, \nu)$ as in Example 3.2, because CPT and EU preferences coalesce.

There is an interesting fact about this example which is connected with Remark 4.3. A closer look to the shape of the distribution function $F^\nu(\cdot)$ in (4.27) shows that if $s \rightarrow 0$ (which corresponds to a more accurate information), then the random variable $\frac{1}{\xi(Y)}$ tends to be more concentrated around 0, i.e. (4.27) tends to $I_{c>0}$. In the other extreme case, as $s \rightarrow \sqrt{T}$ (which corresponds to a *minimal* information) $\frac{1}{\xi(Y)}$ tends to be more concentrated around 1, i.e. (4.27) tends to $I_{c>1}$.

We now remark that in order to ensure well-posedness, one has to compute $\varphi^\nu(c)$ and this is a lengthy but easy task; after that, we should specify a particular form for $T_-(\cdot)$ and check whether $\inf_{c > \frac{1}{\xi(Y)}} k^\nu(c) \geq 1$ as we did in Corollary 4.1. Nonetheless, we observe that we can also find an

⁷Obviously, to define $T_+(1)$ we use the convention $\mathcal{N}(+\infty) = 1$.

estimate about the value $V(x_0, \nu)$ in the well-posed case; if $\inf_{c>0} k^\nu(c) \geq 1$, then we know that $V(x_0, \nu) = x_0^\alpha \varphi^\nu(+\infty)^{1-\alpha}$ and we can compute

$$\varphi^\nu(+\infty) = \mathbb{E}^{\mathbb{Q}} \left[\xi(Y)^{\frac{1}{1-\alpha}} T'_+ \left(F^\nu \left(\frac{1}{\xi(Y)} \right) \right)^{\frac{1}{1-\alpha}} \right] \geq \mathbb{E}^{\mathbb{Q}} \left[\xi(Y)^{\frac{1}{1-\alpha}} \right] \longrightarrow +\infty \quad (4.32)$$

as $s \rightarrow 0^+$ whenever $\inf_{p \in [0,1]} T'_+(p) > 0$. On one hand, this fact suggests that well-posedness can only be assessed for weak information that are not too accurate, i.e. when s^2 is close to T . On the other hand, this condition on $T'_+(\cdot)$ is fulfilled by our particular choice in (4.31) and it is economically meaningful as if $\inf_{p \in [0,1]} T'_+(p) = 0$, then there would be a non-negligible interval of actual probabilities which are highly distorted. Not only, this intuition is implicit in the empirical estimation in [15] where the suggested distortion $T_+(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}$ automatically satisfy $\inf_{p \in [0,1]} T'_+(p) > 0$ for sufficiently high δ (approximately $\delta > 0.28$, whereas in [15] it was estimated $\delta = 0.69$).

Example 4.2 (*Reversed S-shaped probability distortion $T_+(\cdot)$*). We are aware of the fact that empirical observations suggest probability weighting not to be globally convex neither globally concave. While in the previous example an *ad hoc* construction has been performed in order to obtain explicit (and sensible!) expressions, we now suggest a particular reversed S-shaped $T_+(\cdot)$ which may resemble an observable one. Following the same lines as in [8], Example 6.1, and using the framework we established in our Example 4.1, it is not difficult to build such a distortion. Posing $\delta = \frac{s^2 - T}{T}$, for a given set of parameters $a < 0$, $0 < b < -\frac{1}{\delta}$, $c_0 > \frac{s}{\sqrt{T}}$ we obtained

$$T_+(p) = \begin{cases} 4k \left(\frac{s}{\sqrt{T}} \right)^a \delta_a \left[\mathcal{N} \left(\mathcal{N}^{-1} \left(\frac{p+1}{2} \right) \delta_a \right) - \frac{1}{2} \right] & p \in [0, p_0), \\ 4k \left(\frac{s}{\sqrt{T}} \right)^a \delta_a \left[\mathcal{N} \left(\mathcal{N}^{-1} \left(\frac{p_0+1}{2} \right) \delta_a \right) - \frac{1}{2} \right] \\ + 4\tilde{k} \left(\frac{s}{\sqrt{T}} \right)^b \delta_b \left[\mathcal{N} \left(\mathcal{N}^{-1} \left(\frac{p+1}{2} \right) \delta_b \right) - \mathcal{N} \left(\mathcal{N}^{-1} \left(\frac{p_0+1}{2} \right) \delta_b \right) \right] & p \in (p_0, 1), \end{cases} \quad (4.33)$$

where $\delta_a := \sqrt{\frac{1}{\delta} \left(a + \frac{1}{\delta} \right)}$, $\delta_b := \sqrt{\frac{1}{\delta} \left(b + \frac{1}{\delta} \right)}$, $\tilde{k} := kc_0^{a-b}$, $p_0 := F^\nu(c_0)$ and the real number k is uniquely determined by the terminal condition $T_+(1) = 1$. Note that such a $T_+(\cdot)$ is a non decreasing function over $[0, 1]$ and $T'_+(p) \rightarrow +\infty$ as $p \rightarrow 0$ or $p \rightarrow 1$, which is consistent with the empirical estimates. However, it is important to note that the overall construction depends on the weak information (Y, ν) , thus it seems to be completely unrealistic. Effectively, this flaw was still present in Example 6.1 of [8], where the *ad hoc* distortion depended on the market parameters! To conclude, we note that the condition $\inf_{p \in [0,1]} T'_+(p) > 0$ which ensured (4.32) is satisfied for the $T_+(\cdot)$ that we build in (4.33). In fact we have

$$T'_+(p) = T'_+(F^\nu(x)) = \begin{cases} kx^a & \text{if } 0 < x \leq c_0, \\ \tilde{k}x^b & \text{if } x > c_0, \end{cases}$$

which is always greater or equal than $kc_0^a > 0$.

5 Comparison results between EU and CPT agents

In this weak information approach we can consider four different types of investors, depending on the information level (N-agents versus I-agents) and on the valuation criteria (classical expected utility maximizers versus *behavioral* investors *à la* Kahneman and Tversky).

In [2], the authors already compared an EU N-agent with an EU I-agent; the main result is the fact that the insider always gets more than a non informed agent and in Examples 3.1 and 3.2 we also showed some interesting estimates.

Not only, the differences between an EU N-agent and a CPT N-agent are easy to analyze: on one hand, we have seen in Section 3.1 that the optimal policy for a classical N-agent is to choose a constant wealth, i.e. $X^* = x_0 \mathbb{Q}$ a.s., obtaining the optimal value $U(x_0)$, whereas the CPT N-agent's strategy is characterized by her substantial indifference between events of the same probability. This phenomenon thus produce structurally different optimal final wealths, as the *behavioral* agent can even exploit a leverage effect by choosing a negative final wealth with positive probability. Furthermore,

it is obviously seen that this kind of investor can always select $(p^*, x_+^*) = (1, x_0)$, thus obtaining $V(X^*) = u_+(x_0)$. However this strategy is not necessarily the best one, as she has to face Problem (4.6) and find its optimal solution.

5.1 EU versus CPT insiders

To make things sensible, we assume that the two types of investors share the same initial wealth $x_0 > 0$ and the same utility function on gains $u_+(\cdot)$, whereas the CPT agent is also endowed with a value function on losses $u_-(\cdot)$ and probability distortions $T_{\pm}(\cdot)$.

A comparison can be made among the final optimal wealths X^* and $X^{\nu*}$, given by (3.5) and (4.21) respectively. Unfortunately, nothing can be said in full generality; however we see that if $c^{\nu*} < \frac{1}{\xi(Y)}$ then the CPT I-agent selects a negative payoff with positive probability, thus the two policies are substantially different. On the contrary, if $c^{\nu*} = \frac{1}{\xi(Y)}$, then $x_+^{\nu*} = x_0$ and we can say something more, as now we are comparing two positive random variables⁸. In particular, in the CRRA case we can compare the expression appearing in Example 3.1 with (4.24); after some straightforward computations we see that $X^{\nu*} \gtrless X^*$ if and only if

$$T'_+ \left(F^{\nu} \left(\frac{1}{\xi(Y)} \right) \right)^{1/(1-\alpha)} \gtrless \frac{\mathbb{E}^{\mathbb{Q}} \left[\xi(Y)^{1/(1-\alpha)} T'_+ \left(F^{\nu} \left(\frac{1}{\xi(Y)} \right) \right)^{1/(1-\alpha)} \right]}{\mathbb{E}^{\mathbb{Q}} \left[\xi(Y)^{1/(1-\alpha)} \right]}. \quad (5.1)$$

Note that the comparison between $X^{\nu*}$ and X^* concerns two random variables with the same \mathbb{Q} expected value equal to x_0 , whereas the previous expression contains a random variable on the LHS and a constant on the RHS. Thus we deduce that it can not happen that the optimal wealth X^* for a classical I-agent is \mathbb{Q} a.s. greater than $X^{\nu*}$, neither the contrary. The overall results strongly depends on the shape of $T_+(\cdot)$, or equivalently on its derivative.

From a qualitative point of view, we observe that the previous expression is easily reduced to

$$\{X^{\nu*} \gtrless X^*\} \Leftrightarrow \left\{ T'_+ \left(F^{\nu} \left(\frac{1}{\xi(Y)} \right) \right) \gtrless c_1 \right\}$$

for a suitable constant c_1 . Assuming a reversed S-shaped distortion $T_+(\cdot)$ with infinite positive derivative at the endpoints and recalling that $F^{\nu} \left(\frac{1}{\xi(Y)} \right)$ is uniformly distributed over $(0, 1)$ w.r.t. \mathbb{Q}^{ν} , we can see that a CPT I-agent selects a greater final wealth when $F^{\nu}(\cdot)$ takes values near 0 or 1, i.e. when the random variable $\xi(Y)$ takes values close to its *esssup* and *essinf*. In the case of Example 3.2, this situation corresponds to extremely (positive or negative) high values of $\left| W_T^{\mathbb{Q}} \right|$ (thus, the final price should be extremely high or close to its initial value s_0) or to $\left| W_T^{\mathbb{Q}} \right|$ to be quasi-null (hence, the terminal price should be close to $s_0 \exp \left\{ -\frac{\sigma^2}{2} T \right\}$).⁹

Remark 5.1 (*Finite vs Infinite values of the same weak information*). An interesting result is obtained if we compare the optimal values for the two agents; obviously, their magnitudes are not important as we are comparing two different preference paradigms. On the contrary, what is peculiar to this approach is the fact that we can have $u(x_0, \nu) < +\infty$ and $V(x_0, \nu) = +\infty$ even if the two insiders share a common extra information (Y, ν) . To see this, assume the same market setting as in Example 4.1; now, for any fixed initial endowment $x_0 > 0$, Example 3.2 shows that *for every* $s > 0$ we have $u(x_0, \nu) < +\infty$, whereas it tends to infinity only when $s \downarrow 0$. However, if we assume $T_-(\cdot) = id(\cdot)$, then *for every* $s > 0$ Proposition 4.4 implies ill-posedness agent, i.e. $V(x_0, \nu) = +\infty$. In conclusion, a CPT trader reaches “Nirvana” since she fears losses less than a EU investor fears small payoffs.

