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ALGEBRAIC VARIETIES WITH QUASI-PROJECTIVE UNIVERSAL COVER

BENOÎT CLAUDON, ANDREAS HÖRING, AND JÁNOS KOLLÁR

Abstract. We prove that the universal cover of a normal, projective variety X is quasi-projective iff a finite, étale cover of X is a fiber bundle over an Abelian variety with simply connected fiber.

1. Introduction

In his book [Sha74, Sec.IX.4], Shafarevich emphasizes the need to understand universal covers of smooth, projective varieties. Although his conjectures may not hold in general [BK98], they are true for groups with faithful linear representations [EKPR09]. Applications of the general ideas behind the Shafarevich conjectures are discussed in [Kol95]. These methods are especially powerful if the universal cover is easy to describe, as it happens for Abelian varieties, whose universal cover is $\mathbb{C}^n$. This suggests that one should study projective varieties whose universal cover is quasi-projective.

There are two significant results in this direction.

• Nakayama shows [Nak99, Thm.1.4] that the universal cover of a smooth, projective variety X is quasi affine if and only if X has a finite étale cover that is an Abelian variety.

• It is a consequence of the Beauville–Bogomolov decomposition theorem [Bea83] that if X is a Calabi–Yau variety, then its universal cover $\tilde{X}$ is biholomorphic to $Y \times \mathbb{C}^m$ where Y is a compact, simply connected, Calabi–Yau variety. In particular, $\pi_1(X)$ is almost abelian and $\tilde{X}$ is biholomorphic to a quasi-projective variety.

In this paper we prove the following theorem which can be viewed as a common generalization of these results. While we give a complete answer, the proof assumes the validity of the abundance conjecture (1.3). This assumption is already present in [Nak99] explicitly and in [Bea83] implicitly.

1.1. Theorem. *Assume that the abundance conjecture (1.3) holds. Then, for any normal, projective variety X the following are equivalent.*

1.) *The universal cover of X is biholomorphic to a quasi-projective variety.*
2.) *There is a finite, étale, Galois cover $X' \to X$ that is a fiber bundle over an Abelian variety with simply connected fiber.*
3.) *The universal cover $\tilde{X}$ is biholomorphic to a product $\mathbb{C}^m \times F$ where $m \geq 0$ and F is a projective, simply connected variety.*

Note, however, that in general there is no finite, étale, Galois cover $X' \to X$ that is a product of an Abelian variety with F.

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We want to emphasize that in (1.1.1) we do not assume that the deck transformations are algebraic automorphisms of the universal cover and in fact this is not true in general (3.1).

It is clear that (1.1.3) $\Rightarrow$ (1.1.1) and (1.1.2) implies (1.1.3) since every fiber bundle over $\mathbb{C}^m$ with compact analytic fiber is trivial (4.4).

The proof of (1.1.1) $\Rightarrow$ (1.1.2) comes in two independent steps, both of which are more general than needed for (1.1). First we show in (1.2) that the fundamental group of X is almost abelian, that is, it contains an abelian subgroup of finite index. Here we use the abundance conjecture to rule out some possible counter examples.

Once we know that the fundamental group of X is almost abelian, by passing to a finite cover we may assume that it is in fact abelian. Then we prove directly in (1.4) that the Albanese morphism is a fiber bundle. This part does not rely on any conjectural assumptions.

1.2. Proposition. *Let X have the smallest dimension among all normal, projective varieties that have an infinite, quasi-projective, Galois cover $\tilde{X} \to X$ whose Galois group is not almost abelian.*

Then X is smooth and its canonical class K_X is nef but not semi-ample. (That is, $(K_X \cdot C) \geq 0$ for every algebraic curve $C \subset X$ but $\mathcal{O}_X(mK_X)$ is not generated by global sections for any $m > 0$.)

The conclusion would contradict the following, so called *abundance conjecture* [Rei87, Sec.2]. Thus if (1.3) holds then X as in (1.2) can not exist. Thus if a normal, projective variety Y has an infinite, quasi-projective, Galois cover $\tilde{Y} \to Y$ then the Galois group is almost abelian.

1.3. Conjecture. *Let X be a smooth projective variety such that K_X is nef. Then K_X is semi-ample.*

Note that the abundance conjecture is frequently stated for varieties with log canonical singularities, even for log canonical pairs, but we need only the smooth case. The conjecture is known to hold if $\dim X \leq 3$; see [Kol92] for a detailed treatment.

Next we study the quasi-projectivity of Abelian covers.

1.4. Theorem. *Let X be a normal, projective variety and $\alpha : X \to A$ a morphism to an Abelian variety. Let $\pi : \tilde{A} \to A$ be an étale Galois cover with group Γ such that $\tilde{A}$ has no compact analytic subvarieties. By pull-back we obtain $\tilde{\alpha} : \tilde{X} \to \tilde{A}$.*

If $\tilde{X}$ is quasi-projective then $\alpha : X \to A$ is a locally trivial fiber bundle.

In light of the previous statements, we can strengthen Theorem 1.1 as follows:

1.5. Corollary. *Assume that the abundance conjecture (1.3) holds. Let X be a normal, projective variety and $\tilde{X} \to X$ an infinite étale Galois cover such that $\tilde{X}$ is quasi-projective. Then there exist*

1.) a finite, étale, Galois cover $X' \to X$,

2.) a morphism to an Abelian variety $\alpha : X' \to A$ which is a locally trivial fiber bundle and

3.) an étale cover $\pi : \tilde{A} \to A$ such that $\tilde{A}$ has no compact analytic subvarieties

such that $\tilde{X}$ pulls-back from $\tilde{A} \to A$, that is $\tilde{X} \simeq X' \times_A \tilde{A}$.

We do not know if the converse of (1.5) holds or not. Every étale cover $\tilde{A} \to A$ of an Abelian variety is quasi-projective [CC91]. Note, however, that it can happen

that $\tilde{A}$ has no compact analytic subvarieties yet it is not Stein [AK01]. Even if $X \to A$ is a $\mathbb{P}^1$-bundle, we do not know if $X \times_A \tilde{A}$ is quasi-projective or not.

