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► **To cite this version:**

Catherine Donati-Martin, Mylène Maïda. Large deviations for the largest eigenvalue of an Hermitian Brownian motion. 2011. hal-00565889v1

HAL Id: hal-00565889

<https://hal.science/hal-00565889v1>

Preprint submitted on 14 Feb 2011 (v1), last revised 11 Nov 2012 (v2)

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Large deviations for the largest eigenvalue of an Hermitian Brownian motion

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February 14, 2011

Abstract

We establish a large deviation principle for the process of the largest eigenvalue of an Hermitian Brownian motion. By a contraction principle, we recover the LDP for the largest eigenvalue of a rank one deformation of the GUE.

Key words: Large deviations, Dyson Brownian motion, random matrices, stochastic calculus.

AMS 2010 Subject classification: 60F10, 15B52.

1 Introduction

From the fifties, the Gaussian unitary and orthogonal ensembles (GUE/GOE) have been extensively studied. We focus on large deviations and recall some known results. For the global regime, Ben Arous and Guionnet [BAG] established a large deviation principle (LDP) in the scale N^2 for the spectral measure of a Gaussian ensemble of size N . The LDP for the largest eigenvalue of the GOE with a scale N was obtained by Ben Arous, Dembo and Guionnet [BADG]. This result was extended to a rank one perturbation of the GUE/GOE by one of the author in [M]. In both proofs, the fact that the deviations of the spectral measure and those of the largest eigenvalue do not occur in the same scale plays a crucial role and so will be in the proof of our result : in the scale at which we look at the largest eigenvalue, the spectral measure of all but the largest eigenvalue is already well concentrated around the semicircle law.

In 1962, Dyson [D] introduced a dynamical version of the GUE, namely the Hermitian Brownian motion whose set of eigenvalues is a time-dependent Coulomb gas, consisting in particles evolving according to Brownian motions under the influence of their mutual electrostatic repulsions. More precisely, let $(\beta_{ij}, \beta'_{ij})_{1 \leq i < j \leq N}$ be a collection of independent identically distributed standard real

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This work was supported by *Agence Nationale de la Recherche* grants ANR-08-BLAN-0311-03 and ANR-09-BLAN-0084-01

Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; the Hermitian Brownian motion $(H_N)_{t \geq 0}$ is the random process, taking values in the space of $N \times N$ Hermitian matrices, with entries $(H_N)_{kl}$ given, for $k \leq l$ by

$$(H_N)_{kl} = \begin{cases} \frac{1}{\sqrt{2N}}(\beta_{kl} + i\beta'_{kl}), & \text{if } k < l, \\ \frac{1}{\sqrt{N}}\beta_{kk}, & \text{if } k = l. \end{cases}$$

We then set $H_N^\theta(t) = H_N(t) + H_N^\theta(0)$ the Hermitian Brownian motion starting from $H_N^\theta(0) := \text{diag}(\theta, 0, \dots, 0)$, with $\theta \geq 0$ and denote by $\lambda_1^{\theta, N}(t) \geq \lambda_2^{\theta, N}(t) \geq \dots \geq \lambda_N^{\theta, N}(t)$ the set of ordered eigenvalues of $H_N^\theta(t)$. Dyson ([D], see also [G, 12.1]) showed that the eigenvalues of $H_N^\theta(t)$ satisfy the following system of stochastic differential equations (SDE)

$$d\lambda_i(t) = \frac{1}{\sqrt{N}}d\beta_i(t) + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt, \quad t \geq 0, \quad i = 1, \dots, N \quad (1.1)$$

where β_i are independent standard real Brownian motions.

It was rigorously shown in [CL] that this system of SDE admits a unique strong solution and the eigenvalues do not collide.

The process of the eigenvalues is called Dyson Brownian motion. For any $t \geq 0$, the corresponding spectral measure $(\mu_N)_t := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^{\theta, N}(t)}$ converges a.s. to the semicircular distribution σ_t given by

$$d\sigma_t(x) = \frac{1}{2\pi t} \mathbf{1}_{[-2\sqrt{t}, 2\sqrt{t}]} \sqrt{4t - x^2} dx \quad \text{and} \quad \sigma_0 = \delta_0. \quad (1.2)$$

Moreover, the process μ_N satisfies a LDP in the scale N^2 (see [CDG], [G]) in the space $C([0, 1]; \mathcal{P}(\mathbb{R}))$, the set of continuous functions on $[0, 1]$ with values in the set $\mathcal{P}(\mathbb{R})$ of probability measures on \mathbb{R} , equipped with the metric $d(\mu, \nu) = \sup_{t \in [0, 1]} d_{Lip}(\mu_t, \nu_t)$ where

$$d_{Lip}(\mu_t, \nu_t) = \sup_{f \in \mathcal{F}_{Lip}} \left| \int f d\mu_t - \int f d\nu_t \right|$$

where \mathcal{F}_{Lip} denotes the space of bounded Lipschitz functions on \mathbb{R} with Lipschitz and uniform bound less than 1. For any $\mu \in C([0, 1]; \mathcal{P}(\mathbb{R}))$ and $\alpha > 0$, $\mathbb{B}(\mu, \alpha)$ will denote the ball centered at μ with radius α with respect to the metric d .

In this work, we will be interested in showing an LDP for the process of the maximal eigenvalue $(\lambda_1^{\theta, N}(t))_{t \geq 0}$, that is the largest particle of Dyson Brownian motion.

More precisely, for $T > 0$, we consider the processes $(\lambda_1^{\theta, N}(t))_{0 \leq t \leq T}$ as a sequence of random variables with values in the space $C_\theta([0, T], \mathbb{R})$ of continuous functions for $[0, T]$ to \mathbb{R} equal to θ at zero and investigate its LDP in this space endowed with the uniform convergence. For sake of simplicity, we fix $T = 1$ in the sequel.

Our main result is the following:

Theorem 1.1 *Let $(\lambda_1^{\theta, N}(t))_{0 \leq t \leq 1}$ be the process of the largest eigenvalue of an Hermitian Brownian motion $(H_N^\theta(t))_{0 \leq t \leq 1}$. Then the law of $(\lambda_1^{\theta, N}(t))_{0 \leq t \leq 1}$ satisfies a large deviation principle on $C_\theta([0, 1]; \mathbb{R})$ equipped with the topology of uniform convergence, in the scale N , with good rate function*

$$I_\theta(\varphi) = \begin{cases} \frac{1}{2} \int_0^1 \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s} \right) \right)^2 ds, \\ \text{if } \varphi \text{ absolutely continuous and } \varphi(t) \geq 2\sqrt{t} \quad \forall t \in [0, 1], \\ +\infty, \text{ otherwise.} \end{cases} \quad (1.3)$$

Remark 1.2 For the sake of simplicity the theorem above is stated and proven in the paper for the Hermitian Brownian motion but we want to mention that it can be easily extended to the symmetric Brownian motion. With the notations already introduced above, the latter is defined as the random process taking values in the space of $N \times N$ real symmetric matrices so that

$$(S_N)_{kl} = \frac{1}{\sqrt{N}}\beta_{kl}, \text{ if } k \leq l.$$

The process of its eigenvalues satisfies the following system of SDE

$$d\lambda_i(t) = \frac{\sqrt{2}}{\sqrt{N}}d\beta_i(t) + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt, \quad t \geq 0, \quad i = 1, \dots, N$$

and its law satisfies a LDP with good rate function simply given by $\frac{1}{2}I_\theta$.

The proof that will be developed in the sequel can be adapted to the symmetric case with a few minor changes left to the reader.

An easy consequence of Theorem 1.1 is the a.s. convergence of our process, in the topology of uniform convergence for continuous functions on $[0, T]$, towards the function $(f_\theta(t))_{t \geq 0}$ given by:

$$\begin{cases} f_\theta(t) = 2\sqrt{t} & \text{if } \theta = 0, \\ f_\theta(t) = \begin{cases} \theta + \frac{t}{\theta} & t \leq \theta^2 \\ 2\sqrt{t} & t \geq \theta^2 \end{cases} & \text{if } \theta > 0. \end{cases}$$

The case $\theta = 0$ extends the a.s. convergence of the maximal eigenvalue of the GUE of variance a^2/N to $2a$ (see for example [BY]) and in the case $\theta > 0$ a similar result holds for a rank one additive deformations of the GUE (see for example [P]).

In the sequel, we will specify the superscript θ, N in the statements but drop it in the proofs, unless there is any ambiguity. In the static case, the LDP was shown using the explicit expression of the distribution of the N eigenvalues. In the dynamical case, the proof relies on stochastic calculus using that the process of the eigenvalues satisfies the system of SDE (1.1). Roughly speaking, the largest eigenvalue is a solution of a SDE of the form

$$d\lambda_1(t) = \frac{1}{\sqrt{N}}d\beta(t) + b(\lambda_1(t), (\nu_N)_t)dt,$$

with β a standard real Brownian motion, $\nu_N := \frac{1}{N-1} \sum_{i=2}^N \delta_{\lambda_i(t)}$ the empirical distribution of all but the largest eigenvalue and the drift $b(x, \nu)$ that will be explicitated in the sequel being not a continuous function of ν for the weak convergence of probabilities. In the scale of interest, ν_N is close to σ and the rate function I_θ is the one predicted by the Freidlin-Wentzell Theorem (see [DZ, Th.]) for the SDE

$$d\lambda_1(t) = \frac{1}{\sqrt{N}}d\beta(t) + b(\lambda_1(t), \sigma_t)dt,$$

but the main difficulty will be to deal with the singularity of the drift of the SDE.

To prove our main result, we first establish the exponential tightness of the process $(\lambda_1(t))_{0 \leq t \leq 1}$ stated in Proposition 2.1 and proved in Section 2. A short Section 3 will be devoted to the study of I_θ , where we check in particular its lower semicontinuity. Section 4 is devoted to the proof of the lower bound, stated in Proposition 4.1. The upper bound is given in (5.1) and obtained along Section 5. Then Theorem 1.1 will follow from the exponential tightness, the lower bound obtained in Proposition 4.1 and the weak upper bound (5.1) (see [DZ, Chapt. 4] or [AGZ, Cor. D.6 and Th. D.4]).

Finally, in Section 6, we recover by contraction the fixed-time large deviation principles already known and cited at the beginning of this introduction, namely

Theorem 1.3 *The largest eigenvalue of $H_N^\theta(1)$ satisfies a large deviation principle in the scale N , with good rate function K_θ defined as follows:*

- If $\theta \leq 1$,

$$K_\theta(x) = \begin{cases} +\infty, & \text{if } x < 2 \\ \int_2^x \sqrt{z^2 - 4} dz, & \text{if } 2 \leq x \leq \theta + \frac{1}{\theta}, \\ M_\theta(x), & \text{if } x \geq \theta + \frac{1}{\theta}, \end{cases}$$

$$\text{with } M_\theta(x) = \frac{1}{2} \int_2^x \sqrt{z^2 - 4} dz - \theta x + \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{2}\theta^2 + \log \theta.$$

- If $\theta \geq 1$,

$$K_\theta(x) = \begin{cases} +\infty, & \text{if } x < 2 \\ L_\theta(x), & \text{if } x \geq 2, \end{cases}$$

$$\text{with } L_\theta(x) = \frac{1}{2} \int_{\theta + \frac{1}{\theta}}^x \sqrt{z^2 - 4} dz - \theta \left(x - \left(\theta + \frac{1}{\theta} \right) \right) + \frac{1}{4} \left(x^2 - \left(\theta + \frac{1}{\theta} \right)^2 \right).$$

NB : this is a corrected version of Theorem 1.1 in [M], the proof there is correct.

2 Exponential tightness

We want to show the exponential tightness of the process $(\lambda_1(t))_{0 \leq t \leq 1}$ in scale N that is

Proposition 2.1 *For all L , there exists N_0 and a compact set K_L of $C_\theta([0, 1]; \mathbb{R})$ such that:*

$$\forall N \geq N_0, \mathbb{P}(\lambda_1^{\theta, N} \notin K_L) \leq \exp(-LN).$$

From the description of the compact sets of $C([0, 1]; \mathbb{R})$ (Ascoli theorem), it is enough to show (see [RY], [CDG]) the following lemma

Lemma 2.2 *There exists a universal constant $C > 0$ such that for any $\eta > 0$ and there exists δ_0 such that for any $\delta < \delta_0$, for all N , $p \leq N$ and $s \in [0, 1]$,*

$$\mathbb{P} \left(\sup_{s \leq t \leq s + \delta} |\lambda_p(t) - \lambda_p(s)| \geq \eta \right) \leq \exp \left(-CN \frac{\eta^2}{\delta} \right).$$

To get the proposition, for a fixed L , we choose $p = 1$, any η and then δ small enough so that $C \frac{\eta^2}{\delta} > L$.

Proof of lemma 2.2: Let $0 \leq s \leq 1$.