5.2 CPT non-informed agents versus CPT insiders

At last we compare the solutions and the optimal values of the problems faced by a CPT N-agent and an analogous weakly informed investor; we assume that they share the same initial endowment

⁸The comparison is however still difficult as the expression for the CPT I-agent depends on the distortion $T_+(\cdot)$; obviously, if there is no distortion on gains then the two expressions are equal.

⁹Another comparison should concern the portfolios chosen by the two types of investors; the big problem here is still facing with really cumbersome expressions.

$x_0 \in \mathbb{R}$, the same utility functions $u_{\pm}(\cdot)$ and the same probability weightings $T_{\pm}(\cdot)$. We suppose that Assumption 4.3 and 4.6 are in force; hence, we have to choose a concave distortion on gains $T_+(\cdot)$ if we hope to make some explicit comparison. Finally, we assume that I-agent possesses the generic weak information (Y, ν) .

As is standard in asymmetric information models (see e.g. [13]), a fundamental issue consists in proving that an insider always “gets” more than a non-informed trader and the difference between the optimal values of these two investors is usually called *insider’s gain*. Now, we are going to prove that this fact still holds in such a *behavioral* setting; in other words, we show the inequality $V(x_0, \nu) \geq V(X^*)$, whose intuitive meaning is that any additional information is an advantage for the investor, even if she is a *behavioral* one.

Before stating the main result, we need the following preliminary lemma which links $F^{\nu}(\cdot)$ and $F^{\mathbb{Q}}(\cdot)$, the distribution functions of the random variable $\frac{1}{\xi(Y)}$ w.r.t. \mathbb{Q}^{ν} and \mathbb{Q} respectively.

Lemma 5.1. *The following inequality holds:*

$$F^{\nu}(c) \geq F^{\mathbb{Q}}(c), \quad \forall c \in \left[\frac{1}{\xi(Y)}, \overline{\frac{1}{\xi(Y)}} \right]. \quad (5.2)$$

Moreover, if $c \in \left(\frac{1}{\xi(Y)}, \overline{\frac{1}{\xi(Y)}} \right)$ and $\xi \neq 1$ then (5.2) holds strictly.

Proof. It is sufficient to use the following estimation, which holds for every $c \in \left[\frac{1}{\xi(Y)}, \overline{\frac{1}{\xi(Y)}} \right]$:

$$\begin{aligned} F^{\nu}(c) &= 1 - \mathbb{E}^{\mathbb{Q}} \left[\xi(Y) I_{\frac{1}{\xi(Y)} > c} \right] \\ &= F^{\mathbb{Q}}(c) + \mathbb{E}^{\mathbb{Q}} \left[(1 - \xi(Y)) I_{\frac{1}{\xi(Y)} > c} \right] \\ &\geq F^{\mathbb{Q}}(c) + \left(1 - \frac{1}{c}\right) (1 - F^{\mathbb{Q}}(c)) \\ &\geq F^{\mathbb{Q}}(c). \end{aligned}$$

Finally, the strict version is valid thanks to Assumption 2.2, i.e. the equivalence of ν and \mathbb{Q}_Y . \square

At this point we are ready to prove the existence of the *insider’s gain*; note that it is sufficient to find a particular feasible solution to Problem (CPT-I) whose prospect value for the informed trader is greater or equal than $V(X^*)$. A rapid look at Problems (CPT-N) and (CPT-I) shows that they share the same feasible set; hence, we could even choose X^* as the insider’s terminal wealth. This random variable will not be the optimal solution to Problem (CPT-I); however, we are able to prove that $V^{\nu}(X^*) \geq V(X^*)$ and this in turn implies $V(x_0, \nu) \geq V(X^*)$.

Theorem 5.1. *Let Assumption 2.2 hold. Then*

$$V(x_0, \nu) \geq V(X^*).$$

Moreover, if $\xi \neq 1$, $V(X^*) < +\infty$ and the optimal solution (p^*, x_+^*) to Problem (4.6) is such that $p^* \in (0, 1)$, then the inequality is strict.

Proof. First of all, we recall that a *behavioral* N-agent endowed with a concave $T_-(\cdot)$ is indifferent in choosing (4.7) as optimal solution for any given $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} . Now we distinguish two cases, namely when the optimal value for N-agent is finite or not. In the first case, the non-informed trader can select $\tilde{Z} = F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)$; using (4.7), (4.8) and setting $c^* := (F^{\mathbb{Q}})^{-1}(p^*)$, we find

$$X^* = (u'_+)^{-1} \left(\frac{\lambda(c^*, x_+^*)}{T'_+(Z)} \right) I_{\frac{1}{\xi(Y)} \leq c^*} - \frac{x_+^* - x_0}{1 - F^{\mathbb{Q}}(c^*)} I_{\frac{1}{\xi(Y)} < c^*}, \quad (5.3)$$

$$V(X^*) = V_+(X^{*+}) - u_- \left(\frac{x_+^* - x_0}{1 - F^{\mathbb{Q}}(c^*)} \right) T_-(1 - F^{\mathbb{Q}}(c^*)), \quad (5.4)$$

where (p^*, x_+^*) are optimal for Problem (4.6) and $\lambda(c^*, x_+^*)$ is determined by the budget constraint. On the other hand, if the informed investor chooses X^* as her terminal wealth, then she obtains the prospect value

$$V^{\nu}(X^*) = V^{\nu}(X^{*+}) - u_- \left(\frac{x_+^* - x_0}{1 - F^{\nu}(c^*)} \right) T_-(1 - F^{\nu}(c^*)). \quad (5.5)$$

Now, using Lemma 5.1, it is immediate to see that the negative part of the prospect value for N-agent is greater (in absolute value) than that of I-agent. Moreover, using the strict decreasing monotonicity of $u_+ ((u'_+)^{-1}) (\cdot)$, we can explicitly write

$$V_+(X^{*+}) = \int_0^{+\infty} T_+ \left(\mathbb{Q} \left(\left\{ T'_+ \left(F^{\mathbb{Q}} \left(\frac{1}{\xi(Y)} \right) \right) > \frac{\lambda(c^*, x_+^*)}{u'_+(u_+^{-1}(y))} \right\} \cap \left\{ \frac{1}{\xi(Y)} \leq c^* \right\} \right) \right) dy. \quad (5.6)$$

But now it is sufficient to note that $T'_+(\cdot)$ is monotone decreasing, whereas $F^{\mathbb{Q}}(\cdot)$ is monotone increasing; furthermore, $V'_+(X^{*+})$ can be written analogously to $V_+(X^{*+})$ just by replacing \mathbb{Q} with \mathbb{Q}^ν . Therefore, applying again Lemma 5.1 we have $V'_+(X^{*+}) \geq V_+(X^{*+})$, which in turn implies the desired inequality. In the ill-posed case for N-agent, note that we can find a sequence of feasible solutions to Problem (4.6), namely $\{(p^n, x_+^n)\}_{n \in \mathbb{N}}$, where $p^n = (F^{\mathbb{Q}})^{-1}(c^n)$ for some $c^n \in \left[\frac{1}{\xi(Y)}, \frac{1}{\xi(Y)} \right]$, for every $n \in \mathbb{N}$. This sequence will in turn induce a sequence of feasible terminal wealths $\{X^{\nu, n}\}_{n \in \mathbb{N}}$. Now, using the previous argument, it is easily seen that if I-agent chooses that sequence of terminal wealths, then her optimal value will diverge to $+\infty$ too. Finally, the strict version is a consequence of (5.2) holding strictly. \square

We remark that in general the optimal pair (c^*, x_+^*) for N-agent will be different w.r.t. $(c^{\nu*}, x_+^{\nu*})$. Moreover, we have seen that ill-posedness for (CPT-N) implies ill-posedness for (CPT-I). A natural question arising from this observation is whether it is possible to find an example where (CPT-N) is well-posed instead of (CPT-I) being ill-posed. The answer is positive and we are going to exploit some results previously obtained in a single risky asset market driven by an $(\Omega, \mathbb{F}, \mathbb{Q})$ -Brownian motion; see Example 4.1 for the notations.

Proposition 5.1. *Assume preferences of the CRRA case with $x_0 \geq 0$, $T_+(p) = p$ and $T_-(p) = p^\delta$, $0 < \delta < \alpha < 1$. If the weak information of CPT-I agent is given by $Y = W_T^{\mathbb{Q}}$ and $\nu \stackrel{d}{=} \mathcal{N}(0, s^2)$ with $0 < s \leq \sqrt{T}$, then for sufficiently small s Problem (CPT-I) is ill-posed.*

Proof. To start, recall that with these assumptions on the agents' preferences, Corollary 4.1 ensures well-posedness for Problem (CPT-N). Then, ill-posedness for Problem (CPT-I) follows from Proposition 4.6 if we are able to show that

$$\inf_{c > \frac{1}{\xi(Y)}} k^\nu(c) \equiv \inf_{c > \frac{1}{\xi(Y)}} \frac{k_-(1 - F^\nu(c))^\delta}{(1 - F^{\mathbb{Q}}(c))^\alpha \left(\mathbb{E}^{\mathbb{Q}} \left[\xi(Y)^{\frac{1}{1-\alpha}} I_{\frac{1}{\xi(Y)} \leq c} \right] \right)^{1-\alpha}} < 1. \quad (5.7)$$

Now, we apply Jensen's inequality to the convex function $f(x) = x^{\frac{1}{1-\alpha}}$, $\alpha \in (0, 1)$, and we estimate the infimum choosing $\hat{c} = (F^{\mathbb{Q}})^{-1} \left(\frac{1-\alpha}{1-\delta} \right)$; hence we obtain

$$\inf_{c > \frac{1}{\xi(Y)}} k^\nu(c) \leq \inf_{c > \frac{1}{\xi(Y)}} \frac{k_-(1 - F^\nu(c))^\delta}{(1 - F^{\mathbb{Q}}(c))^\alpha F^\nu(c)} = k_- \left(\frac{1-\delta}{\alpha-\delta} \right)^\alpha \frac{(1 - F^\nu(\hat{c}))^\delta}{F^\nu(\hat{c})},$$

where it is important to note that \hat{c} depends both on the preference parameters and on the weak information. At this point it is not difficult to compute

$$F^{\mathbb{Q}}(c) \equiv \mathbb{Q} \left\{ \frac{1}{\xi(Y)} \leq c \right\} = \begin{cases} 0 & \text{if } c \leq \frac{s}{\sqrt{T}}, \\ 2\mathcal{N} \left(\sqrt{\frac{2s^2}{T-s^2}} \ln \left(c \frac{\sqrt{T}}{s} \right) \right) - 1 & \text{if } c > \frac{s}{\sqrt{T}}, \end{cases} \quad (5.8)$$

where as usual $\mathcal{N}(\cdot)$ is the distribution function of the standard Gaussian. Next, we find

$$(F^{\mathbb{Q}})^{-1}(z) = \frac{s}{\sqrt{T}} \exp \left(\frac{T-s^2}{2s^2} \left[\mathcal{N}^{-1} \left(\frac{z+1}{2} \right) \right]^2 \right), \quad z \in [0, 1). \quad (5.9)$$