1.6 (The non-algebraic case)**.** More generally, it is interesting to study compact complex manifolds M whose universal cover $\tilde{M}$ is a Zariski open submanifold of a compact complex manifold $\bar{M}$. Besides the algebraic cases, such examples are given by Hopf manifolds, compact nilmanifolds or more generally any quotient $G\backslash\Gamma$ where G is a (simply connected) non-commutative linear algebraic group and Γ a cocompact lattice (see for instance [Akh95, §3.4 and 3.9], [Win98]). Their classification seems rather difficult.

The problem becomes much more tractable if one assumes that M (and possibly also $\bar{M}$) are Kähler. We expect that in this case (1.1) should hold, but several steps of the proof need to be changed. We plan to discuss these in a subsequent paper.

1.7 (The quasi-projective case)**.** Our methods rely on the study of compact subvarieties of $\tilde{X}$, but it is possible that similar results hold if X is quasi-projective. Very little seems to be known. For instance, we do not know which quasi-projective varieties X have $\mathbb{C}^n$ as their universal cover. The obvious guess is that every such X has a finite, étale, Galois cover $X' \to X$ such that $X' \cong \mathbb{C}^n/\mathbb{Z}^m$ where $m \leq 2n$ and $\mathbb{Z}^m$ acts on $\mathbb{C}^n$ by translations.

The strongest result would be the following analog of (1.5).

Question. Let X be a normal, quasi-projective variety and $\tilde{X} \to X$ an infinite étale Galois cover such that $\tilde{X}$ is quasi-projective. Does there exist

1.) a finite, étale, Galois cover $X' \to X$,
2.) a morphism to a quasi-projective abelian group $\alpha : X' \to A$ that is a locally trivial fiber bundle and
3.) an étale cover $\tilde{A} \to A$ such that $\tilde{A}$ has no compact analytic subvarieties

such that $\tilde{X} \simeq X' \times_A \tilde{A}$?

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2. Algebraic subvarieties of universal covers

Let X be a projective variety and $\pi : \tilde{X} \to X$ an infinite Galois cover with group Γ such that $\tilde{X}$ is biholomorphic to a quasi-projective variety. There is no reason to assume that such a quasi-projective variety is unique. In what follows, we fix one such quasi-projective variety and say that $\tilde{X}$ is quasi-projective. If $Z \subset X$ is a closed subvariety, then its preimage $\tilde{Z} := \pi^{-1}(Z) \subset \tilde{X}$ is a closed, analytic subspace of $\tilde{X}$, but it is rarely quasi-projective.

For instance, let A be an Abelian variety with universal cover $\pi : \mathbb{C}^n \to A$. If $Z \subsetneq A$ is a closed subvariety, then its preimage $\tilde{Z} \subset \mathbb{C}^n$ is never quasi-projective by (1.4). Similarly, let E be a rank 2 vector bundle on A that is not an extension of 2 line bundles. Set $X = \mathbb{P}_A(E)$ with universal cover $\pi : \tilde{X} \to X$. One can see that if $Z \subsetneq X$ is a closed subvariety, then its preimage $\tilde{Z} \subset \mathbb{C}^n$ is never quasi-projective.

We aim to exploit this scarcity of Γ-invariant subvarieties as follows. If $\tilde{X}$ has no positive dimensional compact subvarieties then we are done by [Nak99] (though

this is not how our proof actually goes). Thus let $F \subset \tilde{X}$ be a positive dimensional compact subvariety. Let $\mathrm{Locus}(F, \tilde{X}) \subset \tilde{X}$ denote the union of the images of all finite morphisms $F \to \tilde{X}$. (We are mainly interested in embeddings $F \hookrightarrow \tilde{X}$, but allowing finite maps $F \to \tilde{X}$ works better under finite étale covers. We restrict to finite maps mostly to avoid the constant maps $F \to \tilde{X}$.) It is clear that $\mathrm{Locus}(F, \tilde{X})$ is Γ-invariant. Unfortunately, in general we can only prove that $\mathrm{Locus}(F, \tilde{X})$ is a *countable* union of (locally closed) algebraic subvarieties of $\tilde{X}$. There are, however, 2 special cases where we show that $\mathrm{Locus}(F, \tilde{X})$ is a (possibly reducible and locally closed) algebraic subvariety of $\tilde{X}$. If $\dim \mathrm{Locus}(F, \tilde{X}) < \dim \tilde{X}$, then we use induction to describe $\mathrm{Locus}(F, \tilde{X})$ and arrive at a contradiction. If $\dim \mathrm{Locus}(F, \tilde{X}) = \dim \tilde{X}$, then we obtain a strong structural description of $\tilde{X}$.

2.1. Definition. *Let U and V be normal, quasi-projective varieties, and $U \to V$ a flat, projective morphism with a relatively ample divisor H_V. Let Y be a normal, quasi-projective variety and L the restriction of an ample line bundle on some completion $Y \subset \overline{Y}$ to Y.*

We denote by $\mathrm{FinMor}(U/V, Y, H_V, L, d)$ *the moduli space of finite morphisms $\phi : U_v \to Y$ of degree d, that is, if U_v is a fiber and $\Gamma_\phi \subset U \times Y$ is the graph of ϕ, then*

$$\big(p_U^* H_V + p_Y^* L\big)^{\dim U/V} \cdot \Gamma_\phi = d.$$

Note that our "degree" is not the degree of the image of ϕ, rather the degree of the graph of ϕ. Since the (relative) cycle spaces $\mathrm{Chow}(U/V)$ and $\mathrm{Chow}(\overline{Y})$ are projective (over the base V) and the property of being a graph of a morphism is open in the Zariski topology, we see that $\mathrm{FinMor}(U/V, Y, H_V, L, d)$ is a quasi-projective subvariety of $\mathrm{Chow}(U \times \bar{Y}/V)$. (This would fail if we considered only the degree of the image of ϕ.)