Let us denote by \tilde{H}_N the Hermitian Brownian motion defined, for $u \geq 0$, by $\tilde{H}_N(u) = H_N^\theta(u + s) - H_N^\theta(s)$ and by $(\tilde{\lambda}_i(u))_{u \geq 0}$ its associated eigenvalues, in decreasing order. From a classical relation between eigenvalues (due to Weyl), for $t \geq s$,

$$\lambda_p^\theta(s) + \tilde{\lambda}_N(t - s) \leq \lambda_p^\theta(t) \leq \lambda_p^\theta(s) + \tilde{\lambda}_1(t - s)$$

so that

$$|\lambda_p^\theta(t) - \lambda_p^\theta(s)| \leq \max\left(\tilde{\lambda}_1(t-s), -\tilde{\lambda}_N(t-s)\right) = \|\tilde{H}_N(t-s)\|$$

where $\|\cdot\|$ denotes the operator norm on matrices. For any $\eta > 0$

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t \leq s+\delta} |\lambda_p^\theta(t) - \lambda_p^\theta(s)| \geq \eta\right) &\leq \mathbb{P}\left(\sup_{s \leq t \leq s+\delta} \|\tilde{H}_N(t-s)\| \geq \eta\right) \\ &= \mathbb{P}\left(\sup_{0 \leq u \leq \delta} \|\tilde{H}_N(u)\| \geq \eta\right) \\ &= \mathbb{P}\left(\sqrt{\delta} \sup_{0 \leq u \leq 1} \|H_N^0(u)\| \geq \eta\right), \end{aligned}$$

where we used that \tilde{H}_N has the same law as H_N^0 and the scaling invariance of this law. Now, there exists $C > 0$ such that for M large enough and for all N ,

$$\mathbb{P}\left(\sup_{0 \leq u \leq 1} \|H_N^0(u)\| \geq M\right) \leq \exp(-CNM^2) \quad (2.1)$$

Indeed, from [BADG], Lemma 6.3, we have: for M large enough, for all N ,

$$\mathbb{P}(\|H_N^0(1)\| \geq M) \leq \exp(-NM^2/9); \quad (2.2)$$

Now, $(\|H_N^0(u)\|)_{0 \leq u \leq 1}$ is a positive submartingale and from Doob's inequalities, all the moments of $\sup_{0 \leq u \leq 1} \|H_N^0(u)\|$ are controlled by those of $\|H_N^0(1)\|$ (up to a constant 4). Therefore, the Gaussian concentration inequality (2.2) implies the same type of concentration for $\sup_{0 \leq u \leq 1} \|H_N^0(u)\|$, that is (2.1). \square

3 Some insight on the expected rate function

Before going into the proof of the lower bound, we gather hereafter some useful remarks about the function I_θ defined in Theorem 1.1. In particular, we show in this section that it is lower semi-continuous.

We introduce the following notations : for μ a probability measure on \mathbb{R} and $x \in \mathbb{R}$, we define

$$b(x, \mu) = \int_{-\infty}^x \frac{d\mu(y)}{x-y} \in \mathbb{R}_+ \cup \{\infty\}. \quad (3.1)$$

For $\mu \in \mathcal{P}(\mathbb{R})$, we denote by $r(\mu)$ the right end-point of the support of μ . Let $(\varphi, \mu) \in C_\theta([0, 1]; \mathbb{R}) \times C([0, 1]; \mathcal{P}(\mathbb{R}))$ such that for all $t \in [0, 1]$, $\varphi(t) > r(\mu_t)$. Then, $b(\varphi(t), \mu_t)$ is bounded. We set

$$\mathcal{H} := \{h \in C([0, 1], \mathbb{R})/h \text{ absolutely continuous, } \dot{h} \in \mathbb{L}^2([0, 1])\}$$

with $\mathbb{L}^2([0, 1])$ the set of square-integrable functions from $[0, 1]$ to \mathbb{R} equipped with its usual \mathbb{L}^2 -norm, denoted by $\|\cdot\|_2$. For any $h \in \mathcal{H}$ we define

$$G(\varphi, \mu; h) = h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \varphi(s)\dot{h}(s)ds - \int_0^1 b(\varphi(s), \mu_s)h(s)ds \quad (3.2)$$

$$F(\varphi, \mu; h) := G(\varphi, \mu; h) - \frac{1}{2} \int_0^1 h^2(s)ds \quad (3.3)$$

For $\sigma := (\sigma_t)_{t \geq 0}$ the semicircular process defined in (1.2), the condition $\varphi(t) > r(\mu_t)$ reads $\varphi(t) > 2\sqrt{t}$ and one can check that $F(\varphi, \sigma; h)$ is also well defined under the weaker assumption that $\varphi(t) \geq 2\sqrt{t}$ for all $t \in [0, 1]$. It is indeed well known (see for example [HP, p. 94]) that

$$b(\varphi(t), \sigma_t) = \frac{1}{2t}(\varphi(t) - \sqrt{\varphi^2(t) - 4t}), \quad (3.4)$$

so that $0 \leq b(\varphi(t), \sigma_t) \leq \frac{1}{\sqrt{t}}$ for $\varphi(t) \geq 2\sqrt{t}$.

We now study the properties of F .

Lemma 3.1 *Let $\theta \geq 0$ and $\varphi \in C_\theta([0, 1], \mathbb{R})$ such that for any $t \in [0, 1]$, $\varphi(t) \geq 2\sqrt{t}$ and define*

$$J(\varphi) := \sup_{h \in \mathcal{H}} F(\varphi, \sigma; h). \quad (3.5)$$

Then,

$$J(\varphi) < \infty \Rightarrow \varphi \text{ absolutely continuous and } J(\varphi) = \frac{1}{2} \int_0^1 (\dot{\varphi}(s) - b(\varphi(s), \sigma_s))^2 ds = I_\theta(\varphi)$$

Sketch of proof: The optimization problem for J is a classical one. We just give an outline of the proof and refer for example to [DRYZ] for details on similar computations. Recall that $F(\varphi, \sigma; h) = G(\varphi, \sigma; h) - \frac{1}{2} \int_0^1 h^2(s) ds$ where $h \mapsto G(\varphi, \sigma; h)$ is a linear functional. Replacing h by λh , $\lambda \in \mathbb{R}$ and optimizing in λ yields

$$J(\varphi) = \frac{1}{2} \sup_{h \in \mathcal{H}} \frac{G^2(\varphi, \sigma; h)}{\|h\|_2^2}$$

If $J(\varphi) < \infty$, then the linear functional $G(\varphi, \sigma, \cdot)$ can be extended continuously to $\mathbb{L}^2([0, 1])$ and by Riesz theorem, there exists $k_\varphi \in \mathbb{L}^2([0, 1])$ such that $G(\varphi, \sigma, h) = \int_0^1 h(s) k_\varphi(s) ds$. On the other side, we know that $G(\varphi, \sigma, h)$ can be expressed as in (3.2). This implies that φ is absolutely continuous and $k_\varphi(s) = \dot{\varphi}(s) - b(\varphi(s), \sigma_s)$.

From Cauchy-Schwarz inequality and \mathcal{H} being dense in $\mathbb{L}^2([0, 1])$, $J(\varphi) = \frac{1}{2} \|k_\varphi\|_2^2$. Then the second equality follows from the computation of the Hilbert transform of the semicircular distribution recalled in (3.4) and $\varphi(t) \geq 2\sqrt{t}$. \square

We can now show the following :

Proposition 3.2 *The function $I_\theta : C_\theta([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ is lower semicontinuous.*

Proof: From Lemma 3.1, $I_\theta(\varphi) = \sup_{h \in \mathcal{H}} F(\varphi, \sigma; h)$ where

$$F(\varphi, \sigma; h) = h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \varphi(s) \dot{h}(s) ds - \int_0^1 b(\varphi(s), \sigma_s) h(s) ds - \frac{1}{2} \int_0^1 h^2(s) ds.$$

We shall prove that for fixed $h \in \mathcal{H}$, $\varphi \mapsto F(\varphi, \sigma; h)$ is continuous. From the definition of F , performing an integration by part in the term of the integral in b , it is enough to prove the continuity of $\varphi \mapsto \Lambda(\varphi) := \int_0^1 b(\varphi(s), \sigma_s) ds$ (the other terms are obviously continuous in φ). As we know that $0 \leq b(\varphi(t), \sigma_t) \leq \frac{1}{\sqrt{t}}$ for $\varphi(t) \geq 2\sqrt{t}$, by dominated convergence, if φ_n converges towards φ , $\Lambda(\varphi_n)$ converges to $\Lambda(\varphi)$ pointwise. Now, since the functions involved are increasing, the convergence holds uniformly on the compact $[0, 1]$. \square

4 The lower bound

In this section, we prove the lower bound. For any $\varphi \in C_\theta([0, 1]; \mathbb{R})$, any $\delta > 0$, $B(\varphi, \delta)$ will denote the ball centered at φ with radius δ with respect to the uniform metric, that is the subset of $C([0, 1]; \mathbb{R})$ of functions ψ such that $\sup_{t \in [0, 1]} |\psi(t) - \varphi(t)| < \delta$. We want to show

Proposition 4.1 *Let $\theta \geq 0$ be fixed. For any $\varphi \in C_\theta([0, 1]; \mathbb{R})$,*

$$\lim_{\delta \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P} \left(\lambda_1^{\theta, N} \in B(\varphi, \delta) \right) \geq -I_\theta(\varphi).$$

The bound is trivial if $I_\theta(\varphi) = \infty$ so that it is enough to consider φ such that $I_\theta(\varphi) < \infty$. In fact, following the classical way, we shall prove the above inequality for φ belonging to a well chosen subclass \mathcal{H}_θ dense among the functions of finite entropy. To introduce this subclass, we need a few more notations.

For φ such that $I_\theta(\varphi) < \infty$, we recall from Section 3 that

$$k_\varphi(s) := \dot{\varphi}(s) - b(\varphi(s), \sigma_s) = \dot{\varphi}(s) - \frac{1}{2s}(\varphi(s) - \sqrt{\varphi^2(s) - 4s}). \quad (4.1)$$

and that

$$I_\theta(\varphi) = \frac{1}{2} \int_0^1 k_\varphi^2(s) ds = \frac{1}{2} \|k_\varphi\|_2^2.$$

We define

$$\begin{aligned} \mathcal{H}_\theta &= \{ \varphi \in C_\theta([0, 1]; \mathbb{R}); \varphi(t) > 2\sqrt{t} \ \forall t \in [0, 1]; k_\varphi \text{ smooth} \} \quad \text{for } \theta > 0 \\ \mathcal{H}_0 &= \left\{ \varphi \in C_0([0, 1]; \mathbb{R}); \exists t_0 > 0, \begin{array}{l} \varphi(t) = 2\sqrt{t} \text{ for } t \leq t_0 \\ \varphi(t) > 2\sqrt{t} \text{ for } t > t_0 \end{array} ; k_\varphi \text{ smooth} \right\}, \end{aligned} \quad (4.2)$$

where smooth obviously means infinitely differentiable on $[0, 1]$. For $\varphi \in \mathcal{H}_0$, we denote by $t_0(\varphi) := \sup\{t; \varphi(t) = 2\sqrt{t}\}$ the corresponding threshold.

In Lemma 4.4 will be proven that \mathcal{H}_θ is dense in $\{\varphi \in C_\theta([0, 1]); I_\theta(\varphi) < \infty\}$ after some preliminary considerations in the next subsection.

4.1 Some properties of the functions with finite entropy when $\theta = 0$

As will be seen further, the proof of that \mathcal{H}_θ is dense will be quite straightforward in the case when $\theta > 0$ but more delicate when $\theta = 0$. In this latter case, we first need to understand some features of the functions with finite entropy that we gather here.

We need the following notations : for any φ such that $\varphi(s) \geq 2\sqrt{s}$, $\forall s \in [0, 1]$, we define x_φ by

$$x_\varphi(s) = \frac{\varphi(s) + \sqrt{\varphi^2(s) - 4s}}{2}, \quad \forall s \in [0, 1]. \quad (4.3)$$

so that φ and k_φ can be reexpressed in terms of x_φ . More precisely, $\forall s \in (0, 1]$,

$$\varphi(s) = x_\varphi(s) + \frac{s}{x_\varphi(s)} \quad (4.4)$$

and

$$k_\varphi(s) = 2\dot{x}_\varphi(s) \left(1 - \frac{s}{x_\varphi^2(s)} \right). \quad (4.5)$$

The following lemma gives the behaviour of φ near 0.

Lemma 4.2 ($\theta = 0$) *Let φ satisfy $I_0(\varphi) < \infty$. Then,*

$$\lim_{t \rightarrow 0} \frac{\varphi(t)}{\sqrt{t}} = 2.$$

Proof: Set

$$I^t(\varphi) = \int_0^t \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi^2(s) - 4s} \right) \right)^2 ds.$$

Then, from the finiteness of $I_0(\varphi)$, $\lim_{t \rightarrow 0} I^t(\varphi) = 0$. From Cauchy-Schwarz inequality,

$$\left| \int_0^t \dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi^2(s) - 4s} \right) ds \right| \leq \sqrt{t} (I^t(\varphi))^{1/2}$$

and

$$\left| \frac{\varphi(t)}{\sqrt{t}} - \frac{1}{\sqrt{t}} \int_0^t \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi^2(s) - 4s} \right) ds \right| \leq (I^t(\varphi))^{1/2}.$$

Now, we have, using (4.4),

$$0 \leq \int_0^t \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi^2(s) - 4s} \right) ds = \int_0^t \frac{ds}{x_\varphi(s)} \leq \int_0^t \frac{ds}{\sqrt{s}} = 2\sqrt{t}.$$

Thus, on one hand, $\frac{\varphi(t)}{\sqrt{t}} \geq 2$, whereas $0 \leq \frac{1}{\sqrt{t}} \int_0^t \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi^2(s) - 4s} \right) ds \leq 2$ and the difference of the two terms tends to 0 as t tends to 0. It follows that:

$$\lim_{t \rightarrow 0} \frac{\varphi(t)}{\sqrt{t}} = \lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} \int_0^t \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi^2(s) - 4s} \right) ds = 2.$$

□

The following lemma will be useful in the proof of the lower bound itself.