Using the explicit expression of $F^\nu(\cdot)$ in equation (4.27), we can compute

$$F^\nu(\hat{c}) = 2\mathcal{N} \left(\frac{\sqrt{T}}{s} \mathcal{N}^{-1} \left(\frac{2-\alpha-\delta}{2(1-\delta)} \right) \right) - 1.$$

Now we see that for every choice of $k_- \geq 1$ and $0 < \delta < \alpha < 1$, there exists a $\tilde{s} > 0$ such that for every $s < \tilde{s}$ the inequality in (5.7) is fulfilled. \square

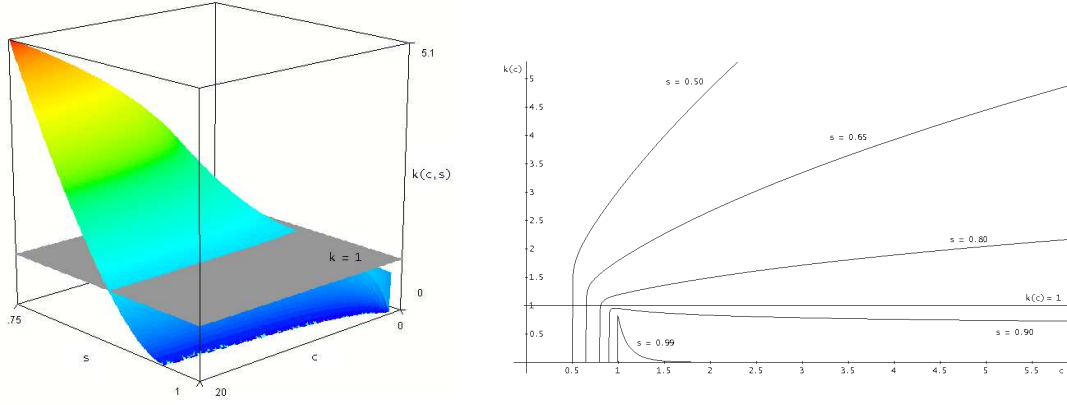


Figure 2: a graphical analysis for the ill-posedness of the CPT insider's problem.

Economically speaking, the meaning of the previous proposition is that there can always exist a particular weak information which ensures well-posedness for N-agent's problems and ill-posedness for the informed investor. Obviously, this extra information must be sufficiently accurate (in our case $s < \bar{s}$) in order to provide an infinite optimal value for I-agent. We recognize that our estimation for (5.7) is effectively rough. For a more detailed analysis, we note that an explicit expression for $k^\nu(c)$ can be provided, even if it is quite cumbersome. However, it is not difficult to perform a graphical analysis whose results are shown in Figure 2. Fixing $\alpha = 0.88$, $\delta = 0.7$ and $T = 1$, k^ν reduces to a function of k_- , s and c ; isolating the loss aversion coefficient k_- , we can now see whether $k_- \leq \sup_{c>s} k(c, s)$ which in turn implies ill-posedness for the CPT insider's problem. In the left-side plot, the 3D surface of $k(c, s)$ shows that even for a quite elevated k_- we still have ill-posedness. On the contrary, if s is sufficiently close to 1, then every $k_- \geq 1$ leads to well-posedness, as the surface lies below the horizontal plane at level $k \equiv 1$ and $k(\cdot, s)$ becomes monotonically decreasing. Finally, for particular values of s , i.e. for specific types of weak information, we depicted in the right-side plot the corresponding curves $k(c)$ which confirm what previously stated.

Remark 5.2. It is worth noticing that with the same hypothesis of Proposition 5.1, the analogous problem for a *classical* informed trader has a completely different solution. Indeed, Problem (EU-I) is well-posed for every $s > 0$ and its optimal value tends to diverge only if $s \downarrow 0$. On the other hand, if we assume $T_-(\cdot) = id$, then Problem (CPT-I) too becomes ill-posed for every $s > 0$, thus showing a substantial lack of robustness.

To conclude this section, we now provide an example where the insider's gain can be explicitly computed and the subsequent results have a clear and intuitive economic explanation.

Example 5.1 (*Explicit evaluation of the insider's gain*). We recover exactly the same setting of Example 4.1 only changing the probability distortion on gains of our informed investors; precisely, this time we assume

$$T_+(p) = 2\mathcal{N}\left(\sqrt{1+2b}\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\right) - 1, \quad b > 0. \quad (5.10)$$

It is important to note that these weighting functions are globally concave over $(0, 1)$ ¹⁰; the concavity allows us to analyze both Problems (CPT-N) and (CPT-I), as now an explicit solution can be found if we check all the conditions in Assumptions 4.3 and 4.6. Firstly, we see that the distortions $T_+(\cdot)$ previously defined are the primitives of

$$T'_+(p) = \sqrt{1+2b} \exp\left(-b\left[\mathcal{N}^{-1}\left(\frac{p+1}{2}\right)\right]^2\right), \quad b > 0. \quad (5.11)$$

As we did in Example 4.1, we check condition (i) of Assumption 4.3, as (ii) is straightforwardly true and (iii) will be controlled *ex post*. Using equation (4.28) and performing the first order derivative, it is immediate to see that $\frac{(F^\nu)^{-1}}{T'_+}(\cdot)$ is non decreasing over $(0, 1]$ for every $b > 0$. Hence, we can

¹⁰Thus, we observe a general overestimation of gain chances.

start making our computations assuming well-posedness for Problem (CPT-I), which implies that of Problem (CPT-N) thanks to Theorem 5.1. For the non-informed investor, we exploit the results of Proposition 4.2, which give us

$$X^* = x_0 \sqrt{\frac{1-\alpha+2b}{(1-\alpha)(1+2b)^{1/(1-\alpha)}}} T'_+(Z)^{1/(1-\alpha)}, \quad (5.12)$$

$$V(X^*) = x_0^\alpha \sqrt{1+2b} \left(\sqrt{\frac{1-\alpha}{1-\alpha+2b}} \right)^{1-\alpha}. \quad (5.13)$$

On the other hand, for the CPT insider we apply Proposition 4.6 which returns

$$X^{\nu*} = x_0 \frac{T-s^2\alpha+2bT}{s^2(1-\alpha)} \exp \left\{ -\frac{T-s^2+2bT}{2Ts^2(1-\alpha)} (W_T^{\mathbb{Q}})^2 \right\}, \quad (5.14)$$

$$V(x_0, \nu) = x_0^\alpha \sqrt{\frac{T(1+2b)}{s^2}} \left(\sqrt{\frac{s^2(1-\alpha)}{T-s^2\alpha+2bT}} \right)^{1-\alpha}. \quad (5.15)$$

The insider's gain is thus given by $V(x_0, \nu) - V(X^*)$; for our purposes, it is more convenient to compute the ratio

$$\frac{V(x_0, \nu)}{V(X^*)} = \frac{\sqrt{T}}{s} \left(\sqrt{\frac{s^2(1-\alpha)}{T-s^2\alpha+2bT}} \right)^{1-\alpha} \geq 1, \quad (5.16)$$

which is increasing in both b , T and decreasing in s , whereas the dependence on α is not monotone. Note that this makes perfectly sense as greater T (or lower s) improve the accuracy of the privileged information; moreover, if $s \uparrow \sqrt{T}$, the ratio (5.16) tends decreasingly to 1. On the contrary, performing the limit $s \downarrow 0$ does not makes sense as Problem (CPT-I) easily becomes ill-posed. On the other hand, as $b \downarrow 0$ we see that $T_+(\cdot)$ converges uniformly to the identity function and, in case of well-posedness, we recover the same results of Example 3.2, where the agent was a *classical* insider. Finally, if $\alpha \uparrow 1$, then (5.16) tends to \sqrt{T}/s ; this is equivalent to say that if the trader becomes risk neutral, then the ratio between the optimal values is nothing but an index of the “goodness” of the extra information.

The comparison between the optimal terminal wealths X^* and $X^{\nu*}$ exhibits the already know flaw of being dependent on the choice of Z ; in particular, if $Z = F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)$ then straightforward computations show that $X^{\nu*} \geq X^*$ if and only if the terminal price of the stock lies in a certain range, whereas if $Z = 1 - F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)$ then we obtain the opposite result.

At last, we remark that it is not possible to merge the previous construction of a concave $T_+(\cdot)$ with the convex one obtained in Example 4.1 in order to get a reversed S-shaped distortion. In fact, condition (i) in Assumption 4.6 will not be fulfilled, as we have

$$T'_+(p) = \begin{cases} c \exp \left\{ -b \left[\mathcal{N}^{-1} \left(\frac{p+1}{2} \right) \right]^2 \right\} & \text{if } 0 \leq p \leq \hat{p}, \\ c \exp \left\{ a \left[\mathcal{N}^{-1} \left(\frac{p+1}{2} \right) \right]^2 \right\} & \text{if } \hat{p} < p \leq 1. \end{cases} \quad (5.17)$$

for some $a, b > 0$, $\hat{p} \in (0, 1)$ and a normalizing constant $c > 0$ such that $T_+(1) = 1$. Using (4.28), we find $\frac{(F^\nu)^{-1}}{T'_+}(\hat{p}+) < \frac{(F^\nu)^{-1}}{T'_+}(\hat{p}-)$, in contrast with the aforementioned condition (i).

6 The goal reaching models and their solutions

A completely different perspective about the target of a financial investor is given by the so-called goal reaching model. Briefly, the trader we consider has the objective of maximizing the probability of her terminal wealth greater than a specific threshold. For a detailed treatment of this model, we refer the reader to [4], where the author solved the problem for a deterministic investment opportunity set using stochastic control theory and HJB equations. However, we will follow the approach developed in [5], which generalizes the solution scheme to a stochastic investment opportunity set. As our agents are supposed to know just the martingale measure \mathbb{Q} instead of the historical probability \mathbb{P} , we will see that some simplifications are in order.

6.1 The non informed agent's problem

We straightforwardly adapt the goal reaching model to the case when the state price density is unknown. For simplicity, we fix a constant threshold $b > 0$ and we assume $x_0 \geq 0$ to be the initial endowment of our agent. Denoting with X the terminal wealth, the problem becomes

$$\begin{aligned} & \text{Maximize} && V_{GR}(X) := \mathbb{Q}\{X \geq b\} \\ & \text{subject to} && \mathbb{E}^{\mathbb{Q}}[X] = x_0, \quad X \geq 0, \quad X \text{ is } \mathcal{F}_T\text{-measurable.} \end{aligned} \tag{GR-N}$$

Now, let X_{GR}^* the optimal solution to Problem (GR-N); it is clear that we should also assume $x_0 < b$ to avoid a trivial problem. Moreover, our N-agent has no convenience in choosing with positive \mathbb{Q} -probability $\{X > b\}$ or $\{0 < X < b\}$, as this will consume her initial budget without raising the objective function. Furthermore, using Chebichev's inequality, we see that $\mathbb{Q}\{X \geq b\} \leq x_0/b$. Hence, the optimal solution will be of the form $X_{GR}^* = bI_A$, with some proper choice of the set $A \in \mathcal{F}_T$ which satisfies the budget constraint. Substantially, we have proved the following result.