In order to simplify the notation, we will abbreviate $\mathrm{FinMor}(U/V, Y, H_V, L, d)$ by $\mathrm{FinMor}(U/V, Y, d)$. We have universal families and morphisms

$$\mathrm{Univ}(U/V, Y, d) \to \mathrm{FinMor}(U/V, Y, d) \quad \text{and} \quad \Phi_d : \mathrm{Univ}(U/V, Y, d) \to Y.$$

Set $\mathrm{FinMor}(U/V, Y) := \bigcup_d \mathrm{FinMor}(U/V, Y, d)$ with universal family

$$\mathrm{FinMor}(U/V, Y) \leftarrow \mathrm{Univ}(U/V, Y) \xrightarrow{\Phi} Y.$$

Note that $\mathrm{FinMor}(U/V, Y)$ and $\mathrm{Univ}(U/V, Y)$ are, in general, countable unions of quasi-projective varieties.

The union of all the images of fibers of $U \to V$ by degree d maps

$$\mathrm{Locus}(U/V, Y, d) := \Phi_d\big(\mathrm{Univ}(U/V, Y, d)\big) \subset Y$$

is a constructible algebraic subset of Y and

$$\mathrm{Locus}(U/V, Y) := \Phi\big(\mathrm{Univ}(U/V, Y)\big) \subset Y$$

is, in general, a countable union of constructible algebraic subsets.

Our main interest is in the case $Y = \tilde{X}$ where $\tilde{X} \to X$ is Galois with group Γ. If we can take L to be Γ-equivariant then each $\mathrm{Locus}(U/V, \tilde{X}, d)$ is Γ-invariant. We see, however, no a priori reason why this should be possible. First, since the Γ-action is holomorphic but in general not algebraic, we do not even know that pulling back by $\gamma \in \Gamma$ maps an algebraic coherent sheaf on $\tilde{X}$ to an algebraic coherent sheaf. Second, even if we know that the Γ-action is algebraic, there need not be any Γ-equivariant ample line bundles.

2.2. The main construction. Let U and V be normal, quasi-projective varieties, and $U \to V$ a flat, projective morphism with a relatively ample divisor H_V. Let X be a projective variety and $\pi : \tilde{X} \to X$ an infinite Galois cover with group Γ such that $\tilde{X}$ is quasi-projective. Let furthermore L be a line bundle that is the restriction of an ample line bundle on some completion $\tilde{X} \subset \overline{X}$ to $\tilde{X}$.

Consider $\mathrm{FinMor}_\ell(U/V, X) \subset \mathrm{FinMor}(U/V, X)$, parametrizing those morphisms $\phi : U_v \to X$ that can be lifted to $\tilde{\phi} : U_v \to \tilde{X}$. Note that Γ acts freely on $\mathrm{FinMor}(U/V, \tilde{X})$ and we have a natural holomorphic map

$$\pi_M : \mathrm{FinMor}(U/V, \tilde{X}) \to \mathrm{FinMor}(U/V, \tilde{X})/\Gamma = \mathrm{FinMor}_\ell(U/V, X).$$

Let $W \subset \mathrm{FinMor}_\ell(U/V, X)$ be an irreducible component and $\tilde{W} := \pi_M^{-1}(W) \subset \mathrm{FinMor}(U/V, \tilde{X})$ its preimage. Every irreducible component of $\tilde{W}$ is quasi-projective, but usually there are infinitely many and Γ permutes them. Thus we do not get any new algebraic variety with Γ-action.

There are, however, two important cases when such a $\tilde{W}$ has finitely many irreducible components, hence is itself quasi-projective. We discuss these in (2.3) and (2.5).

2.3. Lemma. *Let $W \subset \mathrm{FinMor}_\ell(U/V, X)$ be an irreducible component and $W \leftarrow \mathrm{Univ}_W \to X$ the corresponding universal family. Assume that $\mathrm{Univ}_W \to X$ is dominant.*

Then $\tilde{W} := \pi_M^{-1}(W) \subset \mathrm{FinMor}(U/V, \tilde{X})$ is quasi-projective. Moreover, if $V' \subset V$ is an algebraic subvariety and $U' \to V'$ the corresponding family then $\tilde{W} \cap \mathrm{FinMor}(U'/V', \tilde{X})$ is also quasi-projective.

Proof. We denote by $\mathrm{Univ}_{\tilde{W}}$ the fiber product $\mathrm{Univ}_W \times_X \tilde{X}$ and by $\tilde{W}$ the Stein factorisation of the map $\mathrm{Univ}_{\tilde{W}} \to \mathrm{Univ}_W \to W$, so we get a commutative diagram:

$$\begin{tikzcd}
\tilde{W} \arrow[d] & \mathrm{Univ}_{\tilde{W}} \arrow[l] \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\pi"'] \\
W & \mathrm{Univ}_W \arrow[l] \arrow[r] & X
\end{tikzcd}$$

By construction each fiber of $\mathrm{Univ}_{\tilde{W}} \to \tilde{W}$ is also a fiber of $U \to V$. Since $\mathrm{Univ}_W \to X$ is dominant, the image of $\pi_1(\mathrm{Univ}_W) \to \pi_1(X)$ has finite index in $\pi_1(X)$. Therefore $\mathrm{Univ}_{\tilde{W}}$ has only finitely many irreducible components, so $\tilde{W}$ has finitely many irreducible components. Thus $\tilde{W}$ is quasi-projective.

For each fixed d, the space of morphisms $\mathrm{FinMor}(U'/V', \tilde{X}, d)$ is an algebraic subset of $\mathrm{FinMor}(U/V, \tilde{X}, d)$. Since $\tilde{W}$ has finitely many irreducible components, $\tilde{W} \cap \mathrm{FinMor}(U'/V', \tilde{X})$ is a closed algebraic subset of a quasi-projective variety, hence itself quasi-projective. $\square$

The following consequence will be used repeatedly.