Lemma 4.3 ($\theta = 0$) *Let $\varphi \in \mathcal{H}_0$. Then k_φ is positive in a right neighborhood of $t_0(\varphi)$.*

Proof of Lemma 4.3: For $\varphi \in \mathcal{H}_0$, $k_\varphi \equiv 0$ on $[0, t_0(\varphi)]$.

Since $\varphi(s) > 2\sqrt{s}$ for $s > t_0(\varphi)$, we have that $x_\varphi(s) > \sqrt{s}$, for $s > t_0(\varphi)$. As $\dot{x}_\varphi(t_0(\varphi)) = \frac{1}{2\sqrt{t_0(\varphi)}} > 0$ and \dot{x}_φ is continuous (as φ is smooth), $\dot{x}_\varphi(s) > 0$ in a neighborhood of $t_0(\varphi)$ and thus, from (4.5), $k_\varphi(s) > 0$ for $t_0(\varphi) < s < t_0(\varphi) + \varepsilon$ for some $\varepsilon > 0$. □

4.2 Denseness of \mathcal{H}_θ

The goal of this subsection is to establish the following lemma

Lemma 4.4 *Let $\varphi \in C_\theta([0, 1])$ satisfying $I_\theta(\varphi) < \infty$. There exists a sequence $(\varphi_p)_{p \in \mathbb{N}^*}$ of functions in \mathcal{H}_θ such that, as p goes to infinity,*

- φ_p converges to φ in $C_\theta([0, 1], \mathbb{R})$
- $I_\theta(\varphi_p)$ converges to $I_\theta(\varphi)$.

4.2.1 Proof of Lemma 4.4 when $\theta > 0$

Let φ such that $I_\theta(\varphi) < \infty$. As $\varphi(0) = \theta > 0$ and φ is continuous, there exists $t_1 > 0$ such that for any $t \in [0, t_1]$, $\varphi(t) > 2\sqrt{t}$. For any $p \in \mathbb{N}^*$, we define

$$\chi_p(t) = \begin{cases} \varphi(t), & \text{if } t \leq t_1, \\ \varphi(t) + (t - t_1) & \text{if } t_1 \leq t \leq t_1 + \frac{1}{p}, \\ \varphi(t) + \frac{1}{p}, & \text{if } t \geq t_1 + \frac{1}{p}. \end{cases}$$

It is easy to check that χ_p is continuous and for p large enough, for any $t \in [0, 1]$, $\chi_p(t) > 2\sqrt{t}$ and $I_\theta(\chi_p) < \infty$. Moreover, as p goes to infinity, χ_p converges to φ in the uniform norm and k_{χ_p} converges to k_φ in $\mathbb{L}^2([0, 1])$.

It is now enough to check that χ_p can be approximated by a sequence of functions in \mathcal{H}_θ . As we know that for any $t \in [0, 1]$, $\chi_p(t) > 2\sqrt{t}$, we have that $\inf_{s \in (0, 1]} \left(1 - \frac{s}{x_{\chi_p}^2(s)}\right) > 0$. As $k_{\chi_p} \in \mathbb{L}^2([0, 1])$, from (4.5) we get that so does \dot{x}_{χ_p} . It can be approximated by a sequence of smooth functions $\dot{x}_{p,q}$. Set $x_{p,q}(t) = \theta + \int_0^t \dot{x}_{p,q}(s) ds$. The corresponding $\chi_{p,q}$ is defined by

$$\chi_{p,q}(s) = x_{p,q}(s) + \frac{s}{x_{p,q}(s)}$$

and

$$k_{p,q}(s) = 2\dot{x}_{p,q}(s) \left(1 - \frac{s}{x_{p,q}^2(s)}\right),$$

so that $k_{p,q}$ is smooth. For q large enough, for any $s \in [0, 1]$, $x_{p,q}(s) > \sqrt{s}$, so that $\chi_{p,q}(s) > 2\sqrt{s}$. Moreover, as q grows to infinity, the sequence $x_{p,q}$ converges towards x_{χ_p} in uniform norm on $[0, 1]$ so that $\chi_{p,q}$ converges towards χ_p in the same sense and $k_{p,q}$ converges to k_{χ_p} in $\mathbb{L}^2([0, 1])$.

To conclude the proof of the lemma, it is enough to notice that one can find an increasing function ψ from \mathbb{N} to \mathbb{N} such that $\varphi_p := \chi_{p, \psi(p)} \in \mathcal{H}_\theta$ converges towards φ and $k_{\varphi_p} = k_{p, \psi(p)}$ converges to k_φ in $\mathbb{L}^2([0, 1])$.

4.2.2 Proof of Lemma 4.4 when $\theta = 0$

As in the latter paragraph, we establish the proof in two steps: first, we approximate φ by a sequence of functions that are equal to $2\sqrt{t}$ in a neighborhood of 0 and strictly greater than $2\sqrt{t}$ away from 0. Next, we approximate those functions by smooth ones.

Let $r > 0$ and define χ_r by:

$$\chi_r(t) = \begin{cases} 2\sqrt{t}, & t \leq y_r^2 \\ y_r + \frac{t}{y_r}, & y_r^2 \leq t \leq r \\ \varphi(t) + r(t - r), & t \geq r \end{cases}$$

with $y_r = \frac{\varphi(r) - \sqrt{\varphi^2(r) - 4r}}{2} \leq \sqrt{r}$ so that χ_r is continuous.

$$\|\varphi - \chi_r\| \leq \sup_{s \leq r} |\varphi(s) - \chi_r(s)| \vee r \leq 2 \sup_{s \leq r} |\varphi(s) - 2\sqrt{s}| \vee r \xrightarrow[r \rightarrow 0]{} 0$$

using Lemma 4.2. It remains to show that $I_0(\varphi) - I_0(\chi_r)$ tends to 0. If we set $J_r(f) = \int_0^r k_f^2(s)ds$ and $J'_r(f) = \int_r^1 k_f^2(s)ds$, we get

$$I_0(\varphi) - I_0(\chi_r) = (J_r(\varphi) - J_r(\chi_r)) + (J'_r(\varphi) - J'_r(\chi_r))$$

with $J_r(\varphi) \rightarrow 0$ as $r \rightarrow 0$.

$$J_r(\chi_r) = \int_{y_r^2}^r \left(\frac{1}{y_r} - \frac{y_r}{s} \right)^2 ds = \frac{r}{y_r^2} - \frac{y_r^2}{r} - 2 \ln \left(\frac{r}{y_r^2} \right) \xrightarrow{r \rightarrow 0} 0$$

since y_r/\sqrt{r} tends to 1 thanks to Lemma 4.2.

On the other hand, if we define $h_{\varphi,r}$ by

$$h_{\varphi,r}(t) := r - \frac{1}{2t} \left(r(t-r) + \sqrt{\varphi^2(t) - 4t} - \sqrt{(\varphi(t) + r(t-r))^2 - 4t} \right)$$

then, by Cauchy-Schwarz inequality, we have

$$|J'_r(\varphi) - J'_r(\chi_r)|^2 \leq \int_r^1 (2k_\varphi(s) + h_{\varphi,r}(t))^2 dt \int_r^1 (h_{\varphi,r}(t))^2 dt$$

Therefore it is enough to show that $\int_r^1 (h_{\varphi,r}(t))^2 dt$ goes to zero as r goes to zero. To show that, we notice that, for $t \in [r, 1]$,

$$\left| \frac{r(t-r)}{2t} \right| \leq \frac{r}{2}. \quad (4.6)$$

Moreover,

$$\begin{aligned} \left| \frac{1}{2t} \left(\sqrt{\varphi^2(t) - 4t} - \sqrt{(\varphi(t) + r(t-r))^2 - 4t} \right) \right| &= \frac{1}{2t} \frac{2r\varphi(t)(t-r) + r^2(t-r)^2}{\sqrt{\varphi^2(t) - 4t} + \sqrt{(\varphi(t) + r(t-r))^2 - 4t}} \\ &\leq \frac{\sqrt{2r\varphi(t)(t-r)}}{2t} + \frac{|r(t-r)|}{2t} \\ &\leq Cr^{1/4} + \frac{r}{2}, \end{aligned} \quad (4.7)$$

where we used that, from Lemma 4.2, $t \mapsto \frac{\varphi(t)}{\sqrt{t}}$ is bounded on $[0, 1]$ by a constant C . Putting (4.6) and (4.7) together, we get $\int_r^1 (h_{\varphi,r}(t))^2 dt$ goes to zero as r goes to zero.

Now $\dot{\chi}_r(s) = \frac{1}{\sqrt{s}}$ on $[0, y_r^2]$ and $\dot{\chi}_r \in \mathbb{L}^2([y_r^2, 1])$ since $k_{\chi_r} \in \mathbb{L}^2([y_r^2, 1])$. For any $r > 0$, there exists a sequence of function $\dot{\chi}_{r,q}$ smooth on $]0, 1]$ such that $\dot{\chi}_{r,q}(s) = \frac{1}{\sqrt{s}}$ on $[0, y_r^2]$ and $\dot{\chi}_{r,q}$ tends to $\dot{\chi}_r$ in $\mathbb{L}^2([y_r^2/2, 1])$. Setting $\chi_{r,q}(t) = \int_0^t \dot{\chi}_{r,q}(s)ds$, we have:

- $\chi_{r,q}$ tends to χ_r in uniform norm.
- $k_{\chi_{r,q}}$ is smooth.
- $k_{\chi_{r,q}}$ converges to k_{χ_r} in $\mathbb{L}^2([0, 1])$ so that $I_0(\chi_{r,q})$ converges to $I_0(\chi_r)$

Putting everything together, we conclude that there exists an increasing function ψ such that the sequence of functions $\varphi_p = \chi_{p,\psi(p)}$ satisfies the requirements of Lemma 4.4. \square

4.3 Almost sure convergence of $\lambda_1^{\theta, N}$ and ν_N under \mathbb{P}^{k_φ}

The strategy of the proof of the lower bound will be classical : we will make a proper change of measure so that the function φ becomes “typical” under the new measure. We will therefore need to study more precisely the behavior of $\lambda_1^{\theta, N}$ under the new measure \mathbb{P}^{k_φ} .

To be more precise, for $h \in \mathbb{L}^2([0, 1])$, we define the exponential martingale M^h such that for any $t \in [0, 1]$,

$$M_t^h = \exp \left[N \left(\int_0^t h(s) \frac{1}{\sqrt{N}} d\beta_1(s) - \frac{1}{2} \int_0^t h^2(s) ds \right) \right], \quad (4.8)$$

where β_1 is the standard Brownian motion appearing in the SDE for $\lambda_1^{\theta, N}$ (see (1.1)). We denote by $(\mathcal{F}_t)_{t \geq 0}$ its canonical filtration.

We now introduce \mathbb{P}^{k_φ} the probability defined by $\mathbb{P}^{k_\varphi} := M_1^{k_\varphi} \# \mathbb{P}$, meaning that for any $t \leq 1$, the Radon-Nikodym derivative of \mathbb{P}^{k_φ} with respect to \mathbb{P} on \mathcal{F}_t is given by $M_t^{k_\varphi}$ and we also denote by \mathbb{E}^{k_φ} the expectation under \mathbb{P}^{k_φ} . Recall that ν_N is the empirical distribution of all but the largest eigenvalue defined in the introduction. For any $r > 0, \alpha > 0$, we also define

$$\mathbb{B}_r(\sigma, \alpha) := \mathbb{B}(\sigma, \alpha) \cap \left\{ \mu \in C([0, 1], \mathcal{P}(\mathbb{R})); \forall s, \text{supp}(\mu_s) \subset]-\infty, 2\sqrt{s} + r] \right\}.$$

The goal of this subsection will be to show

Proposition 4.5 *For any $r > 0, \delta > 0, \alpha > 0$ and $\varphi \in \mathcal{H}_\theta$,*

$$\mathbb{P}^{k_\varphi}(\lambda_1^{\theta, N} \in B(\varphi, \delta); \nu_N \in \mathbb{B}_r(\sigma, \alpha)) \xrightarrow{N \rightarrow \infty} 1$$

The proof of the proposition relies on some lemmata.

