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TWISTED MOMENTS OF AUTOMORPHIC L-FUNCTIONS

YUK-KAM LAU, EMMANUEL ROYER, AND JIE WU

ABSTRACT. We study the moments of the symmetric power L-functions of primitive forms at the edge of the critical strip twisted by the square of the value of the standard L-function at the center of the critical strip. We give a precise expansion of the moments as the order goes to infinity.

CONTENTS

1. Introduction	1
2. Preliminary results	6
2.1. Combinatorial results	6
2.2. Analytical results	7
3. Evaluation of the moments	9
3.1. Moments in the all level case	9
3.2. Moments for levels without small prime factors	14
4. Behavior for the asymptotic real moments	16
4.1. Behavior of the main term	16
4.2. Behavior of the constant term	21
References	25

1. INTRODUCTION

Let ρ be a representation on SU(2). For any $g \in$ SU(2) we define the polynomial

(1) $D(X,\rho,g) = \det(I - X\rho(g))^{-1}.$

Endowing SU(2) with its Haar measure, Cogdell & Michel remarked that

$$\int_{SU(2)} D(X,\rho,g)^{z} dg = 1 + \left[\frac{z^{2}}{2} \operatorname{FrSc}(\rho)^{2} + \frac{z}{2} \operatorname{FrSc}(\rho)\right] X^{2} + O_{z} (X^{3})$$

[CM04, (2.26)] for any complex number z, where $FrSc \rho$ is the Frobenius-Schur indicator of ρ . The coefficient of X^2 is then

 $\begin{cases} 0 & \text{if } \rho \text{ is not self-dual,} \\ \frac{z(z-1)}{2} & \text{if id appears once in the irreducible decomposition of Sym}^2 \rho, \\ \frac{z(z+1)}{2} & \text{if id appears once in the irreducible decomposition of } \wedge^2 \rho. \end{cases}$

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For $\rho = \text{St}$ (the standard representation of SU(2)), this coefficient is $\frac{z(z-1)}{2}$. In particular, the Euler product (indexed over the set *P* of all prime numbers)

$$\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)}D\left(p^{-1/2},\mathrm{St},g\right)^{z}\mathrm{d}g$$

converges only for $z \in \{0, 1\}$.

Let $k \ge 2$ be a (fixed) even integer. For any squarefree integer N such that the set of primitive forms of weight k over $\Gamma_0(N)$ is not empty, we denote by $H_k^*(N)$ this set. To any $f \in H_k^*(N)$ we associate an *L*-function defined by the Euler product

$$L(s,f) = \prod_{p \in \mathscr{P}} \det \left(I - X \operatorname{St}(g_f(p)) p^{-s} \right)^{-1}$$

where for any prime number *p*, the matrix

$$g_f(p) = \begin{pmatrix} \alpha_f(p) & 0\\ 0 & \beta_f(p) \end{pmatrix}$$

is made up of the local parameters in p associated to f. For any prime p not dividing N, this matrix belongs to SU(2) and for the $\omega(N)$ prime numbers dividing *N* we have $\alpha_f(p) = \pm p^{-1/2}$ and $\beta_f(p) = 0$. Hence it tempts naturally to model the moments of L-functions for the primitive forms in $H_k^*(N)$ (over the discrete harmonic measure) with Euler product of polynomial of type (1) with g in SU(2) endowed with its Haar measure. As in [CM04], denote by \sum^h the harmonic average. It is apparent that

$$\lim_{N \to +\infty} \sum_{f \in \mathrm{H}_{k}^{*}(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^{\mathrm{0}} = \prod_{p \in \mathscr{P}} \int_{\mathrm{SU}(2)} D\left(p^{-1/2}, \mathrm{St}, g\right)^{\mathrm{0}} \mathrm{d}g$$

and it follows from [RW07, Theorem A and Proposition B] that

$$\lim_{N \to +\infty} \sum_{f \in \mathrm{H}_{k}^{*}(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^{\mathrm{I}} = \prod_{p \in \mathscr{P}} \int_{\mathrm{SU}(2)} D\left(p^{-1/2}, \mathrm{St}, g\right)^{\mathrm{I}} \mathrm{d}g.$$

The generalization to high power moments sounds problematic, and in fact, there is a convergence problem on the right side. For z = 2, the lack of convergence of the product in the representation side comes from the term 1/p so a natural remedy is natural to consider the normalized form

$$\prod_{p\in\mathscr{P}}\int_{\mathrm{SU}(2)}\left(1-\frac{1}{p}\right)D\left(p^{-1/2},\mathrm{St},g\right)^2\mathrm{d}g.$$

To fix ideas, we assume temporarily N to be prime. It turns out that the remedy is appropriate; in fact,

$$\sum_{f \in \mathcal{H}_{k}^{*}(N)}^{h} L\left(\frac{1}{2}, f\right)^{2} \sim \left[\prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p}\right) \int_{\mathrm{SU}(2)} D\left(p^{-1/2}, \mathrm{St}, g\right)^{2} \mathrm{d}g\right] \log N \quad (N \to +\infty)$$
$$\sim \mathrm{e}^{-\gamma} \prod_{p \leqslant N} \int_{\mathrm{SU}(2)} D\left(p^{-1/2}, \mathrm{St}, g\right)^{2} \mathrm{d}g \quad (N \to +\infty),$$

where γ is the Euler constant. In other words, we may model $L(1/2, f)^2$ by the product over prime numbers $p \leq N$ of the random variables $g \mapsto D(p^{-1/2}, \text{St}, g)^2$ with a correction factor $e^{-\gamma}$.

Our result is actually more precise and we compute all the complex moments of $L(1, \text{Sym}^m f)$ twisted by $L(1/2, f)^2$ without too heavy restriction on the level N. To give our results, we need a few notation.

For any integer $m \ge 1$, the *m*th symmetric power *L*-function of $f \in H_k^*(N)$ is

$$L(s, \operatorname{Sym}^m f) = \prod_{p \in \mathscr{P}} \det(I - \operatorname{Sym}^m \rho(g_f(p)) p^{-s})^{-1}.$$

If $m \in \{1, 2, 4\}$ it is known to have all the required properties to be an *L*-function in the sense of [IK04, §5.1] and to have no Landau-Siegel zero [GJ78, Kim03, KS02]. For other values of *m*, we impose two standard hypothesis - Hypothesis Sym^k *f* and LSZ^k *f* in [CM04]. Therefore, our results are unconditional for $m \in \{1, 2, 4\}$ and rest on the standard conjectures for all other cases. We write, γ_{∞} for the gamma factor of L(s, f) which depends only on the weight of *f*. Explicitly it is given by

$$\gamma_{\infty}(s) = \pi^{-s} \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right).$$

Let

$$F^{z}(w, s; X) = (1 - X^{1+2w}) \int_{SU(2)} D(X^{1/2+w}, St, g)^{2} D(X^{1+s}, Sym^{m}, g)^{z} dg$$

and

$$C^{z}(w,s;X) = \begin{cases} (1+X^{2+2w})(1-X^{2+2w})^{-2}(1-X^{1+m/2+s})^{-z} & \text{if } 2 \mid m, \\ \frac{(1+X^{1+w})^{-2}(1-X^{1+m/2+s})^{-z} + (1-X^{1+w})^{-2}(1+X^{1+m/2+s})^{-z}}{2} & \text{if } 2 \nmid m. \end{cases}$$

The function $C^{z}(w, s; X)$ will be used as a correction factor to $F^{z}(w, s; X)$. Moreover we define

$$A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^{m};N\right) = \prod_{\substack{p\in\mathscr{P}\\p\nmid N}} F^{z}\left(0,0;\frac{1}{p}\right) \prod_{\substack{p\in\mathscr{P}\\p\mid N}} C^{z}\left(0,0;\frac{1}{p}\right)$$

and

$$B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^{m};N\right) = \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0} \left(\prod_{\substack{p \in \mathscr{P} \\ p \nmid N}} F^{z}\left(w,0;\frac{1}{p}\right) \prod_{\substack{p \in \mathscr{P} \\ p \mid N}} C^{z}\left(w,0;\frac{1}{p}\right)\right).$$

Finally denote by $\varphi(n)$ (resp. $\mu(n)$) the Euler function (resp. Möbius) and by \log_k the *k*-fold iterated logarithm.

Below are our main results.

Theorem A–Let $m \in \{1, 2, 4\}$. There exists two positive real numbers c_m and δ_m such that for any sufficiently large squarefree N,

$$\begin{split} \frac{N}{\varphi(N)} & \sum_{f \in \mathbf{H}_{k}^{*}(N)}^{\mathbf{h}} L\left(\frac{1}{2}, f\right)^{2} L\left(1, \operatorname{Sym}^{m} f\right)^{z} \\ &= A^{2, z} \left(\frac{1}{2}, 1; \operatorname{St}, \operatorname{Sym}^{m}; N\right) \log N \\ &+ 2A^{2, z} \left(\frac{1}{2}, 1; \operatorname{St}, \operatorname{Sym}^{m}; N\right) \left(\gamma + \frac{\gamma_{\infty}'}{\gamma_{\infty}} \left(\frac{1}{2}\right) + \sum_{p \mid N} \frac{\log p}{p - 1}\right) \\ &+ B^{2, z} \left(\frac{1}{2}, 1; \operatorname{St}, \operatorname{Sym}^{m}; N\right) + O_{m} \left(\exp\left(-\delta_{m} \frac{\log N}{\log_{2} N}\right)\right) \end{split}$$

uniformly in

$$|z| \leqslant c_m \frac{\log N}{\log_2 N \log_3 N}$$

This theorem is proved in Section 3.1. The dependance on the level can be easily depicted when *N* has no small prime factors. Consider the set of numbers

$$\mathcal{N}(h) = \left\{ N \in \mathbb{Z}_{>0} : \mu(N)^2 = 1 \text{ and } P^-(N) \ge h(N) \right\}$$

for some function *h* where $P^{-}(N)$ is the smallest prime factor of *N* with the convention $P^{-}(1) = +\infty$. We write

$$A^{2,z}\left(\frac{1}{2},1;\operatorname{St},\operatorname{Sym}^{m}\right) = A^{2,z}\left(\frac{1}{2},1;\operatorname{St},\operatorname{Sym}^{m};1\right)$$
$$= \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p}\right) \int_{\operatorname{SU}(2)} D\left(\frac{1}{p^{1/2}},\operatorname{St},g\right)^{2} D\left(\frac{1}{p},\operatorname{Sym}^{m},g\right)^{z} \mathrm{d}g$$

and

$$B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^{m}\right) = B^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^{m},1\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}w} \prod_{|w=0} \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p^{1+2w}}\right) \int_{\mathrm{SU}(2)} D\left(\frac{1}{p^{1/2+w}},\mathrm{St},g\right)^{2} D\left(\frac{1}{p},\mathrm{Sym}^{m},g\right)^{z} \mathrm{d}g.$$

Corollary B–Let $m \in \{1,2,4\}$. There exists a positive real number c_m such that for any sufficiently large squarefree $N \in \mathcal{N}(\log^2)$,

$$\sum_{f \in \mathcal{H}_{k}^{*}(N)}^{h} L\left(\frac{1}{2}, f\right)^{2} L(1, \operatorname{Sym}^{m} f)^{z} = (1 + o_{m}(1))A^{2, z}\left(\frac{1}{2}, 1; \operatorname{St}, \operatorname{Sym}^{m}\right) \log N$$

uniformly in

$$|z| \leqslant c_m \frac{\log N}{\log_2 N \log_3 N}.$$

This is shown in Section 3.2.