2.4. Lemma. *Let X be a projective variety and $\pi : \tilde{X} \to X$ an infinite Galois cover with group Γ such that $\tilde{X}$ is quasi-projective. Let $X^0 \subset X$ be a dense, Zariski open subset and $g^0 : X^0 \to Z^0$ a flat, proper morphism with connected general fiber F such that π induces a finite covering $\tilde{F} \to F$. Let $\tilde{g}^0 : \tilde{X}^0 \to \tilde{Z}^0$ be the corresponding flat, proper morphism with general fiber $\tilde{F}$. Then (at least) one of the following holds:*

1.) $\tilde g^0$ extends to a locally trivial, Γ-equivariant fibration $\tilde g : \tilde X \to \tilde Z$, or
2.) $\tilde X$ contains a closed Γ-invariant subvariety that is disjoint from a general fiber of $\tilde g^0$.

Proof. Let L be a line bundle that is the restriction of an ample line bundle on some completion $\tilde X \subset \overline{X}$ to $\tilde X$.

By assumption π induces a finite covering $\tilde F_v \to F_v$, say of degree m, on the fibers of g. Let $U \to V$ be a flat, proper morphism whose fibers are the degree m étale covers of the fibers of g. Let $W \subset \mathrm{FinMor}(U/V, X)$ be an irreducible component parametrizing morphisms $U_v \to X$ whose image is a fiber of g^0. Then $\mathrm{Univ}_W \to X$ is dominant and we can use Lemma 2.3.

We fix an actual g-fiber $\tilde F_z$. By Lemma 2.3 applied to $M(\tilde F_z, X, d)$, we see that $\tilde W \cap M(\tilde F_z, \tilde X)$ is algebraic. Thus the image of the universal family over $\tilde W \cap M(\tilde F_z, X)$ gives a constructible, Γ-invariant subset $\mathrm{Locus}_W(\tilde F_z, \tilde X) \subset \tilde X$. If $\mathrm{Locus}_W(\tilde F_z, \tilde X)$ is not Zariski dense, then its closure is a Γ-invariant, closed, algebraic subset that is disjoint from a general fiber.

Otherwise the morphism $\tilde g^0$ is a locally trivial fiber bundle with fiber $\tilde F_z$ over a Zariski open subset of Z^0. Let $X^* \subset \tilde X$ be the largest open set over which $\tilde g^0$ extends to a locally trivial fiber bundle. Then X^* is Γ-invariant, hence if $X^* \neq \tilde X$ then $\tilde X \setminus X^*$ is a Γ-invariant, closed algebraic subset that is disjoint from a general fiber. Otherwise $X^* = \tilde X$ which shows (1). □

Since $\tilde g : \tilde X \to \tilde Z$ is Γ-equivariant, the Γ-action on $\tilde X$ descends to a Γ-action on $\tilde Z$. If $\tilde F$ has no fixed point free automorphisms, then the Γ-action on $\tilde Z$ is fixed point free, but in general it can have finite stabilizers. In some cases we will show that a finite index subgroup of Γ acts freely on $\tilde Z$, but this does not seem to be automatic.

2.5. Lemma. *Let X be a projective variety and $\pi : \tilde X \to X$ an infinite Galois cover with group Γ such that $\tilde X$ is quasi-projective. Let $X^0 \subset X$ be a dense, Zariski open subset and $g^0 : X^0 \to Z^0$ a proper, birational morphism. Let $E^0 \subset \mathrm{Ex}(g^0)$ be a maximal dimensional irreducible component of the exceptional set, $F \subset E^0$ a general fiber of $g^0|_{E^0}$ and $E \subset X$ the closure of E^0. Assume that $\dim E + \dim F \geq \dim X$ and π induces a finite covering $\tilde F \to F$.*

Then $\pi^{-1}\bigl(E\bigr)$ is an algebraic subvariety of $\tilde X$.

Proof. Let $\bar X \supset \tilde X$ be a smooth, algebraic compactification. It is clear that $\pi^{-1}\bigl(E\bigr)$ is a closed analytic subspace of $\tilde X$; let $\tilde E_i$ be its irreducible components.

Assume first that each $\tilde E_i$ is algebraic. By (2.6), there is a subvariety $G \subset F$ such that the intersection number $(G \cdot E)_X \neq 0$. Set $G_i := \tilde E_i \cap \pi^{-1}(G)$. Then $\bigl(G_i \cdot \tilde E_i\bigr)_{\bar X} \neq 0$ for every i and $\bigl(G_i \cdot \tilde E_j\bigr)_{\bar X} \neq 0$ for $i \neq j$. Thus the homology classes of the closures $[\bar E_i] \in H_*\bigl(\bar X, \mathbb{Q}\bigr)$ are linearly independent, and therefore $\pi^{-1}\bigl(E\bigr)$ has only finitely many irreducible components. Each is algebraic by assumption, thus $\pi^{-1}\bigl(E\bigr)$ is an algebraic subvariety of $\tilde X$.

Thus it remains to show that each $\tilde E_i$ is algebraic.

Let $E^1 \subset E^0$ be an open subset such that $f^1 := f|_{E^1} : E^1 \to f(E^1)$ is proper and flat. By assumption π induces a finite covering $\tilde F_v \to F_v$ say of degree m, on each connected component of a fiber of f^1. Let $U \to V$ be a flat, proper morphism whose fibers are the degree m étale covers of these F_v.

Note that if a morphism $\phi : \tilde{F}_v \to X$ maps to a fiber of f^1 then so does every small deformation of it. Thus, for each i, there is an irreducible component $W_i \subset \mathrm{FinMor}(U/V, \tilde{X})$ such that $\Phi\big(\mathrm{Univ}_{W_i}\big) \subset \tilde{X}$ is a Zariski dense constructible subset of E_i. Therefore every E_i is algebraic. □

2.6. Lemma. *Let $f : X \to Y$ be a projective, birational morphism, X smooth. Let $E \subset \mathrm{Ex}(f)$ be a maximal dimensional irreducible component of the exceptional set and $F \subset E$ a general fiber of $f|_E$. The following are equivalent.*

1.) There is a subvariety $G \subset F$ such that the intersection number $(G{\cdot}E)_X \neq 0$.
2.) $\dim E + \dim F \geq \dim X$.