Proposition 6.1. *For Problem (GR-N), we have:*

- if $x_0 = 0$, then $X_{GR}^* = 0$ \mathbb{Q} a.s. and $V_{GR}(X_{GR}^*) = 0$;
- if $0 < x_0 < b$, then $X_{GR}^* = bI_A$ for every $A \in \mathcal{F}_T$ such that $\mathbb{Q}(A) = x_0/b$ and $V_{GR}(X_{GR}^*) = x_0/b$;
- if $x_0 \geq b$, then $X_{GR}^* = x_0$ \mathbb{Q} a.s. and $V_{GR}(X_{GR}^*) = 1$.

6.2 The insider's problem

Passing to the analogous model for an insider, all we have to do is to frame the problem using the privileged weak information (Y, ν) and the updated probability measure \mathbb{Q}^ν . Thus, we have

$$\begin{aligned} & \text{Maximize} && V_{GR}^\nu(X) := \mathbb{Q}^\nu\{X \geq b\} \\ & \text{subject to} && \mathbb{E}^\nu \left[\frac{1}{\xi(Y)} X \right] = x_0, \quad X \geq 0, \quad X \text{ is } \mathcal{F}_T\text{-measurable,} \end{aligned} \tag{GR-I}$$

with optimal solution $X_{GR}^{\nu*}$ and corresponding optimal value $V_{GR}(x_0, \nu)$. We immediately see that Problem (GR-I) is nothing but Problem (9) in [5], where \mathbb{P} is replaced by \mathbb{Q}^ν and the state price density ρ is replaced by $\frac{1}{\xi(Y)}$; not only, we observe the law-invariant nature of the objective function as it happened for the CPT agents' problems. Now, it remains to adapt the main result for the goal reaching problem in [5], namely Theorem 1, to get the solution.

Proposition 6.2. *For Problem (GR-I), we have:*

- if $x_0 = 0$, then $X_{GR}^{\nu*} = 0$ \mathbb{Q}^ν a.s. and $V_{GR}(x_0, \nu) = 0$;
- if $0 < x_0 < b$, then $X_{GR}^{\nu*} = b^\nu I_{\frac{1}{\xi(Y)} \leq c^\nu}$, where $c^\nu = (F^\mathbb{Q})^{-1} \left(\frac{x_0}{b} \right) > 0$ and $V_{GR}(x_0, \nu) = F^\nu(c^\nu)$.
- if $x_0 \geq b$, then $X_{GR}^{\nu*} = x_0$ \mathbb{Q}^ν a.s. and $V_{GR}(x_0, \nu) = 1$.

Comparison results between the non-informed investor and an insider are quite evident. Firstly, the presence of the insider's gain in non trivial cases can be assessed simply using Lemma 5.1, which gives

$$V_{GR}(x_0, \nu) = F^\nu(c^\nu) \geq F^\mathbb{Q}(c^\nu) = \frac{x_0}{b} = V_{GR}(X^*).$$

Finally, we can easily compare the optimal terminal wealths if N-agent chooses $A = \left\{ \frac{1}{\xi(Y)} \leq c^\nu \right\}$. In such a case, we see that $X_{GR}^* = X_{GR}^{\nu*}$ but they lead to different optimal values as the evaluation criteria for the two investors are different.

7 The Yaari's model and its solution

The last model we are going to analyze is the one proposed by Yaari in [16]. Loosely speaking, it exhibits an important connection point with the CPT model as a probability distortion $w(\cdot)$ is applied; however, in that model gains are not distinct from losses, so what is important for the trader is the level of terminal wealth X . Moreover, the value function is simply the identity, hence no distortion in payments is allowed. We now frame and solve the problems relative to a non informed investor and an insider respectively; in particular, we adopt the approach developed in [5]. At last, we will provide an example where the insider's gain is explicitly computed. From now on, the following hypothesis on the distortion $w(\cdot)$ will be in force.

Assumption 7.1 (see [5], Assumption 6). $w(\cdot) : [0, 1] \rightarrow [0, 1]$ is continuous and strictly increasing with $w(0) = 0$, $w(1) = 1$. Furthermore, $w(\cdot)$ is continuously differentiable on $(0, 1)$.

7.1 The non informed agent's problem

For our N-agent, we adapt the solution scheme proposed in [5], Section 3.2. Assuming an initial endowment $x_0 > 0$, a standard formulation of this model would be

$$\begin{aligned} \text{Maximize} \quad & V_Y(X) := \int_0^{+\infty} w(\mathbb{Q}\{X > x\}) \\ \text{subject to} \quad & \mathbb{E}^{\mathbb{Q}}[X] = x_0, \quad X \geq 0, \quad X \text{ is } \mathcal{F}_T\text{-measurable.} \end{aligned} \tag{YA-N}$$

Once again, we note that the objective function is law-invariant, in the sense that if X is a feasible solution to (YA-N) with distribution function $F_X(\cdot)$, then for every $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} we have $V_Y(X) = V_Y((F_X)^{-1}(Z))$ (see e.g. Lemma A.1). Moreover, the structure of the objective function may be a source of ill-posedness, similarly to what happened for the CPT model. A straightforward adaptation of the proof in [5], Theorem 2, shows the next result.

Proposition 7.1. Under Assumption 7.1, Problem (YA-N) is ill-posed if $\liminf_{z \downarrow 0} w'(z) = +\infty$, and well-posed if $\limsup_{z \downarrow 0} w'(z) < +\infty$.

In particular, $w(z) = z^\gamma$ leads to well-posedness if $\gamma > 1$, whereas $\gamma < 1$ implies ill-posedness. We also recall that in Yaari's model, a convex distortion is equivalent to risk aversion. Not only, using Jensen's inequality we see that if $w(\cdot)$ is convex, then Problem (YA-N) has the trivial solution $X_Y^* = x_0$ \mathbb{Q} a.s. with $V_Y(X_Y^*) = x_0$ (we will use this fact in Example 7.1). Therefore, there remains some other interesting shapes of $w(\cdot)$ to analyze. Following [5], we impose this technical condition.

Assumption 7.2 (see [5], Assumption 7). $M(z) := w'(1-z)$ is continuous on $(0, 1)$, and there exists $z_0 \in (0, 1)$ such that $M(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$.

In other words, the previous assumption describes an S-shaped distortion function; hence, our N-agent underestimate the probabilities of small terminal wealths, whereas she overestimates those of high levels of X ; substantially, she is overconfident. Using exactly the same argument as in the proofs of Proposition 1 and Theorem 3 in [5], we state the main result of this section.

Proposition 7.2. Suppose Assumption 7.2 holds. Define

$$z(\lambda) := \inf \{z \in (0, z_0) : M(z) = \lambda\}, \tag{7.1}$$

$$h(\lambda) := w(1 - z(\lambda)) - \lambda(1 - z(\lambda)), \tag{7.2}$$

and let λ^* be the unique positive root of $h(\cdot)$. Then, for every $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} , we have $X_Y^* = b^* I_{z(\lambda^*) < Z \leq 1}$, where $b^* = \frac{x_0}{1 - z(\lambda^*)}$ is determined by the budget constraint. Moreover, $V_Y(X_Y^*) = b^* w(1 - z(\lambda^*))$.

7.2 The insider's problem

For a weakly informed trader who follows the tenets of Yaari's dual theory of choice, the optimization problem can be naturally framed as

$$\begin{aligned} \text{Maximize} \quad & V_Y^\nu(X) := \int_0^{+\infty} w(\mathbb{Q}^\nu\{X > x\}) \\ \text{subject to} \quad & \mathbb{E}^\nu \left[\frac{1}{\xi(Y)} X \right] = x_0, \quad X \geq 0, \quad X \text{ is } \mathcal{F}_T\text{-measurable.} \end{aligned} \tag{YA-I}$$

We will call $X_Y^{\nu^*}$ its optimal solution, with optimal value $V_Y(x_0, \nu)$. Once again, we recover the same structure as in [5], Problem (11) and we replace ρ with $\frac{1}{\xi(Y)}$ and \mathbb{P} with \mathbb{Q}^ν . Before giving the solution, we impose the technical hypothesis

Assumption 7.3 (see [5], Assumption 7). $M^\nu(z) := \frac{w'(1-z)}{(F^\nu)^{-1}(1-z)}$ is continuous on $(0, 1)$, and there exists $z_0 \in (0, 1)$ such that $M(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$.

The solution to Problem (YA-I) is completely given by the next result.

Proposition 7.3. *Suppose Assumption 7.3 holds. Define*

$$z^\nu(\lambda^\nu) := \inf \{z \in (0, z_0) : M^\nu(z) = \lambda^\nu\}, \quad (7.3)$$

$$h^\nu(\lambda) := \int_{z^\nu(\lambda)}^1 [w'(1-z) - \lambda(F^\nu)^{-1}(1-z)] dz. \quad (7.4)$$

Let λ^{ν^*} be the unique positive root of $h^\nu(\cdot)$. Then, $X_Y^{\nu^*} = b^\nu I_{\frac{1}{\xi(Y)} \leq c^\nu}$, where c^ν is the unique root of

$$\varphi^\nu(c) := xw(F^\nu(x)) - w'(F^\nu(x)) \int_0^x s dF^\nu(s) \quad (7.5)$$

over $\left((F^\nu)^{-1}(1-z_0), \frac{1}{\xi(Y)}\right)$ and b^ν is implicitly defined by the budget constraint $\mathbb{E}^\nu \left[\frac{1}{\xi(Y)} X_Y^{\nu^*} \right] = x_0$. Moreover, $V_Y(x_0, \nu) = \lambda^{\nu^*} x_0$.

Proof. use the same arguments as in [5], Proposition 1 and Theorem 3. □

Before giving an explicit example, we show that I-agent always gets a higher optimal value w.r.t. N-agent. In fact, Problems (YA-N) and (YA-I) share the same feasible set; therefore, the non-informed agent can choose $Z = F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)$ and the insider can select the corresponding X_Y^* as her terminal wealth. Hence, using Lemma 5.1, we can compute

$$\begin{aligned} V_Y^\nu(X_Y^*) &= \int_0^{+\infty} w \left(\mathbb{Q}^\nu \left\{ b^* I_{z(\lambda^*) < F^{\mathbb{Q}}\left(\frac{1}{\xi(Y)}\right)} > x \right\} \right) dx \\ &= \int_0^{b^*} w \left(F^\nu \left((F^{\mathbb{Q}})^{-1}(1-z(\lambda^*)) \right) \right) dx \\ &\geq \int_0^{b^*} w(1-z(\lambda^*)) dx \\ &= V_Y(X_Y^*), \end{aligned}$$

which obviously implies $V_Y(x_0, \nu) \geq V_Y(X^*)$. This time too the comparison between the optimal terminal wealths is not very sensible, as it strongly depends on the choice of Z .