It is interesting to evaluate the asymptotic behavior of the main term

$$A^{2,z}\left(\frac{1}{2},1;\mathrm{St},\mathrm{Sym}^{m}\right)$$

and the constant term $B^{2,z}(\frac{1}{2},1; \operatorname{St}, \operatorname{Sym}^m)$ as the exponent $z \to +\infty$ in real numbers.

Let X_m be the Chebyshev polynomial of second kind whose restriction on [-2,2] is defined by

$$X_m(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

They come up naturally in theory of modular forms since, if $\{\chi_{\text{Sym}^m}: m \in \mathbb{Z}_{\geq 0}\}$ is the set of irreducible characters of SU(2), then

$$\chi_{\operatorname{Sym}^m}(g) = X_m(\operatorname{tr} g).$$

Let us introduce some auxiliary functions.

(2)
$$g_m(t) \coloneqq \log \int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\mathrm{tr}\,g)} \mathrm{d}g = \log \left(\frac{2}{\pi} \int_0^{\pi} \mathrm{e}^{tX_m(2\cos\theta)} \sin^2\theta \,\mathrm{d}\theta\right) \quad (t \ge 0),$$

(3)
$$\widetilde{g}_m(t) := \begin{cases} g_m(t) & \text{if } 0 \leq t < 1, \\ g_m(t) - (m+1)t & \text{if } t \geq 1, \end{cases}$$

and

(4)
$$h_m(t) \coloneqq \frac{\int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\mathrm{tr}\,g)} \,\mathrm{tr}\,g \,\mathrm{d}g}{2\int_{\mathrm{SU}(2)} \mathrm{e}^{t\chi_m(\mathrm{tr}\,g)} \,\mathrm{d}g} = \frac{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} \cos\theta \sin^2\theta \,\mathrm{d}\theta}{\int_0^\pi \mathrm{e}^{tX_m(2\cos\theta)} \sin^2\theta \,\mathrm{d}\theta} \quad (t \ge 0),$$

(5)
$$\widetilde{h}_m(t) \coloneqq \begin{cases} h_m(t) & \text{if } 0 \leq t < 1, \\ h_m(t) - 1 & \text{if } t \geq 1. \end{cases}$$

Theorem C-Let $J \ge 1$ and $m \ge 1$ be two fixed integers. Then we have

$$\log A^{2,z} \left(\frac{1}{2}, 1; \text{St}, \text{Sym}^{m}\right)$$

= $z \left\{ (m+1)\log_{2} z + (m+1)\gamma + \sum_{j=1}^{J} \frac{a_{j}}{(\log z)^{j}} + O\left(\frac{1}{(\log z)^{J+1}}\right) \right\}$

uniformly for $z \ge 3$, where γ is the Euler constant and

$$a_j \coloneqq \int_0^{+\infty} \frac{\widetilde{g}_m(t)}{t^2} (\log t)^{j-1} \,\mathrm{d}t.$$

The implied constant depends on J and m only.

Theorem C is proved in Section 4.1.

$$B^{2,z}\left(\frac{1}{2},1;\operatorname{St},\operatorname{Sym}^{m}\right) \ll A^{2,z}\left(\frac{1}{2},1;\operatorname{St},\operatorname{Sym}^{m}\right)\log z$$

uniformly for $z \ge 3$ if m is even; and

$$B^{2,z}\left(\frac{1}{2},1;\operatorname{St},\operatorname{Sym}^{m}\right) = A^{2,z}\left(\frac{1}{2},1;\operatorname{St},\operatorname{Sym}^{m}\right)\left\{b_{m} + O\left(e^{-\sqrt{\log z}}\right)\right\}\sqrt{z}$$

uniformly for $z \ge 3$ if m is odd, where

$$b_m \coloneqq -4\left(2 + \int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}} \,\mathrm{d}t\right) \neq 0.$$

The implied constants depend on m only.

Section 4.2 is devoted to its proof.

It is surprising that the asymptotic behavior of $\log B^{2,z}(\frac{1}{2}, 1; \text{St}, \text{Sym}^m)$ changes dramatically according as the parity of *m*.

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2. PRELIMINARY RESULTS

For every $g \in SU(2)$, define $\lambda_{Sym^m}^{z,v}(g)$ by the expansion

$$D(X, \operatorname{Sym}^{m}, g)^{z} = \sum_{\nu=0}^{+\infty} \lambda_{\operatorname{Sym}^{m}}^{z, \nu}(g) X^{\nu}.$$

We have from [RW07, (46) and (36)],

$$\lambda_{\operatorname{Sym}^{m}}^{z,\nu}(g) = \sum_{u=0}^{m\nu} \mu_{\operatorname{Sym}^{m},\operatorname{Sym}^{u}}^{z,\nu} \chi_{\operatorname{Sym}^{u}}(g)$$

with

(6)
$$\mu_{\operatorname{Sym}^m,\operatorname{Sym}^u}^{z,\nu} = \int_{\operatorname{SU}(2)} \lambda_{\operatorname{Sym}^m}^{z,\nu}(g) \chi_{\operatorname{Sym}^u}(g) \, \mathrm{d}g.$$

One should remark $\mu_{\text{Sym}^m,\text{Sym}^u}^{z,v} = 0$ for n > mv. Recall that $\{\chi_{\text{Sym}^m} : m \in \mathbb{Z}_{\geq 0}\}$ is explicitly defined by the generating series

(7)
$$\sum_{m \ge 0} \chi_{\operatorname{Sym}^m}(g) T^m = \frac{1}{(1 - \alpha T)(1 - \overline{\alpha} T)} = D(T, \operatorname{St}, g)$$

where α and $\overline{\alpha}$ are the eigenvalues of *g*. It follows from the study of Cogdell & Michel [CM04] (see also [RW07, eq. (38), (39) and (52)]) that

(8)
$$\mu_{\operatorname{Sym}^m,\operatorname{Sym}^u}^{z,0} = \delta(u,0),$$

(9)
$$\mu_{\text{Sym}^m,\text{Sym}^u}^{z,1} = z\delta(u,m),$$

(10)
$$|\mu_{\operatorname{Sym}^{m},\operatorname{Sym}^{u}}^{z,v}| \leqslant \binom{(m+1)|z|+v-1}{v}.$$

2.1. **Combinatorial results.** The aim of this short section is to prove the two following useful equalities:

(11)
$$\sum_{u \ge 0} \frac{\tau(p^u)}{p^{(1+w)u}} \sum_{\substack{v \ge 0\\ u \equiv mv \pmod{2}}} \frac{\tau_z(p^v)}{p^{(1+m/2+s)v}} = C^z\left(w, s; \frac{1}{p}\right),$$

(12)
$$\sum_{u \ge 0} \frac{\tau(p^u)}{p^{(1/2+w)u}} \sum_{v \ge 0} \frac{\mu_{\text{Sym}^m,\text{Sym}^u}^{z,v}}{p^{(1+s)v}} = F^z\left(w,s;\frac{1}{p}\right).$$

Thanks to (10) and the binomial theorem, the series in (12) is absolutely convergent for $\Re s > -1/2$ and $\Re w > -1/2$.

Equality (11) follows directly from the following expressions:

$$\sum_{\substack{u \ge 0\\ u \text{ odd}}} (u+1)X^u = \frac{2X}{(1-X^2)^2}, \qquad \sum_{\substack{u \ge 0\\ u \text{ even}}} (u+1)X^u = \frac{1+X^2}{(1-X^2)^2}$$

and

$$\sum_{\substack{\nu \ge 0 \\ \nu \equiv r \pmod{2}}} {\binom{\nu+z-1}{\nu}} X^{\nu} = \frac{(1-X)^{-z} + (-1)^r (1+X)^{-z}}{2}$$

for any $r \in \{0, 1\}$.

From (6) we deduce

$$\sum_{u \ge 0} \frac{\tau(p^u)}{p^{(1/2+w)u}} \sum_{v \ge 0} \frac{\mu_{\text{Sym}^m, \text{Sym}^u}^{z, v}}{p^{(1+s)v}} = \int_{\text{SU}(2)} D\left(\frac{1}{p^{1+s}}, \text{Sym}^m, g\right)^z \sum_{u \ge 1} \frac{(u+1)\chi_{\text{Sym}^u}(g)}{p^{(1/2+w)u}} \, \mathrm{d}g.$$

Let $g \in SU(2)$ and let $\alpha, \overline{\alpha}$ be its eigenvalues. We use (7) to get

$$\sum_{u \ge 1} (u+1)\chi_{\text{Sym}^{u}}(g) T^{u} = \frac{d}{dT} \frac{T}{(1-\alpha T)(1-\overline{\alpha}T)} = (1-T^{2})D(T,\text{St},g)^{2}.$$

This gives (12).

2.2. Analytical results.

Lemma 2.1-Let $m \ge 1$ and $z_m = (m+1)\min\{n \in \mathbb{Z}_{\ge 0} : n \ge |z|\}$. (a) For $\sigma \ge 3/4$ and $r \ge 1/3$, we have

$$\prod_{p|N} \sum_{u \ge 0} \frac{\tau(p^u)}{p^{ru}} \sum_{\substack{v \ge 0\\ u \equiv mv \pmod{2}}} \frac{\tau_{|z|}(p^v)}{p^{(\sigma+m/2)v}} \leqslant e^{c[|z|+S_r(N)]}$$

where

$$S_r(N) = \begin{cases} 1 & \text{if } r > 1/2 \\ \log_3(N) & \text{if } r = 1/2 \\ (\log N)^{1-2r} / \log_2 N & \text{if } r < 1/2 \end{cases}$$

and the constant c > 0 does not depend on σ .