Lemma 4.6 *Under \mathbb{P}^{k_φ} , μ_N and ν_N converge as N goes to infinity to the semicircular process σ .*

Proof: a) It is well known that μ_N is exponentially tight in scale N^2 , under \mathbb{P} (see [CDG], [G]). Let $A \in C([0, 1], \mathcal{P}(\mathbb{R}))$, then

$$\begin{aligned} \mathbb{P}^{k_\varphi}(\mu_N \in A) &= \mathbb{E} \left(M_1^{k_\varphi} \mathbf{1}_{\mu_N \in A} \right) \\ &\leq \mathbb{E} \left((M_1^{k_\varphi})^2 \right)^{1/2} (\mathbb{P}(\mu_N \in A))^{1/2} \\ &= \exp \left(\frac{N}{2} \int_0^1 k_\varphi^2(s) ds \right) \mathbb{P}(\mu_N \in A)^{1/2} \end{aligned} \quad (4.9)$$

From the exponential tightness of μ_N under \mathbb{P} , there exists a compact K_L in $C([0, 1], \mathcal{P}(\mathbb{R}))$ such that

$$\mathbb{P}(\mu_N \in K_L^c) \leq \exp(-N^2 L).$$

Therefore, from (4.9)

$$\mathbb{P}^{k_\varphi}(\mu_N \in K_L^c) \leq \exp(-N^2 L/4)$$

for N large enough. This proves the exponential tightness of μ_N under \mathbb{P}^{k_φ} and thus its a.s. pre-compactness in $C([0, 1], \mathcal{P}(\mathbb{R}))$.

b) From Girsanov’s theorem, we have that under \mathbb{P}^{k_φ} , the process $(\lambda_i^{\theta, N}(t))_{t \leq 1, i=1, \dots, N}$ satisfies the system of stochastic differential equations:

$$\begin{cases} d\lambda_i(t) = \frac{1}{\sqrt{N}} d\beta_i(t) + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt, & i = 2, \dots, N \\ d\lambda_1(t) = \frac{1}{\sqrt{N}} d\beta_1(t) + k_\varphi(t) dt + \frac{1}{N} \sum_{j \neq 1} \frac{1}{\lambda_1(t) - \lambda_j(t)} dt \end{cases} \quad (4.10)$$

where $(\beta_i)_{1 \leq i \leq N}$ are independent Brownian motions under \mathbb{P}^{k_φ} .

The proof of the convergence of μ_N towards σ follows the same proof as in Rogers and Shi [RS] (see also Guionnet [G], Chan [C]): any limit point μ_t satisfies a deterministic evolution equation (the term in k_φ disappears in the limit)

$$\int f(x) d\mu_t(x) = \int f(x) d\mu_0(x) + \frac{1}{2} \int_0^t \int \frac{f'(x) - f'(y)}{x - y} d\mu_s(x) d\mu_s(y) ds$$

for which uniqueness holds. When $\mu_0 = \delta_0$ as in our setting, μ_t is the semicircular law σ_t . Therefore μ_N converges a.s. to the semicircle process σ . Since $d(\mu_N, \nu_N) \leq \frac{2}{N}$, the same convergence holds for ν_N . \square

Lemma 4.7 *For any $r > 0, \alpha > 0$,*

$$\mathbb{P}^{k_\varphi}(\nu_N \in \mathbb{B}_r(\sigma, \alpha)) \xrightarrow{N \rightarrow \infty} 1.$$

Proof: Since we already know the convergence of ν_N towards σ under \mathbb{P}^{k_φ} , it is enough to prove that under \mathbb{P}^{k_φ} ,

$$\limsup_{N \rightarrow \infty} \lambda_2(t) \leq 2\sqrt{t}. \quad (4.11)$$

We define $(\lambda_i^{(\varepsilon)}(t), i = 1, \dots, N)$ the strong solution of the system of SDE (4.10) with initial conditions $\lambda_1^{(\varepsilon)}(0) = \theta + \varepsilon$ and $\lambda_i^{(\varepsilon)}(0) = \frac{\varepsilon}{i}$, for $i = 2, \dots, N$, so that in particular $\lambda_2(t) = \lambda_2^{(0)}(t)$.

We also introduce $(\overline{\lambda_i^{(\varepsilon)}}(t), i = 2, \dots, N)$ the strong solution of the system of SDE:

$$d\overline{\lambda_i^{(\varepsilon)}}(t) = \frac{1}{\sqrt{N}} d\beta_i(t) + \frac{1}{N} \sum_{j=2, j \neq i}^N \frac{1}{\overline{\lambda_i^{(\varepsilon)}}(t) - \overline{\lambda_j^{(\varepsilon)}}(t)} dt, \quad i = 2, \dots, N, \quad (4.12)$$

with initial conditions $\overline{\lambda_i^{(\varepsilon)}}(0) = \frac{\varepsilon}{i}$, for $i = 2, \dots, N$.

The process $(\overline{\lambda_i^{(0)}}(t), i = 2, \dots, N)$ is distributed as the eigenvalues of $\sqrt{\frac{N-1}{N}} H_{N-1}(t)$ where H_{N-1} is a standard Hermitian Brownian motion of size $N - 1$. Therefore, $\lim_{N \rightarrow \infty} \overline{\lambda_2^{(0)}}(t) = 2\sqrt{t}$ a.s..

Our goal is now to compare $\lambda_2^{(0)}(t)$ with $\overline{\lambda_2^{(0)}}(t)$.

The first step is to show that for any $\varepsilon > 0$ fixed, N fixed, for all $t \in [0, 1]$, $\lambda_2^{(\varepsilon)}(t) \leq \overline{\lambda_2^{(\varepsilon)}}(t)$. Let $R > 0$ large enough so that $\frac{1}{R} < \frac{\varepsilon}{N^2}$ and

$$T_R = \inf \left\{ t \geq 0, \forall i, j = 2, \dots, N, i \neq j, \left| \lambda_i^{(\varepsilon)}(t) - \lambda_j^{(\varepsilon)}(t) \right| \vee \left| \overline{\lambda_i^{(\varepsilon)}}(t) - \overline{\lambda_j^{(\varepsilon)}}(t) \right| \leq \frac{1}{R} \right\}$$

On $[0, T_R]$, the drift of the two systems of SDE are uniformly Lipschitz and we can apply Example 2.1 of [Ber] to get that $\lambda_2^{(\varepsilon)}(t) \leq \overline{\lambda_2^{(\varepsilon)}}(t)$.

Moreover, from [CL], we know that T_R goes to infinity as R goes to infinity. In particular, if we choose R large enough for T_R to be larger than 1, our inequality holds for any $t \in [0, 1]$.

Now, the solutions of (4.12), resp. (4.10), are continuous with respect to the initial condition (see [Ce]); thus, letting $\varepsilon \rightarrow 0$, we obtain $\lambda_2^{(0)}(t) \leq \overline{\lambda_2^{(0)}}(t)$ a.s. Putting everything together, we have that $\limsup_{N \rightarrow \infty} \lambda_2(t) \leq 2\sqrt{t}$. \square

Lemma 4.8 *Let $\varphi \in \mathcal{H}_\theta$.*

- 1) For $\theta > 0$, the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ on $[0, 1]$ with initial value $y(0) = \theta$ admits a unique solution larger than $2\sqrt{t}$, namely φ .
- 2) For $\theta = 0$, the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ on $[0, 1]$ with value at time $t_0(\varphi)$ $y(t_0(\varphi)) = 2\sqrt{t_0(\varphi)}$ admits a unique solution larger than $2\sqrt{t}$, namely φ .

Proof: Let us first check that in both cases there is a unique solution larger than $2\sqrt{t}$. We recall that for $x \geq 2\sqrt{t}$,

$$b(x, \sigma_t) = \frac{1}{2\pi t} \int \frac{1}{x-y} \sqrt{4t-y^2} dy = \frac{1}{2t}(x - \sqrt{x^2 - 4t}).$$

It is easy to see that $x \mapsto b(x, \sigma_t)$ is decreasing on $[2\sqrt{t}, \infty[$. Let x, y two solutions of $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ such that for any $t \in [0, 1]$, $x(t), y(t) \geq 2\sqrt{t}$. Then,

$$(x(t) - y(t))^2 = 2 \int_0^t (x(s) - y(s))(b(x(s), \sigma_s) - b(y(s), \sigma_s)) ds \leq 0.$$

In the first case, it is very easy to check that φ is a solution. In the second case, notice that for $t \leq t_0(\varphi)$, $k_\varphi(t) = 0$ and we know that $t \mapsto 2\sqrt{t}$ is a solution of $dy(t) = b(y(t), \sigma_t)dt$ with initial condition $y(0) = 0$. \square

The last lemma to complete the proof of Proposition 4.5 is the following

Lemma 4.9 For any $\theta \geq 0$ and $\varphi \in \mathcal{H}_\theta$, under \mathbb{P}^{k_φ} , the process $\lambda_1^{\theta, N}$ converges a.s. to φ .

Proof : As in Lemma 4.6, from the exponential tightness in scale N of λ_1 under \mathbb{P} , we deduce the exponential tighness of λ_1 under \mathbb{P}^{k_φ} and the a.s. pre-compactness of λ_1 . Let $x(t)$ be a limit point. There exists $f : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing such that $\lambda_1^{\theta, f(N)}(t)$ converge to $x(t)$. In the sequel we omit the superscript $\theta, f(N)$.

The crucial step of the proof, which is similar for any value of θ is to show that $x(t) \geq \varphi(t)$.

From the a.s. convergence of $(\mu_N)_t$ towards σ_t and using that $\sigma_t([2\sqrt{t} - \varepsilon, 2\sqrt{t}]) > 0$, it follows that $\liminf_N \lambda_1(t) \geq 2\sqrt{t}$ and thus $x(t) \geq 2\sqrt{t}$. From Itô's formula, we get

$$\begin{aligned} ((\varphi(t) - \lambda_1(t))^+)^2 &= -\frac{2}{\sqrt{N}}(\varphi(t) - \lambda_1(t))^+ d\beta_1(t) \\ &\quad + 2 \int_0^1 (\varphi(t) - \lambda_1(t))^+ [b(\varphi(t), \sigma_t) - b_N(\lambda_1(t), (\nu_N)_t)] dt \\ &\quad + \frac{1}{N} \mathbf{1}_{\varphi(t) - \lambda_1(t) \geq 0} dt. \end{aligned}$$

The first and last term converge to zero and we decompose the second term in three $2(A^1(t) + A^2(t) + A^3(t))$ where

$$A^1(t) = A_{f(N)}^1(t) = \int_0^1 (\varphi(t) - \lambda_1(t))^+ [b(\varphi(t), \sigma_t) - b(\lambda_1(t), \sigma_t)] dt$$

and

$$A^2(t) = \int_0^1 (\varphi(t) - \lambda_1(t))^+ [b(\lambda_1(t), \sigma_t) - b(\lambda_1(t), (\nu_{f(N)})_t)] dt$$

and

$$A^3(t) = \int_0^1 (\varphi(t) - \lambda_1(t))^+ [b(\lambda_1(t), (\nu_{f(N)})_t) - b_N(\lambda_1(t), (\nu_{f(N)})_t)] dt.$$

Passing to the limit, we obtain:

$$((\varphi(t) - x(t))^+)^2 = 2 \lim_{N \rightarrow \infty} (A^1(t) + A^2(t) + A^3(t)).$$

We now use the continuity on \mathbb{R} and the monotony on $[2\sqrt{t}, \infty[$ of the function $x \mapsto b(x, \sigma_t)$, the lower semicontinuity of $(x, \mu) \mapsto b(x, \mu)$ to conclude that :

$$\lim_{N \rightarrow \infty} A^1(t) = \int_0^1 (\varphi(t) - x(t))^+ [b(\varphi(t), \sigma_t) - b(x(t), \sigma_t)] dt \leq 0$$

and

$$\lim_{N \rightarrow \infty} A^2(t) \leq 0.$$

We also easily get that

$$\lim_{N \rightarrow \infty} A^3(t) = 0.$$

Therefore, we obtain that $x(t) \geq \varphi(t)$.

In the case when $\theta > 0$, we therefore get that $x(t)$ is well separated from the support of σ_t , we can argue as before, using (4.11), and obtain that $\lim_{N \rightarrow \infty} b_N(\lambda_1(t), (\nu_N)_t) = b(x(t), \sigma_t)$. Therefore, letting $N \rightarrow \infty$ in the equation of λ_1 , we obtain that x is a solution of the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ and therefore equal to φ .

In the case when $\theta = 0$, we have to treat first the case $t \leq t_0$. On this interval, $k_\varphi = 0$, so that $\lambda_1(t)$ converges to $2\sqrt{t}$, that is $\varphi(t)$.

For any $t > t_0$, $x(t)$ is well separated from the support of σ_t , and we get as before $\lim_{N \rightarrow \infty} b_N(\lambda_1(t), (\nu_N)_t) = b(x(t), \sigma_t)$ so that x is a solution of the differential equation $dy(t) = (k_\varphi(t) + b(y(t), \sigma_t))dt$ with initial condition $x(t_0) = 2\sqrt{t_0}$ and therefore equal to φ . \square

Proposition 4.5 is straightforward from Lemmata 4.6 to 4.9.

4.4 Lower bound for a non null initial condition: $\theta > 0$

We want to show Proposition 4.1 under the assumption that $\theta > 0$.

Thanks to Lemma 4.4 above, it is enough to show this lower bound under the additionnal hypothesis that $\varphi \in \mathcal{H}_\theta$. We make this assumption in the sequel. We set $r := \frac{1}{2} \inf_{s \in [0, 1]} (\varphi(s) - 2\sqrt{s}) > 0$.