Example 7.1 (*Evaluation of the insider's gain in Yaari's model*). In the single risky asset setting as in Example 3.2, we assume that the weak information of I-agent is given by $Y = F_W(W_T^{\mathbb{Q}})$ and $\nu(dx) = [(2-2a)x + a] dx$, $a \in (0, 1)$, where $F_W(\cdot)$ is the cdf of the random variable $W_T^{\mathbb{Q}}$. Note that $Y \sim U(0, 1)$ w.r.t. \mathbb{Q} and the economic intuition behind this example is that the insider has a weak knowledge about the terminal price, as the distortion applied by $F_W(\cdot)$ is irrelevant due to its strict monotonicity. Furthermore, the parameter a is an index of the goodness of the extra information; in particular, (Y, ν) becomes more valuable as $a \rightarrow 0^+$. On the contrary, if $a \rightarrow 1^-$ we recover the *minimal* information case. Intuitively, lowering a says that it is more probable to observe higher prices. At this point, we can immediately compute

$$\frac{1}{\xi(Y)} = \frac{1}{(2-2a)F_W(W_T^{\mathbb{Q}}) + a}, \quad \frac{1}{\xi(Y)} = \frac{1}{2-a}, \quad \overline{\frac{1}{\xi(Y)}} = \frac{1}{a}. \quad (7.6)$$

Now, we assume a risk averse investor endowed with probability distortion $w(z) = z^\gamma$, $\gamma > 1$; as noticed in Section 7.1, for such a convex $w(\cdot)$ we already know that $X_Y^* = x_0$ \mathbb{Q} a.s. and $V_Y(X_Y^*) =$

x_0 . Next, we check the validity of Assumption 7.3; using equation (7.6) together with the uniform distribution of Y , we find

$$M(z) = \gamma(1-z)^{\gamma-1} \sqrt{4z(a-1) + (a-2)^2}, \quad z_0 = \frac{a^2(1-\gamma) + 2(1-a)}{2(1-a)(2\gamma-1)}. \quad (7.7)$$

Then, we look for a root of $\varphi^\nu(\cdot)$ as defined in Proposition 7.3. It turns out that an admissible c^ν is only obtained under an additional condition over the parameters γ and a . More precisely, we have

$$c^\nu = \frac{2\gamma-1}{2-a} \quad \text{if} \quad \gamma < \frac{a^2-2a+2}{a^2}. \quad (7.8)$$

The final step is to find the optimal solution $X_Y^{\nu*}$ together with its optimal value. Using the budget constraint we have

$$X_Y^{\nu*} = b^{\nu*} I_{W_T^{\mathbb{Q}} \geq (F_W)^{-1}\left(\frac{a\gamma-1}{(a-1)(2\gamma-1)}\right)} \quad (7.9)$$

$$V_Y(x_0, \nu) = x_0 \frac{\gamma^\gamma (\gamma-1)^{\gamma-1} (2-a)^{2\gamma-1}}{(1-a)^{\gamma-1} (2\gamma-1)^{2\gamma-1}}. \quad (7.10)$$

where $b^{\nu*} = x_0 \frac{(1-a)(2\gamma-1)}{(2-a)(\gamma-1)}$. We remark that our insider will obtain $b^{\nu*}$ if the terminal prices are higher than a certain threshold which is decreasing in both a and γ as economic intuition suggests. Not only, $b^{\nu*}$ too is decreasing in both parameters and the \mathbb{Q} -probability of obtaining $b^{\nu*}$ is nothing but $\frac{(\gamma-1)(2-a)}{(2\gamma-1)(1-a)}$, increasing in both γ and a . Finally, we note that $V_Y(x_0, \nu) \geq x_0$ obviously holds; however, there is no clear dependence of $V_Y(x_0, \nu)$ in the parameters.

8 Conclusions

In this paper, we framed and solved portfolio optimization problems for a wide variety of investor types. Namely, expected utility, cumulative prospect theory, goal reaching and Yaari's dual theory maximizers were analyzed, either if they only had the martingale measure as their information set, either if they possessed a privileged weak information. We remark that the results we obtained are new to the literature, to the best of our knowledge. In particular, we have seen that for an insider, all we had to do was to correctly write down her problem and then exploit already known results ([8] in the CPT case, [5] in the goal reaching and Yaari-type cases). Moreover, the optimal terminal payoff was determined in an almost surely fashion. On the contrary, for the non informed investor, the problem needed to be completely solved due to the fact that in our case the state price density is degenerate. It turns out that the law-invariant nature of the performance criteria and the knowledge of the martingale measure lead to a family of optimal solutions. Hence, we lost the almost sure characterization but we recovered uniqueness *in distribution*. Notably, the information level does not affect the structure of the optimal terminal wealths; hence they can be compared together with the corresponding optimal values. In particular, for a non expected utility trader we recovered a terminal payoff which resembled a gamble on the final price level, and that payoff can even be negative in the CPT case.

Other new features contained in this paper concern an explicit evaluation of the optimal terminal wealth of a CPT and a Yaari-type insider. In particular, we proposed two new classes of probability distortions, a convex and a concave one and a new example of weak information which turns out to be economically meaningful (see Examples 4.1, 5.1 and 7.1). Nonetheless, we generally proved the existence of the insider's gain in every preference paradigm; notably, the CPT case results the most involved one. We were also able to provide easy-to-read expressions for such an insider's gain (see Examples 5.1 and 7.1). Finally, we recall that ill-posedness of our problems was an important issue especially in the CPT case. In some involved examples, we also performed graphical analysis which helped us to understand how well-posedness can be retained when some parameters of the model are changed. The partial and the strong information setting are left to future research.

A A Choquet maximization problem

Our aim is to solve a general utility maximization problem which includes a Choquet capacity:

$$\begin{aligned} & \text{Maximize} && V_1(X) = \int_0^{+\infty} T(\mathbb{P}\{u(X) > y\}) dy \\ & \text{subject to} && \mathbb{E}^\mathbb{P}[X] = a, X \geq 0, \end{aligned} \tag{A.1}$$

where $a \geq 0$, $T : [0, 1] \rightarrow [0, 1]$ is a strictly increasing, differentiable function with $T(0) = 0$, $T(1) = 1$, and $u(\cdot)$ is a strictly concave, strictly increasing, twice differentiable function with $u(0) = 0$, $u'(0) = +\infty$, $u'(+\infty) = 0$. Note that the only difference with the Choquet maximization problem solved in [8], Appendix C, is the absence of the atom-less weighting function ξ in the constraint; unfortunately, this amounts to the impossibility to use their results and the need to prove a variant of them.

We will denote with X^* the optimal solution to Problem (A.1). The case $a = 0$ is trivial, as it implies $X^* = 0$ with optimal value $V_1(X^*) = 0$; therefore assume $a > 0$. First of all we have the following result, which states the *law-invariance* of the problem.

Lemma A.1. *Suppose Problem (A.1) admits a feasible solution X whose distribution function is $G(\cdot)$; then for every random variable $Z \sim U(0, 1)$ w.r.t. \mathbb{P} we have $V_1(X) = V_1(G^{-1}(Z))$.*

Proof. As is easily hinted by the structure of Problem (A.1), the only relevant feature of the optimal solution is its distribution. Formally, for any such Z we can compute

$$\mathbb{E}^\mathbb{P}[G^{-1}(Z)] = \int_0^{+\infty} \mathbb{P}\{G^{-1}(Z) > y\} dy = \int_0^{+\infty} \mathbb{P}\{X > y\} dy = \mathbb{E}^\mathbb{P}[X] = a,$$

thus the random variable $G^{-1}(Z)$ is feasible and we have

$$\begin{aligned} V_1(X) &= \int_0^{+\infty} T(\mathbb{P}\{u(X) > y\}) dy = \int_0^{+\infty} T(\mathbb{P}\{X > u^{-1}(y)\}) dy \\ &= \int_0^{+\infty} T(1 - \mathbb{P}\{X \leq u^{-1}(y)\}) dy = \int_0^{+\infty} T(1 - G(u^{-1}(y))) dy \\ &= \int_0^{+\infty} T(\mathbb{P}\{Z > G(u^{-1}(y))\}) dy = \int_0^{+\infty} T(\mathbb{P}\{u(G^{-1}(Z)) > y\}) dy \\ &= V_1(G^{-1}(Z)) \end{aligned}$$

as claimed. □

We note immediately the difference between our Lemma A.1 and Lemma C.1 in [8]: we do not have an *almost sure* result; however we proved that for *any* such Z the previous equivalence holds, thus it is clearly true even for an optimal X^* : in this way we can choose Z as we prefer. This is a general feature of our results, i.e. replacing the *almost sureness* with a weaker condition on the distribution functions which gives us an additional degree of freedom. From now on, we follow [8] just adding some slight modifications; let's introduce the problem

$$\begin{aligned} & \text{Maximize} && v_1(G) := \int_0^{+\infty} T(\mathbb{P}\{u(G^{-1}(Z)) > y\}) dy \\ & \text{subject to} && \begin{cases} \mathbb{E}^\mathbb{P}[G^{-1}(Z)] = a, \\ G \text{ is the distribution function of a non negative random variable,} \end{cases} \end{aligned} \tag{A.2}$$

which changes the domain of our problem from a random variable set to a function set; specifically, the function $G(\cdot)$ must be non-decreasing, càdlàg and satisfy $G(0-) = 0$, $G(+\infty) = 1$. From Lemma A.1 we deduce the equivalence between the two previous Problems (A.1) and (A.2).

Proposition A.1. *If G^* is optimal for Problem (A.2), then for any $Z \sim U(0, 1)$ w.r.t. \mathbb{P} the random variable $X^* := (G^*)^{-1}(Z)$ is optimal for Problem (A.1). Conversely, if X^* is optimal for (A.1), then its distribution function G^* is optimal for (A.2).*

Performing the same calculations as in [8] and setting

$$\Gamma := \{g : [0, 1] \rightarrow \mathbb{R}^+, g \text{ is non-decreasing, left continuous, with } g(0) = 0\},$$

we can rewrite Problem (A.2) into

$$\begin{aligned} & \text{Maximize} && \bar{v}_1(g) := \mathbb{E}[u(g(Z))T'(1-Z)] \\ & \text{subject to} && \mathbb{E}^{\mathbb{P}}[g(Z)] = a, g \in \Gamma. \end{aligned} \tag{A.3}$$

But thanks to the assumptions on $T(\cdot)$ and $u(\cdot)$ we now have a concave optimization problem in $g(\cdot)$ and we can use Lagrange method. Thus, for a given $\lambda \in \mathbb{R}$, we can solve

$$\begin{aligned} & \text{Maximize} && \bar{v}_1^\lambda(g) := \mathbb{E}[u(g(Z))T'(1-Z) - \lambda g(Z)] \\ & \text{subject to} && g \in \Gamma, \end{aligned} \tag{A.4}$$

and then determine λ via the original constraint. As noticed in [8], if we ignore the constraint and apply standard maximization techniques we find $g(z) = (u')^{-1}(\lambda/T'(1-z))$. Moreover, if $T'(z)$ is non-increasing in $z \in (0, 1]$, then $g(z)$ is non-decreasing in $z \in [0, 1)$ and therefore it solves Problem (A.4). However, if $T'(z)$ is not non-increasing then we are not able to find an explicit solution¹¹. We remark that if $T(z)$ is twice continuously differentiable, then $T'(z)$ non-increasing amounts to require a concave $T(\cdot)$; in particular $T(\cdot) = id$ satisfies this condition.