(b) For $\sigma > 1$ and $r \ge 1/3$ we have

$$\prod_{p \nmid N} \sum_{u \ge 0} \frac{\tau(p^u)}{p^{ru}} \sum_{v \ge 0} \frac{|\mu_{\operatorname{Sym}^m, \operatorname{Sym}^u}^{z, v}|}{p^{\sigma v}} \le \exp\left(c_{\sigma}(z_m + 3)\right),$$

where $c_{\sigma} > 0$ is a constant depending on σ . (c) For $\sigma \in [3/4, 1]$ and $r \in [1/3, 1]$ we have

$$\prod_{p \nmid N} \sum_{u \ge 0} \frac{\tau(p^u)}{p^{ru}} \sum_{\nu \ge 0} \frac{|\mu_{\operatorname{Sym}^m, \operatorname{Sym}^u}^{z, \nu}|}{p^{\sigma \nu}} \\ \leq \exp\left(c(z_m + 3) \left[\frac{(z_m + 3)^{-1 + 1/\sigma} - 1}{(1 - \sigma)\log(z_m + 3)} + \log_2(z_m + 3) \right] \right)$$

where c > 0 is a constant not depending on σ .

Proof. (a) Let

$$A_m(p) = \sum_{u \ge 0} \frac{\tau(p^u)}{p^{ur}} \sum_{\substack{v \ge 0\\ u \equiv mv \pmod{2}}} \frac{\tau_{|z|}(p^v)}{p^{(\sigma+m/2)v}}.$$

If m is even then by (2.1),

(13)
$$A_m(p) = \sum_{u \text{ even } \nu} \sum_{v} = \left(1 + \frac{1}{p^{2r}}\right) \left(1 - \frac{1}{p^{2r}}\right)^{-2} \left(1 - \frac{1}{p^{\sigma + m/2}}\right)^{-|z|}.$$

If *m* is odd, then we get

$$A_m(p) = \sum_{u \text{ even } v} \sum_{v \text{ even } + \sum_{u \text{ odd } v}} \sum_{v \text{ odd}} \leqslant \sum_{u \text{ even } v} \sum_{v \text{ even } + \sum_{u \text{ even } v}} \sum_{v \text{ odd}} \leqslant \sum_{u \text{ even } v} \sum_{v}.$$

In both cases, we are led to the bound in the right side of (13). Since $\sigma + m/2 \ge 5/4$ and $r \ge 1/3$, this yields

$$\prod_{p|N} A_m(p) \ll \exp\left(c\left(|z| + \sum_{p|N} \frac{1}{p^{2r}}\right)\right) \leq \exp\left(c[|z| + S_r(N)]\right).$$

(b) The proof is similar to [RW07, Page 728]. We separate the product into two parts according to $p^{\sigma} \leq z_m + 3$ or $p^{\sigma} > z_m + 3$. Using (9) and (10), we have

(14)
$$\prod_{p^{\sigma}>z_m+3} \leqslant \exp\left(\sum_{p^{\sigma}>z_m+3} \left(\frac{z_m}{p^{\sigma+rm}} + \sum_{\nu \geqslant 2} \frac{1}{p^{\sigma\nu}} \sum_{u \geqslant 0} \frac{\tau(p^u)}{p^{ru}} |\mu_{\operatorname{Sym}^m, \operatorname{Sym}^u}^{z,\nu}|\right)\right)$$

and

$$\sum_{v \ge 2} \leqslant \sum_{u \ge 0} \frac{u+1}{p^{ru}} \sum_{v \ge 2} \binom{z_m + v - 1}{v} \frac{1}{p^{\sigma v}}$$

with

$$\sum_{\nu \ge 2} {\binom{z_m + \nu - 1}{\nu}} \frac{1}{p^{\sigma \nu}} \leqslant \frac{z_m(z_m + 1)}{p^{2\sigma}} \sum_{\nu \ge 2} {\binom{z_m + \nu - 1}{\nu - 2}} \frac{1}{p^{\sigma(\nu - 2)}}$$

so that

(15)
$$\sum_{\nu \ge 2} \leqslant \left(1 - \frac{1}{p^r}\right)^{-2} \left(\frac{z_m + 1}{p^{\sigma}}\right)^2 \left(1 - \frac{1}{p^{\sigma}}\right)^{-z_m - 2} \leqslant 4 \left(1 - \frac{1}{2^{1/3}}\right)^{-2} \left(\frac{z_m + 1}{p^{\sigma}}\right)^2$$

since $p^{\sigma} > z_m + 3$. Reporting (15) in (14) leads to

$$\prod_{p^{\sigma}>z_m+3}\leqslant \exp\left(c(z_m+3)^{1/\sigma}\right).$$

Now we deal with $p^{\sigma} < z_m + 3$. Using (8), (9) and (10), we have

$$\sum_{u \ge 0} \frac{\tau(p^u)}{p^{ru}} \sum_{v \ge 0} \frac{|\mu_{\operatorname{Sym}^m, \operatorname{Sym}^u}^{z, v}|}{p^{\sigma v}} \leqslant 1 + \frac{z_m}{p^{\sigma}} + \sum_{v \ge 2} \frac{1}{p^{\sigma v}} \sum_{u \ge 0} \frac{\tau(p^u)}{p^{ru}} \binom{z_m + v - 1}{v}.$$

8

The right hand side, denoted by *R*, satisfies

$$R = 1 + \frac{z_m}{p^{\sigma}} + \sum_{v \ge 2} \frac{1}{p^{\sigma v}} {\binom{z_m + v - 1}{v}} + \sum_{v \ge 2} \frac{1}{p^{\sigma v}} \sum_{u \ge 1} \frac{\tau(p^u)}{p^{ru}} {\binom{z_m + v - 1}{v}} \\ = \left(1 - \frac{1}{p^{\sigma}}\right)^{-z_m} + \frac{1}{p^{\sigma + r}} \sum_{u \ge 0} \frac{u + 2}{p^{ru}} \sum_{v \ge 1} {\binom{z_m + v}{v + 1}} \frac{1}{p^{\sigma v}} \\ \leqslant \left(1 - \frac{1}{p^{\sigma}}\right)^{-z_m} + \frac{2z_m}{p^{\sigma + r}} \left(1 - \frac{1}{p^{r}}\right)^{-2} \sum_{v \ge 1} {\binom{z_m + v}{v}} \frac{1}{p^{\sigma v}} \\ \leqslant \left(1 - \frac{1}{p^{\sigma}}\right)^{-z_m} + \frac{2z_m}{p^{\sigma + r}} \left(1 - \frac{1}{p^{r}}\right)^{-2} \left(1 - \frac{1}{p^{\sigma}}\right)^{-z_m - 1} \\ \leqslant \left(1 - \frac{1}{p^{\sigma}}\right)^{-z_m - 1} \left(1 + c\frac{z_m}{p^{\sigma + r}}\right)$$
(16)

for some absolute constant c > 0. Since σ and $\sigma + r$ are greater than 1 it follows that

$$\prod_{p^{\sigma} < z_m + 3} \leq \exp\left(c_{\sigma}(z_m + 1)\right).$$

(c) As for establishing (15) we have an absolute constant c such that

(17)
$$\prod_{p^{\sigma}>z_m+3} \leqslant \exp\left(\sum_{p^{\sigma}>z_m+3} \frac{z_m}{p^{\sigma+rm}} + c\frac{(z_m+1)^2}{p^{2\sigma}}\right)$$
$$\leqslant \exp\left(c\frac{(z_m+3)^{1/\sigma}}{\log(z_m+3)}\right).$$

From (16) we have

$$\prod_{p^{\sigma} < z_m + 3} \leq \exp\left(c(z_m + 1)\sum_{p^{\sigma} < z_m + 3} \frac{1}{p^{\sigma}} + \frac{1}{p^{\sigma+r}}\right)$$

and using

$$\sum_{p \leqslant y} \frac{1}{p^{\sigma}} \ll \log_2 y + \frac{y^{1-\sigma} - 1}{(1-\sigma)\log y}$$

valid uniformly for $1/2 \le \sigma \le 1$ and $y \ge e^2$ [TW03, Lemma 3.2] we get

(18)
$$\prod_{p^{\sigma} < z_m + 3} \leq \exp\left(c(z_m + 3)\left[\frac{(z_m + 3)^{(1-\sigma)/\sigma} - 1}{(1-\sigma)\log(z_m + 3)} + \log_2(z_m + 3)\right]\right)$$

The result is a consequence of (17) and (18).

3. EVALUATION OF THE MOMENTS

3.1. **Moments in the all level case.** We fix *G* any function which is holomorphic and bounded in some sufficiently wide vertical strip $|\Re s| \ll 1$, even and normalized by G(0) = 1. (Note G'(0) = 0.)

Let $z \in \mathbb{C}$ and $x \ge 1$. Define

(19)
$$\omega_{\text{Sym}^{m}f}^{z}(x) = \sum_{n=1}^{+\infty} \frac{\lambda_{\text{Sym}^{m}f}^{z}(n)}{n} e^{-n/x}$$

for all $f \in \mathrm{H}_{k}^{*}(N)$. We prove the following lemma.