From our assumptions on φ , there exists $\delta > 0$ small enough such that:

$$\forall \chi \in B(\varphi, \delta), \forall \mu \in \mathbb{B}_r(\sigma, \alpha), \forall s \in [0, 1] \text{ and } y \in \text{supp}(\mu_s), \chi(s) - y \geq \frac{r}{4}. \quad (4.13)$$

For $h \in \mathcal{H}$ and $(\varphi, \mu) \in C_\theta([0, 1]; \mathbb{R}) \times C([0, 1]; \mathcal{P}(\mathbb{R}))$ such that for all $t \in [0, 1]$, $\varphi(t) > r(\mu_t)$, we can define

$$\begin{aligned} G_N(\varphi, \mu; h) &= h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \varphi(s)\dot{h}(s)ds - \int_0^1 b_N(\varphi(s), \mu_s)h(s)ds \\ F_N(\varphi, \mu; h) &:= G_N(\varphi, \mu; h) - \frac{1}{2} \int_0^1 h^2(s)ds, \end{aligned} \quad (4.14)$$

where $b_N = \frac{N-1}{N}b$.

Therefore, similarly to (4.8), we have $M_1^h = \exp(NF_N(\lambda_1, \nu_N; h))$.

We get

$$\begin{aligned}
\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) &\geq \mathbb{P}(\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}_r(\sigma, \alpha)) \\
&= \mathbb{E} \left(\mathbf{1}_{\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}_r(\sigma, \alpha)} \frac{M_1^{k_\varphi}}{M_1^{k_\varphi}} \right) \\
&= \mathbb{E}^{k_\varphi} \left(\mathbf{1}_{\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}_r(\sigma, \alpha)} \exp(-NF_N(\lambda_1, \nu_N; k_\varphi)) \right) \\
&\geq \exp \left(-N \sup_{(\psi, \mu) \in C_{\alpha, \delta, r}} F_N(\psi, \mu; k_\varphi) \right) \mathbb{P}^{k_\varphi}(\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}_r(\sigma, \alpha))
\end{aligned}$$

where

$$C_{\alpha, \delta, r} = B(\varphi, \delta) \times \mathbb{B}_r(\sigma, \alpha). \quad (4.15)$$

Therefore

$$\begin{aligned}
\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) &\geq - \sup_{(\psi, \mu) \in C_{\alpha, \delta, r}} F(\psi, \mu; k_\varphi) \\
&+ \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}^{k_\varphi}(\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}_r(\sigma, \alpha)).
\end{aligned} \quad (4.16)$$

From the property (4.13) above, the fonction $(\psi, \mu) \mapsto F(\psi, \mu; k_\varphi)$ is continuous on $C_{\alpha, \delta, r}$ and we checked in Lemma 3.1 that $F(\varphi, \sigma; k_\varphi) = I_\theta(\varphi)$.

Moreover, from Proposition 4.5, we get that the last term in (4.16) is equal to zero.

We have thus obtained that for $\varphi \in \mathcal{H}_\theta$,

$$\lim_{\delta \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1^\theta \in B(\varphi, \delta)) \geq -I_\theta(\varphi). \quad (4.17)$$

4.5 Lower bound for a null initial condition

We want to show Proposition 4.1 under the assumption that $\theta = 0$.

Again from Lemma 4.4, we can assume that $\varphi \in \mathcal{H}_0$ and we set $t_0(\varphi)$ as defined in (4.2). Then, $k_\varphi = 0$ on $[0, t_0(\varphi)]$. We choose ε given by Lemma 4.3 and we denote by $r := r(\varepsilon) = \frac{1}{2} \inf_{s \in [t_0(\varphi) + \varepsilon, 1]} (\varphi(s) - 2\sqrt{s}) > 0$. As in the case when $\theta > 0$, we perform a change of measure via the martingale M^{k_φ} . Recall that F_N is defined by (4.14). We define $F_N^{(\varepsilon)}$ by

$$\begin{aligned}
F_N^{(\varepsilon)}(\varphi, \mu; k_\varphi) &= k_\varphi(1)\varphi(1) - \int_{t_0(\varphi)}^1 \varphi(s) \dot{k}_\varphi(s) ds - \int_{t_0(\varphi) + \varepsilon}^1 b_N(\varphi(s), \mu_s) k_\varphi(s) ds \\
&\quad - \frac{1}{2} \int_0^1 k_\varphi^2(s) ds,
\end{aligned}$$

in other words,

$$F_N(\varphi, \mu; k_\varphi) = F_N^{(\varepsilon)}(\varphi, \mu; k_\varphi) - \int_{t_0(\varphi)}^{t_0(\varphi) + \varepsilon} b_N(\varphi(s), \mu_s) k_\varphi(s) ds.$$

Therefore, for such ε , $F_N \leq F_N^{(\varepsilon)}$ and we obtain (as in the previous subsection)

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \geq \exp \left(-N \sup_{(\psi, \mu) \in C_{\alpha, \delta, r}} F_N^{(\varepsilon)}(\psi, \mu; k_\varphi) \right) \mathbb{P}^{k_\varphi}(\lambda_1 \in B(\varphi, \delta); \nu_N \in \mathbb{B}_r(\sigma, \alpha))$$

where $C_{\alpha, \delta, r}$ is defined in (4.15) and, using Proposition 4.5,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \geq - \sup_{(\psi, \mu) \in C_{\alpha, \delta, r}} F_N^{(\varepsilon)}(\psi, \mu; k_\varphi).$$

Now, for $\delta = \delta(\varepsilon)$ small enough, $F_N^{(\varepsilon)}$ is continuous on $C_{\alpha, \delta, r}$, since

$$\forall \psi \in B(\varphi, \delta), \forall \mu \in \mathbb{B}_r(\sigma, \alpha), \forall s \in [t_0(\varphi) + \varepsilon, 1] \text{ and } y \in \text{supp}(\mu_s), \psi(s) - y \geq \frac{r}{4}.$$

and therefore

$$\lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \geq -F^{(\varepsilon)}(\varphi, \sigma; k_\varphi),$$

where

$$F^{(\varepsilon)}(\varphi, \sigma; k_\varphi) = \frac{1}{2} \int_{t_0(\varphi) + \varepsilon}^1 (\dot{\varphi}(s) - b(\varphi(s), \sigma_s))^2 ds - \int_{t_0(\varphi)}^{t_0(\varphi) + \varepsilon} \varphi(s) \dot{k}_\varphi(s) ds - \frac{1}{2} \int_0^\varepsilon k_\varphi^2(s) ds.$$

This last quantity tends to $I_0(\varphi)$ as ε tends to 0. \square

5 The upper bound

We first prove the following

Proposition 5.1 *Let $\theta \geq 0$ and $\varphi \in C_\theta([0, 1]; \mathbb{R})$ such that there exists $t_0 \in [0, 1]$ so that $\varphi(t_0) < 2\sqrt{t_0}$. Then*

$$\lim_{\delta \downarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1^{\theta, N} \in B(\varphi, \delta)) = -\infty.$$

We proceed as in [BADG]. From [CDG], we know that the process μ_N satisfies a LDP in the scale N^2 with a good rate function whose unique minimizer is the semicircular process σ for which we know that the support of σ_t is $[-2\sqrt{t}, 2\sqrt{t}]$.

Let $\delta_0 = 2\sqrt{t_0} - \varphi(t_0)$. By continuity of φ , there exists $\varepsilon > 0$, such that for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\varphi(t) < 2\sqrt{t} - \frac{\delta_0}{2}$.

For any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, there exists f_t such that $f_t(y) = 0$ if $y \leq \varphi(t)$ and $\int f_t(x) d\sigma_t(x) > 0$. We let $F := \{\mu \in C([0, 1]; \mathcal{P}(\mathbb{R})) / \int f_t(x) d\mu_t(x) = 0 \forall t \in [t_0 - \varepsilon, t_0 + \varepsilon]\}$, which is a closed set.

For any $\delta < \frac{\delta_0}{2}$,

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}(\mu_N \in F).$$

As $\sigma \notin F$, $\limsup_{N \rightarrow \infty} \frac{1}{N^2} \ln \mathbb{P}(\mu_N \in F) < 0$, which gives the Proposition. \square

We thus consider the case where $\varphi(t) \geq 2\sqrt{t}$ and as a first step, we prove the upper bound for a function φ which satisfies $\varphi(t) > 2\sqrt{t}$ for all $t \in [0, 1]$ (this implies in particular that $\theta > 0$).

5.1 The upper bound for functions φ well separated from $t \mapsto 2\sqrt{t}$

Proposition 5.2 *Let $\varphi \in C_\theta([0, 1]; \mathbb{R})$ such that for any $t \in [0, 1]$, $\varphi(t) > 2\sqrt{t}$. Then*

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1^{\theta, N} \in B(\varphi, \delta)) \leq -I_\theta(\varphi). \quad (5.1)$$

Let $\varphi \in C_\theta([0, 1]; \mathbb{R})$ such that $\varphi(t) > 2\sqrt{t}$ for all $t \in [0, 1]$. We recall that $r := \frac{1}{2} \inf_t (\varphi(t) - 2\sqrt{t}) > 0$. The strategy of the proof will heavily rely on the fact that only a finite number of eigenvalues can deviate strictly above $t \mapsto 2\sqrt{t}$.

More precisely, we have

Proposition 5.3 For any $\eta > 0$, $L > 0$, there exists $K := K(\eta, L)$ (independent of N) such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\exists t \in [0, 1], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) \leq -L. \quad (5.2)$$

We will first prove a fixed time version of the same result stated in the following lemma:

Lemma 5.4 For any $\eta > 0$, $L > 0$, there exists $K := K(\eta, L)$ such that for any $t \in [0, 1]$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_{K+1}^{N, \theta}(t) > 2\sqrt{t} + \eta) \leq -L.$$

Proof of Proposition 5.3: We fix $\eta > 0$ and $L > 0$. If C is the universal constant of Lemma 2.2, let R be such that $C\eta^2 R > 18L$ and choose a subdivision $(t_k)_{1 \leq k \leq R}$ of the interval $[0, 1]$ such that for all $1 \leq k \leq R$, $|t_k - t_{k+1}| \leq \frac{2}{R}$. Now, for any $K \in \mathbb{N}^*$

$$\begin{aligned} \mathbb{P}(\exists t \in [0, 1], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) &= \mathbb{P}(\cup_k (\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta)) \\ &\leq R \max_{1 \leq k \leq R} \mathbb{P}(\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) \end{aligned} \quad (5.3)$$

Then

$$\begin{aligned} \mathbb{P}(\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) &\leq \mathbb{P}\left(\lambda_{K+1}(t_k) > 2\sqrt{t_k} + \frac{\eta}{2}\right) \\ &\quad + \mathbb{P}\left(\exists t \in [t_k, t_{k+1}], \lambda_{K+1}(t) > 2\sqrt{t} + \eta ; \lambda_{K+1}(t_k) \leq 2\sqrt{t_k} + \frac{\eta}{2}\right) \\ &\leq \mathbb{P}\left(\lambda_{K+1}(t_k) > 2\sqrt{t_k} + \frac{\eta}{2}\right) + \mathbb{P}\left(\sup_{t_k \leq t < t_{k+1}} |\lambda_{K+1}(t) - \lambda_{K+1}(t_k)| \geq \frac{\eta}{3}\right) \end{aligned}$$

From Lemma 5.4, we can find $K := K(\eta, L)$ such that

$$\limsup_N \frac{1}{N} \ln \mathbb{P}\left(\lambda_{K+1}(t_k) > 2\sqrt{t_k} + \frac{\eta}{2}\right) \leq -L.$$

From Lemma 2.2 applied for $p = K(\eta, L)$,

$$\limsup_N \frac{1}{N} \ln \mathbb{P}\left(\sup_{t_k \leq t < t_{k+1}} |\lambda_{K+1}(t) - \lambda_{K+1}(t_k)| \geq \frac{\eta}{3}\right) = -L.$$

As R is independent of N , (5.3) gives the lemma. \square

Proof of Lemma 5.4:

The first observation is that, as $H_N^\theta(t) = H_N^0(t) + \text{diag}(\theta, 0, \dots, 0)$, with $\theta \geq 0$, by Weyl's inequalities, $\lambda_{K+1}^\theta(t) \leq \lambda_K^0(t)$ so that

$$\mathbb{P}(\lambda_{K+1}^\theta(t) > 2\sqrt{t} + \eta) \leq \mathbb{P}\left(\lambda_K^0(t) > 2\sqrt{t} + \eta\right) = \mathbb{P}\left(\lambda_K^0(1) > 2 + \frac{\eta}{\sqrt{t}}\right) \leq \mathbb{P}(\lambda_K^0(1) > 2 + \eta).$$

Therefore, Lemma 5.4 will be a direct consequence of the fact that for any $p \geq 1$, the law of $(\lambda_1^0(1), \dots, \lambda_p^0(1))$ satisfies a LDP in the scale N with good rate function

$$F : (x_1, \dots, x_p) \mapsto 1_{x_1 \geq x_2 \geq \dots \geq x_p} \sum_{i=1}^p K_0(x_i),$$

with K_0 the individual rate function at time 1 as defined in Theorem 1.3. This is a particular case of Theorem 2.10 in [BGM] in the case when the potential V is just Gaussian ($V(x) = x^2$) therein. From this, if we define $K \geq \frac{L}{K_0(2+\eta)}$, we deduce that,

$$\begin{aligned} & \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_{K+1}^{\theta, N}(t) > 2\sqrt{t} + \eta) \\ & \leq \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_K^{0, N}(1) > 2 + \eta) = \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_1^{0, N}(1) > 2 + \eta, \dots, \lambda_K^{0, N}(1) > 2 + \eta) \\ & \leq -KK_0(2 + \eta) \leq -L. \quad \square \end{aligned}$$

Lemma 5.5 *Let K fixed as in the above proposition. Let $j \in \{1, \dots, K\}$. We denote by $\mu_N^{(j)}$ the spectral measure of the $N - j$ smallest eigenvalues $\mu_N^{(j)} = \frac{1}{N - j} \sum_{p=j+1}^N \delta_{\lambda_p}$. Then,*

$$d(\mu_N^{(j)}, \mu_N) \leq \frac{2K}{N}.$$

Therefore, if $\mu_N \in \mathbb{B}(\sigma, \alpha)$, $\mu_N^{(j)} \in \mathbb{B}(\sigma, 2\alpha)$ for $N \geq N_0$.