Denote $R_u(x) := -\frac{xu''(x)}{u'(x)}$, $x > 0$, the index of Relative Risk Aversion (RRA for short) of the function $u(\cdot)$. We have

Proposition A.2. *Assume that $T'(z)$ is non-increasing in $z \in (0, 1]$ and $\liminf_{x \rightarrow +\infty} R_u(x) > 0$. Then for any $Z \sim U(0, 1)$ w.r.t. \mathbb{P} , the following claims are equivalent:*

- (i) Problem (A.3) is well-posed for any $a > 0$.
- (ii) Problem (A.3) admits a unique optimal solution for any $a > 0$.
- (iii) $\mathbb{E}^{\mathbb{P}} \left[u \left((u')^{-1} \left(\frac{1}{T'(1-Z)} \right) \right) T'(1-Z) \right] < +\infty$.
- (vi) $\mathbb{E}^{\mathbb{P}} \left[u \left((u')^{-1} \left(\frac{\lambda}{T'(1-Z)} \right) \right) T'(1-Z) \right] < +\infty \forall \lambda > 0$.

Furthermore, when one of the above (i)-(iv) holds, the optimal solution to Problem (A.3) is

$$g^*(x) \equiv (G^*)^{-1}(x) = (u')^{-1} \left(\frac{\lambda}{T'(1-x)} \right), x \in [0, 1),$$

where $\lambda > 0$ is the one satisfying $\mathbb{E}^{\mathbb{P}}[(G^*)^{-1}(1-Z)] = a$.

Proof. As in the proof of Proposition C.2 in [8], we can define a new probability measure $\tilde{\mathbb{P}}$ such that $d\tilde{\mathbb{P}} = T'(1-Z) d\mathbb{P}$ and a random variable $\zeta := \frac{1}{T'(1-Z)}$ which is positive \mathbb{P} -a.s.. We can now rewrite Problem (A.3) as follows.

$$\begin{aligned} & \text{Maximize} && \bar{v}_1(g) := \mathbb{E}^{\tilde{\mathbb{P}}}[u(g(Z))] \\ & \text{subject to} && \mathbb{E}^{\tilde{\mathbb{P}}}[\zeta g(Z)] = a, g \in \Gamma. \end{aligned}$$

By [7], Theorem 5.4, we get the result. □

We remark that the claim (ii) (as it appears in Proposition C.2 of [8]) is still valid because the optimal solution $g^*(\cdot)$ to Problem (A.3) determines the inverse of a distribution function whereas the optimal solution X^* to Problem (A.1) is not unique \mathbb{P} -a.s. as it depends on the choice of Z ; however, X^* is unique *in law*. Moreover we could also replace $1-Z$ with Z in every explicit expression containing an expected value, since $Z \sim U(0, 1)$ too. Now we can state the main result of this section.

Theorem A.1. *Assume that $T'(z)$ is non-increasing in $z \in (0, 1]$ and $\liminf_{x \rightarrow +\infty} R_u(x) > 0$; for any fixed $Z \sim U(0, 1)$ w.r.t. \mathbb{P} define $X(\lambda) := (u')^{-1} \left(\frac{\lambda}{T'(1-Z)} \right)$ for $\lambda > 0$. If $V_1(X(1)) < +\infty$, then $X(\lambda)$ is an optimal solution of Problem (A.1), where λ is the one satisfying $\mathbb{E}^{\mathbb{P}}[X(\lambda)] = a$. If $V_1(X(1)) = +\infty$, then Problem (A.1) is ill-posed.*

¹¹If one restricts the domain over the step functions $g \in \Gamma$, then solving (A.4) is equivalent to solve a non-linear programming problem in \mathbb{R}^n , which once again does not have an easy explicit solution.

We the obvious adaptations of the proofs, we can also state a necessary condition of optimality as in [8].

Lemma A.2. *If $g(\cdot)$ is optimal for Problem (A.4), then either $g \equiv 0$ or $g(x) > 0 \forall x > 0$.*

Theorem A.2. *If X^* is an optimal solution for Problem (A.1) with some $a > 0$, then $\mathbb{P}\{X^* = 0\} = 0$.*

Note that these last results do not depend on the choice of Z ; they will be useful in order to state monotonicity properties of the value function of a CPT non-informed agent.

B A Choquet minimization problem

In this section we solve a general utility minimization problem including a Choquet capacity:

$$\begin{aligned} \text{Minimize} \quad & V_2(X) := \int_0^{+\infty} T(\mathbb{P}\{u(X) > y\}) dy \\ \text{subject to} \quad & \mathbb{E}^{\mathbb{P}}[X] = a, X \geq 0, \end{aligned} \tag{B.1}$$

where $a, T(\cdot)$ satisfy the same hypothesis of those employed in Problem (A.1) and $u(\cdot)$ is strictly increasing, concave and $u(0) = 0$. Once again the only difference with the Choquet minimization problem solved in [8], Appendix D, is the absence of the atom-less weighting function ξ .

We will denote as usual with X^* the optimal solution to Problem (B.1). Note that there is always a feasible solution, namely $X = a$ \mathbb{P} -a.s.; hence the optimal value of Problem (B.1) is a finite non-negative number. Proceeding as in Appendix A we can show the following *law-invariance* lemma.

Lemma B.1. *Suppose Problem (B.1) admits a feasible solution X whose distribution function is $G(\cdot)$; then for every random variable $Z \sim U(0, 1)$ w.r.t. \mathbb{P} we have $V_1(X) = V_1(G^{-1}(Z))$.*

Thus, we can look for a solution to the following problem.

$$\begin{aligned} \text{Minimize} \quad & \bar{v}_2(g) := \mathbb{E}[u(g(Z))T'(1 - Z)] \\ \text{subject to} \quad & \mathbb{E}^{\mathbb{P}}[g(Z)] = a, g \in \Gamma, \end{aligned} \tag{B.2}$$

where $g(\cdot)$ represents the inverse of a distribution function $G(\cdot)$, i.e. $g(\cdot) = G^{-1}(\cdot)$. As already pointed out in [8], Problem (B.2) is a difficult one in that we have to minimize a concave objective function in a function space. Again we can seek among *corner point solutions* and by straightforward modifications of the proof of Proposition D.2 in [8], we can prove the next result.

Proposition B.1. *Assume that $u(\cdot)$ is strictly concave at 0. Then the optimal solution for Problem (B.2), if it exists, must be in the form $g(t) = \frac{a}{1-b}I_{(b,1)}(t)$, $t \in [0, 1]$.*

Obviously by left continuity of $g(\cdot)$ we can extend the optimal $g(\cdot)$ over $[0, 1]$ by setting $g(1) := \frac{a}{1-b}$. Moreover such a $g(\cdot)$ is uniformly bounded in $t \in [0, 1]$, so it follows by the preceding lemma that an X^* optimal for Problem (B.1) is uniformly bounded from above. Thanks to the previous proposition, we can reduce our problem to find an optimal real number $b \in [0, 1]$; therefore we introduce the following

$$\begin{aligned} \text{Minimize} \quad & \bar{v}_2(b) := \mathbb{E}[u(g(Z))T'(1 - Z)] \\ \text{subject to} \quad & g(\cdot) = \frac{a}{1-b}I_{(b,1]}(\cdot), \quad 0 \leq b < 1. \end{aligned} \tag{B.3}$$

Adapting the proofs of Proposition D.3 and Theorem D.1 in [8], we can state

Proposition B.2. *Problems (B.2) and (B.3) have the same infimum values.*

Theorem B.1. *Problems (B.1) and (B.3) have the same infimum values. If, in addition, $u(\cdot)$ is strictly concave at 0, then Problem (B.1) admits an optimal solution if and only if the following problem*

$$\min_{0 \leq b < 1} u\left(\frac{a}{1-b}\right) T(1-b)$$

admits an optimal solution b^ , in which case the optimal solution to Problem (B.1) is of the form $X^* = \frac{a}{1-b^*}I_{(b^*,1]}(Z)$ for any choice of $Z \sim U(0, 1)$ w.r.t. \mathbb{P} .*

C The solution of a CPT non-informed agent's problem

We will now proceed to completely solve Problem (CPT-N); the scheme of the solution is nothing but the one already showed in [8]. Some results will be just restated without proofs as they need only slight and straightforward adaptations. As already noted, the constraint $\mathbb{E}^{\mathbb{Q}}[X] = x_0$ will impose the main changes. Recall Problem (CPT-N):

$$\begin{aligned} & \text{Maximize} && V(X) = V_+(X^+) - V_-(X^-) \\ & \text{subject to} && \mathbb{E}^{\mathbb{Q}}[X] = x_0, X \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{Q} \text{ a.s. lower bounded,} \end{aligned}$$

where

$$V_+(X^+) := \int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X^+) > y\}) dy, \quad V_-(X^-) := \int_0^{+\infty} T_-(\mathbb{Q}\{u_-(X^-) > y\}) dy.$$

As noticed in [8], Proposition 3.1, to avoid systematic ill-posedness we will impose

Assumption C.1. $V_+(X) < +\infty$ for any nonnegative, \mathcal{F}_T -measurable random variable X satisfying $\mathbb{E}^{\mathbb{Q}}[X] < +\infty$.