Lemma 3.1 – For all x, z and N we have

$$\sum_{f \in \mathcal{H}_{k}^{*}(N)}^{h} L\left(\frac{1}{2}, f\right)^{2} \omega_{\operatorname{Sym}^{m} f}^{z}(x) = 2 \sum_{q \ge 1}^{\infty} \frac{\tau(q)}{\sqrt{q}} V_{N}\left(\frac{q}{N}\right) \sum_{n \ge 1}^{\infty} \frac{e^{-n/x}}{n} \tau_{z}(n_{N})$$
$$\times \left(\prod_{p \mid n^{(N)}} \mu_{\operatorname{Sym}^{m}, \operatorname{Sym}^{\nu_{p}(q)}}^{z, \nu_{p}(n)}\right) \delta\left(q^{(N)} \mid n^{(N)m}\right) \frac{\Box(n_{N}^{m}q_{N})}{\sqrt{n_{N}^{m}q_{N}}} + O(\operatorname{Err})$$

where

(20)
$$V_N(y) = \frac{1}{2i\pi} \int_{(2)} \zeta^{(N)} (1+2w) \left(\frac{\gamma_{\infty}(1/2+w)}{\gamma_{\infty}(1/2)}\right)^2 \frac{G(w)}{w} y^{-w} dw$$

and

Err =
$$\frac{\tau(N)^2 \log N \log_2 N}{N^{1/4}} x^{m/4} (\log x)^{z_m+1} (z_m+m+1)!.$$

Proof. Let $L(s, f \boxplus f) = L(s, f)^2$. This is an *L*-function in the sense of [IK04, §5.1]. In particular the gamma factor is $\gamma_{\infty}(s)^2$, the sign of the functional equation is 1, the conductor is N^2 and the *n*-th Dirichlet coefficient is

$$\lambda_{f \boxplus f}(n) = \sum_{\substack{(q,r) \in \mathbb{Z}_{\geq 0}^2 \\ qr^2 = n}} \mathbb{1}^{(N)}(r) \lambda_f(q) \tau(q).$$

Therefore we can apply [IK04, Theorem 5.3] to obtain

(21)
$$L\left(\frac{1}{2},f\right)^2 = 2\sum_{q\ge 1}\frac{\lambda_f(q)\tau(q)}{\sqrt{q}}V_N\left(\frac{q}{N}\right)$$

where

$$V_N(y) = \sum_r \frac{\mathbb{1}^{(N)}(r)}{r} \int_{(3)} (yr^2)^{-u} G(u) \left(\frac{\gamma_{\infty}(1/2+u)}{\gamma_{\infty}(1/2)}\right)^2 \frac{\mathrm{d}u}{u}$$
$$= \int_{(3)} y^{-u} \zeta^{(N)}(1+2u) G(u) \left(\frac{\gamma_{\infty}(1/2+u)}{\gamma_{\infty}(1/2)}\right)^2 \frac{\mathrm{d}u}{u}.$$

We have to evaluate

$$T = \sum_{f \in \mathcal{H}_k^*(N)}^{h} \lambda_f(q) \lambda_{\operatorname{Sym}^m f}^z(n).$$

Similarly to [RW07, Lemma 12] we have

(22)
$$T = \frac{\tau_{z}(n_{N})}{\sqrt{n_{N}^{m}q_{N}}} \Box(n_{N}^{m}q_{N})\delta\left(q^{(N)} \mid n^{(N)m}\right) \prod_{p \mid q^{(N)}} \mu_{\text{Sym}^{m},\text{Sym}^{\nu_{p}(q)}}^{z,\nu_{p}(n)} + O\left(\frac{\tau(N)^{2}\log_{2}N}{N}n^{m/4}q^{1/4}\tau(q)\log(Nnq)\tau_{(m+1)|z|}(n)\right)$$

From (19), (21) and (22) we deduce

$$\sum_{f \in \mathcal{H}_k^*(N)}^h L\left(\frac{1}{2}, f\right)^2 \omega_{\operatorname{Sym}^m f}^z(x) = P + E$$

where *P* is the announced principal term and

(23)
$$E = \frac{\tau(N)^2 \log_2 N}{N} \sum_q \frac{\tau(q)^2}{q^{1/4}} \log(Nq) V_N\left(\frac{q}{N}\right) \sum_n \frac{\tau_{(m+1)|z|}(n) \log n}{n^{1-m/4}} e^{-n/x}.$$

We proved in [RW07, Proof of Lemma 16] that the summation over n is

(24)
$$\sum_{n} \ll x^{m/4} (\log x)^{z_m + 1} (z_m + m + 1)!.$$

Moreover, by (20) and since

$$\sum_{q} \frac{\tau(q)^2 \log(Nq)}{q^s} = \left[\log(N) - \frac{\mathrm{d}}{\mathrm{d}s}\right] \frac{\zeta^4(s)}{\zeta(2s)}$$

we get, after having moved the integration line in V_N from (2) to (7/10) and crossed a pole at w = 3/4 the majoration

(25)
$$\sum_{q} \frac{\tau(q)^2 \log(Nq)}{q^{1/4}} V_N\left(\frac{q}{N}\right) \ll N^{3/4} \log N.$$

The announced error term is a consequence of (23) with (24) and (25). $\hfill \Box$

We study the principal term exhibited in Lemma 3.1 in the following lemma. Lemma 3.2 – For any squarefree integer N, any $z \in \mathbb{C}$ and any $x \in \mathbb{R}$ such that

$$\frac{1}{100m}\log N \leqslant \log x \leqslant \frac{1}{12}\log N$$

we have

$$\begin{split} &\sum_{q \ge 1} \frac{\tau(q)}{\sqrt{q}} V_N\left(\frac{q}{N}\right) \sum_{n \ge 1} \frac{\mathrm{e}^{-n/x}}{n} \tau_z(n_N) \left(\prod_{p \mid n^{(N)}} \mu_{\mathrm{Sym}^m, \mathrm{Sym}^{v_p(q)}}^{z, v_p(n)}\right) \delta\left(q^{(N)} \mid n^{(N)m}\right) \frac{\Box(n_N^m q_N)}{\sqrt{n_N^m q_N}} \\ &= \frac{\varphi(N)}{N} A^{2, z} \left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m; N\right) \left(\frac{1}{2} \log N + \gamma + \frac{\gamma'_\infty}{\gamma_\infty} \left(\frac{1}{2}\right) + \sum_{p \mid N} \frac{\log p}{p - 1}\right) \\ &+ \frac{1}{2} B^{2, z} \left(\frac{1}{2}, 1; \mathrm{St}, \mathrm{Sym}^m; N\right) + O(\mathrm{Err}) \end{split}$$

where

$$\operatorname{Err} = \exp\left(c\left[\log_2 N - \frac{\log N}{\log(z_m + 3)} + (z_m + 3)\log(z_m + 3)\right]\right).$$

Proof. We write Σ for the sum to be evaluated: (26)

$$\Sigma = \frac{1}{(2i\pi)^2} \int_{(1)} \int_{(1)} N^w \zeta^{(N)} (1+2w) \left(\frac{\gamma_{\infty}(1/2+w)}{\gamma_{\infty}(1/2)}\right)^2 H_N^z(w,s) G(w) \frac{\mathrm{d}w}{w} \Gamma(s) x^s \,\mathrm{d}s$$

with

$$\begin{split} H_N^z(w,s) &= \\ &\sum_q \frac{\tau(q)}{q^{w+1/2} q_N^{1/2}} \sum_n \frac{\tau_z(n_N)}{n^{s+1} n_N^{m/2}} \delta\left(q^{(N)} \,|\, n^{(N)m}\right) \Box(n_N^m q_N) \prod_{p \mid q^{(N)}} \mu_{\operatorname{Sym}^m, \operatorname{Sym}^{v_p(q)}}^{z, v_p(n)}. \end{split}$$

. .

Writing $a = n^{(N)}$, $b = n_N$, $c = q^{(N)}$ and $d = q_N$ we have $H_N^z(w, s) = AB$ where

$$A = \sum_{b \mid N^{\infty}} \frac{\tau_{z}(b)}{b^{1+m/2+s}} \sum_{d \mid N^{\infty}} \frac{\tau(d)}{d^{w+1}} \Box(db^{m})$$

=
$$\prod_{p \mid N} \sum_{u \ge 0} \frac{\tau(p^{u})}{p^{u(w+1)}} \sum_{\substack{v \ge 0\\u \equiv mv \pmod{2}}} \frac{\tau_{z}(p^{v})}{p^{(s+1+m/2)v}}$$

=
$$C^{z} \left(w, s; \frac{1}{p}\right)$$

by (11) and

$$B = \sum_{(a,N)=1} \frac{1}{a^{s+1}} \sum_{c|a^{m}} \frac{\tau(c)}{c^{w+1/2}} \prod_{p|c} \mu_{\text{Sym}^{m},\text{Sym}^{v_{p}(c)}}^{z,v_{p}(a)}$$

=
$$\prod_{p \nmid N} \sum_{u \ge 0} \frac{\tau(p^{u})}{p^{(1/2+w)u}} \sum_{v \ge 0} \frac{\mu_{\text{Sym}^{m},\text{Sym}^{u}}^{z,v}}{p^{(1+s)v}}$$

=
$$F^{z} \left(w, s; \frac{1}{p}\right)$$

by (12). (Recall that $\mu_{\text{Sym}^m,\text{Sym}^u}^{z,v}$ vanishes when u > mv.)

In (26) we shift the *w*-contour to $\Re w = -1/6$ encountering a simple pole at 0 and obtain

(27)
$$\Sigma = P + \frac{1}{2i\pi} \int_{(1)} \Sigma^{-}(s) \Gamma(s) x^{s} ds$$

with

$$P = \frac{\varphi(N)}{N} \frac{1}{2i\pi} \int_{(1)} \left[\left(\frac{1}{2} \log N + \gamma + \sum_{p|N} \frac{\log p}{p-1} + \frac{\gamma'_{\infty}(1/2)}{\gamma_{\infty}(1/2)} \right) H_N^z(0, s) + \frac{1}{2} \frac{\partial}{\partial w}_{|(0,s)} H_N^z(w, s) \right] \Gamma(s) x^s \, \mathrm{d}s.$$

We bound $|\Sigma^{-}|$ as follows. We use lemma 2.1 choosing $\sigma = 2$ and r = 5/6 in (a), r = 1/3 in (b) to get

$$|\Sigma^{-}(s)| \ll N^{-1/6} \exp\left[c\left(\frac{(\log N)^{1/3}}{\log_2 N} + z_m\right)\right]$$

hence

$$\Sigma = P + O\left\{xN^{-1/6}\exp\left[c\left(\frac{(\log N)^{1/3}}{\log_2 N} + z_m\right)\right]\right\}.$$

We now treat the integral in the defining expression for *P*. For this, we replace the segment $[1 - i\log^2 x, 1 + i\log^2 x]$ by the union of the three segments $[1 - i\log^2 x, -\sigma - i\log^2 x]$, $[-\sigma - i\log^2 x, -\sigma + i\log^2 x]$, $[-\sigma + i\log^2 x, 1 + i\log^2 x]$ with $\sigma = 1/\log(|z|+3)$. We shall show that the residue Res of the pole of Γ at 0 provides the main contribution whereas the integral on the new contour enters the error term.