Proof: Let $f \in \mathcal{F}_{Lip}$,

$$\mu_N^{(j)}(f) - \mu_N(f) = \frac{j}{N(N - j)} \sum_{p>j} f(\lambda_p) - \frac{1}{N} \sum_{p=1}^j f(\lambda_p)$$

and

$$|\mu_N^{(j)}(f) - \mu_N(f)| \leq \frac{j}{N} + \frac{j}{N} \leq \frac{2K}{N}. \quad \square$$

Proof of Proposition 5.2: Let $\varphi \in C_\theta([0, 1]; \mathbb{R})$ such that for any $t \in [0, 1]$, $\varphi(t) > 2\sqrt{t}$. We recall that $r = \frac{1}{2} \inf(\varphi(t) - 2\sqrt{t})$. For $k \in \mathbb{N}^*$, let $\delta > 0$ such that $\delta < \frac{r}{4K}$ and $\alpha > 0$. We have

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}(A_{N, \delta, \alpha, K}) + \mathbb{P}(\exists t \in [0, 1], \lambda_{K+1}(t) > 2\sqrt{t} + r) + \mathbb{P}(\mu_N \notin \mathbb{B}(\sigma, \alpha)) \quad (5.4)$$

with

$$A_{N, \delta, \alpha, K} := \left\{ \lambda_1 \in B(\varphi, \delta); \forall p > K, \forall t, \lambda_p(t) \leq 2\sqrt{t} + r; \mu_N \in \mathbb{B}(\sigma, \alpha) \right\}$$

We now choose a subdivision $(t_k)_{1 \leq k \leq R}$ as in the proof of Proposition 5.3, such that $|t_k - t_{k+1}| \leq \frac{2}{R}$. We choose R such that $\sup_{|t-s| \leq \frac{2}{R}} |\varphi(s) - \varphi(t)| \leq \frac{\delta}{6}$. From the choice of the parameters, for each $t \in [0, 1]$, if we are on $A_{N, \delta, \alpha, K}$, there exists at least a gap between two eigenvalues (among the K largest ones) larger than δ . For $1 \leq k \leq R$, we can define a random variable I_k with values in $\{1, \dots, K\}$ by

$$I_k := \inf\{i \leq K, \lambda_i(t_k) - \lambda_{i+1}(t_k) > \delta\}.$$

We write

$$A_{N, \delta, \alpha, K} \subset \bigcup_{\mathbf{i} \in \{1, \dots, K\}^R} A_{N, \mathbf{i}, \delta, \alpha} \quad (5.5)$$

where, for $\mathbf{i} = (i_k, k \leq R)$,

$$A_{N,\mathbf{i},\delta,\alpha} = \{\lambda_1 \in B(\varphi, \delta), \forall k \leq R, \forall i < i_k, \lambda_i(t_k) - \lambda_{i+1}(t_k) \leq \delta, \lambda_{i_k}(t_k) - \lambda_{i_k+1}(t_k) > \delta, \mu_N \in \mathbb{B}(\sigma, \alpha)\}.$$

As, for $i \leq i_k$, $\lambda_i(t_k) \geq \varphi(t_k) - i\delta$.

$$A_{N,\mathbf{i},\delta,\alpha} \subset \{\lambda_1 \in B(\varphi, \delta), \forall k \leq R, \forall i < i_k, \lambda_i(t_k) \geq \varphi(t_k) - i\delta, \lambda_{i_k}(t_k) - \lambda_{i_k+1}(t_k) > \delta, \mu_N \in \mathbb{B}(\sigma, \alpha)\}.$$

Now, if we let

$$B_{N,\mathbf{i},\delta,\alpha} = \left\{ \lambda_1 \in B(\varphi, \delta), \forall i < i_k, \forall t \in [t_k, t_{k+1}[, \lambda_i(t) \geq \varphi(t) - (i+1)\delta, \lambda_{i_k}(t) - \lambda_{i_k+1}(t) > \frac{2}{3}\delta, \mu_N \in \mathbb{B}(\sigma, \alpha) \right\},$$

then

$$A_{N,\mathbf{i},\delta,\alpha} \subset B_{N,\mathbf{i},\delta,\alpha} \cup \left\{ \exists k, \exists t \in [t_k, t_{k+1}[, \exists i \leq i_k + 1, |\lambda_i(t) - \lambda_i(t_k)| > \frac{\delta}{6} \right\} \quad (5.6)$$

and the second term will again be controlled by Lemma 2.2.

We now work on $B_{N,\mathbf{i},\delta,\alpha}$. Let $j \leq K$ and $X_j(t) := \frac{1}{j} \sum_{i=1}^j \lambda_i(t)$ is a solution of the SDE

$$dX_j(t) = \frac{1}{\sqrt{N}} \frac{1}{j} \sum_{i \leq j} d\beta_i(s) + \frac{1}{N} \frac{1}{j} \sum_{i \leq j} \sum_{p > j} \frac{1}{\lambda_i(t) - \lambda_p(t)} dt \quad (5.7)$$

We denote by $dB^j(s) = \frac{1}{\sqrt{j}} \sum_{i=1}^j d\beta_i(s)$, which is a standard Brownian motion. Let $h \in \mathcal{H}$, we define the exponential martingale

$$\begin{aligned} M_t^h &= \exp \left[N \left(\int_0^t \sum_k h(s) 1_{[t_k, t_{k+1}[}(s) \frac{1}{\sqrt{N}} \frac{1}{\sqrt{i_k}} dB^{i_k}(s) - \frac{1}{2} \int_0^t \sum_k 1_{[t_k, t_{k+1}[}(s) \frac{1}{i_k} h^2(s) ds \right) \right] \\ &\quad \frac{1}{\sqrt{N}} \frac{1}{\sqrt{i_k}} dB^{i_k}(t) = dX_{i_k}(t) - \frac{1}{N} \frac{1}{i_k} \sum_{i \leq i_k} \sum_{p > i_k} \frac{1}{\lambda_i(t) - \lambda_p(t)} dt \\ &\quad = dX_{i_k}(t) - \frac{N - i_k}{N} \frac{1}{i_k} \sum_{i \leq i_k} \int \frac{\mu_t^{(i_k)}(dx)}{\lambda_i(t) - x} dt \end{aligned}$$

and

$$\begin{aligned} M_1^h &= \exp \left[N \sum_k \left(\left[h_s X_{i_k}(s) \right]_{t_k}^{t_{k+1}} - \frac{N - i_k}{N} \frac{1}{i_k} \sum_{i \leq i_k} \int_{t_k}^{t_{k+1}} \int \frac{\mu_t^{(i_k)}(dx)}{\lambda_i(t) - x} h(t) dt - \int_{t_k}^{t_{k+1}} \dot{h}(s) X_{i_k}(s) ds \right. \right. \\ &\quad \left. \left. - \frac{1}{2i_k} \int_{t_k}^{t_{k+1}} h^2(s) ds \right) \right] \end{aligned}$$

We recall from Lemma 5.5 that if $\mu_N \in \mathbb{B}(\sigma, \alpha)$, $\mu_N^{(i_k)} \in \mathbb{B}(\sigma, 2\alpha)$.

M_1^h can be written as a functional

$$M_1^h = \exp \left(NF_N(\lambda_1, \dots, \lambda_K, \mu_N^{(1)}, \dots, \mu_N^{(K)}; h) \right).$$

We denote by

$$\begin{aligned} \Lambda_{\mathbf{i},\delta,\alpha} &= \{(\psi_1, \dots, \psi_K, \nu_1, \dots, \nu_K) : \psi_1 \in B(\varphi, \delta), \forall i < i_k, \forall t \in [t_k, t_{k+1}[, \psi_i(t) \geq \varphi(t) - (i+1)\delta, \\ &\quad \psi_{i_k}(t) - \psi_{i_k+1}(t) > \frac{2}{3}\delta; \nu_i \in \mathbb{B}(\sigma, 2\alpha), \text{supp}(\nu_{i_k}(\cdot)) \subset] - \infty, \psi_{i_k+1}(\cdot) \} \end{aligned}$$

where in the above set, the functions are such that $\psi_1 \geq \psi_2 \dots \geq \psi_K$. We denote by $\underline{\psi} = (\psi_1, \dots, \psi_K)$ and $\underline{\nu} = (\nu_1, \dots, \nu_K)$. Then,

$$\begin{aligned} \mathbb{P}(B_{N,i,\delta,\alpha}) &= \mathbb{E}[1_{B_{N,i,\delta,\alpha}} \frac{M_1^h}{M_1^h}] \\ &\leq \exp(-N \inf_{(\underline{\psi}, \underline{\nu}) \in \Lambda_{i,\delta,\alpha}} F_N(\underline{\psi}, \underline{\nu}; h)) \mathbb{E}[M_1^h] \\ &\leq \exp(-N \inf_{(\underline{\psi}, \underline{\nu}) \in \Lambda_{i,\delta,\alpha}} F_N(\underline{\psi}, \underline{\nu}; h)) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{N} \ln \mathbb{P}(B_{N,i,\delta,\alpha}) &\leq - \inf_{(\underline{\psi}, \underline{\nu}) \in \Lambda_{i,\delta,\alpha}} F_N(\underline{\psi}, \underline{\nu}; h). \\ \limsup \frac{1}{N} \ln \mathbb{P}(B_{N,i,\delta,\alpha}) &\leq - \inf_{(\underline{\psi}, \underline{\nu}) \in \Lambda_{i,\delta,\alpha}} F_i(\underline{\psi}, \underline{\nu}; h) \end{aligned}$$

where

$$\begin{aligned} F_i(\underline{\psi}, \underline{\nu}; h) &= [h_s \Psi_{i_k}(s)]_{t_k}^{t_{k+1}} - \frac{1}{i_k} \sum_{i \leq i_k} \int_{t_k}^{t_{k+1}} \int \frac{\nu_t^{(i_k)}(dx)}{\psi_i(t) - x} h(t) dt - \int_{t_k}^{t_{k+1}} \dot{h}(s) \Psi_{i_k}(s) ds \\ &\quad - \frac{1}{2i_k} \int_{t_k}^{t_{k+1}} h^2(s) ds \end{aligned}$$

with $\Psi_j = \frac{1}{j} \sum_{i \leq j} \psi_i$.

Let us take $\alpha \rightarrow 0$. The function $\underline{\nu} \mapsto F_i(\underline{\psi}, \underline{\nu}; h)$ is continuous on the set $\Lambda_{i,\delta,\alpha}$ since for $i \leq i_k$,

$$\psi_i(t) - x \geq \frac{2}{3} \delta \quad \forall x \in \text{supp}((\nu_{i_k})_t).$$

We obtain

$$\lim_{\alpha \rightarrow 0} \limsup \frac{1}{N} \ln \mathbb{P}(B_{N,i,\delta,\alpha}) \leq - \inf_{\underline{\psi} \in \Lambda_{i,\delta}} F_i(\underline{\psi}, \underline{\sigma}; h)$$

where $\underline{\sigma} = (\sigma, \dots, \sigma)$ and $\Lambda_{i,\delta}$ is defined as in $\Lambda_{i,\delta,\alpha}$ without the conditions on ν_i . Now, take $\delta \rightarrow 0$, the above functional is continuous in $\underline{\psi}$ and

$$\lim_{\delta \rightarrow 0} \lim_{\alpha \rightarrow 0} \limsup \frac{1}{N} \ln \mathbb{P}(B_{N,i,\delta,\alpha}) \leq -F_i(\underline{\varphi}, \underline{\sigma}; h) \quad (5.8)$$

where

$$F_i(\underline{\varphi}, \underline{\sigma}; h) = h(1)\varphi(1) - h(0)\varphi(0) - \int_0^1 \int_{\mathbb{R}} \frac{\sigma_t(dx)}{\varphi(t) - x} h(t) dt - \sum_k \frac{1}{2i_k} \int_{t_k}^{t_{k+1}} h^2(s) ds$$

and

$$-F_i(\underline{\varphi}, \underline{\sigma}; h) \leq -F(\varphi, \sigma; h)$$

where F is defined by (3.3).