We now split Problem (CPT-N) into its *positive* and *negative* part, also defining their respective optimal values $v_+(A, x_+)$ and $v_-(A, x_+)$ as usual; after that we merge them back:

- *Positive Part Problem:* given the pair (A, x_+) , with $A \in \mathcal{F}_T$ and $x_+ \geq x_0^+$,

$$\begin{aligned} & \text{Maximize} && V_+(X) = \int_0^{+\infty} T_+(\mathbb{Q}\{u_+(X) > y\}) dy \\ & \text{subject to} && \mathbb{E}^{\mathbb{Q}}[X] = x_+, \quad X \geq 0 \text{ } \mathbb{Q} \text{ a.s.}, \quad X = 0 \text{ } \mathbb{Q} \text{ a.s. on } A^C. \end{aligned} \tag{C.1}$$

- *Negative Part Problem:* given the pair (A, x_+) , with $A \in \mathcal{F}_T$ and $x_+ \geq x_0^+$,

$$\begin{aligned} & \text{Minimize} && V_-(X) = \int_0^{+\infty} T_-(\mathbb{Q}\{u_-(X) > y\}) dy \\ & \text{subject to} && \begin{cases} \mathbb{E}^{\mathbb{Q}}[X] = x_+ - x_0, & X \geq 0 \text{ } \mathbb{Q} \text{ a.s.}, & X = 0 \text{ } \mathbb{Q} \text{ a.s. on } A, \\ X \text{ is upper bounded } \mathbb{Q} \text{ a.s.} \end{cases} \end{aligned} \tag{C.2}$$

- *Merged Problem:*

$$\begin{aligned} & \text{Maximize} && v_+(A, x_+) - v_-(A, x_+) \\ & \text{subject to} && \begin{cases} A \in \mathcal{F}_T, & x_+ \geq x_0^+, \\ x_+ = 0 \text{ if } \mathbb{Q}(A) = 0, & x_+ = x_0 \text{ if } \mathbb{Q}(A) = 1. \end{cases} \end{aligned} \tag{C.3}$$

With only a few and simple adaptations, we can prove the following two results.

Proposition C.1 ([8], Proposition 5.1). *Problem (CPT-N) is ill-posed if and only if Problem (C.3) is ill-posed.*

Proposition C.2 ([8], Proposition 5.2). *Given X^* , define $A^* := \{\omega : X^* \geq 0\}$ and $x_+^* := \mathbb{E}^{\mathbb{Q}}[(X^*)^+]$. Then X^* is optimal for Problem (CPT-N) if and only if (A^*, x_+^*) are optimal for Problem (C.3) and $(X^*)^+$ and $(X^*)^-$ are respectively optimal for Problems (C.1) and (C.2) with parameters (A^*, x_+^*) .*

Therefore, the original Problem (CPT-N) for N-agent is again equivalent to the set of Problems (C.1)-(C.3). The next step is the crucial one, as it completely changes the structure of the solution of our problem. From now on, we will not be able to recover the *almost sure* characterization results obtained in [8]; on the other hand, we can avoid every technical detail related to the *comonotonicity* and *anti-comonotonicity* of the random variables employed in the solution (see [8], Appendix B, where a series of so-called *quantile* problems is solved).

The fact is that the density ρ permitted a huge simplification of the overall procedure, making it possible to look for a solution where the set A was of the form $\{\rho \leq c\}$ for some real number $c \in [\underline{\rho}, \bar{\rho}]$. Now we can find a quite similar result adapting the proof of Theorem 5.1 in [8]; this will substantially reduce the complexity of Problem (C.3).

Theorem C.1. For any feasible (A, x_+) of Problem (C.3) such that $\mathbb{Q}(A) = p$ and for every (Ω, \mathcal{F}) random variable $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} , we have

$$v_+(\bar{A}, x_+) - v_-(\bar{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+), \quad (\text{C.4})$$

where $\bar{A} := \{Z \leq p\}$.

Proof. Fix such a random variable Z . The cases $x_+ = x_0^+$ and $p = 0$ or $p = 1$ are trivial, so we assume that $x_+ > x_0^+$ and $p \in (0, 1)$. Define $B := A^C$ and $\bar{A} := \{Z \leq p\}$ and set

$$\begin{aligned} A_1 &= A \cap \{Z \leq p\}, & A_2 &= A \cap \{Z > p\}, \\ B_1 &= B \cap \{Z \leq p\}, & B_2 &= B \cap \{Z > p\}. \end{aligned}$$

Note that $\mathbb{Q}(A_1 \cup A_2) = \mathbb{Q}(A_1 \cup B_1) = p$, so that $\mathbb{Q}(A_2) = \mathbb{Q}(B_1)$. If $\mathbb{Q}(A_2) = 0$ then the result is trivial, so suppose $\mathbb{Q}(A_2) > 0$. Choose a feasible solution X_1 for Problem (C.1) with parameters (A, x_+) ; we will prove that $V_+(X_1) \leq v_+(\bar{A}, x_+)$ (the proof for a feasible solution X_2 for Problem (C.2) is analogous). To this end, define $f_1(t) := \mathbb{Q}\{X_1 \leq t | A_2\}$, $g_1(t) := \mathbb{Q}\{Z \leq t | B_1\}$, $t \in [0, 1]$, $Z_1 := g_1(Z)$ and $Y_1 := f_1^{-1}(Z_1)$. Note that Z has no atom w.r.t. \mathbb{Q} , which in turn implies that it has no atom w.r.t. $\mathbb{Q}(\cdot | B_1)$. Moreover one can show that $Z_1 \sim U(0, 1)$ w.r.t. $\mathbb{Q}(\cdot | B_1)$, implying $\mathbb{Q}\{Y_1 \leq t | B_1\} = \mathbb{Q}\{Z_1 \leq f_1(t) | B_1\} = f_1(t)$. To see this note that

$$g_1(t) = \frac{\mathbb{Q}\{A^C \cap (Z \leq t) \cap (Z \leq p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}} = \frac{\mathbb{Q}\{A^C \cap (Z \leq t \wedge p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}},$$

so we can compute

$$\mathbb{Q}\{Z_1 \leq t | B_1\} = \frac{\mathbb{Q}\{A^C \cap (Z_1 \leq t) \cap (Z \leq p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}} = \frac{\mathbb{Q}\{A^C \cap (Z \leq g_1^{-1}(t) \wedge p)\}}{\mathbb{Q}\{A^C \cap (Z \leq p)\}} = g_1(g_1^{-1}(t)) = t.$$

Consequently, $\mathbb{E}^{\mathbb{Q}}[X_1 I_{A_2}] = \mathbb{Q}(A_2) \mathbb{E}^{\mathbb{Q}}[X_1 | A_2] = \mathbb{E}^{\mathbb{Q}}[Y_1 I_{B_1}]$. Now set $\bar{X}_1 := X_1 I_{A_1} + Y_1 I_{B_1}$. Then $\mathbb{E}^{\mathbb{Q}}[X_1] = \mathbb{E}^{\mathbb{Q}}[\bar{X}_1]$, so \bar{X}_1 is feasible for Problem (C.1) with parameters (\bar{A}, x_+) . Finally it is obviously seen that $\mathbb{Q}\{\bar{X}_1 > t\} = \mathbb{Q}\{X_1 > t\}$, therefore by the definition of $V_+(\cdot)$ it follows that $V_+(\bar{X}_1) \geq V_+(X_1)$. Combining this with the similar result for the *Negative Part Problem* we get the desired inequality (C.4). \square

The meaning of Theorem C.1 is that a non-informed agent only cares about the probability of events, no matter what structure they have or what economic phenomenon they represent. As it will appear clear in what follows, for such an agent investing in a risky asset is not so different with respect to tossing a coin or betting on horses!¹²

We can now proceed similarly to Jin and Zhou, using $v_+(p, x_+)$ and $v_-(p, x_+)$ to denote $v_+(\{\omega : Z \leq p\}, x_+)$ and $v_-(\{\omega : Z \leq p\}, x_+)$ respectively; note that we can choose Z as we prefer, and the previous definition is in some sense *independent* of Z thanks to the above Theorem. Accordingly, we replace Problem (C.3) by the easier constrained optimization problem in \mathbb{R}^2 :

$$\begin{aligned} &\text{Maximize} && v_+(p, x_+) - v_-(p, x_+) \\ &\text{subject to} && \begin{cases} p \in [0, 1], & x_+ \geq x_0^+, \\ x_+ = 0 & \text{if } p = 1, & x_+ = x_0 & \text{if } p = 0. \end{cases} \end{aligned} \quad (\text{C.5})$$

Using Theorem C.1 we obtain the general structure of the solution to Problem (CPT-N), which is indeed similar to the one find in [8]. However, on one hand it must reflect the freedom in choosing Z and on the other hand the fact that once a particular Z is chosen, then the optimal solution of Problem (CPT-N) is allowed to depend only on Z , in order to avoid possible correlation effects. In what follows we will consider such a Z fixed and denote with X^* the optimal solution depending on Z .

¹²Recall that also in the original framework in [8], the optimal policy for an investor was to behave like a gambler, but she would choose a terminal gain accompanied with a high price of the underlying stock, opposite to a final loss if the terminal price would have fallen below a certain threshold.

Theorem C.2. Given X^* and Z , define $p^* := \mathbb{Q}\{X^* \geq 0\}$, $x_+^* := \mathbb{E}^{\mathbb{Q}}[(X^*)^+]$. Then X^* is optimal for Problem (CPT-N) if and only if (p^*, x_+^*) is optimal for Problem (C.5) and $(X^*)^+ I_{Z \leq p^*}$ and $(X^*)^- I_{Z > p^*}$ are respectively optimal for Problems (C.1) and (C.2) with parameters $(\{\omega : Z \leq p^*\}, x_+^*)$.

The next step consists in solving the *positive* and the *negative* part Problems (C.1) and (C.2) using the results obtained in Appendix A and B respectively. In order to obtain a more explicit result, we impose the following conditions.

Assumption C.2. $T'_+(z)$ is non-increasing for $z \in (0, 1]$, $\liminf_{x \rightarrow +\infty} -\frac{xu''_+(x)}{u'_+(x)} > 0$ and for any $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} we have $\mathbb{E}^{\mathbb{Q}} \left[u_+ \left((u'_+)^{-1} \left(\frac{1}{T'_+(Z)} \right) \right) T'_+(Z) \right] < +\infty$.

At this point we can perform the same procedure used in [8], Section 6.1; thanks to the freedom in choosing Z we will show that a *canonical* choice linked to Z will give us more tractable expressions.

Theorem C.3. Let Assumption C.2 hold. For any $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} and for a given $p \in [0, 1]$, set $A := \{\omega : Z \leq p\}$; let $x_+ \geq x_0^+$ be given. Then:

- (i) if $x_+ = 0$, then the optimal solution of Problem (C.1) is $X^* = 0$ and $v_+(p, x_+) = 0$;
- (ii) if $x_+ > 0$, $p = 0$ then there is no feasible solution to Problem (C.1) and $v_+(p, x_+) = -\infty$;
- (iii) if $x_+ > 0$, $p \in (0, 1]$ then the optimal solution to (C.1) is $X^*(\lambda) = (u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) I_{Z \leq p}$ with the optimal value $v_+(p, x_+) = \mathbb{E}^{\mathbb{Q}} \left[u_+ \left((u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) \right) T'_+(Z) I_{Z \leq p} \right]$, where $\lambda > 0$ is the unique real number satisfying $\mathbb{E}^{\mathbb{Q}}[X^*(\lambda)] = x_+$.