Proof. Note first that although X and E are not assumed compact, the intersection number $(G \cdot E)_X$ is defined where the subscript indicates that we compute the intersection number in X. If it is nonzero then $\dim E + \dim G = \dim X$, thus (1) implies (2).

To see the converse, note that if we take a general hyperplane section of Y and replace X by its preimage, the inequality in (2) remains valid. Thus, after taking $\dim f(E)$ hyperplane sections, we can suppose that $E = F$ maps to a point and $2 \dim F \geq \dim X$. Next we take hyperplane sections of X. After r steps, eventually we are reduced to consider $f_r : X_r \to Y_r$ such that the exceptional set E_r maps to a point and $2 \dim E_r = \dim X_r$. Set $G = E_r$. Then $\big(G \cdot E\big)_X = \big(G \cdot G\big)_{X_r}$ and $(-1)^{\dim G}\big(G \cdot G\big)_{X_r} > 0$ by [dCM02, Thm.2.4.1]. □

3. Proofs of the main results

Proof of Proposition 1.2. Let X be a normal, projective variety and $\tilde{X} \to X$ a quasi-projective Galois cover with group Γ. We study where X fits into the birational classification plan of varieties and we show that many cases would lead to a lower dimensional normal, projective variety Y and a quasi-projective Galois cover $\tilde{Y} \to Y$ with group Γ_Y that is a finite index subgroup of Γ.

After several such tries, we see that there is no place for the smallest dimensional example, unless Γ is almost Abelian.

Step 1: X is smooth. First we claim that $\tilde{X}$ has no nontrivial, closed, subvariety invariant under a finite index subgroup $\Gamma' \subset \Gamma$. Indeed, given such $\tilde{W}$ with irreducible components $\tilde{W}_i$, each of them is invariant under a finite index subgroup $\Gamma_i \subset \Gamma$. Taking the normalization $\tilde{W}_i^n$, we would get a smaller dimensional example $W_i^n := \tilde{W}_i^n/\Gamma_i$ as in (1.2); a contradiction.

Since $\mathrm{Sing}\,\tilde{X} \subset \tilde{X}$ is algebraic and Γ-invariant, we conclude that X is smooth.

Step 2: K_X is nef. If K_X is not nef, there is an extremal contraction $g : X \to Z$ [KM98, Thm.3.7].

Assume first that g is not birational and let $F \subset X$ denote a smooth fiber. Then F is a smooth Fano variety, in particular it is rationally connected [Cam92, KMM92]. Rationally connected manifolds are simply connected, so the fibre F lifts to $F \hookrightarrow \tilde{X}$. Thus, by (2.4), there is a locally trivial fiber bundle $\tilde{X} \to \tilde{Z}$ with fiber F. The variety $\tilde{Z}$ is quasi-projective by (4.8). Note that F does not admit fixed point free actions by any finite group: the étale quotient would also be rationally connected, so simply connected. Therefore the stabilizer $\mathrm{stab}_\Gamma(F_z)$ is trivial for every fiber F_z of $\tilde{X} \to \tilde{Z}$. Hence the Γ-action descends to a free Γ-action on $\tilde{Z}$; a contradiction to the minimality of the dimension of X.

Assume next that g is birational. Let $E \subset \mathrm{Ex}(g)$ be a maximal dimensional irreducible component of the exceptional set and $F \subset E$ a general fiber of $g|_E$. By the Ionescu-Wiśniewski inequality (see for instance [AW97, Thm.2.3]) one has $\dim F + \dim E \geq \dim X$. By [Kol93, 7.5] and [Tak03, Thm.1.2], the map $\pi_1(X) \to \pi_1(Z)$ is an isomorphism, thus the embedding $F \hookrightarrow X$ lifts to $F \hookrightarrow \tilde{X}$.

Therefore (2.5) implies that $\pi^{-1}(E) \subset \tilde{X}$ is an algebraic and Γ-invariant subset of $\tilde{X}$. This is again a contradiction, thus K_X is nef.

Step 3: The Iitaka fibration. If K_X is not semiample, then we are done. Otherwise K_X is semiample and for sufficiently divisible $m > 0$, the sections of $\mathcal{O}_X(mK_X)$ define a morphism (called the Iitaka fibration) $\tau : X \to I(X)$ with connected fibers such that $\mathcal{O}_X(mK_X) \cong \tau^* M$ for some ample line bundle M. By the adjunction formula, $mK_F \sim 0$ for any smooth fiber $F \subset X$.

Ideally we would like to apply (2.4) to $\tau : X \to I(X)$ and conclude that $X \cong I(X)$. However, in general $\tilde{X} \to X$ induces an infinite cover of $\tilde{F} \to F$, hence (2.4) does not apply. Instead we first study compact subspaces of $\tilde{F}$ and then move on to the case when $\tilde{F}$ has no positive dimensional compact subspaces.

Step 4: Excluding compact subspaces of $\tilde{F}$. Here we prove that there is a finite, étale cover $F' \to F$ that is an Abelian variety and there is no positive dimensional subvariety $B \subset F$ such that $\pi_1\big(B\big) \to \Gamma$ has finite image.

This could be done in one step, but it may be more transparent to handle the two assertions separately.

As a consequence of the Beauville-Bogomolov decomposition theorem, a suitable finite, étale, Galois cover $F' \to F$ with group G admits a G-equivariant morphism $F' \to A$ where A is an Abelian variety and the fibers are simply connected. Thus $F \to A/G$ is a morphism whose general fibers have torsion canonical class and finite fundamental group. Moreover, at least over a dense open subset of $X^0 \subset X$, these maps give a proper, smooth morphism $X^0 \to Z^0$ whose fibers D have torsion canonical class and finite fundamental group. Thus, by (2.4), there is a locally trivial fiber bundle $\tilde{\tau} : \tilde{X} \to \tilde{Z}$ with fiber $\tilde{D}$ for some finite étale cover $\tilde{D} \to D$.