We have proved that for any $K \in \mathbb{N}^*$, any $h \in \mathcal{H}$, and any subdivision $(t_k)_{1 \leq k \leq R}$ of $[0, 1]$,

$$\lim_{\delta \rightarrow 0} \lim_{\alpha \rightarrow 0} \limsup \frac{1}{N} \ln \mathbb{P}(B_{N,i,\delta,\alpha}) \leq -F(\varphi, \sigma; h).$$

We now go back to the decompositions (5.5) and (5.6). Let us first treat the case when $I_\theta(\varphi) < \infty$. We choose $L = -2I_\theta(\varphi)$ and K as given in Proposition 5.3 so that

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\exists t \in [0, 1], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) \leq -2I_\theta(\varphi).$$

Moreover

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\mu_N \notin \mathbb{B}(\sigma, \alpha)) = -\infty.$$

and from Lemma 2.2, if we choose R , the number of points of the subdivision such that $R > \frac{26I_\theta(\varphi)}{C\delta^2}$,

$$\limsup_N \frac{1}{N} \ln \mathbb{P}\left(\exists k, \exists t \in [t_k, t_{k+1}], \exists i \leq i_k + 1 |\lambda_{i_k}(t) - \lambda_{i_k}(t_k)| > \frac{\delta}{6}\right) \leq -2I_\theta(\varphi).$$

We thus obtain, for any $h \in \mathcal{H}$,

$$\lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -\inf(F(\varphi, \sigma; h), 2I_\theta(\varphi)).$$

Optimizing in h gives

$$\lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -I_\theta(\varphi).$$

In the case where $I_\theta(\varphi) = \infty$, for any L , we can associate K as in Proposition 5.3 such that

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\exists t \in [0, 1], \lambda_{K+1}(t) > 2\sqrt{t} + \eta) \leq -L \tag{5.9}$$

In the same way as above, with $R > \frac{18L}{C\delta^2}$, we then show that

$$\lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -L$$

and since the left-hand side does not depend on L ,

$$\lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) = -\infty.$$

□

We now extend Proposition 5.2 to any function φ such that $\varphi(t) \geq 2\sqrt{t}$.

5.2 The upper bound for functions φ not well separated from $t \mapsto 2\sqrt{t}$

Proposition 5.6 *Let $\varphi \in C_\theta([0, 1]; \mathbb{R})$ such that for any $t \in [0, 1]$, $\varphi(t) \geq 2\sqrt{t}$. Then*

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1^\theta \in B(\varphi, \delta)) \leq -I_\theta(\varphi).$$

Proof of Proposition 5.6: For any $\epsilon > 0$, let $J_\epsilon = \{t \in [0, 1], \varphi(t) > 2\sqrt{t} + \epsilon\}$. J_ϵ is an open set in $[0, 1]$ and \bar{J}_ϵ is compact so that we can find a set V_ϵ of the form $V_\epsilon = \cup_{i=1}^{N_\epsilon} a_i(\epsilon), b_i(\epsilon)[$ such that

$$\bar{J}_\epsilon \subset V_\epsilon \subset J_{\epsilon/2}.$$

Then, on V_ϵ , $\varphi(t) > 2\sqrt{t}$. For a function f on $[0, 1]$, we denote by $f|_A$ its restriction to a subset A of $[0, 1]$. Then,

$$\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}(\lambda_1|_{V_\epsilon} \in B(\varphi|_{V_\epsilon}, \delta)).$$

From Proposition 5.2,

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 |_{V_\epsilon} \in B(\varphi|_{V_\epsilon}, \delta)) \leq - \sum_{i=1}^{N_\epsilon} I_\theta(\varphi|_{[a_i(\epsilon), b_i(\epsilon)]})$$

where

$$I_\theta(\varphi|_{[a,b]}) = \frac{1}{2} \int_a^b \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s} \right) \right)^2 ds,$$

this quantity may be infinite. Let $\epsilon \rightarrow 0$, by monotone convergence,

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq - \frac{1}{2} \int_J \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s} \right) \right)^2 ds$$

where $J = \{t \in [0, 1], \varphi(t) > 2\sqrt{t}\}$ (the right-hand side can be infinite).

Assume that φ is differentiable almost everywhere (a.e.) on $[0, 1]$. Since $\varphi(t) \geq 2\sqrt{t}$, if φ is differentiable in s_0 such that $\varphi(s_0) = 2\sqrt{s_0}$: then, $\dot{\varphi}(s_0) = \frac{1}{\sqrt{s_0}}$ and

$$\dot{\varphi}(s_0) - \frac{1}{2s_0} \left(\varphi(s_0) - \sqrt{\varphi(s_0)^2 - 4s_0} \right) = 0.$$

Thus,

$$\int_J \left(\dot{\varphi}(s) - \frac{1}{2s} \left(\varphi(s) - \sqrt{\varphi(s)^2 - 4s} \right) \right)^2 ds = I_\theta(\varphi).$$

If φ is not differentiable a.e., then $I_\theta(\varphi) = \infty$. Consider first the case $\theta > 0$. From the lower semicontinuity of I_θ , for all $C > 0$, there exists ϵ such that

$$B(\varphi, \epsilon) \subset \{\psi; I_\theta(\psi) > C\}.$$

Define

$$\begin{cases} \psi(t) = \varphi(t) & \text{on } \bar{J}_\epsilon \\ \psi(t) = 2\sqrt{t} + \epsilon & \text{on } (\bar{J}_\epsilon)^c \end{cases}$$

Then, $\psi \in B(\varphi, \epsilon)$ and

$$\begin{aligned} \int_{(\bar{J}_\epsilon)^c} \left(\dot{\psi}(s) - \frac{1}{2s} \left(\psi(s) - \sqrt{\psi(s)^2 - 4s} \right) \right)^2 ds &= \int_{(\bar{J}_\epsilon)^c} \left(\frac{1}{2s} \left(\epsilon - \sqrt{\epsilon^2 + 4\sqrt{s}\epsilon} \right) \right)^2 ds \\ &\leq K\epsilon \end{aligned}$$

for some constant K . The last inequality follows from the fact that since $\theta > 0$, $(\bar{J}_\epsilon)^c \subset [a, 1]$ for a strictly positive a . Therefore, for ϵ small enough, $I_\theta(\psi|_{\bar{J}_\epsilon}) = I_\theta(\varphi|_{\bar{J}_\epsilon}) \geq \frac{C}{2}$. Moreover, $I_\theta(\varphi|_{V_\epsilon}) \geq I_\theta(\varphi|_{\bar{J}_\epsilon})$ so that we get

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq -\frac{C}{2}.$$

Since the inequality is true for all C ,

$$\lim_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\lambda_1 \in B(\varphi, \delta)) = -\infty.$$

Now for $\theta = 0$, if $I_0(\varphi|_{[a,1]}) < \infty$ for all $a > 0$, then, φ would be a.e. differentiable. Therefore, we can assume that there exists a a such that $I_0(\varphi|_{[a,1]}) = \infty$ and argue as before, using that $\mathbb{P}(\lambda_1 \in B(\varphi, \delta)) \leq \mathbb{P}(\lambda_1|_{[a,1]} \in B(\varphi|_{[a,1]}, \delta))$. \square

6 Contraction principle

The goal of this section is to get from Theorem 1.1 a new proof of the following results concerning the deviations of the largest eigenvalue at fixed time (say $t = 1$).

Note that this fixed time result has been used in the preceding section for the proof of the upper bound in the case $\theta = 0$, the goal here is to extend it to any $\theta > 0$.

Proof: As $\varphi \mapsto \varphi(1)$ is continuous, by contraction principle (Theorem 4.2.1 in [DZ]), we get that $\lambda_1(1)$ satisfies a LDP with good rate function $J_{0,\theta}$, where, for any $\eta \in [0, 1[$, we denote by

$$I_\eta(\varphi) = \int_\eta^1 f(t, \varphi(t), \dot{\varphi}(t)) dt,$$

with

$$f(t, x, y) = \frac{1}{2} \left(y - \frac{1}{2t} \left(x - \sqrt{x^2 - 4t} \right) \right)^2$$

and, for $x \geq 2$, $\theta \geq 2\sqrt{\eta}$,

$$J_{\eta,\theta}(x) = \inf_{\substack{\varphi \text{ s.t. } \varphi(\eta) = \theta, \\ \varphi(1) = x}} I_\eta(\varphi).$$

As $I_\eta(\varphi)$ is a good rate function, the infimum in the above problem is reached, and we denote by φ^η an infimum. We first show the following lemma :

Lemma 6.1

For any $\eta \in]0, 1[$, if, for any $t \in [\eta, 1]$, $\varphi^\eta(t) > 2\sqrt{t}$, then,

$$\varphi^\eta(t) = \frac{x - \theta}{1 - \eta} (t - \eta) + \theta.$$

Proof: Let $\eta \in [0, 1[$ be fixed. We denote by $\epsilon := \inf_{t \in [\eta, 1]} (\varphi^\eta(t) - 2\sqrt{t}) > 0$.

It is easy to check that the infimum of I_η is finite, therefore, we know that φ^η is absolutely continuous with $\dot{\varphi}^\eta \in \mathbb{L}^1$. Following the proof of Theorem 4 in Chapter 9.2.3 in [I-T], we first show that it is a solution to the DuBois-Reymond equation, i.e. there exists a constant r such that for any $t \in [\eta, 1]$,

$$\frac{\partial f}{\partial y}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) - \int_\eta^t \frac{\partial f}{\partial x}(s, \varphi^\eta(s), \dot{\varphi}^\eta(s)) ds = r. \quad (6.1)$$

Let $x(\cdot)$ be a function such that $|x(\cdot)| \leq \frac{\epsilon}{2}$, $x(\eta) = 0$, $\dot{x}(\cdot)$ is in \mathbb{L}^∞ and its integral between η and 1 is zero. We set, for $t \in [\eta, 1]$ and $\xi \in [0, 1]$,

$$d(t, \xi) = x(t) \frac{\partial f}{\partial x}(t, \varphi^\eta(t) + \xi x(t), \dot{\varphi}^\eta(t) + \xi \dot{x}(t)) + \dot{x}(t) \frac{\partial f}{\partial y}(t, \varphi^\eta(t) + \xi x(t), \dot{\varphi}^\eta(t) + \xi \dot{x}(t))$$

so that, for all $\lambda \in [0, 1]$,

$$\int_0^\lambda d(t, \xi) d\xi = f(t, \varphi^\eta(t) + \lambda x(t), \dot{\varphi}^\eta(t) + \lambda \dot{x}(t)) - f(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)).$$

By continuity, for all $t \in [\eta, 1]$,

$$e(t, \lambda) := \frac{1}{\lambda} \int_0^\lambda d(t, \xi) d\xi \xrightarrow{\lambda \rightarrow 0} d(t, 0).$$

Moreover, as

$$\frac{\partial f}{\partial y}(t, x, y) = y - \frac{1}{2t}(x - \sqrt{x^2 - 4t})$$

and

$$\frac{\partial f}{\partial x}(t, x, y) = -\frac{1}{2t} \left(y - \frac{1}{2t}(x - \sqrt{x^2 - 4t}) \right) \left(1 - \frac{x}{\sqrt{x^2 - 4t}} \right),$$

and for any $\xi \in [0, 1]$, $\inf_{t \in [\eta, 1]} (\varphi^\eta(t) + \xi x(t) - 2\sqrt{t}) \geq \frac{\epsilon}{2}$, we get that there exists $q_{\eta, \epsilon}$ integrable such that for any $t \in [\eta, 1]$ for any $\xi \in [0, 1]$, $|d(t, \xi)| \leq q_{\eta, \epsilon}(t)$. Therefore, by dominated convergence,

$$\lim_{\lambda \rightarrow 0} \int_{\eta}^1 e(t, \lambda) dt = \int_{\eta}^1 d(t, 0) dt.$$

By definition

$$\lim_{\lambda \rightarrow 0} \int_{\eta}^1 e(t, \lambda) dt = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (I_{\eta}(\varphi^\eta + \lambda x) - I_{\eta}(\varphi^\eta)),$$

and

$$\int_{\eta}^1 d(t, 0) dt = \int_{\eta}^1 x(t) \frac{\partial f}{\partial x}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) + \dot{x}(t) \frac{\partial f}{\partial y}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) dt.$$

But, as we know that φ^η is a minimizer, it gives that the latter integral is equal to zero. By integration by parts, we get that

$$\int_{\eta}^1 \dot{x}(t) \left(\frac{\partial f}{\partial y} f(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) - \int_{\eta}^t \frac{\partial f}{\partial x}(s, \varphi(s), \dot{\varphi}(s)) ds \right) dt = 0,$$

where we used that $x(\eta) = x(1) = 0$. As this relation is linear in $x(\cdot)$ we can relax the condition that its norm is less than $\frac{\epsilon}{2}$.

As the relation above holds for a dense subset of the functions $y \in L^\infty$ such that $\int_{\eta}^1 y(t) dt = 0$, we deduce that φ^η is a solution of the DuBois-Reymond equation (6.1).