We write

(28)
$$P - \operatorname{Res} = A_0 + A_1 + A_2 + B_0 + B_1 + B_2$$

where

$$\operatorname{Res} = \frac{\varphi(N)}{N} \left(\frac{\log N}{2} + \gamma + \sum_{p|N} \frac{\log p}{p-1} + \frac{\gamma'_{\infty}}{\gamma_{\infty}} \left(\frac{1}{2} \right) \right) H_N^z(0,0) + \frac{\varphi(N)}{2N} \frac{\partial}{\partial w}|_{(0,0)} H_N^z(w,s),$$
$$A_0 = \frac{\varphi(N)}{N} \left(\frac{\log N}{2} + \gamma + \sum_{p|N} \frac{\log p}{p-1} + \frac{\gamma'_{\infty}}{\gamma_{\infty}} \left(\frac{1}{2} \right) \right) \frac{1}{2i\pi} \int_{1\pm i\log^2 x}^{1\pm i\infty} H_N^z(0,s) \Gamma(s) x^s \, \mathrm{d}s,$$
$$B_0 = \frac{\varphi(N)}{2N} \frac{1}{2i\pi} \int_{1\pm i\log^2 x}^{1\pm i\infty} \frac{\partial}{\partial w}|_{(0,s)} H_N^z(w,s) \Gamma(s) x^s \, \mathrm{d}s,$$

and A_1 (resp. B_1) has the same integrand as A_0 (resp. B_0) but the contour is $[1 - i \log^2 x, -\sigma - i \log^2 x]$ and A_2 (resp. B_2) has the same integrand as A_0 (resp. B_0) but the contour is $[-\sigma - i \log^2 x, -\sigma + i \log^2 x]$.

From lemma 2.1 (a) and (b) and Stirling formula [IK04, (5.113)] we have

(29)
$$A_0 \ll \frac{\varphi(N) \log N}{N} e^{-\log^2 x + c(z_m + 3)}$$

From lemma 2.1 (a) and (c) and Stirling formula we have

(30)
$$A_1 \ll \frac{\varphi(N) \log N}{N} e^{-\log^2 x + c(z_m + 3) \log_2(z_m + 3)}.$$

and

(31)
$$A_2 \ll \frac{\varphi(N) \log N}{N} \exp\left(-\frac{\log x}{\log(z_m+3)}\right) e^{c(z_m+3)\log_2(z_m+3)}$$

The contribution of B_0 , B_1 and B_2 are easily seen to be dominated by the ones of A_0 , A_1 and A_2 thanks to Cauchy integral formula. Reporting (29), (30) and (31) in (28) and the result in (27) we obtain that Σ is the announced principal term (the residue Res) up to an error term

$$\ll \exp\left(c\left(-\frac{\log N}{\log(z_m+3)} + (z_m+3)\log(z_m+3) + \log_2 N\right)\right).$$

This completes the proof.

We have now the ingredients to prove theorem A. As in [RW07, pages 743] we have

(32)
$$\sum_{f \in \mathrm{H}^*_k(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 L(1, \mathrm{Sym}^m f)^z = \sum_{f \in \mathrm{H}^*_k(N)}^{\mathrm{h}} L\left(\frac{1}{2}, f\right)^2 \omega_{\mathrm{Sym}^m f}^z(x) + O(\mathrm{Err})$$

where

$$\operatorname{Err} = x^{-1/\log_2 N} e^{D|z|\log_3 N} \log^4 N + e^{D|z|\log_2 N - \frac{1}{2}\log^2 N} + N^{-1/4} \log^{D|z|} N$$
$$\ll \exp\left(D|z|\log_2 N - \alpha \frac{\log N}{\log_2 N}\right)$$

by setting $x = N^{\alpha}$. We have also used

$$\sum_{f \in \mathcal{H}_k^*(N)}^{h} L\left(\frac{1}{2}, f\right)^2 \ll \log N$$

which follows from (21) and Petersson trace formula [ILS00, Corollary 2.10] or [RW07, Lemma 10].

Reporting lemma 3.2, 3.1 in (32) and assuming

$$|z| \leqslant \varepsilon \frac{\log N}{\log_2 N \log_3 N}$$

for $\varepsilon > 0$ small enough (regarding to α) we obtain the theorem.

3.2. Moments for levels without small prime factors. Corollary B is a consequence of the following lemma.

Lemma 3.3–We have

$$\frac{\varphi(N)}{N}A^{2,z}\left(\frac{1}{2},1;\text{St},\text{Sym}^{m},N\right) = A^{2,z}\left(\frac{1}{2},1;\text{St},\text{Sym}^{m}\right)[1+o_{m}(1)]$$

and

$$\frac{\varphi(N)}{N}B^{2,z}\left(\frac{1}{2}, 1; \text{St}, \text{Sym}^{m}, N\right) = B^{2,z}\left(\frac{1}{2}, 1; \text{St}, \text{Sym}^{m}\right)[1 + o_{m}(1)] + A^{2,z}\left(\frac{1}{2}, 1; \text{St}, \text{Sym}^{m}\right)o_{m}(1)$$

uniformly for

(33)
$$\begin{cases} N \in \mathcal{N}(\log^2) \\ |z| \ll_m \frac{\log N}{\log_2 N \log_3 N} \end{cases}$$

Proof. To prove the first equality, we write

(34)
$$\frac{\varphi(N)}{N} A^{2,z} \left(\frac{1}{2}, 1; \text{St}, \text{Sym}^m, N\right) = A^{2,z} \left(\frac{1}{2}, 1; \text{St}, \text{Sym}^m\right) \frac{E_1(N)}{E_2(N)}$$

with

$$E_1(N) = \prod_{p|N} C^z \left(0, 0; \frac{1}{p} \right),$$

$$E_2(N) = \prod_{p|N} \int_{SU(2)} D(p^{-1/2}, St, g)^2 D(p^{-1}, Sym^m, g)^z dg.$$

First, we deal with $E_1(N)$. For *m* even we have

$$\begin{split} E_1(N) &= \left(1 + O\left(\frac{\omega(N)}{P^-(N)^2}\right)\right) \left(1 + O\left(\frac{(|z|+1)\omega(N)}{P^-(N)^{1+m/2}}\right)\right) \\ &= 1 + O\left(\frac{(|z|+1)\omega(N)}{P^-(N)^{\min(2,1+m/2)}}\right) \end{split}$$

as soon as the function inside the error term is bounded. If m is odd then

$$\begin{split} C^{z}\left(0,0;\frac{1}{p}\right) &= \frac{1}{2}\left(1+\frac{2}{p}+O\left(\frac{1}{p^{2}}\right)\right)\left(1+\frac{z}{p^{1+m/2}}+O\left(\frac{(|z|+1)^{2}}{p^{2+m}}\right)\right) \\ &+ \frac{1}{2}\left(1-\frac{2}{p}+O\left(\frac{1}{p^{2}}\right)\right)\left(1-\frac{z}{p^{1+m/2}}+O\left(\frac{(|z|+1)^{2}}{p^{2+m}}\right)\right) \\ &= 1+O\left(\frac{(|z|+1)^{2}}{p^{2+m/2}}\right) \end{split}$$

so that

(35)
$$E_1(N) = 1 + O\left(\frac{(|z|+1)^2 \omega(N)}{P^-(N)^{2+m/2}}\right).$$

From (35) we deduce that

(36)
$$E_1(N) = 1 + o_m(1)$$

if N and z satisfy (33).

To study $E_2(N)$ we define

(37)
$$e(z,p) = \int_{SU(2)} D(p^{-1/2}, St, g)^2 D(p^{-1}, Sym^m, g)^z dg$$
$$= \sum_{\nu_1=0}^{+\infty} p^{-\nu_1} \sum_{\nu_2=0}^{+\infty} p^{-\nu_2/2} \sum_{u=0}^{\min(m\nu_1, \nu_2)} \mu_{Sym^m, Sym^u}^{z, \nu_1} \mu_{St, Sym^u}^{2, \nu_2}$$

by orthogonality. Using (8) and (9) we compute the contribution of $v_1 = 1$ and $v_2 = 2$ to (37) and with (10) we obtain

$$\begin{split} |e(z,p)-1| \leqslant \sum_{\nu_2=2}^{+\infty} \binom{3+\nu_2}{\nu_2} \frac{1}{p^{\nu_2/2}} + \frac{|z|}{p} \sum_{\nu_2=m}^{+\infty} \binom{3+\nu_2}{\nu_2} \frac{1}{p^{\nu_2/2}} \\ &+ \sum_{\nu_1=2}^{+\infty} \binom{(m+1)|z|+\nu_1-1}{\nu_1} \frac{1}{p^{\nu_1}} \sum_{\nu_2=0}^{+\infty} \binom{3+\nu_2}{\nu_2} \frac{1}{p^{\nu_2/2}} \\ \ll_m \frac{1}{p} + \frac{|z|}{p^{1+m/2}} + \frac{|z|(|z|+1)}{p^2}. \end{split}$$

It follows that

(38)
$$E_2(N) = 1 + O\left(\frac{\omega(N)}{P^-(N)} \left(1 + \frac{|z|}{P^-(N)^{m/2}} + \frac{(|z|+1)^2}{P^-(N)}\right)\right) = 1 + o_m(1)$$

if N and z satisfy (33). The first result of the lemma follows from (34), (36) and (38).