Proof. Cases (i) and (ii) are trivial; to prove (iii) we follow an argument similar to that in the proof of Theorem 6.1 in Jin and Zhou (2007). Define $T_A(x) := \frac{T_+(x\mathbb{Q}(A))}{T_+(\mathbb{Q}(A))} = \frac{T_+(xp)}{T_+(p)}$, $x \in [0, 1]$ and the conditional probability measure $\mathbb{Q}_A := \mathbb{Q}(\cdot|A)$. Now consider Problem (C.1) in the conditional probability space $(\Omega \cap A, \mathcal{F} \cap A, \mathbb{Q}_A)$, i.e.

$$\begin{aligned} \text{Maximize} \quad & V_+(Y) = T_+(p) \int_0^{+\infty} T_A(\mathbb{Q}_A\{u_+(Y) > y\}) dy \\ \text{subject to} \quad & \mathbb{E}^{\mathbb{Q}_A}[Y] = \frac{x_+}{p}, \quad Y \geq 0. \end{aligned} \tag{C.6}$$

We can apply Theorem A.1 to Problem (C.6) choosing *any* random variable $\tilde{Z} \sim U(0, 1)$ w.r.t. \mathbb{Q}_A ; note that every required assumption for Theorem A.1 is still fulfilled. At this point, in order to simplify calculations as much as possible, we see that once Z is chosen there is a *canonical* choice of \tilde{Z} : $\tilde{Z} = 1 - g(Z)$, where $g(z) := \mathbb{Q}\{Z \leq t|A\}$. In fact we can show that if $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} , then \tilde{Z} has the same distribution w.r.t. \mathbb{Q}_A . To see this, note that

$$\mathbb{Q}_A\{\tilde{Z} \leq t\} = \frac{\mathbb{Q}\{\tilde{Z} \leq t, A\}}{p} = \frac{\mathbb{Q}\{1 - g(Z) \leq t, Z \leq p\}}{p} = \frac{\mathbb{Q}\{Z \geq g^{-1}(1 - t), Z \leq p\}}{p},$$

but we can explicitly compute

$$g(t) = \frac{\mathbb{Q}\{Z \leq t \wedge p\}}{p} = \frac{t \wedge p}{p},$$

therefore we obtain $\mathbb{Q}_A\{\tilde{Z} \leq t\} = t$, $t \in (0, 1)$. Using such a choice of \tilde{Z} we can find that an optimal solution to Problem (C.1) is $X^* = (u'_+)^{-1} \left(\frac{\bar{\lambda} T_+(p)}{p T_+(pg(Z))} \right) I_{Z \leq p}$, where $\bar{\lambda}$ is uniquely determined by the constraint. We now observe that on the set $\{Z \leq p\}$ we have $g(Z) = Z/p$; finally we set $\lambda := \frac{\bar{\lambda} T_+(p)}{p}$ to find our results. \square

Comparing this result with the analogous in [8], we see that the link between the two solutions is substantially made by the replacement of the set $\{\rho \leq c\}$ with $\{Z \leq p\}$; in particular $c = \underline{\rho}$ corresponds to $p = 0$ and $c = \bar{\rho}$ corresponds to $p = 1$. Thanks to freedom in choosing Z , once again we see that a non-informed agent is only interested in probabilities, not in events.

With a simple adaptation of the proof of [8], Proposition 6.2, we can also state the strict monotonicity of the optimal value $v_+(\cdot, x_+)$ w.r.t. p .

Proposition C.3. *If $x_+ > 0$ and $Z \sim U(0,1)$ w.r.t. \mathbb{Q} , then Problem (C.1) admits an optimal solution with parameters $(\{Z \leq p\}, x_+)$ only if $v_+(\bar{p}, x_+) > v_+(p, x_+)$ for any $\bar{p} > p$.*

We now proceed to solve the *negative* part Problem (C.2). We follow again the arguments applied in [8], Section 7, combining them with our results in Appendix B.

Theorem C.4. *Assume that $u_-(\cdot)$ is strictly concave at 0. For any $Z \sim U(0,1)$ w.r.t. \mathbb{Q} and for a given $p \in [0,1]$ set $A := \{\omega : Z \leq p\}$. Let $x_+ \geq x_0^+$ be given. Then:*

- (i) *if $p = 1, x_+ = x_0$ then the optimal solution of Problem (C.2) is $X^* = 0$ and $v_-(p, x_+) = 0$;*
- (ii) *if $p = 1, x_+ \neq x_0$ then there is no feasible solution to Problem (C.2) and $v_-(p, x_+) = +\infty$;*
- (iii) *if $p \in [0,1)$ then $v_-(p, x_+) = \inf_{0 \leq b < 1} u_- \left(\frac{x_+ - x_0}{(1-p)(1-b)} \right) T_-((1-p)(1-b))$. Moreover, Problem (C.2) with parameters (A, x_+) admits an optimal solution X^* if and only if the minimization problem*

$$\min_{0 \leq b < 1} u_- \left(\frac{x_+ - x_0}{(1-p)(1-b)} \right) T_-((1-p)(1-b)) \quad (\text{C.7})$$

admits an optimal solution b^ , in which case $X^* = \frac{x_+ - x_0}{(1-p)(1-b^*)} I_{Z > (1-p)b^* + p}$.*

Proof. Cases (i) and (ii) are trivial; to prove (iii) we define $T_{A^C}(x) := \frac{T_-(x\mathbb{Q}(A^C))}{T_-(\mathbb{Q}(A^C))} = \frac{T_-(x(1-p))}{T_-(1-p)}$, $x \in [0,1]$ and the conditional probability measure $\mathbb{Q}_{A^C} := \mathbb{Q}(\cdot | A^C)$. Let's consider Problem (C.2) in the conditional probability space $(\Omega \cap A^C, \mathcal{F} \cap A^C, \mathbb{Q}_{A^C})$:

$$\begin{aligned} \text{Minimize} \quad & V_-(Y) = T_-(1-p) \int_0^{+\infty} T_{A^C}(\mathbb{Q}_{A^C}\{u_-(Y) > y\}) dy \\ \text{subject to} \quad & \mathbb{E}^{\mathbb{Q}_{A^C}}[Y] = \frac{x_+ - x_0}{1-p}, \quad Y \geq 0, \quad Y \mathbb{Q}_{A^C} \text{ a.s. bounded.} \end{aligned} \quad (\text{C.8})$$

Now we apply Theorem B.1 to Problem (C.8), choosing *any* random variable $\tilde{Z} \sim U(0,1)$ w.r.t. \mathbb{Q}_{A^C} . Once again, when Z is chosen there is a *canonical* choice of \tilde{Z} : $\tilde{Z} = g(Z)$, where $g(z) := \mathbb{Q}\{Z \leq t | A^C\}$. Indeed, if $Z \sim U(0,1)$ w.r.t. \mathbb{Q} then \tilde{Z} has the same distribution w.r.t. \mathbb{Q}_{A^C} . To see this, observe that

$$\mathbb{Q}_{A^C}\{\tilde{Z} \leq t\} = \frac{\mathbb{Q}\{\tilde{Z} \leq t, A^C\}}{1-p} = \frac{\mathbb{Q}\{g(Z) \leq t, Z > p\}}{1-p} = \frac{\mathbb{Q}\{Z \leq g^{-1}(t), Z > p\}}{1-p},$$

but we can compute

$$g(t) = \frac{\mathbb{Q}\{Z \leq t, Z > p\}}{1-p} = \frac{t-p}{1-p} \wedge 0,$$

therefore we obtain $\mathbb{Q}_{A^C}\{\tilde{Z} \leq t\} = t, t \in (0,1)$. Using such a choice of \tilde{Z} and recalling that an optimal solution to Problem (C.8) is automatically bounded (if it exists), we can find that an optimal solution to Problem (C.2) is

$$X^* = \frac{x_+ - x_0}{(1-p)(1-b^*)} I_{Z > p} I_{g(Z) \in (b^*, 1]} = \frac{x_+ - x_0}{(1-p)(1-b^*)} I_{Z > g^{-1}(b^*)} = \frac{x_+ - x_0}{(1-p)(1-b^*)} I_{Z > (1-p)b^* + p},$$

thanks to the fact that on the set $\{Z > p\}$ we have $g(Z) > 0$ and $g^{-1}(t) = [(1-p)t + p] \vee 1$. \square

At last we have to merge these results to obtain the overall solution to Problem (CPT-N). As in [8] we take an intermediate step using the following problem

$$\begin{aligned} \text{Maximize} \quad & v_+(p, x_+) - u_- \left(\frac{x_+ - x_0}{1-p} \right) T_-(1-p) \\ \text{subject to} \quad & \begin{cases} p \in [0,1], & x_+ \geq x_0^+, \\ x_+ = 0 \text{ if } p = 1, & x_+ = x_0 \text{ if } p = 0, \end{cases} \end{aligned} \quad (\text{C.9})$$

where we set $u_- \left(\frac{x_+ - x_0}{1-p} \right) T_-(1-p) := 0$ if $p = 1$ and $x_+ = x_0$. By simply adapting the proofs in [8], Lemma 8.1 and Proposition 8.1, we claim:

Lemma C.1. *For any feasible pair (p, x_+) for Problem (C.5), $u_- \left(\frac{x_+ - x_0}{1-p} \right) T_-(1-p) \geq v_-(p, x_+)$.*

Proposition C.4. *Problems (C.5) and (C.9) have the same supremum values.*

Finally we state the main result of this section:

Theorem C.5. *Assume that $u_-(\cdot)$ is strictly concave at 0. We have the following results:*

- (i) *if X^* is optimal for Problem (CPT-N), then $p^* := \mathbb{Q}\{X^* \geq 0\}$, $x_+^* := \mathbb{E}^{\mathbb{Q}}[(X^*)^+]$ are optimal for Problem (C.9);*
- (ii) *if (p^*, x_+^*) is optimal for Problem (C.9) and X_+^* is optimal for Problem (C.1) with parameters $(\{Z \leq p^*\}, x_+^*)$, where $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} , then $X^* := (X^*)^+ I_{Z \leq p^*} - \frac{x_+^* - x_0}{1 - p^*} I_{Z > p^*}$ is optimal for Problem (CPT-N).*

To conclude, if Assumption C.2 is in force, then for any $Z \sim U(0, 1)$ w.r.t. \mathbb{Q} we have

$$X^* = (u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) I_{Z \leq p^*} - \frac{x_+^* - x_0}{1 - p^*} I_{Z > p^*},$$

$$V(X^*) = \mathbb{E}^{\mathbb{Q}} \left[u_+ \left((u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) \right) T'_+(Z) I_{Z \leq p^*} \right] - u_- \left(\frac{x_+^* - x_0}{1 - p^*} \right) T_-(1 - p^*),$$

where (p^*, x_+^*) are optimal for Problem (C.9) and λ satisfies $\mathbb{E}^{\mathbb{Q}} \left[(u'_+)^{-1} \left(\frac{\lambda}{T'_+(Z)} \right) I_{Z \leq p^*} \right] = x_+^*$. We finally note that all this construction can be considered as an adaptation of the model set up in [8] if we had started with prices following a geometric or an arithmetic Brownian motion, as is often supposed in the finance literature.

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