The next step is to check that φ^η is a C^2 solution of the Euler-Lagrange equation, that is, for any $t \in [\eta, 1]$,

$$\frac{d}{dt} \frac{\partial f}{\partial y}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) - \frac{\partial f}{\partial x}(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) = 0. \quad (6.2)$$

We denote by

$$g(t, y) = f(t, \varphi^\eta(t), y) - y \int_{\eta}^t \frac{\partial f}{\partial x}(s, \varphi^\eta(s), \dot{\varphi}^\eta(s)) ds - ry.$$

With our expression of f , $g(t, \cdot)$ is a convex quadratic polynomial, therefore it has a unique minimizer $y(t)$, which is solution of the equation

$$\frac{\partial g}{\partial y}(t, y(t)) = 0.$$

We can compute y explicitly, it is given by

$$y(t) = \frac{\varphi^\eta(t)}{2t} - \frac{1}{2t} \sqrt{\varphi^\eta(t)^2 - 4t} + \frac{1}{2t} \frac{\varphi^\eta(t)}{\sqrt{\varphi^\eta(t)^2 - 4t}} + r.$$

As we know that φ^η is absolutely continuous, so is y . But, by unicity of the minimizer, we have that $y = \dot{\varphi}^\eta$ so that we get that φ^η is continuously differentiable. Therefore $\frac{\partial g}{\partial y}$ is continuously differentiable in both variables and from the implicit function theorem, we get that $y(t)$ is continuously differentiable, so that φ^η is twice continuously differentiable. Differentiating (6.1) we get (6.2).

Straightforward computations leads to $\ddot{\varphi}^\eta \equiv 0$ and thus

$$\varphi^\eta(t) = \frac{x - \theta}{1 - \eta}(t - \eta) + \theta.$$

□

Another useful lemma will be the following

Lemma 6.2 *For any $\eta \in [0, 1[$, if there exists $t_0 \in [\eta, 1]$ such that $\varphi^\eta(t_0) = 2\sqrt{t_0}$, then*

$$I_\eta(\varphi^\eta) \geq \int_2^x \sqrt{u^2 - 4} \, du.$$

Proof: We have

$$I_\eta(\varphi^\eta) \geq \int_{t_0}^1 f(t, \varphi^\eta(t), \dot{\varphi}^\eta(t)) dt.$$

For any φ such that $I_\eta(\varphi) < \infty$, for $t \geq \eta$, we denote

$$K_t(\varphi) = \frac{1}{2} \int_t^1 \left(\dot{\varphi}(s) - \frac{1}{2s}(\varphi(s) + \sqrt{\varphi^2(s) - 4s}) \right)^2 ds.$$

If we let $y(s) = \frac{\varphi(s)}{\sqrt{s}}$, one has

$$K_t(\varphi) = I_t(\varphi) - \int_{y(t)}^{y(1)} \sqrt{u^2 - 4} \, du.$$

But $K_t \geq 0$ so that

$$I_\eta(\varphi^\eta) \geq \int_{\frac{\varphi^\eta(t_0)}{\sqrt{t_0}}}^x \sqrt{u^2 - 4} \, du = \int_2^x \sqrt{u^2 - 4} du.$$

□

From there, we can prove

Lemma 6.3 *For any $\eta \in [0, 1[$, if $\theta = 2\sqrt{\eta}$ or $\left(2\sqrt{\eta} < \theta < 1 + \eta \text{ and } x \leq \frac{\theta + \sqrt{\theta^2 - 4\eta}}{2} + \frac{2}{\theta + \sqrt{\theta^2 - 4\eta}} \right)$, then*

$$J_{\eta, \theta}(x) = \int_2^x \sqrt{u^2 - 4} \, du.$$

Proof: We first start with the case when $\eta > 0$. Let us assume that $\theta = 2\sqrt{\eta}$ or $2\sqrt{\eta} < \theta < 1 + \eta$ and $x \leq \frac{\theta + \sqrt{\theta^2 - 4\eta}}{2} + \frac{2}{\theta + \sqrt{\theta^2 - 4\eta}}$. Let φ^η be a minimizer. By Lemma 6.2, we have that $J_{\eta, \theta}(x) \geq \int_2^x \sqrt{u^2 - 4} \, du$. But we can exhibit functions for which this bound is reached.

For $\theta = 2\sqrt{\eta}$ and $x \geq 2$, we let t^* be such that $\sqrt{t^*} := \frac{x + \sqrt{x^2 - 4}}{2}$ and

$$\varphi_\eta^*(t) = \begin{cases} 2\sqrt{t} & \text{if } \eta \leq t \leq t^* \\ 2\sqrt{t^*} + \frac{1}{\sqrt{t^*}}(t - t^*) & \text{if } t^* \vee \eta \leq t \leq 1 \end{cases}$$

Taking into account that

$$\int_2^x \sqrt{z^2 - 4} dz = \frac{1}{2}x\sqrt{x^2 - 4} - 2 \ln \left(\frac{x + \sqrt{x^2 - 4}}{2} \right), \quad (6.3)$$

we get that φ_η^* realises the infimum.

For $2\sqrt{\eta} < \theta < 1 + \eta$ and $x \leq \frac{\theta + \sqrt{\theta^2 - 4\eta}}{2} + \frac{2}{\theta + \sqrt{\theta^2 - 4\eta}}$, we let s^* be such that $\sqrt{s^*} = \frac{\theta + \sqrt{\theta^2 - 4\eta}}{2}$, and

$$\varphi_\eta^*(t) = \begin{cases} \theta + \frac{1}{\sqrt{s^*}}(t - \eta) & \text{if } \eta \leq t \leq s^* \\ 2\sqrt{t} & \text{if } s^* \leq t \leq t^* \\ 2\sqrt{t^*} + \frac{1}{\sqrt{t^*}}(t - t^*) & \text{if } t^* \leq t \leq 1 \end{cases}$$

realises the infimum.

Now, the extension to the case when $\eta = 0$ is easy to obtain. Let us assume that $\theta = 0$ or $0 < \theta < 1$ and $x < \theta + \frac{1}{\theta}$. Let φ be a path such that $I(\varphi) < \infty$. It implies in particular φ is continuous and therefore it is easy to check that there exists $\eta > 0$ such that $\varphi(\eta) = 2\sqrt{\eta}$ or $2\sqrt{\eta} < \varphi(\eta) < 1 + \eta$ and $x \leq \frac{\varphi(\eta) + \sqrt{\varphi(\eta)^2 - 4\eta}}{2} + \frac{2}{\varphi(\eta) + \sqrt{\varphi(\eta)^2 - 4\eta}}$. From the first part of the proof applied to $\theta = \varphi(\eta)$, we get that $I_0(\varphi) \geq I_\eta(\varphi) \geq \int_2^x \sqrt{u^2 - 4} du$. To conclude the proof, it is easy to check that φ_0^* realises this infimum. \square

We now go to

Lemma 6.4 *For any $\eta \in]0, 1[$, if $\theta > 1 + \eta$ or $\left(2\sqrt{\eta} < \theta < 1 + \eta \text{ and } x > \frac{\theta + \sqrt{\theta^2 - 4\eta}}{2} + \frac{2}{\theta + \sqrt{\theta^2 - 4\eta}}\right)$, then*

$$J_{\eta, \theta}(x) = I_\eta(d_{\eta, \theta, x}),$$

where for any $t \in [\eta, 1]$,

$$d_{\eta, \theta, x}(t) = \frac{(x - \theta)}{1 - \eta}(t - \eta) + \theta.$$

Proof: From Lemma 6.1, we know that $d_{\eta, \theta}$ realises the infimum over the functions that do not touch $t \mapsto 2\sqrt{t}$ and from Lemma 6.2 that the infimum over the functions that do touch $t \mapsto 2\sqrt{t}$ is always greater than $\int_2^x \sqrt{u^2 - 4} du$.

In the case when $2\sqrt{\eta} < \theta < 1 + \eta$ and $x > \frac{\theta + \sqrt{\theta^2 - 4\eta}}{2} + \frac{2}{\theta + \sqrt{\theta^2 - 4\eta}}$, it is a direct computation to check that $I_\eta(d_{\eta, \theta, x}) \leq \int_2^x \sqrt{u^2 - 4} du$.

We now assume that $\theta > 1 + \eta$. Let φ be a function that do touch $t \mapsto 2\sqrt{t}$ and t_0 be such that $\varphi(t_0) = 2\sqrt{t_0}$ and for $t \in [\eta, t_0[$, $\varphi(t) > 2\sqrt{t}$. Obviously, $I_\eta(\varphi) = \int_\eta^{t_0} f(t, \varphi(t), \dot{\varphi}(t)) dt + I_{t_0}(\varphi)$. We know for above that $I_{t_0}(\varphi) > \int_2^x \sqrt{u^2 - 4} du$.

Furthermore, by the same reasoning as in the proof of Lemma 6.1, we can show that the optimum in the first integral is reached at the affine function $h(t) = \frac{2\sqrt{t_0} - \theta}{t_0 - \eta}(t - \eta) + \theta$. A quite tedious computation allows us to get that, for any $t_0 \in [\eta, 1]$,

$$I_\eta(d_{\eta, \theta, x}) \leq \int_2^x \sqrt{u^2 - 4} du + \int_\eta^{t_0} f(t, h(t), \dot{h}(t)) dt$$

and this concludes the proof. \square

The last point, to complete the proof of the Theorem is to extend the above lemma to the case when $\eta = 0$. More precisely, we have,

Lemma 6.5 *If $\theta \geq 1$ or $(0 < \theta < 1 \text{ and } x \geq \theta + \frac{1}{\theta})$, let $\varphi_*(t) = (x - \theta)t + \theta$ for $t \in [0, 1]$, then for any φ , $I_0(\varphi_*) \leq I_0(\varphi)$*

This would conclude the proof by the following computation. We set $\alpha = x - \theta$.

$$\begin{aligned} I(\varphi_*) &= \frac{1}{2} \int_0^1 \left(\alpha - \frac{1}{2t}(\theta + \alpha t - \sqrt{(\theta + \alpha t)^2 - 4t}) \right)^2 dt \\ &= \frac{1}{2} \int_0^1 \left(\alpha - \frac{2}{\theta + \alpha t + \sqrt{(\theta + \alpha t)^2 - 4t}} \right)^2 dt \end{aligned}$$

The change of variable $u = \theta + \alpha t + \sqrt{(\theta + \alpha t)^2 - 4t}$ leads after some tedious computations

$$I(\varphi_*) = -\ln \left(\frac{x + \sqrt{x^2 - 4}}{2\theta} \right) + \frac{1}{4}x\sqrt{x^2 - 4} + \frac{1}{4}x^2 - \theta x + \frac{\theta^2}{2} + \frac{1}{2}.$$

This agrees with the formulae giving M_θ and L_θ since we have (6.3). \square

Let us now go to the proof of Lemma 6.5. Let φ be a function on $[0, 1]$ such that $I_0(\varphi) < \infty$. From Lemma 4.4, one can always assume that φ is smooth. Let $\varepsilon > 0$ be fixed. Let η small enough, such that $\int_0^\eta f(t, \varphi_*(t), \dot{\varphi}_*(t)) dt \leq \varepsilon$ and $\varphi(\eta) > 1$ or

$$2\sqrt{\eta} < \varphi(\eta) \leq 1 + \eta \text{ and } x > \frac{\varphi(\eta) + \sqrt{\varphi(\eta)^2 - 4\eta}}{2} + \frac{2}{\varphi(\eta) + \sqrt{\varphi(\eta)^2 - 4\eta}}.$$

Then

$$\begin{aligned} I_0(\varphi) - I_0(\varphi_*) &= \int_0^\eta f(t, \varphi(t), \dot{\varphi}(t)) dt - \int_0^\eta f(t, \varphi_*(t), \dot{\varphi}_*(t)) dt \\ &\quad + \int_\eta^1 f(t, \varphi(t), \dot{\varphi}(t)) dt - \int_\eta^1 f(t, d_{\eta, \varphi(\eta), x}(t), \dot{d}_{\eta, \varphi(\eta), x}(t)) dt \\ &\quad + \int_\eta^1 f(t, d_{\eta, \varphi(\eta), x}(t), \dot{d}_{\eta, \varphi(\eta), x}(t)) dt - \int_\eta^1 f(t, \varphi_*(t), \dot{\varphi}_*(t)) dt. \end{aligned}$$

On the first line, the first term is positive and the second one is greater than $-\varepsilon$. The second line is positive, as we know from Lemma 6.4 that $d_{\eta, \varphi(\eta), x}$ realises the infimum in this case. Now, using the continuity of φ and the fact that $\frac{\theta - \varphi(\eta)}{\eta}$ is bounded, it is a straightforward computation to check that the last term can be bounded by a function of η going to zero with η . Therefore, for η small enough, $I_0(\varphi) - I_0(\varphi_*) \geq -2\varepsilon$. But this is true for any ε so that $I_0(\varphi) - I_0(\varphi_*) \geq 0$. \square

Acknowledgments: We would like to thank Marc Yor who suggested this problem to us. We are really indebted to Ofer Zeitouni, who helped us with a crucial suggestion leading to the argument for the upper bound developed in Section 5. We thank for his hospitality the American Institute of Mathematics (AIM) where the discussion with OZ took place.

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