We consider now $B^{2,z}(\frac{1}{2},1; \text{St}, \text{Sym}^m, N)$. We begin in considering

$$F_N^z(w,0) = \left(\prod_{\substack{p \in \mathcal{P} \\ p \nmid N}} F^z\left(w,0;\frac{1}{p}\right) \prod_{\substack{p \in \mathcal{P} \\ p \mid N}} C^z\left(w,0;\frac{1}{p}\right)\right)$$

with enough uniformity in some fixed neighbourhood of w to be authorized to apply Cauchy integral formula. We write $F_N^z(w,0) = F_1^z(w,0)Q_N(w)$ with

$$Q_N(w) = Q_N^{(1)}(w) / Q_N^{(2)}(w)$$

and

$$Q_N^{(1)}(w) = \prod_{p \mid N} C^z \left(w, 0, \frac{1}{p} \right), \qquad Q_N^{(2)}(w) = \prod_{p \mid N} F^z \left(w, 0, \frac{1}{p} \right).$$

As for $E_1(N)$ and $E_2(N)$ we compute

(39)
$$Q_N^{(1)}(w) = 1 + O_{\varepsilon} \left(\frac{\omega(N)}{P^-(N)^{1-\varepsilon}} \left(1 + \frac{|z|}{P^-(N)^{m/2+\varepsilon}} \right) \right)$$

and

(40)
$$\frac{N}{\varphi(N)}Q_N^{(2)}(w) = 1 + O_{\varepsilon}\left(\frac{\omega(N)}{P^-(N)^{1-2\varepsilon}}\left(1 + \frac{|z|}{P^-(N)^{1/2+\varepsilon}} + \frac{(|z|+1)^2}{P^-(N)^{1+2\varepsilon}}\right)\right)$$

the constant implied by the error term being independant of w such that $\Re w > -\varepsilon$. It follows in particular that

(41)
$$Q_N(0) = 1 + o_m(1)$$

if *N* and *z* satisfy (33). Denote $C(0, \varepsilon)$ the circle of centre 0 and radius ε . We have

(42)
$$\frac{\mathrm{d}}{\mathrm{d}w}|_{w=0}F_N^z(w,0) = \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0}F_1^z(w,0)Q_N(0) + F_1^z(0,0) \cdot \frac{1}{2i\pi}\int_{C(0,\varepsilon)}Q_N(w)\frac{\mathrm{d}w}{w^2}$$

and from the uniformity in w in (39) and (40) we deduce

(43)
$$\frac{1}{2i\pi} \int_{C(0,\varepsilon)} Q_N(w) \frac{\mathrm{d}w}{w^2} = o(1).$$

Reporting (41) and (43) in (42) we obtain the second result of the lemma. \Box

4. BEHAVIOR FOR THE ASYMPTOTIC REAL MOMENTS

4.1. **Behavior of the main term.** The aim of this section is to prove Theorem C. In fact we shall establish a more general result (see Proposition 4.1 below). Write

(44)
$$D_m(\theta, t) := D(t, \operatorname{Sym}^m, g) = \prod_{j=0}^m \left(1 - e^{i(m-2j)\theta} t\right)^{-1},$$

and

$$F_m^{\ell,z}(w,s;t) := \left(1 - t^{1+2w}\right)^{\frac{\ell(\ell-1)}{2}} \frac{2}{\pi} \int_0^{\pi} D_1\left(\theta, t^{1/2+w}\right)^{\ell} D_m\left(\theta, t^{1+s}\right)^z \sin^2\theta \,\mathrm{d}\theta.$$

so that

$$F^{z}(w,s;t) = F_{m}^{2,z}(w,s;t).$$

Proposition 4.1–Let $J \ge 1$, $\ell \ge 0$ and $m \ge 1$ be three fixed integers. Then we have

$$\sum_{p \leq y} \log F_m^{\ell, z} \left(0, 0; \frac{1}{p} \right) = z \left\{ (m+1) \log_2 z + (m+1)\gamma + \sum_{j=1}^J \frac{a_j}{(\log z)^j} + O\left(\frac{1}{(\log z)^{J+1}}\right) \right\}$$

uniformly for $y \ge z^{3/2} \ge 10$, where γ is the Euler constant and a_j is defined as in Theorem C.

Since

$$A^{2,z}\left(\frac{1}{2},1;\operatorname{St},\operatorname{Sym}^{m}\right) = \prod_{p\in\mathscr{P}} F_{m}^{2,z}\left(0,0;\frac{1}{p}\right),$$

Theorem C is an immediate consequence of Proposition 4.1 by taking $\ell = 2$ and making $y \rightarrow +\infty$.

In order to prove this proposition, we first establish some preliminary lemmas.

Lemma 4.2–Let $g_m(t)$ and $\tilde{g}_m(t)$ be defined as in (2) and (3). Then

$$\widetilde{g}_m(t) \ll \begin{cases} t^2 & \text{if } 0 \leqslant t < 1, \\ \log(2t) & \text{if } t \ge 1, \end{cases}$$

and

$$\widetilde{g}_m'(t) \ll \begin{cases} t & \text{if } 0 \leqslant t < 1, \\ t^{-1} & \text{if } t \geqslant 1. \end{cases}$$

Proof. When $t \ge 0$, we can write

$$e^{tX_m(2\cos\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t\sin((m+1)\theta)}{\sin\theta}\right)^n.$$

From this we deduce, for $0 \leq t < 1$,

$$\widetilde{g}_m(t) = \log\left(1 + \sum_{n=2}^{\infty} \frac{t^n}{n!} \frac{2}{\pi} \int_0^{\pi} \left(\frac{\sin((m+1)\theta)}{\sin\theta}\right)^n \sin^2\theta \,\mathrm{d}\theta\right)$$
$$= \log\left(1 + t^2 + O(t^3)\right) \approx t^2$$

and

$$\widetilde{g}'_m(t) \asymp t.$$

Let C_m be the maximum of $2X'_m(x)$ in [-2,2]. Then, since $X_m(2) = m + 1$, we have

$$0 \leqslant m + 1 - X_m(2\cos\theta) \leqslant C_m(1 - \cos\theta)$$

for every $\theta \in [0, \pi]$. Thus for $t \ge 1$, we have by (3) and (5),

$$\widetilde{g}'_m(t) = -\frac{\int_0^{\pi} e^{tX_m(2\cos\theta)}(m+1-X_m(2\cos\theta))\sin^2\theta \,\mathrm{d}\theta}{\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta \,\mathrm{d}\theta} \ll_m |\widetilde{h}_m(t)|.$$

Now (54) of Lemma 4.6 below implies $\tilde{g}'_m(t) \ll t^{-1}$ for $t \ge 1$. From this we immediately deduce $\tilde{g}_m(t) \ll \log(2t)$ for $t \ge 1$.

Lemma 4.3– Let $m \ge 1$ be a fixed integer. Then we have

(45)
$$\int_0^{\pi} e^{tX_m(2\cos\theta)} \cos\theta \sin^2\theta \,\mathrm{d}\theta \ll t \int_0^{\pi} e^{tX_m(2\cos\theta)} \sin^2\theta \,\mathrm{d}\theta$$

and

(46)
$$\frac{2}{\pi} \int_0^{\pi} e^{tX_m(2\cos\theta)} \cos^2\theta \sin^2\theta \,\mathrm{d}\theta = \left\{\frac{1}{4} + O(t)\right\} \frac{2}{\pi} \int_0^{\pi} e^{tX_m(2\cos\theta)} \sin^2\theta \,\mathrm{d}\theta$$

uniformly for $t \ge 0$. The implied constants depend on *m* only.

Proof. First we note that these estimates are trivial for $t \ge 1$, so we suppose that $0 \le t \le 1$. In view of the following relations:

$$\frac{2}{\pi} \int_0^{\pi} \cos^n \theta \sin^2 \theta \, \mathrm{d}\theta = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n = 0 \\ 2 \frac{(2r-1)!!}{(2r+2)!!} & \text{if } n = 2r \end{cases}$$

(with $n!! := n \cdot (n-2) \cdots$) and $e^{tX_m(2\cos\theta)} = 1 + O(t)$, it follows that

$$\int_0^{\pi} e^{tX_m(2\cos\theta)}\cos\theta\sin^2\theta\,d\theta \ll t\int_0^{\pi}|\cos\theta|\sin^2\theta\,d\theta$$
$$\ll t\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta\,d\theta.$$

Similarly

$$\frac{2}{\pi} \int_0^{\pi} e^{tX_m(2\cos\theta)} \cos^2\theta \sin^2\theta \,\mathrm{d}\theta = \frac{1}{4} + O(t)$$

which implies (46).

Lemma 4.4–Let $\ell \ge 0$ and $m \ge 1$ be two fixed integers. Suppose $z \ge 4$ is real. Then we have

(47)
$$\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = -(m+1)z\log\left(1-\frac{1}{p}\right) + O\left(\log z\right)$$

uniformly for $2 \leq p \leq \sqrt{z}$; and

(48)
$$\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = g_m\left(\frac{z}{p}\right) + O\left(\frac{z}{p^{3/2}}\right)$$

uniformly for $p \ge \sqrt{z} \ge 2$. The implied constants depend on ℓ and m only.

Proof. We have

$$\prod_{j=0}^{m} \left(1 - \frac{\mathrm{e}^{\mathrm{i}(m-2j)\theta}}{p} \right) = \sum_{\nu=0}^{m+1} \frac{(-1)^{\nu}}{p^{\nu}} \sum_{0 \leq j_1 < \dots < j_{\nu} \leq m} \mathrm{e}^{\mathrm{i}(\nu m - 2j_1 - \dots - 2j_{\nu})\theta}.$$

Since the left-hand side is real and

$$\sum_{0\leqslant j_1<\cdots< j_v\leqslant m} \mathrm{e}^{\mathrm{i}(vm-2j_1-\cdots-2j_v)\theta} = 1 \qquad (v=0,m+1),$$

it follows that, with notation $\mathbf{j}_{v} = (j_1, \dots, j_{v})$ and $\ell_{\mathbf{j}_{v}}^m = vm - 2j_1 - \dots - 2j_{v}$,

$$\begin{split} \prod_{j=0}^{m} \left(1 - \frac{\mathrm{e}^{\mathrm{i}(m-2j)\theta}}{p} \right) &= \sum_{\nu=0}^{m+1} \frac{(-1)^{\nu}}{p^{\nu}} \sum_{0 \leq j_1 < \dots < j_{\nu} \leq m} \cos\left(\ell_{j_{\nu}}^m \theta\right) \\ &= \left(1 - \frac{1}{p} \right)^{m+1} + \sum_{\nu=1}^{m} \frac{(-1)^{\nu-1}}{p^{\nu}} \sum_{0 \leq j_1 < \dots < j_{\nu} \leq m} \left\{ 1 - \cos\left(\ell_{j_{\nu}}^m \theta\right) \right\} \\ &= \left(1 - \frac{1}{p} \right)^{m+1} + \sum_{\nu=1}^{m} \frac{(-1)^{\nu-1}}{p^{\nu}} \sum_{0 \leq j_1 < \dots < j_{\nu} \leq m} 2\sin^2\left(\ell_{j_{\nu}}^m \theta/2\right). \end{split}$$