By Lemmas 4.5 and 4.6, by passing to a finite cover of $\tilde{X}$ we can assume that $\tilde{X} \cong \tilde{D} \times \tilde{Z}$ and the product decomposition is unique. Thus the Γ-action on $\tilde{X}$ gives a homomorphism of Γ to $\mathrm{Aut}\big(\tilde{D}\big)$ with finite image. So there is a finite index subgroup $\Gamma_0 \subset \Gamma$ that acts trivially on $\tilde{D}$. Then for any $p \in \tilde{D}$, the section $\{p\} \times \tilde{Z}$ is quasi-projective and Γ_0-invariant; again a contradiction.

Thus now we know that a general fiber F of $\tau : X \to I(X)$ has a finite, étale cover $F' \to F$ that is an Abelian variety.

Assume next that for every general fiber F there are positive dimensional subvarieties $B_i \subset F'$ such that $\pi_1\big(B_i\big) \to \Gamma$ has finite image. The largest dimensional such subvarieties are an Abelian subvariety $B' \subset F'$ and its translates.

Consider the relative Γ-Shafarevich map for $X \to I(X)$ [Kol93, 3.10]. We get a dense open set X^0 and a smooth, proper morphism $\rho : X^0 \to Y^0$ such that $\pi_1\big(B\big) \to \Gamma$ has finite image for every fiber B of ρ. (Moreover, $X \to I(X)$ factors through ρ and ρ is universal with these properties). The preimage of B in F' is a translate of B'.

As before, we can apply (2.4) to $X^0 \to Y^0$. Thus we obtain a locally trivial fiber bundle $\tilde{X} \to \tilde{Y}$ with fiber $\tilde{B}$. By Lemma 4.5 we may assume that $\tilde{X} \to \tilde{Y}$ is topologically trivial. In particular, $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) + \pi_1(\tilde{B})$.

Thus, by passing to a finite Galois cover of $\tilde{B}$, we can assume that $\pi_1\big(\tilde{B}\big) \to \Gamma$ is the constant map, hence Γ is a quotient of $\pi_1(\tilde{Y})$. Thus the free Γ-action on $\tilde{X}$ descends to a free Γ-action on $\tilde{Y}$. This again contradicts the assumption on the minimality of $\dim X$.

Thus we conclude that the general fiber F of the Iitaka fibration $\tau : X \to I(X)$ has a finite étale cover $F' \to F$ that is an Abelian variety and a very general fiber F has no positive dimensional subvariety $B \subset F$ such that $\pi_1\big(B\big) \to \pi_1(X) \to \Gamma$ has finite image. Thus, in the terminology of [Kol93], X has generically large fundamental group on F.

Step 5: Abelian schemes. This part of the proof closely follows [Nak99]. By [Kol93, 5.9 and 6.3], X has a finite étale cover $X_1 \to X$ that is birational to a smooth projective variety X_2 such that the Iitaka fibration $\tau_2 : X_2 \to I(X_2)$ is smooth with Abelian fibers and general type base. We are thus in position to apply the Kobayashi-Ochiai theorem (4.2): the image $\mathrm{Im}\big(\pi_1(F) \to \Gamma\big)$ has finite index in Γ. We obtain the final contradiction since this implies that Γ is almost abelian. $\square$

Proof of Theorem 1.4. Consider the Stein factorization $X \to B \to A$.

If there is a map $B \to Y$ such that Y is of general type, we know by (4.3) that a finite étale cover is a product of a variety of general type Y' and an Abelian variety. By (4.2) the group Γ induces a finite covering on Y', so $B \times_A \tilde{A}$ has a finite cover that is a product of a variety of general type and a cover of an Abelian variety. In particular $B \times_A \tilde{A}$, hence $\tilde{A}$, contains compact analytic subvarieties. It follows by (4.3) that B is an Abelian variety.

Assume that $g : B \to A$ is not surjective. Then $g(B) \subsetneq A$ is an Abelian subvariety and by Poincaré's theorem there is an Abelian subvariety $C \subset A$ such that $C \cap g(B)$ is finite. Moreover, $\pi_1(B) \to \pi_1(A)$ has infinite index image but $\pi_1(C) + \pi_1(B) \to \pi_1(A)$ has finite index image. By assumption $\tilde{A} \to A$ induces an infinite degree cover of C, thus $\pi^{-1}(B)$ has infinitely many connected components. This is impossible since $\tilde{X}$ is quasi-projective. Thus $g : B \to A$ is surjective. Therefore we can replace A with B and assume to start with that $\alpha : X \to A$ is surjective with connected fibers.

Consider first the case when α is birational. If X is singular then consider $\big(\mathrm{Sing}\, X\big)^n \to A$, the normalization of the singular locus mapping to A. By induction on the dimension, (1.4) applies. This map is, however, not even surjective. Thus X is smooth. Let $E \subset X$ denote an irreducible component of the exceptional divisor $\mathrm{Ex}(\alpha)$ such that $\dim \alpha(E)$ has maximal possible dimension. By (2.5), its preimage $\tilde{E} \subset \tilde{X}$ is quasi-projective. By induction on the dimension we get that $\alpha|_E : E \to A$ is a locally trivial fiber bundle. But this map is not even surjective. Thus α has no exceptional divisors and therefore it is an isomorphism.

If α is not birational, let $F \subset X$ be a general fiber. Suppose that there exists a closed Γ-invariant subvariety $Z \subset \tilde{X}$ that is disjoint from F. Let Z^n be the normalisation, then Z^n/Γ is a normal, projective variety with a morphism to the abelian variety A. Since Z is quasi-projective we see again by induction on the dimension that $Z^n/\Gamma \to A$ is surjective. Thus Z meets F, a contradiction.