Introducing the notation

$$\widetilde{D}_m(\theta, p^{-1}) := 1 + \left(1 - \frac{1}{p}\right)^{-(m+1)} \sum_{\nu=1}^m \frac{(-1)^{\nu-1}}{p^{\nu}} \sum_{0 \le j_1 < \dots < j_\nu \le m} 2\sin^2\left(\ell_{j_\nu}^m \theta/2\right),$$

we can write

$$\prod_{j=0}^{m} \left(1 - \frac{\mathrm{e}^{\mathrm{i}(m-2j)\theta}}{p} \right) = \left(1 - \frac{1}{p} \right)^{m+1} \widetilde{D}_m(\theta, p^{-1})$$

and

$$F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = \left(1 - \frac{1}{p}\right)^{-(m+1)z + \ell(\ell-1)/2} \check{F}_m^{\ell,z}(p)$$

with

$$\check{F}_m^{\ell,z}(p) \coloneqq \frac{2}{\pi} \int_0^{\pi} D_1\left(\theta, p^{-1/2}\right)^{\ell} \widetilde{D}_m\left(\theta, p^{-1}\right)^{-z} \sin^2\theta \,\mathrm{d}\theta.$$

Observing the nonnegativity of the integrand, we infer that for some suitably small positive constant δ ,

$$\begin{split} \check{F}_{m}^{\ell,z}(p) &\geq \frac{2}{\pi} \left(1 - \frac{1}{\sqrt{2}} \right)^{2\ell} \int_{0}^{\delta\sqrt{p/z}} \left(1 + \frac{c_{m}\theta^{2}}{p} \right)^{-z} \theta^{2} \,\mathrm{d}\theta \\ &\gg \int_{0}^{\delta\sqrt{p/z}} \left(1 + \frac{c_{m}\delta^{2}}{z} \right)^{-z} \theta^{2} \,\mathrm{d}\theta \\ &\gg \left(1 + \frac{c_{m}\delta^{2}}{z} \right)^{-z} \left(\frac{p}{z} \right)^{3/2} \\ &\gg_{m} \left(\frac{p}{z} \right)^{3/2} \end{split}$$

for $p \leq z$. On the other hand, it is obvious that

$$\left|\widetilde{D}_m(\theta, p^{-1})^{-1}\right| \leq 1$$
 and $\check{F}_m^{\ell, z}(p) \ll 1$

uniformly for $p \leq \sqrt{z}$. By combining these estimates, we find that

$$\log F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = \log\left(1-\frac{1}{p}\right)^{-(m+1)z+\ell(\ell-1)/2} + \log \check{F}_m^{\ell,z}(p)$$
$$= (m+1)z\log\left(1-\frac{1}{p}\right)^{-1} + O\left(\log z\right)$$

for $p \leq \sqrt{z}$.

Next we prove (48). In view of (44) and (9), it is easy to see that

(49)
$$D_m(\theta, p^{-1})^z = e^{(z/p)X_m(2\cos\theta)} \left\{ 1 + O\left(\frac{z}{p^2}\right) \right\} \qquad (p \ge \sqrt{z}),$$

where the implied constant depends on *m* at most. Thus for $p \ge \sqrt{z}$, we can write

$$F_m^{\ell,z}\left(0,0;\frac{1}{p}\right) = \left\{1 + O\left(\frac{z}{p^2}\right)\right\} \left(1 - \frac{1}{p}\right)^{\ell(\ell-1)/2} \widetilde{F}_m^{\ell,z}(p)$$

with

$$\widetilde{F}_m^{\ell,z}(p) \coloneqq \frac{2}{\pi} \int_0^{\pi} D_1\left(\theta, p^{-1/2}\right)^{\ell} \mathrm{e}^{(z/p)X_m(2\cos\theta)} \sin^2\theta \,\mathrm{d}\theta.$$

Since

(50)
$$D_1(\theta, p^{-1/2})^{\ell} = 1 + \frac{2\ell\cos\theta}{p^{1/2}} + \frac{2(\ell+1)\ell\cos^2\theta - \ell}{p} + O\left(\frac{1}{p^{3/2}}\right)$$

where the implied constant depends on ℓ at most, (45) and (46) of Lemma 4.3 allow us to deduce that

$$\widetilde{F}_{m}^{\ell,z}(p) = \left\{ 1 + \frac{\ell(\ell-1)}{2p} + O\left(\frac{z}{p^{3/2}}\right) \right\} \frac{2}{\pi} \int_{0}^{\pi} e^{(z/p)X_{m}(2\cos\theta)} \sin^{2}\theta \,\mathrm{d}\theta.$$

Inserting it into the preceding relation, we easily obtain (48).

Now we are ready to prove Proposition 4.1. From (47) and (48), we deduce that for $y \ge z^{3/2}$,

$$\sum_{p \leqslant y} \log F_m^{\ell, z} \left(0, 0; \frac{1}{p} \right) = (m+1) z \sum_{p \leqslant \sqrt{z}} \log \left(1 - \frac{1}{p} \right)^{-1} + \sum_{\sqrt{z}$$

In view of (2), (3) and the following estimate

$$\sum_{\sqrt{z}$$

the last asymptotic formula can be written as

(51)
$$\sum_{p \leqslant y} \log F_m^{\ell, z} \left(0, 0; \frac{1}{p} \right)$$
$$= (m+1)z \sum_{p \leqslant z} \log \left(1 - \frac{1}{p} \right)^{-1} + \sum_{\sqrt{z}$$

By the prime number theorem, it follows that

(52)
$$\sum_{\sqrt{z}$$

where

$$R_1 := \int_{\sqrt{z}}^{y} \widetilde{g}_m\left(\frac{z}{u}\right) \mathrm{d}O\left(u\mathrm{e}^{-2\sqrt{\log u}}\right).$$

In view of Lemma 4.2, a simple partial integration gives us

$$R_1 \ll z \mathrm{e}^{-\sqrt{\log z}}.$$

In order to evaluate the last integral of (52), we use the change of variables t = z/u to write

$$\int_{\sqrt{z}}^{y} \frac{\widetilde{g}_m(z/u)}{\log u} du = z \int_{z/y}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2 \log(z/t)} dt$$
$$= z \int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2 \log(z/t)} dt + O(R_2)$$

where

$$R_2 := z \int_{z/y}^{1/\sqrt{z}} \frac{\left| \tilde{g}_m(t) \right|}{t^2 \log(z/t)} \, \mathrm{d}t \ll \frac{z^{1/2}}{\log z}$$

by using Lemma 4.2. On the other hand, we have

$$\begin{split} \int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2 \log(z/t)} \, \mathrm{d}t &= \frac{1}{\log z} \int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2 (1 - (\log t)/\log z)} \, \mathrm{d}t \\ &= \sum_{j=1}^J \frac{1}{(\log z)^j} \int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2} (\log t)^{j-1} \, \mathrm{d}t + O_J \left(\frac{1}{(\log z)^{J+1}}\right). \end{split}$$

Extending the interval of integration $[1/\sqrt{z}, \sqrt{z}]$ to $(0, \infty)$ and bounding the contributions of $(0, 1/\sqrt{z}]$ and $[\sqrt{z}, \infty)$ by using Lemma 4.2, we have

$$\int_{1/\sqrt{z}}^{\sqrt{z}} \frac{\widetilde{g}_m(t)}{t^2} (\log t)^{j-1} dt = a_j + O\left(\frac{(\log z)^j}{\sqrt{z}}\right).$$

Combining these estimates, we find that

(53)
$$\sum_{\sqrt{z}$$

Now the desired result follows from (51), (53) and the prime number theorem in the form

$$\sum_{p \leq z} \log\left(1 - \frac{1}{p}\right)^{-1} = \log_2 z + \gamma + O\left(e^{-2\sqrt{\log z}}\right).$$

This completes the proof.

4.2. Behavior of the constant term. The aim of this section is to prove Theorem D. We shall prove a slightly more general result, i.e. Proposition 4.5. Clearly Theorem D is its simple consequence with the choice of $\ell = 2$.

Let $\ell \ge 0$ and $m \ge 1$ be two fixed integers. Define

$$B_m(w) = B_m(w, z, p) := \frac{2}{\pi} \int_0^{\pi} D_1 \left(\theta, p^{-(1/2+w)}\right)^{\ell} D_m \left(\theta, p^{-1}\right)^{z} \sin^2 \theta \, \mathrm{d}\theta$$

so that

$$F_m^{\ell,z}(w,0;p^{-1}) = (1-p^{-(1+2w)})^{\ell(\ell-1)/2} B_m(w).$$

Proposition 4.5–Let $\ell \ge 0$. We have

$$\sum_{p \leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0} \log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right) \ll \log z$$

uniformly for $y \ge z \ge 10$ if m is even; and

$$\sum_{p \leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0} \log F_m^{\ell,z} \left(w, 0; \frac{1}{p}\right) = \sqrt{z} \left\{ b_{\ell,m} + O\left(\mathrm{e}^{-\sqrt{\log z}}\right) \right\}$$

uniformly for $y \ge z e^{2\sqrt{\log z}} \ge 10$ if *m* is odd, where

$$b_{\ell,m} := -2\ell \left(2 + \int_0^{+\infty} \frac{\widetilde{h}_m(t)}{t^{3/2}} \,\mathrm{d}t \right).$$

The implied constant depends on ℓ and m only.

We need preliminary lemmas.

Lemma 4.6–Let $h_m(t)$ and $\tilde{h}_m(t)$ be defined as in (4) and (5). Then

(54)
$$\widetilde{h}_m(t) \ll \begin{cases} t & \text{if } 0 \leq t < 1, \\ t^{-1} & \text{if } t \geq 1, \end{cases} \qquad \widetilde{h}'_m(t) \ll \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ t^{-1} & \text{if } t \geq 1. \end{cases}$$

Further if m is even, then

$$(55) h_m(t) = 0 (t \ge 0).$$

Proof. Equation (55) follows from

$$h_m(t) = \int_{-\pi/2}^{\pi/2} e^{tX_m(2\cos\theta)} \cos\theta \sin^2\theta \,d\theta \qquad (m \text{ even})$$

by parity. The estimates of (54) with $0 \le t \le 1$ are equivalent to (45). Next we prove $\tilde{h}_m(t) \ll t^{-1}$ for $t \ge 1$, i.e.