Thus we know by (2.4) that $\tilde{X}$ is a locally trivial fiber bundle $\tilde{\tau} : \tilde{X} \to \tilde{Z}$ with fiber F. By (4.8), $\tilde{Z}$ is quasi-projective and $\tilde{\tau}$ factors through $\tilde{\alpha}$. By taking the quotient we obtain

$$\alpha : X \xrightarrow{\tau} Z \xrightarrow{\alpha_Z} A.$$

Note that by construction $Z = \tilde{Z}/\Gamma$ is a normal complex space and $\tau : X \to Z$ is proper and equidimensional. Thus by (4.8), Z is a projective variety. We already saw that these imply that α_Z is an isomorphism. Thus $\tilde{Z} = \tilde{A}$, $\tilde{\alpha} : \tilde{X} \to \tilde{A}$ is a locally trivial fiber bundle and so is $\alpha : X \to A$. □

Proof of Corollary 1.5. Let Γ be the Galois group of the cover $\tilde{X} \to X$. By (1.2) the group Γ is almost abelian, so there exists an intermediate finite étale, Galois cover $X' \to X$ such that $\tilde{X} \to X'$ is Galois with a Galois group Γ' that is free abelian. Let A be the Albanese torus of X'. By (4.7) the group Γ' is a quotient of $H_1(A, \mathbb{Z})$. The maximal sub-Hodge structure contained in the kernel of $H_1(A, \mathbb{Z}) \to \Gamma'$ corresponds to an Abelian subvariety $B \subset A$. Thus up to replacing A by A/B we can suppose that this sub-Hodge structure is zero. Hence if $\pi : \tilde{A} \to A$ is the Galois cover corresponding to the group Γ', the quasi-projective variety $\tilde{A}$ does not have compact analytic subvarieties. Conclude with (1.4). □

Next we give examples of fiber bundles $X \to E$ over elliptic curves whose universal cover $\tilde{X} \to \tilde{E} = \mathbb{C}$ is quasi-projective, yet the deck transformations can not be chosen algebraic.

3.1. Example. We start with a noncompact example.

Let Y be a $\mathbb{C}^*$-bundle over an elliptic curve E. Pulling it back to $\tilde{E} \cong \mathbb{C}$ we get the trivial bundle $\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}$.

Note that every algebraic morphism $\mathbb{C} \to \mathbb{C}^*$ is constant, hence every algebraic automorphism of $\mathbb{C} \times \mathbb{C}^*$ that commutes with the first projection is of the form $(x, y) \mapsto (x, cy)$. In particular, they preserve the flat structure. Thus if $Y \to E$ is a quotient of $\mathbb{C} \times \mathbb{C}^*$ by algebraic deck transformations then the flat structure on $\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}$ descends to a flat structure on $Y \to E$. In particular, $c_1(Y) = 0$ and $Y \to E$ is topologically trivial.

By contrast, the biholomorphisms $(x, y) \mapsto (x, e^{g(x)}y)$ of $\mathbb{C} \times \mathbb{C}^*$ commute with the first projection, and every $\mathbb{C}^*$-bundle over E is the quotient of $\mathbb{C} \times \mathbb{C}^*$ by holomorphic deck transformations.

To get compact examples out of these, let F be a projective variety such that the connected component of $\mathrm{Aut}(F)$ is $\left(\mathbb{C}^*\right)^m$ for some $m > 0$. (For instance, F can be the blow up of $\mathbb{P}^2$ at 3 non-collinear points.) By the above arguments, if $X \to E$ is a locally trivial F-bundle that is a quotient of $\mathbb{C} \times F$ by algebraic deck transformations, then $X \to E$ is topologically trivial after a finite cover $E' \to E$. (In fact, $X \to E$ is itself topologically trivial if $\mathrm{Aut}(F) = \left(\mathbb{C}^*\right)^m$.) On the other hand, F-bundles obtained from a line bundle with nonzero Chern class do not have this property.

4. Auxiliary results

Here we collect various theorems that were used during the proofs. The most important one is a consequence of the Kobayashi-Ochiai theorem which asserts that a meromorphic map from a quasi-projective variety to a variety of general type can not have essential singularities.

4.1. Theorem. [KO75, Thm.2] *Let Y be a projective variety of general type, V a complex manifold, and $B \subset V$ a proper closed analytic subset. Let $f : V \setminus B \dashrightarrow Y$ be a nondegenerate meromorphic map. (That is, such that the tangent map $T_{V \setminus B} \to$*

T_Y is surjective at at least one point $v \in V \setminus B$.) Then f extends to a meromorphic map $V \dashrightarrow Y$.

Since a fiber of a meromorphic map $V \dashrightarrow Y$ has only finitely many irreducible components, this immediately implies the following.

4.2. Corollary. *Let X be a quasi-projective variety and $f : X \dashrightarrow Y$ nondegenerate meromorphic map from X to a variety of general type. Let $F \subset X$ be an irreducible component of any fiber of f.*

Let $\tilde{X} \to X$ be an étale Galois cover with group Γ. If $\tilde{X}$ is Zariski open in a compact complex manifold then $\operatorname{im}[\pi_1(F) \to \pi_1(X) \to \Gamma]$ *has finite index in Γ.* □

In a special case, the above conclusion can be strengthened much further.

4.3. Theorem. [Kaw81, Thm.13] *Let A be an Abelian variety and $X \to A$ a finite morphism from a normal projective variety to A. If X does not map onto a variety of general type, then X is an Abelian variety. Otherwise a finite étale cover of X is a product of a variety of general type and an Abelian variety.* □

The splitting mentioned in the statement of Theorem 1.1 is a straightforward consequence of deep results of Grauert.

4.4. Theorem. *Let $f : X \to Y$ be a locally trivial proper fibration between complex manifolds. If the universal cover of Y is Stein and contractible, then the universal cover of X splits as a product:*

$$\tilde{X} \simeq \tilde{F} \times \tilde{Y}.$$

Proof. Since f is locally trivial and proper, it is a fiber bundle with fiber F and group $G = \operatorname{Aut}(F)$ (a complex Lie group). Consider the fiber product

$$\tilde{X}_f = X \times_Y \tilde{Y};$$

it is a connected cover of X which is also a fiber bundle over $\tilde{Y}$ (fiber F and group G). We can now apply [Gra58, Satz 6]: this fiber bundle has to be trivial and this gives a splitting

$$\tilde{X}_f \simeq F \times \tilde{Y}.$$

Since $\tilde{X}_f$ is an intermediate cover, $\tilde{X}$ has to split as well. □

If the base of a fiber bundle is not known to be contractible, one can still prove global topological triviality of certain fiber bundles.

4.5. Lemma. *Let V be a complex manifold and $\pi : U \to V$ a complex analytic fiber bundle with compact fiber F. Assume that the structure group G is compact (hence its connected component G° is a complex torus). Assume furthermore that there is a closed subspace $W \subset U$ such that $\pi : W \to V$ is generically finite.*

Then there is a finite étale cover $\sigma : V' \to V$ such that the pull-back $\sigma^* U \to V'$ is globally trivial as a C^∞-fiber bundle.

If G is finite then $\sigma^* U \to V'$ is complex analytically trivial.

Proof. We have a monodromy representation $\pi_1(V) \to G/G^\circ$. By passing to the cover of V corresponding to its kernel, we may assume that the structure group G is a complex torus. If $\dim G = 0$ then we have a trivial bundle.

In general, G is diffeomorphic to $\left(S^1\right)^{2d}$, thus C^∞-fiber bundles with structure group G are classified by

$$c_1(U/V) \in H^1\big(V, G\big) = H^1\big(V, S^1\big)^{2d} = H^2\big(V, \mathbb{Z}\big)^{2d}.$$

Let $Z \subset V$ denote the closed subspace over which $\pi : W \to V$ has positive dimensional fibers. Then Z has complex codimension ≥ 2, thus $H^2\big(V, \mathbb{Z}\big) = H^2\big(V \setminus Z, \mathbb{Z}\big)$. Therefore we can replace V by $V \setminus Z$ and assume that $\pi : W \to V$ is finite.

After base change to W, the fiber bundle has a section, thus its Chern class is trivial. This is equivalent to

$$c_1(U/V) \in \ker\Big[H^2\big(V, \mathbb{Z}\big) \xrightarrow{\pi^*} H^2\big(W, \mathbb{Z}\big)\Big]^{2d}.$$

With $\mathbb{Q}$-coefficients, the map π^* is an injection, thus $c_1(U/V)$ is torsion in $H^2\big(V, \mathbb{Z}\big)^{2d}$. The torsion in $H^2\big(V, \mathbb{Z}\big)$ comes from the torsion in $H_1\big(V, \mathbb{Z}\big)$, hence it is killed after a suitable finite étale cover of V. □

Examples where the assumptions of (4.5) hold are given by the following. (See [Uen75, §14] for a more modern exposition.)

4.6. Lemma. [Mat58, Mat63] *Let X be a normal, projective variety that is not birationally ruled. Let L be an ample line bundle and let $\mathrm{Aut}(X, L)$ denote the group of those automorphisms $\tau : X \to X$ such that $\tau^* L$ is numerically equivalent to L. Then the identity component $\mathrm{Aut}^\circ(X, L) \subset \mathrm{Aut}(X, L)$ is an Abelian variety and the quotient $\mathrm{Aut}(X, L)/\mathrm{Aut}^\circ(X, L)$ is finite.* □

Recall that for a normal projective variety X, the Albanese morphism is defined as the universal map from X to abelian varieties. The following result is a straightforward consequence of the analytic construction of the Albanese morphism for smooth projective varieties but still holds for normal ones.

4.7. Lemma. *Let X be a normal, projective variety and $\mathrm{alb} : X \to \mathrm{Alb}(X)$ the Albanese morphism. Then $\mathrm{alb}_* : H_1(X, \mathbb{Z}) \to H_1(\mathrm{Alb}(X), \mathbb{Z})$ is surjective with finite kernel.*

Proof. Take a resolution of singularities $g : Y \to X$. Set $A_Y := \mathrm{Alb}(Y)$.

Let $\tilde{X} \to X$ be the Galois cover corresponding to $H_1(X, \mathbb{Z})/(\text{torsion})$. It induces a Galois cover $\tilde{Y} \to Y$ whose Galois group is again $H_1(X, \mathbb{Z})/(\text{torsion})$. Thus there is a Galois cover $\tilde{A}_Y \to A_Y$ such that $\tilde{Y} = \tilde{A}_Y \times_{A_Y} Y$.

Let $F_x \subset Y$ be any fiber of g. By construction, $\tilde{Y} \to Y$ is trivial on F_x, hence $\tilde{A}_Y \to A_Y$ is trivial on $\mathrm{alb}_Y(F_x)$.

Let $B_Y \subset A_Y$ be the smallest Abelian subvariety such that every $\mathrm{alb}_Y(F_x)$ is contained in a translate of B_Y. Then $\tilde{A}_Y \to A_Y$ is trivial on B_Y, hence $\tilde{A}_Y$ is a pull back of the corresponding cover $\tilde{A}_X \to A_X := A_Y/B_Y$.

By construction, every fiber of g maps to a point in A_X, thus $Y \to A_Y$ descends to a morphism $X \to A_X$ and $\tilde{X} = \tilde{A}_X \times_{A_X} X$. Thus $A_X = \mathrm{Alb}(X)$ and we are done. □

4.8. Lemma. *Let $f : X \to Z$ be a proper, equidimensional morphism of normal complex spaces. Assume that X is quasi-projective. Then Z has a unique quasi-projective structure such that f is an algebraic morphism.*

Proof. The map f determines a natural morphism $Z \to \mathrm{Chow}(X)$ which maps Z biholomorphically onto a connected component of $\mathrm{Chow}(X)$. We thus need to identify Z with its image. □

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Benoît Claudon, Institut Élie Cartan Nancy, Université Henri Poincaré Nancy 1, B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France
E-mail address: Benoit.Claudon@iecn.u-nancy.fr

Andreas Höring, Université Pierre et Marie Curie and Albert-Ludwig Universität Freiburg
Current address: Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstraße 1, 79104 Freiburg im Breisgau, Germany
E-mail address: hoering@math.jussieu.fr

János Kollár, Department of Mathematics, Fine Hall, Washington Road, Princeton University, Princeton, NJ 08544-1000, USA
E-mail address: kollar@math.princeton.edu