(56)
$$\frac{\int_0^{\pi} e^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta \,\mathrm{d}\theta}{\int_0^{\pi} e^{tX_m(2\cos\theta)}\sin^2\theta \,\mathrm{d}\theta} \ll \frac{1}{t}.$$

From the power series expansion, we have

$$X_m(2\cos\theta) = (m+1) - \frac{m(m+1)(m+2)}{6}\theta^2 + O_m(\theta^4),$$

and hence there exists $\delta = \delta_m \in (0, \pi/(3(m+1)))$ such that for all $0 \leq \theta \leq \delta$,

(57)
$$(m+1) - \frac{(m+2)^3}{6}\theta^2 < X_m(2\cos\theta) < (m+1) - \frac{1}{6}\theta^2.$$

Since $\theta \mapsto X_m(2\cos\theta)$ is continuous on the compact $[\delta, 2]$ where its values are strictly less than m + 1, there exists $\alpha_m \in (0, m + 1)$ such that

(58)
$$|X_m(2\cos\theta)| \leq \alpha_m \quad (\delta \leq \theta \leq \pi/2).$$

We give a lower bound to the denominator of the fraction in (56). As the integrand is nonnegative, we infer from (57) that

(59)
$$\int_{0}^{\pi} e^{tX_{m}(2\cos\theta)} \sin^{2}\theta \, d\theta \ge \int_{0}^{0} e^{tX_{m}(2\cos\theta)} \sin^{2}\theta \, d\theta \\ \gg e^{(m+1)t} \int_{0}^{\delta} e^{-c_{m}t\theta^{2}}\theta^{2} \, d\theta \gg_{m} \frac{e^{(m+1)t}}{t^{3/2}}$$

where the implied constant in \gg_m depends on m only. For the numerator in the left-hand side of (56), we write

$$\int_0^{\pi} e^{tX_m(2\cos\theta)} (1-\cos\theta) \sin^2\theta \, d\theta = \int_0^{\pi/2} e^{tX_m(2\cos\theta)} (1-\cos\theta) \sin^2\theta \, d\theta$$
$$+ \int_0^{\pi/2} e^{-tX_m(2\cos\theta)} (1+\cos\theta) \sin^2\theta \, d\theta$$

Since $X_m(2\cos\theta) \ge 0$ for $\theta \in [0, \pi/(2(m+1))]$, we deduce with (58) that

$$\int_0^{\pi/2} \mathrm{e}^{-tX_m(2\cos\theta)} (1+\cos\theta) \sin^2\theta \,\mathrm{d}\theta \ll \int_0^{\pi/(2(m+1))} \mathrm{d}\theta + \int_{\pi/(2(m+1))}^{\pi/2} \mathrm{e}^{t\alpha_m} \,\mathrm{d}\theta$$
$$\ll \mathrm{e}^{\alpha_m t},$$

which is negligible in comparison with (59). Splitting at $\theta = \delta$ and applying (57) and (58), we have

$$\int_0^{\pi/2} \mathrm{e}^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta\,\mathrm{d}\theta \ll \mathrm{e}^{(m+1)t}\int_0^{\delta} \mathrm{e}^{-\frac{1}{6}t\theta^2}\theta^4\,\mathrm{d}\theta + \int_{\delta}^{\pi/2} \mathrm{e}^{\alpha_m t}\,\mathrm{d}\theta$$
$$\ll t^{-5/2}\mathrm{e}^{(m+1)t} + \mathrm{e}^{\alpha_m t}.$$

The desired estimate in (56) follows with (59) and the fact $\alpha_m < m + 1$.

A direct differentiation shows that

$$\widetilde{h}'_{m}(t) = \frac{\int_{0}^{\pi} e^{tX_{m}(2\cos\theta)} X_{m}(2\cos\theta) \sin^{2}\theta \,d\theta \int_{0}^{\pi} e^{tX_{m}(2\cos\theta)} (1-\cos\theta) \sin^{2}\theta \,d\theta}{\left(\int_{0}^{\pi} e^{tX_{m}(2\cos\theta)} \sin^{2}\theta \,d\theta\right)^{2}} - \frac{\int_{0}^{\pi} e^{tX_{m}(2\cos\theta)} X_{m}(2\cos\theta) (1-\cos\theta) \sin^{2}\theta \,d\theta}{\int_{0}^{\pi} e^{tX_{m}(2\cos\theta)} \sin^{2}\theta \,d\theta} \qquad (t \ge 1).$$

Using the nonnegativity, we see that

$$\int_0^{\pi} e^{tX_m(2\cos\theta)} X_m(2\cos\theta) \sin^2\theta \, \mathrm{d}\theta \ll \int_0^{\pi} e^{tX_m(2\cos\theta)} \sin^2\theta \, \mathrm{d}\theta,$$

and

$$\int_0^{\pi} e^{tX_m(2\cos\theta)} X_m(2\cos\theta)(1-\cos\theta)\sin^2\theta \,d\theta$$
$$\ll \int_0^{\pi} e^{tX_m(2\cos\theta)}(1-\cos\theta)\sin^2\theta \,d\theta.$$

Therefore (56) implies $\widetilde{h}'_m(t) \ll t^{-1}$ for $t \ge 1.$

Lemma 4.7–Let $\ell \ge 0$ and $m \ge 1$ be two fixed integers. Then we have

(60)
$$\frac{B'_m(0)}{B_m(0)} \ll \frac{\log p}{p^{1/2}}$$

uniformly for all p and $z \ge 1$; and

(61)
$$\frac{B'_m(0)}{B_m(0)} = -2\ell \frac{\log p}{p^{1/2}} h_m\left(\frac{z}{p}\right) - \ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right)$$

uniformly for $p \ge z^{2/3}$. The implied constants depend on ℓ and m only. Proof. We have

(62)
$$B'_{m}(0) = -2\ell \frac{2}{\pi} \int_{0}^{\pi} D_{1} \left(\theta, p^{-1/2}\right)^{\ell+1} \left(\frac{\cos\theta}{p^{1/2}} - \frac{1}{p}\right) (\log p) D_{m} \left(\theta, p^{-1}\right)^{z} \sin^{2}\theta \, \mathrm{d}\theta$$

hence

$$B'_m(0) \ll \frac{\log p}{p^{1/2}} \int_0^{\pi} D_m(\theta, p^{-1})^z \sin^2 \theta \, \mathrm{d}\theta.$$

This implies (60), since

$$B_m(0) = \left\{ 1 + O\left(\frac{1}{p^{1/2}}\right) \right\} \frac{2}{\pi} \int_0^{\pi} D_m \left(\theta, p^{-1}\right)^z \sin^2 \theta \, \mathrm{d}\theta.$$

In view of (50), it follows that

(63)
$$D_1(\theta, p^{-1/2})^{\ell+1}\left(\frac{\cos\theta}{p^{1/2}} - \frac{1}{p}\right) = \frac{\cos\theta}{p^{1/2}} + \frac{2(\ell+1)\cos^2\theta - 1}{p} + O\left(\frac{1}{p^{3/2}}\right).$$

By using it, (49) and (46) of Lemma 4.3, we can deduce, for $p \ge \sqrt{z}$,

$$B'_{m}(0) = -2\ell \frac{\log p}{p^{1/2}} \frac{2}{\pi} \int_{0}^{\pi} e^{(z/p)X_{m}(2\cos\theta)} \cos\theta \sin^{2}\theta \,d\theta$$
$$-\left\{\ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^{2}}\right)\right\} \frac{2}{\pi} \int_{0}^{\pi} e^{(z/p)X_{m}(2\cos\theta)} \sin^{2}\theta \,d\theta.$$

Under the same condition, thanks to (49) and (45), we have

$$B_m(0) = \left\{ 1 + O\left(\frac{1}{p} + \frac{z}{p^{3/2}}\right) \right\} \frac{2}{\pi} \int_0^{\pi} e^{(z/p)X_m(2\cos\theta)} \sin^2\theta \, \mathrm{d}\theta.$$

Combining these, we obtain (61).

Lemma 4.8– Let $\ell \ge 0$ and $m \equiv 0 \pmod{2}$ be two fixed integers. Then we have

(64)
$$\frac{B'_m(0)}{B_m(0)} \ll \frac{\log p}{p}$$

uniformly for all p and $z \ge 1$, and

(65)
$$\frac{B'_m(0)}{B_m(0)} = -\ell(\ell-1)\frac{\log p}{p} + O\left(\frac{\log p}{p^{3/2}} + \frac{z\log p}{p^2}\right)$$

uniformly for $p \ge z^{2/3}$. The implied constants depend on ℓ and m only.

Proof. Equation (64) follows from (62) and (63) since, by parity consideration we have a^{π}

$$\int_0^{\pi} (\cos\theta) D_m (\theta, p^{-1})^z \sin^2\theta \, \mathrm{d}\theta = 0.$$

Equation (65) is an immediate consequence of (61) since $h_m(t) = 0$ when *m* is even.

Now we are ready to prove Proposition 4.5. If m is even, we apply Lemma 4.8 to

$$\sum_{p \leq y} \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0} \log F_m^{\ell,z}\left(w,0;\frac{1}{p}\right) = \sum_{p \leq y} \ell(\ell-1) \frac{\log p}{p-1} + \sum_{p \leq y} \frac{B_m'(0)}{B_m(0)}$$

and obtain

$$\sum_{p \leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0} \log F_m^{\ell,z} \left(w, 0; \frac{1}{p}\right) = \sum_{p \leqslant z} \ell(\ell-1) \frac{\log p}{p-1} + \sum_{p \leqslant z} \frac{B'_m(0)}{B_m(0)} + \sum_{z$$

When *m* is odd, by using (60) of Lemma 4.7 for $p \le z^{2/3}$ and (61) for $z^{2/3} , we obtain$

$$\sum_{p \leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0} \log F_m^{\ell,z} \left(w, 0; \frac{1}{p}\right) = -2\ell \sum_{z^{2/3}$$

so that

(66)
$$\sum_{p \leqslant y} \frac{\mathrm{d}}{\mathrm{d}w}|_{w=0} \log F_m^{\ell,z} \left(w, 0; \frac{1}{p} \right) = -2\ell \left\{ \sum_{p \leqslant z} \frac{\log p}{p^{1/2}} + \sum_{z^{2/3}$$

By using the prime number theorem, it follows by integration by parts that

$$\sum_{z^{2/3}
$$= \sqrt{z} \int_0^{+\infty} \frac{\tilde{h}_m(t)}{t^{3/2}} \,\mathrm{d}t + O\left(\sqrt{z}\mathrm{e}^{-\sqrt{\log z}}\right)$$$$

with the help of Lemma 4.6, provided $y \ge z e^{2\sqrt{\log z}}$. Combining these yields

(67)
$$\sum_{z^{2/3}$$

Now the required result is a simple consequence of (66) and(67) and the prime number theorem.

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